Chapter 15

Queuing Systems

1

(a) Efficiency $= 100 - 29 = 71\%$

(b) For average waiting time ≤ 3 minutes, at least 5 cashiers are needed

For efficiency $\geq 90\%$, the associated idleness percentage is $\leq 10\%$. The corresponding number of cashiers is at most 2.

Conclusion:

The two conditions cannot be satisfied simultaneously. At least one of the two conditions must be relaxed.

2

$C_A = \$18$ per hour

$C_B = \$25$ per hour

Length of queue $A = 4$ jobs

Length of queue $B = .7 \times 4 = 2.8$ jobs

Cost of $A = \$18 + 4 \times \$10 = \$58$ per hour

Cost of $B = \$25 + 2.8 \times \$10 = \$53$ per hour

Decision:

Select Model B.

1

Situation	Customer	Server
a	Plane	Runway
b	Passenger	Taxi
c	Machinist	Clerk at tool crib
d	Letter	Clerk
e	Student	Registrar's office
f	Cases	Judge
g	Shopper	Cashier
h	Car	Parking space

2

Situation	Calling Source	Customers arrival
a	∞	Individual
b	∞	Individual
c	∞	Individual
d	∞	Bulk
e	∞	Individual
f	∞	Individual
g	∞	Individual
h	∞	Individual

Situation	Interarrival time	Service time
a	Probabilistic	Time to clear runway
b	Probabilistic	Ride time
c	Probabilistic	Time to receive tool
d	Deterministic	Time to process letter
e	Probabilistic	Time to process registrn
f	Probabilistic	Trial time
g	Probabilistic	check-out time
h	Probabilistic	Parking time.

Situation	Queue Capacity	Queue Discipline
a	∞	FIFO
b	∞	FIFO
c	∞	FIFO
d	∞	Random
e	∞	FIFO
f	∞	FIFO
g	∞	FIFO
h	0	None

3

#	Queueing situation	Customers
1	Arrival of orders	Orders
2	Processing (single machine)	Rush orders
3	Processing (single machine)	Regular jobs
4	Processing (Prod. line)	Rush jobs
5	Processing (Prod. line)	Regular jobs
6	Receipt of completed jobs	Completed orders
7	Tool crib	Tools
8	Machine breakdown	machines

#	Servers	Discipline	Service time	Queue length	Source
1	Foreman	Priority	Sorting time	∞	∞
2	machine	FIFO	Prod. time	∞	∞
3	machine	FIFO	Prod. time	∞	∞
4	Prod. line	FIFO	Prod. time	∞	∞
5	Prod. line	FIFO	Prod. time	∞	∞
6	Shipping facilities	FIFO	Loading time	finite	∞
7	Tool crib	Priority	Exchange time	finite	finite
8	Repair persons	Priority	Repair time	finite	finite

4

(a) T. (b) T. (c) T.

5

(a) None.
(b) None.
(c) None.
(d) None.
(e) Jockey or balk
(f) None
(g) Jockey
(h) None

Set 15.3a

1

(a) Av. interarrival time (in time units)

$$= \frac{1}{\text{arrival rate } \lambda \text{ (in customers/unit time)}}$$

(b) Let $\bar{I}$ = av. interarrival time

(i) $\lambda = \frac{60}{10} = 6$ arrivals/hr

$\bar{I} = 10$ minutes $= \frac{1}{6}$ hour

(ii) $\lambda = \frac{60}{3} = 20$ arrivals/hr

$\bar{I} = \frac{6}{2} = 3$ minutes $= \frac{1}{20}$ hr

(iii) $\lambda = \frac{10}{30} \times 60 = 20$ arrivals/hr

$\bar{I} = \frac{30}{10} = 3$ minutes $= \frac{1}{20}$ hour

(iv) $\lambda = 1/.5 = 2$ arrivals/hour

$\bar{I} = .5$ hour

(c) Let $\bar{S}$ = av. service time

(i) $\mu = \frac{60}{12} = 5$ services/hour

$\bar{S} = 12$ minutes $= .2$ hour

(ii) $\mu = \frac{60}{7.5} = 8$ services/hr

$\bar{S} = 7.5$ min $= .125$ hr

(iii) $\mu = \frac{5}{30} \times 60 = 10$ services/hr

$\bar{S} = \frac{30}{5} = 6$ min $= 1/10$ hr

(iv) $\mu = \frac{1}{.3} = 3.33$ services/hr

$\bar{S} = .3$ hour

2

(a) $\lambda_{hour} = .2$ failures/hr

$\lambda_{week} = .2 \times 24 \times 7 = 33.6$ failures/wk

(b) P{at least one failure in 2 hours}

$= P\{\text{time betn. failures} \le 2\}$

$= P\{t \le 2\} = 1 - e^{-.2 \times 2} \cong .33$

(c) $P\{t > 3 \text{ hrs}\} = 1 - P\{t \le 3\} = e^{-.2 \times 3} \cong .55$

(d) $P\{t \le 1 \text{ hour}\} = 1 - e^{-.2 \times 1} = .18$

3

$\lambda = \frac{1}{.05} = 20$ arrivals/hr

(a) $f(t) = \lambda e^{-\lambda t}$

$= 20 e^{-20t}, \quad t > 0$

(b) $P\{t > \frac{15}{60}\} = P\{t > .25\}$

$= e^{-20 \times .25}$

$= .00674$

(c) $P\{t \le \frac{3}{60}\} = P\{t \le .05\}$

$= 1 - e^{-20 \times .05} = .632$

$P\{t > \frac{5}{60}\} = e^{-\frac{20 \times 5}{60}} = .189$

(d) $t = 45 - 10 = 35$ minutes

Av. # of arrivals in 35 min.

$= 20 \times \frac{35}{60} = 11.67$ arrivals

4

$\lambda = \frac{1}{6}$ arrivals/hr

$P\{t \geqslant 1\} = e^{-1/6 \times 1} = .846$

$P\{t \le .5\} = 1 - e^{-1/6 \times .5}$

$= 1 - e^{-1/12} = .08$

5

(a) $\lambda = \frac{60}{10} = 6$ arrivals/hr

(b) $P\{t \ge \frac{15}{60}\} = e^{-6 \times \frac{15}{60}} = .223$

(c) $P\{t \le \frac{20}{60}\} = 1 - e^{-6 \times \frac{20}{60}} = .865$

6

(a) $P\{t \le \frac{2}{60}\} = 1 - e^{-35(2/60)}$

$= .6886$

(b) $P\{\frac{2}{60} \le t \le \frac{3}{60}\}$

$= P\{t \le 3/60\} - P\{t \le \frac{2}{60}\}$

$= (1 - e^{-35 \times 3/60}) - (1 - e^{-35 \times 2/60})$

$= e^{-70/60} - e^{-105/60} = .1376$

(c) $P\{t \geqslant 3/60\} = e^{-35(3/60)}$

$= .1738$

7

$\lambda = \frac{60}{1.5} = 40$ arrivals/hr

Jim's Payoff	$-2¢$	$+2¢$
Prob.	$P\{t \geq 1\}$	$P\{t \leq 1\}$

$P\{t \geq 1\} = e^{-40(1/60)} = .5134$

$P\{t \leq 1\} = 1 - .5134 = .4866$

Jim's exp. payoff/arriving customer

$= -2 \times .5134 + 2 \times .4866$

$= -.0536$ cent

Jim's exp. payoff/8 hours

$= -.0536(8\lambda)$

$= -.0536 \times 8 \times 40$

$\cong -17.15$ cent

Conclusion: Jim will pay Ann an average of 17 cents every 8 hrs

8

Jim's payoff	2	0	-2
Probability	$P\{t \leq 1\}$	$P\{1 \leq t \leq 1.5\}$	$P\{t \geq 1.5\}$

$P\{t \leq 1\} = .4866$

$P\{t \geq 1.5\} = e^{-40(1.5/60)}$

$= .3679$

2	0	-2
.4866	.1455	.3679

Jim's expected payoff/8 hours

$= [2 \times .4866 + 0 \times .1455 - 2 \times .3679] \times 40 \times 8$

$\cong 76$ cents

9

$2¢$	$3¢$	$-5¢$	$-6¢$
$t \leq 1$	$1 \leq t \leq 1.5$	$1.5 \leq t \leq 2$	$t \geq 2$

$\lambda = 40$ arrivals/hr

$P\{t \leq 1\} = 1 - e^{-40/60} = .4866$

$P\{1 \leq t \leq 1.5\} = e^{-40(1/60)} - e^{-40(1.5/60)}$

$= .1455$

$P\{1.5 \leq t \leq 2\} = e^{-40(1.5/60)} - e^{-40(2/60)}$

$= .1043$

$P\{t \geq 2\} = e^{-40(2/60)} = .2636$

Jim's exp. payoff/8 hours

$= 8 \times 40 (2 \times .4866 + 3 \times .1455 - 5 \times .1043 - 6 \times .2636)$

$\simeq -222$ cents

Jim pays Ann an average of \$2.22/8 hours.

10

(a) $\lambda = \frac{60}{6} = 10$ customers/hr

$P\{t \leq 4 \text{ min}\} = 1 - e^{-10(4/60)}$

$= .4866$

(b)

$$\%\text{discount} = \begin{cases} 10\%, & \text{if } t \leq 4 \\ 6\%, & \text{if } 4 < t \leq 5 \\ 2\%, & \text{if } t > 5 \end{cases}$$

$P\{t \leq 4\} = .4866$

$P\{4 < t \leq 5\} = e^{-10(4/60)} - e^{-10(5/60)}$

$= .0788$

$P\{t > 5\} = e^{-10(5/60)} =$

$= .4346$

Expected % discount

$= 10 \times .4866 + 6 \times .0788 + 2 \times .4346$

$= 6.208\%$

11

$$\lambda = \frac{365 \times 24}{9000} = .973 \text{ failure/yr}$$

$$P\{t \le 1\} = 1 - e^{-.973 \times 1}$$
$$= .622$$

12

Lack-of-memory property applies.

(a) The waiting time for the green bus is exponential with mean 10 minutes:

$$f(t) = .1\, e^{-.1t}, \quad t \ge 0$$

(b) The waiting time for the red bus is exponential with mean 7 minutes:

$$f(t) = \frac{1}{7} e^{-t/7}, \quad t \ge 0$$

13

$$E\{t\} = \int_0^\infty t\, \lambda e^{-\lambda t} dt$$
$$= -\int_0^\infty t\, d e^{-\lambda t}$$
$$= -\left(t e^{-\lambda t} - \int_0^\infty e^{-\lambda t} dt\right)$$
$$= -\left(t e^{-\lambda t} - \frac{1}{\lambda} e^{-\lambda t}\right)\Big]_0^\infty$$
$$= 1/\lambda$$

$$E\{t^2\} = \lambda \int_0^\infty t^2 e^{-\lambda t} dt$$
$$= -\int_0^\infty t^2 d e^{-\lambda t}$$
$$= -\left[t^2 e^{-\lambda t} - \int_0^\infty 2t\, e^{-\lambda t} dt\right]$$

continued...

$$= -\left[t^2 e^{-\lambda t} - \frac{2}{\lambda}\int_0^\infty t\, \lambda e^{-\lambda t} dt\right]_0^\infty$$
$$= +\frac{2}{\lambda^2}$$

$$Var\{t\} = E\{t^2\} - E^2\{t\}$$
$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2$$
$$= \frac{1}{\lambda^2}$$

1

TORA input = (5, 0, 0, ∞, ∞)

$$p_{n\ge 5}(t=1hr) = 1 - [p_0(1)+\cdots+p_4(1)]$$
$$= 1 - e^{-5}\left(1+5+\frac{5^2}{2!}+\frac{5^3}{3!}+\frac{5^4}{4!}\right)$$
$$= 1 - .44049 = .55951$$

2

$\lambda = 1$ trip/month

(a) $\lambda t = 3$: TORA input = (3, 0, 0, ∞, ∞)

$$p_0(3) = \frac{(1\times 3)^0 e^{-1\times 3}}{0!} = .049787$$

(b) $\lambda t = 12$: TORA input = (12, 0, 0, ∞, ∞)

$$p_{n\le 8}(t=12) = p_0(12)+\cdots+p_8(12)$$
$$= \frac{12^0 e^{-12}}{0!} + \frac{12^1 e^{-12}}{1!} + \cdots + \frac{12^8 e^{-12}}{8!}$$
$$= .15503$$

(c) $p_0(1) = \frac{1^0 e^{-1}}{0!} = e^{-1} = .3679$

TORA input = (1, 0, 0, ∞, ∞)

3

$\lambda = 2$ arrivals/minute

(a) $\lambda t = 2\times 5 = 10$ arrivals

(b) $\lambda t = 2\times .5 = 1$

TORA input = (1, 0, 0, ∞, ∞)

$$p_0(t=.5) = e^{-2\times .5} = .3679$$

(c) $1 - p_0(t=.5) = 1 - .3679 = .6321$

(d) $\lambda t = 2\times 3 = 6$ arrivals

TORA input = (6, 0, 0, ∞, ∞)

$$p_0(t=3) = \frac{(2\times 3)^0 e^{-2\times 3}}{0!} = .00248$$

4

$\lambda = 1/5 = .2$ arrival/min

(a) $p_2(t=7) = \frac{(.2\times 7)^2 e^{-.2\times 7}}{2!} = .24167$

TORA input = (1.4, 0, 0, ∞, ∞)

(b) $p_1(t=5) = \frac{(.2\times 5)^1 e^{-.2\times 5}}{1!} = .36788$

5

$\lambda = 25$ books per day

(a) $\lambda t = 25\times 30 = 750$ books $= 7.5$ shelves

(b) 10 bookcases $= 10\times 5\times 100 = 5000$ books

$$p_{n>5000}(t=30) = 1 - [p_0(30)+\cdots+p_{5000}(30)] \cong 0$$

6

(a) $\lambda_{green} = .1$ stop/min, $\lambda_{red} = 1/7$ stop/min

$$\lambda_{combined} = .1 + \frac{1}{7} = .24286 \text{ stop/min}$$

$$p_2(5) = \frac{(.24286\times 5)^2 e^{-.24286\times 5}}{2!} = .219$$

The two buses could be 2 G, 2R or 1G and 1R.

(b) $P\{t\le 2\} = 1 - e^{-.243\times 2} = .3849$

7

$$E\{n|t\} = \sum_{n=1}^{\infty} n\frac{(\lambda t)^n e^{-\lambda t}}{n!}$$
$$= \lambda t e^{-\lambda t}\sum_{n=1}^{\infty}\frac{(\lambda t)^{n-1}}{(n-1)!}$$
$$= \lambda t e^{-\lambda t} e^{+\lambda t} = \lambda t$$

$$E\{n^2|t\} = \sum_{n=0}^{\infty} n^2\frac{(\lambda t)^n e^{-\lambda t}}{n!}$$
$$= \sum_{n=1}^{\infty} n^2\frac{(\lambda t)^n e^{-\lambda t}}{n!}$$
$$= \lambda t e^{-\lambda t}\sum_{n=1}^{\infty}\frac{n(\lambda t)^{n-1}}{(n-1)!}$$
$$= \lambda t e^{-\lambda t}\frac{\partial}{\partial\lambda t}\left(\lambda t\sum_{n=1}^{\infty}\frac{(\lambda t)^{n-1}}{(n-1)!}\right)$$
$$= \lambda t e^{-\lambda t}\frac{d}{\partial\lambda t}\left(\lambda t e^{\lambda t}\right)$$
$$= \lambda t e^{-\lambda t}\left(\lambda t e^{\lambda t} + e^{\lambda t}\right)$$
$$= (\lambda t)^2 + \lambda t$$

Thus,

$$var\{n|t\} = (\lambda t)^2 + \lambda t - (\lambda t)^2$$
$$= \lambda t$$

8

$$p_0'(t) = -\lambda p_0(t) \qquad (1)$$

$$p_n'(t) = -\lambda p_n(t) + \lambda p_{n-1}(t) \qquad (2)$$

From (1)

$$d p_0(t) = -\lambda p_0(t) dt$$

which yields

$$p_0(t) = A e^{-\lambda t}$$

Because $p_0(0) = 1 \Rightarrow A = 1$, $p_0(t) = e^{-\lambda t}$

For $n=1$:

$$p_1'(t) = -\lambda p_1(t) + \lambda p_0(t)$$

$$= -\lambda p_1(t) + \lambda e^{-\lambda t}$$

or

$$p_1'(t) + \lambda p_1(t) = \lambda e^{-\lambda t}$$

This yields the solution:

$$p_1(t) = e^{-\int \lambda dt} \left\{ \int \lambda e^{-\lambda t} e^{-\int \lambda dt} dt + C \right\}$$

$$= \lambda t e^{-\lambda t} + C$$

Because $p_1(0) = 0$, $C = 0$, and

$$p_1(t) = \frac{\lambda t e^{-\lambda t}}{1!}$$

<u>Induction proof</u>:

Given

$$p_i(t) = \frac{(\lambda t)^i e^{-\lambda t}}{i!}$$

then

$$p_{i+1}'(t) + \lambda p_{i+1}(t) = \lambda \frac{(\lambda t)^i e^{-\lambda t}}{i!}$$

The solution is

continued...

$$p_{i+1}(t) = e^{-\int \lambda dt} \left\{ \frac{\lambda (\lambda t)^i e^{-\lambda t} e^{\int \lambda dt}}{i!} dt + C \right\}$$

$$= \frac{e^{-\lambda t} (\lambda t)^{i+1}}{(i+1)!} + C$$

Because $p_{i+1}(0) = 0$, $C = 0$, and

$$p_{i+1}(t) = \frac{e^{-\lambda t} (\lambda t)^{i+1}}{(i+1)!}$$

2

$\mu = 3$ dozens/day, $N=18$

TORA input data $= (0, \mu t, 1, 18, 18)$

(a) $\mu = 3\times3 = 9$

$p_0(t=3) = .00532$ (from TORA)

(b) $\mu t = 3\times2 = 6$

$\sum_{n=0}^{18} n\, p_n(2) = 11.955$

(c) This part can be solved using Poisson or exponential distributions:

Poisson: $\mu t = 3\times1 = 3$

Probability $= p_0(1) + p_1(1) + \cdots + p_{17}(1)$

$= .9502$ (from TORA)

Exponential: mean $= 1/3$ day

P{purchasing at least one dozen in 1 day}

= P{time between purchases ≤ 1}

$= 1 - e^{-3\times1} = .9502$

(d) Exponential: $P\{t \le .5\} = 1 - e^{-3\times.5} = .7769$

Poisson: $p_0(.5) + p_1(.5) + \cdots + p_{17}(.5) = .7769$

(e) $p_0(1) = 0$ $(\mu t = 3\times1 = 3)$

3

$N = 40$, $\mu = 10$ calls/hr

TORA input $(0, \mu t, 1, 40, 40)$

(a) $p_{n>0}(t=4) = 1 - p_0(4)$

$= 1 - .521 = .479$

(b) $E\{n|t=4\} = \sum_{n=0}^{40} n\, p_n(4) \simeq 2.5$ blocks

$\cong 25$ tickets

4

$N=48$, $\mu = \frac{4\times10}{8} = 5$ cans/hr

$\mu t = 5\times4 = 20$ cans

$p_0(4) \cong .000005$ (from TORA)

$N=48$, $\mu t = 5\times8 = 40$, $p_0(8) = .11958$

5

$\mu = 1/1 = 1$ withdrawal/week

$N=5$, $\mu t = 4$

$p_0(4) = .37116$

6

$N = 80$ items, $\mu = 5$ items/day

(a) $\mu t = 5\times2 = 10$ items

$p_{70}(2) = .1251$

(b) $\mu t = 5\times4 = 20$ items

$p_0(4) = .00001$

(c) $\mu t = 5\times4 = 20$ items

$E\{n|4\text{ days}\} = \sum_{n=0}^{80} n\, p_n(4) \simeq 60$ items

Av. # of withdrawals $= 80-60$

$= 20$ items

7

$\mu = 1/1 = 1$ breakdown/day

$N = 10$, $\mu t = 1\times2 = 2$

From TORA, $p_0(2) = .00005$

8

(a) $N = 25$, $\mu = 3$/day

$t = 6$ days, $\mu t = 18$

Av. stock remaining after 6 days

$= E\{n|t=6\} = 7.11$

Av. order size $= 25 - 7.11$

$\simeq 18$ items

(b) $t = 4$, $\mu t = 3\times4 = 12$

$p_0(4) = .00069$

(c) $t = 6$, $\mu t = 3\times6 = 18$

$p_{n\le14}(6) = p_0(6) + \cdots + p_{14}(6) = .9696$

9

P{time betn. departures $> T$}

= P{no departures during T}

= P{N left after time T}

$= p_N(T)$

$P\{t > T\} = p_N(T) = \frac{(\mu T)^0 e^{-\mu T}}{0!}$

$= e^{-\mu T}$

10

$p_N'(t) = -\mu p_N(t)$ (1)

$p_n'(t) = -\mu p_n(t) + \mu p_{n+1}(t), \quad 0 < n < N$ (2)

From (1), we get

$$p_N(t) = c\, e^{-\mu t}$$

Given $p_N(0) = 1$, then $c = 1$ and

$$p_N(t) = e^{-\mu t}$$

Next, consider (2) for $n = N-1$

$$p_{N-1}'(t) = -\mu p_{N-1}(t) + \mu p_N(t)$$
$$= -\mu p_{N-1}(t) + \mu e^{-\mu t}$$

Thus,

$$p_{N-1}(t) = e^{-\int \mu t}\left\{\int \mu e^{-\mu t} e^{\int \mu dt} dt + C\right\}$$
$$= e^{-\mu t} \mu t + C$$

Because $p_{N-1}(0) = 0$, $C = 0$ and $p_{N-1}(t) = (\mu t)e^{-\mu t}$

<u>Induction proof:</u>

Given $p_{n+1}(t) = \dfrac{(\mu t)^{N-n-1} e^{-\mu t}}{(N-n-1)!}$, then

$$p_n'(t) = -\mu p_n(t) + \frac{\mu (\mu t)^{N-n-1} e^{-\mu t}}{(N-n-1)!}$$

Solution gives

$$p_n(t) = \frac{(\mu t)^{N-n} e^{-\mu t}}{(N-n)!}$$

1

(a) $P\{0 \text{ counter open}\} = p_0 = \frac{1}{55}$

$P\{1 \text{ counter open}\} = p_1 + p_2 + p_3$

$= \frac{1}{55}(2 + \quad + 8) = \frac{14}{55}$

$P\{2 \text{ counters open}\} = p_4 + p_5 + p_6$

$= \frac{1}{55}(8+8+8) = \frac{24}{55}$

$P\{3 \text{ counters open}\} = p_7 + p_8 + \cdots$

$= 1 - (p_0 + \cdots + p_6)$

$= 1 - (\frac{1}{55} + \frac{14}{55} + \frac{24}{55}) = \frac{16}{55}$

(b) Av. # busy counters

$= 0 \times \frac{1}{55} + 1 \times \frac{14}{55} + 2 \times \frac{24}{55} + 3 \times \frac{16}{55}$

$=$ 2 counters

(c) Av. # idle counters $= 3 - 2 = 1$

2

$\lambda = 1/5 = .2$ arrival /min

$= 12$ arrivals /hr

(a)

$$\mu_n = \begin{cases} 5 \text{ customers/hr}, & n = 0,1,2 \\ 10 \text{ customers/hr}, & n = 3,4 \\ 15 \text{ customers/hr}, & n = 5,6 \\ 20 \text{ customers/hr}, & n \geq 7 \end{cases}$$

$p_1 = \frac{12}{5} p_0 = 2.4 p_0$

$p_2 = (\frac{12}{5})^2 p_0 = 5.76 p_0$

$p_3 = (\frac{12}{5})^2 (\frac{12}{10}) p_0 = 6.912 p_0$

$p_4 = (\frac{12}{5})^2 (\frac{12}{10})^2 p_0 = 8.2944 p_0$

$p_5 = (\frac{12}{5})^2 * (\frac{12}{10})^2 (\frac{12}{15}) p_0 = 6.63552 p_0$

$p_6 = (\frac{12}{5})^2 (\frac{12}{10})^2 (\frac{12}{15})^2 p_0 = 5.308416 p_0$

$p_{n\geq 7} = (\frac{12}{5})^2 (\frac{12}{10})^2 (\frac{12}{15})^2 (\frac{12}{20})^{n-6} p_0 = 5.308416 (.6)^{n-6} p_0$

From $\sum_{n=0}^{\infty} p_n = 1$, we get $p_0 = .002587$

$p_1 = .05421$, $p_2 = .13010$, $p_3 = .15612$

$p_4 = .18735$, $p_5 = .14988$, $p_6 = .1199$

$p_{n\geq 7} = .1199 (.6)^{n-6}$

(b) $p_{n\geq 7} = 1 - (p_0 + p_1 + \cdots + p_6) = .8$

Continued...

(c) $P\{0 \text{ counter}\} \Rightarrow p_0 = .002587$

$P\{1 \text{ counter}\} = p_1 + p_2 = .18431$

$P\{2 \text{ counters}\} = p_3 + p_4 = .34347$

$P\{3 \text{ counters}\} = p_5 + p_6 = .26978$

$P\{4 \text{ counters}\} = p_7 + p_8 + \cdots = .199853$

Av. # idle counters

$= 4 - (1 \times .18431 + 2 \times .34347 + 3 \times .26978 + 4 \times .199853) \cong 1.52$

3

$$\mu_n = \begin{cases} 5n, & n = 1,2 \\ 15, & n = 3,4,\ldots \end{cases}$$

$p_1 = (\frac{10}{5}) p_0 = 2 p_0$

$p_2 = (\frac{10}{5})(\frac{10}{10}) p_0 = 2 p_0$

$p_{n\geq 3} = (\frac{10}{5})(\frac{10}{10})(\frac{10}{15})^{n-2} p_0 = 2 (\frac{2}{3})^{n-2} p_0$

Thus,

$p_0 + 2p_0 + 2p_0 + [2(\frac{2}{3}) + 2(\frac{2}{3})^2 + \cdots] p_0 = 1$

which gives $p_0 = .1111$

(a) Prob that 3 counters are in use

$= p_{n\geq 3} = 1 - (p_0 + p_1 + p_2)$

$= 1 - (.1111 + .2222 + .2222)$

$= .4445$

(b) $p_{n \leq 2} = p_0 + p_1 + p_2 = .5555$

4

$$\lambda_n = \begin{cases} 12 \text{ cars/hr}, & n = 0,1,\ldots,10 \\ 0 & n \geq 11 \end{cases}$$

$\mu_n = 60/6 = 10$ cars/hr

$p_n = (\frac{12}{10})^n p_0$, $n = 1,2,\ldots,10$

$= 0$, $n \geq 11$

$p_0 (1 + 1.2 + 1.2^2 + \cdots + 1.2^{10}) = p_0 \frac{1 - 1.2^{11}}{1 - 1.2}$

Thus, $p_0 = .0311$

Continued...

4 continued

(a) $p_{10} = \left(\frac{12}{10}\right)^{10} p_0 = .19259$

(b) $p_{n\geq 1} = 1 - p_0 = 1 - .0311 = .9689$

(c) Av. length of the line

$= 0p_0 + 1p_1 + \cdots + 10p_{10}$

$= 1\times.03732 + 2\times.04479 + 3\times.05375$
$+ 4\times.0645 + 5\times.0774 + 6\times.09288$
$+ 7\times.11145 + 8\times.13374 + 9\times.16049$
$+ 10\times.19259 = 6.71071$

5

$\lambda_n = 6$ arrivals/hr, $n = 0, 1, \ldots, 8$

$= 5$ arrivals/hr, $n = 9, 10, \cdots 11, 12$

$\mu_n = n/.5 = 2n$/hr, $n = 1,2,3,4$

$= 10$/hr, $n \geq 5$

n	p_n
0	0.049526
1	0.148578
2	0.222866
3	0.222866
4	0.16715
5	0.10029
6	0.060174
7	0.036104
8	0.021663
9	0.010831
10	0.005416
11	0.002708
12	0.001354

$p_1 = \frac{6}{2} p_0 = 3p_0$

$p_2 = \frac{6}{2}\cdot\frac{6}{4} p_0 = 4.5 p_0$

$p_3 = \frac{6}{2}\cdot\frac{6}{4}\cdot\frac{6}{6}\cdot p_0 = 4.5 p_0$

$p_4 = \frac{6}{2}\cdot\frac{6}{4}\cdot\frac{6}{6}\cdot\frac{6}{8} p_0 = 3.375 p_0$

$p_5 = \frac{6}{2}\cdot\frac{6}{4}\cdot\frac{6}{6}\cdot\frac{6}{8}\cdot\frac{6}{10} p_0 = 2.025 p_0$

$p_6 = \frac{6}{2}\cdot\frac{6}{4}\cdot\frac{6}{6}\cdot\frac{6}{8}\cdot\frac{6}{10}\cdot\frac{6}{10} p_0 = 1.215 p_0$

$p_7 = \frac{6}{2}\cdot\frac{6}{4}\cdot\frac{6}{6}\cdot\frac{6}{8}\cdot\frac{6}{10}\cdot\frac{6}{10}\cdot\frac{6}{10} p_0 = .729 p_0$

$p_8 = \frac{6}{2}\cdot\frac{6}{4}\cdot\frac{6}{6}\cdot\frac{6}{8}\left(\frac{6}{10}\right)^4 p_0 = .4374 p_0$

$p_{n\geq 9} = \frac{6}{2}\cdot\frac{6}{4}\cdot\frac{6}{6}\cdot\frac{6}{8}\left(\frac{6}{10}\right)^4\left(\frac{5}{10}\right)^{n-8} p_0 = .4374(.5)^{n-8} p_0$

From $\sum_{n=0}^{12} p_n = 1$, we get $p_0 = .0495$

(a) $p_{12} = .4374\times.5^4\times.0495 = .00135$

(b) $p_{n\geq 5} = 1 - (p_0 + p_1 + \cdots + p_4) = .2385$

(c) Av. # busy tables $= 0p_0 + 1p_1 + 2p_2 + 3p_3$
$+ 4p_4 + 5p_{n\geq 5} = 2.9768$

5 continued

(d) $1p_6 + 2p_7 + \cdots + 7p_{12}$

$= 1\times.0662 + 2\times.0361 + 3\times.0217$
$+ 4\times.0108 + 5\times.0054 +$
$6\times.0027 + 7\times.00135$

$= .2935$ pair

6

$\lambda = 4$ customers/hr

$$\lambda_n = \begin{cases} 4, & n = 0, 1, \ldots, 4 \\ 0, & n \geq 5 \end{cases}$$

$\mu_n = \frac{60}{15} = 4$ customers/hr

(a) $p_1 = \frac{4}{4} p_0$

$p_2 = \left(\frac{4}{4}\right)^2 p_0$

$p_3 = \left(\frac{4}{4}\right)^3 p_0$

$p_4 = \left(\frac{4}{4}\right)^4 p_0$

$p_0 + p_1 + \cdots + p_4 = 1 \implies p_0 = 1/5$

$p_0 = p_1 = p_2 = p_3 = p_4 = 1/5$

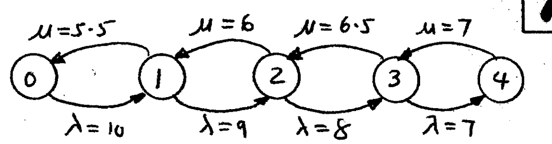
(b) expected # in shop $=$

$0p_0 + 1p_1 + 2p_2 + 3p_3 + 4p_4$

$= \frac{1}{5}(1+2+3+4) = 2$

(c) $p_4 = .2$

7



(a) $5.5p_1 = 10p_0$

$10p_0 + 6p_2 = (5.5+9)p_1$

$9p_1 + 6.5p_3 = (6+8)p_2$

$8p_2 + 7p_4 = (6.5+7)p_3$

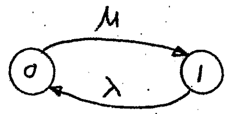
(b) $p_1 = 1.82 p_0$, $p_2 = 2.727 p_0$

$p_3 = 3.3566 p_0$, $p_4 = 3.3566 p_0$

$p_0 + p_1 + \cdots + p_4 = 1 \implies p_0 = .08882$

$p_1 = .1614$, $p_2 = .2422$, $p_3 = .2981$, $p_4 =.$

8



(a) $\mu p_1 = \lambda p_0$

$p_1 = \frac{\lambda}{\mu} p_0$

(b) $p_0 + \frac{\lambda}{\mu} p_0 = 1$

$p_0 = \frac{1}{1+\rho}$, $\rho = \lambda/\mu$

$p_1 = \frac{\rho}{1+\rho}$

(c) $L_s = 0p_0 + 1p_1 = \frac{\rho}{1+\rho}$

(d) $\lambda_{eff} = \lambda p_0 = \frac{\lambda}{1+\rho}$

(e) $W_q = \frac{L_s}{\lambda_{eff}} - \frac{1}{\mu}$

$= \frac{\rho/(1+\rho)}{\lambda/(1+\rho)} - \frac{1}{\mu} = 0$

9

$\lambda_{n-1} p_{n-1} + \mu_{n+1} p_{n+1} =$

$\lambda_{n-1}\left(\frac{\lambda_0}{\mu_1} \cdot \frac{\lambda_1}{\mu_2} \cdots \frac{\lambda_{n-2}}{\mu_{n-1}}\right) +$

$\mu_{n+1}\left(\frac{\lambda_0}{\mu_1} \cdot \frac{\lambda_1}{\mu_2} \cdots \frac{\lambda_n}{\mu_{n+1}}\right)$

$= \mu_n\left(\frac{\lambda_0}{\mu_1} \frac{\lambda_1}{\mu_2} \cdots \frac{\lambda_{n-1}}{\mu_n}\right) +$

$\lambda_n\left(\frac{\lambda_0}{\mu_1} \frac{\lambda_1}{\mu_2} \cdots \frac{\lambda_{n-1}}{\mu_n}\right)$

$= \mu_n p_n + \lambda_n p_n$

$= (\mu_n + \lambda_n) p_n$

Set 15.6a

1

(a) $L_q = \sum_{n=6}^{8}(n-5)p_n$

$= 1p_6 + 2p_7 + 3p_8$

$= 1\times.05847 + 2\times.03508 + 3\times.02105$

$= .19177$

(b) $W_q = \dfrac{L_q}{\lambda_{eff}}$

$= \dfrac{.1917}{5.8737} = .03265$ hour

$W_s = W_q + \dfrac{1}{\mu}$

$= .03264 + \dfrac{1}{2} = .53265$ hour

(c) $\lambda_{lost} = \lambda p_8$

$= 6\times.02105 = .1263$ car/hr

Number lost/8 hrs $= .1263\times8 = 1.01$ cars

(d) Average number of empty spaces

$= c - (L_s - L_q)$

$= c - \sum_{n=0}^{8} np_n + \sum_{n=c+1}^{8}(n-c)p_n$

$= \left(c\sum_{n=0}^{8} p_n - c\sum_{n=c+1}^{8} p_n\right) - \left(\sum_{n=0}^{8} np_n - \sum_{n=c+1}^{8} np_n\right)$

$= c\sum_{n=0}^{c} p_n - \sum_{n=0}^{c} np_n$

$= \sum_{n=0}^{c-1}(c-n)p_n$

2

(a)

$\lambda_n = 6$ cars/hr, $n = 0, 1, \ldots, 6$

$\mu_n = \begin{cases} \left(\frac{4}{3}\right)n, & n = 1, 2, \ldots, 6 \\ 8, & n = 7, 8, 9, 10 \end{cases}$

$p_n = \left(\frac{6}{4/3}\right)^n \frac{1}{n!} p_0, \; n = 0, 1, \ldots, 6$

continued...

2 continued

$p_n = \dfrac{\left(\frac{6}{4/3}\right)^n}{6!\,6^{n-6}} p_0, \; n = 7, 8, 9, 10$

$p_0\left(1 + \frac{9/2}{1!} + \frac{(9/2)^2}{2!} + \frac{(9/2)^3}{3!} + \frac{(9/2)^4}{4!} + \frac{(9/2)^5}{5!} + \frac{(9/2)^6}{6!} + \frac{(9/2)^7}{6!\,6} + \frac{(9/2)^8}{6!\,6^2} + \frac{(9/2)^9}{6!\,6^3} + \frac{(9/2)^{10}}{6!\,6^4}\right) = 1$

Thus, $p_0 = .0004$

n	p_n	n	p_n
1	.00304	6	.10027
2	.01141	7	.12534
3	.02852	8	.15667
4	.05348	9	.19584
5	.08022	10	.24480

(b) $\lambda_{eff} = \lambda(1 - p_{10}) = 10(1 - .2448)$

$= 7.552$ cars/hr

(c) $L_s = 0p_0 + 1p_1 + 2p_2 + \cdots + 10p_{10}$

$= 7.6941$ cars

(d) $W_s = \dfrac{L_s}{\lambda_{eff}} = \dfrac{7.6941}{7.552} = 1.0155$ cars

$W_q = 1.0155 - \dfrac{1}{4/3} = .2655$

(e) $L_q = \lambda_{eff} W_q$

$= .2655\times7.552$

$= 2.005$ cars

Average number of occupied spaces $= L_s - L_q$

$= 7.6941 - 2.005$

$= 5.6891$ spaces

1

(a) % utilization $= 100(1-p_0)$

$$= 100\,\frac{\lambda}{\mu}$$

$$= 100\left(\frac{4}{6}\right) = 66.67\%$$

(b) $p_{n\geq 1} = 1 - p_0 = \frac{\lambda}{\mu} = \frac{4}{6} = .6667$

(c) $p_{n\leq 7} = p_0 + p_1 + \cdots + p_7$

$$= 1 - \left(\frac{\lambda}{\mu}\right)^8 = 1 - \left(\frac{4}{6}\right)^8 = .961$$

(d) $p_0 + p_1 + \cdots + p_K \geq .99$

From Figure 17-6, $K = 11$

Also, we can determine K from

$$1 - \rho^{K+1} \geq .99$$

$$(K+1) \geq \frac{\ln .01}{\ln(4/6)} = 11.$$

or $K \geq 11.350 - 1 = 10.358$

Thus, $K \geq 11$

Note that the desired number of parking spaces is almost doubled (from 5 to 11) to accommodate the increase in the acceptance percentage from 90% to 99%.

2

$\lambda = 1/5 = .2$ job/day

$\mu = 1/4 = .25$ job/day

From the TORA output on the next column,

(a) $p_0 = .2$

(b) Av. income/month $= \$50\,\mu t$

$$= 50 \times .25 \times 30$$

$$= \$375$$

(c) Av. number of jobs awaiting completion $= L_q = 3.2$ jobs

cost $= 3.2 \times \$40 = \$128$

Continued...

Lambda =	0.20000	Mu =	0.25000
Lambda eff =	0.20000	Rho/c =	0.80000
Ls =	4.00000	Lq =	3.20000
Ws =	20.00000	Wq =	16.00000

n	Probability, pn	Cumulative, Pn	n	Probability, pn	Cumulative, Pn
0	0.20000	0.20000	23	0.00118	0.99528
1	0.16000	0.36000	24	0.00094	0.99622
2	0.12800	0.48800	25	0.00076	0.99698
3	0.10240	0.59040	26	0.00060	0.99758
4	0.08192	0.67232	27	0.00048	0.99807
5	0.06554	0.73786	28	0.00039	0.99845
6	0.05243	0.79028	29	0.00031	0.99876
7	0.04194	0.83223	30	0.00025	0.99901
8	0.03355	0.86578	31	0.00020	0.99921
9	0.02684	0.89263	32	0.00016	0.99937
10	0.02147	0.91410	33	0.00013	0.99949
11	0.01718	0.93128	34	0.00010	0.99959
12	0.01374	0.94502	35	0.00008	0.99968
13	0.01100	0.95602	36	0.00006	0.99974
14	0.00880	0.96482	37	0.00005	0.99979
15	0.00704	0.97185	38	0.00004	0.99983
16	0.00563	0.97748	39	0.00003	0.99987
17	0.00450	0.98199	40	0.00003	0.99989
18	0.00360	0.98559	41	0.00002	0.99991
19	0.00288	0.98847	42	0.00002	0.99993
20	0.00231	0.99078	43	0.00001	0.99995
21	0.00184	0.99262	44	0.00001	0.99996
22	0.00148	0.99410			

3

$\lambda = 1/4 = .25$ case/wk

$\mu = 1/1.5 = .66667$ case/wk

M/M/c/GD/N/K Queueing Model

Input Data			
λ =	0.25		0.66667
c =	1		
Sys. Lim., N=	infinity	Source limit, K =	infinity
Output Results			
λ_{eff} =	0.2500		0.3750
Ls =	0.6000	Lq =	0.2250
Ws =	2.4000	Wq =	0.9000

n	Pn	CPn	1-CPn
0	0.625002	0.625002	0.374998
1	0.234375	0.859376	0.140624
2	0.087890	0.947266	0.052734
3	0.032959	0.980225	0.019775
4	0.012359	0.992584	0.007416
5	0.004635	0.997219	0.002781
6	0.001738	0.998957	0.001043
7	0.000652	0.999609	0.000391
8	0.000244	0.999853	0.000147
9	0.000092	0.999945	0.000055
10	0.000034	0.999979	0.000021
11	0.000013	0.999992	0.000008
12	0.000005	0.999997	0.000003
13	0.000002	0.999999	0.000001
14	0.000001	1.000000	0.000000

(a) $L_q = .225$ case

(b) $1 - p_0 = 1 - .625 = .375$ or 37.5%

(c) $W_s = 2.4$ weeks

4

Present situation:

$\lambda = 90$ cars/hr

$\mu = \frac{3600}{38} = 94.7368$ cars/hr

New situation:

$\lambda = 90$ cars per hour

$\mu = \frac{3600}{30} = 120$ cars per hour

Continued...

Comparative Analysis

Scenario	c	Lambda	Mu	L'da eff	p0	Ls	Lq	Ws	Wq
1	1	90.00000	94.73680	90.00000	0.05000	19.00017	18.05017	0.21111	0.20056
2	1	90.00000	120.00000	90.00000	0.25000	3.00000	2.25000	0.03333	0.02500

L_s(present) = 19 cars

% of idle time (new) = p_0(new) × 100

= 100 × .25 = 25 %

The device can be justified based on the number of waiting customers, L_s, in the present system, but not on the basis of % idle time in the new one.

5

Scenario 1-- (M/M/1):(GD/infinity/infinity)

Lambda = 0.40000 Mu = 0.66667
Lambda eff = 0.40000 Rho/c = 0.60000

Ls = 1.49998 Lq = 0.89998
Ws = 3.74995 Wq = 2.24996

n	Probability, pn	Cumulative, Pn	n	Probability, pn	Cumulative, Pn
0	0.40000	0.40000	11	0.00145	0.99782
1	0.24000	0.64000	12	0.00087	0.99869
2	0.14400	0.78400	13	0.00052	0.99922
3	0.08640	0.87040	14	0.00031	0.99953
4	0.05184	0.92224	15	0.00019	0.99972
5	0.03110	0.95335	16	0.00011	0.99983
6	0.01866	0.97201	17	0.00007	0.99990
7	0.01120	0.98320	18	0.00004	0.99994
8	0.00672	0.98992	19	0.00002	0.99996
9	0.00403	0.99395	20	0.00001	0.99998
10	0.00242	0.99637			

(a) $p_0 = .4$

(b) $L_q = .9$ car

(c) $W_q = 2.25$ minutes

(d) $p_{n \geq 11} = 1 - CP_{10} = 1 - .99637 = .0036$

6

Scenario 1-- (M/M/1):(GD/infinity/infinity)

Lambda = 10.00000 Mu = 12.00000
Lambda eff = 10.00000 Rho/c = 0.83333

Ls = 5.00000 Lq = 4.16667
Ws = 0.50000 Wq = 0.41667

n	Probability, pn	Cumulative, Pn	n	Probability, pn	Cumulative, Pn
0	0.16667	0.16667	27	0.00121	0.99393
1	0.13889	0.30556	28	0.00101	0.99494
2	0.11574	0.42130	29	0.00084	0.99579
3	0.09645	0.51775	30	0.00070	0.99649
4	0.08038	0.59812	31	0.00059	0.99707
5	0.06698	0.66510	32	0.00049	0.99756
6	0.05582	0.72092	33	0.00041	0.99797
7	0.04651	0.76743	34	0.00034	0.99831
8	0.03876	0.80619	35	0.00028	0.99859
9	0.03230	0.83849	36	0.00024	0.99882
10	0.02692	0.86541	37	0.00020	0.99902
11	0.02243	0.88784	38	0.00016	0.99918
12	0.01869	0.90654	39	0.00014	0.99932
13	0.01558	0.92211	40	0.00011	0.99943
14	0.01298	0.93509	41	0.00009	0.99953
15	0.01082	0.94591	42	0.00008	0.99961
16	0.00901	0.95493	43	0.00007	0.99967
17	0.00751	0.96244	44	0.00005	0.99973
18	0.00626	0.96870	45	0.00005	0.99977
19	0.00522	0.97392	46	0.00004	0.99981
20	0.00435	0.97826	47	0.00003	0.99984
21	0.00362	0.98189	48	0.00003	0.99987
22	0.00302	0.98491	49	0.00002	0.99989
23	0.00252	0.98742	50	0.00002	0.99991
24	0.00210	0.98952	51	0.00002	0.99992
25	0.00175	0.99126	52	0.00001	0.99994
26	0.00146	0.99272	53	0.00001	0.99995

(a) $p_0 + p_1 + p_2 = .4213$

continued...

(b) $1 - CP_2 = 1 - .4213 = .5787$

(c) $W_q = .417$ hour

(d) Let N = spaces (including car being served)

$CP_{N-1} \geq .9$

Because $CP_{11} = .88784$ and $CP_{12} = .90654$,

$N - 1 \geq 12 \Rightarrow N \geq 13.$

7

In general, $L_s < L_q + 1$. The reason is that $p_0 > 0$, usually. Consider

$$L_q = \sum_{n=1}^{\infty}(n-1)p_n$$

$$= \sum_{n=1}^{\infty} n p_n - \sum_{n=1}^{\infty} p_n$$

$$= L_s - (1 - p_0)$$

The closer p_0 is to zero, the more likely $L_s \cong L_q + 1$ will hold.

8

Consider

$$L_q = \sum_{n=1}^{\infty}(n-1)p_n$$

$$= \sum_{n=1}^{\infty}(n-1)(1-\rho)\rho^n$$

$$= (1-\rho)\rho^2 \frac{d}{d\rho}\left(\sum_{n=1}^{\infty}\rho^{n-1}\right)$$

$$= (1-\rho)\rho^2 \frac{d}{d\rho}\sum_{n=0}^{\infty}\rho^n$$

$$= (1-\rho)\rho^2 \frac{d}{d\rho}\left(\frac{1}{1-\rho}\right)$$

$$= \rho^2(1-\rho)\frac{1}{(1-\rho)^2}$$

$$= \frac{\rho^2}{1-\rho}$$

9

(a) $P\{j \text{ in queue} \mid j \geq 1\}$

$$= p\{n \text{ in system} \mid n \geq 2\}$$

$$= \frac{p_n}{\sum_{j=2}^{\infty} p_j}$$

Thus,

$$\text{expected number} = \sum_{n=2}^{\infty}(n-1)\frac{p_n}{\sum_{j=2}^{\infty} p_j}$$

$$= \frac{\sum_{n=2}^{\infty} np_n - \sum_{n=2}^{\infty} p_n}{\sum_{n=2}^{\infty} p_n}$$

$$= \frac{\sum_{n=1}^{\infty} np_n - p_1}{\sum_{n=2}^{\infty} p_n} - 1$$

$$= \frac{\frac{\rho}{1-\rho} - \rho(1-\rho)}{1-[(1-\rho)+\rho(1-\rho)]} - 1$$

$$= \frac{1}{1-\rho}$$

(b) Exp. number in queue given the system is not empty

$$= \sum_{n=1}^{\infty}(n-1)\left(\frac{p_n}{\sum_{j=1}^{\infty} p_j}\right)$$

$$= \frac{\sum_{n=1}^{\infty} np_n - \sum_{n=1}^{\infty} p_n}{\sum_{j=1}^{\infty} p_j}$$

$$= \frac{\left(\frac{\rho}{1-\rho}\right) - \rho}{\rho}$$

$$= \frac{\rho}{1-\rho}$$

Continued...

Thus,

Exp. waiting time in queue for those who must wait

$$= \frac{\rho/(1-\rho)}{\lambda}$$

$$= \frac{1}{\mu-\lambda}$$

1

$$w(\tau)=(\mu-\lambda)e^{-(\mu-\lambda)\tau},\ \tau>0$$

$$\left.\begin{aligned}\lambda &= 1/4 = .25/\text{wk}\\ \mu &= 1/1.5 = .667\ /\text{wk}\end{aligned}\right\}(\mu-\lambda)=.417$$

$$\rho=\lambda/\mu=\frac{1.5}{4}=.375$$

$$w(\tau)=.417e^{-.417\tau},\ \tau>0$$

$$P\{\tau>1\}=e^{-.417\times 1}=.659$$

2

(a) Standard deviation $=\frac{1}{\mu-\lambda}=\frac{1}{6-4}=.5$

(b) $w(\tau)=(\mu-\lambda)e^{-(\mu-\lambda)\tau},\ \tau>0$

$$P\left\{\frac{1}{2(\mu-\lambda)}\le\tau\le\frac{3}{2(\mu-\lambda)}\right\}$$
$$=(1-e^{-1.5})-(1-e^{-.5})$$
$$=e^{-.5}-e^{-1.5}$$
$$=.3834$$

3

$W_s\le 10$ minutes, $\lambda=4/\text{hr}$

$$\frac{1}{(\mu-\lambda)}\le\frac{10}{60}\ \text{hr}$$

or $\mu-\lambda\ge 6$

or $\mu\ge 6+\lambda=10/\text{hr}$

4

$P\{\tau>\frac{10}{60}\}\le .1$, or

$$e^{-\frac{1}{6}(\mu-4)}\le .1$$

$$\mu-4\ge 13.8$$

$$\mu\ge 17.8/\text{hr}$$

5

$P\{\tau>5\}=e^{-(\mu-\lambda)t}=e^{-.267\times 5}=.2636$

where $\lambda=.4/\text{min}$, $\mu=.667/\text{min}$

Exp. # customers in a 12-hr day

$=\lambda\times 12\times 60=.4\times 12\times 60=288$ cust.

Exp. cost $=288\times .2636\times .5=\$37.95$

6

Let

$w_{n+1}(t|n)=$ conditional pdf for waiting in queue given there are n customers ahead

$=$ n-fold convolution of the exponential pdf

$$=\frac{\mu(\mu t)^{n-1}e^{-\mu t}}{(n-1)!}$$

$w(t)=$ absolute pdf of waiting time in queue

$g(t,n)=$ joint pdf of t and n

$$=w_{n+1}(t|n)\,p_n$$
$$=\frac{\mu(\mu t)^{n-1}e^{-\mu t}}{(n-1)!}\rho^n(1-\rho)$$

(a) For $t>0$

$$w(t)=\sum_{n=1}^{\infty}g(t,n)$$
$$=\frac{\mu\rho e^{-\mu t}(1-\rho)}{e^{-\mu\rho t}}\sum_{n=1}^{\infty}\frac{(\mu\rho t)^{n-1}e^{-\mu\rho t}}{(n-1)!}$$
$$=\mu\rho(1-\rho)e^{-\mu(1-\rho)t},\ t>0$$

For $t=0$, $w(0)=p_0=(1-\rho)$

$$w(t)=\begin{cases}1-\rho, & t=0\\ \mu\rho(1-\rho)e^{-\mu(1-\rho)t}, & t>0\end{cases}$$

(b) $W_q=E\{t\}$

$$=\int_0^{\infty}t\,w(t)\,dt$$
$$=0\,w(0)+\int_{0^+}^{\infty}t\,w(t)\,dt$$
$$=\int_{0^+}^{\infty}\mu\rho t(1-\rho)e^{-\mu(1-\rho)t}dt$$
$$=\frac{\rho}{\mu(1-\rho)}$$

1

(a) $p_0 = .3654$

(b) $W_q = .207$ hour

(c) Average number of empty spaces $= 4 - L_q$

$= 4 - .788$

$= 3.212$ spaces

(d) $p_5 = .04812$

(e) $W_s \leq 10$ minutes

Title: 17.6d-1
Comparative Analysis

Scenario	c	Lambda	Mu	L'da eff	p0	Ls	Lq	Ws	Wq
1	1	4.00000	6.00000	3.80752	0.36541	1.42256	0.78797	0.37362	0.20695
2	1	4.00000	7.00000	3.89179	0.44403	1.11691	0.56094	0.28699	0.14413
3	1	4.00000	8.00000	3.93651	0.50794	0.90476	0.41270	0.22984	0.10484
4	1	4.00000	9.00000	3.96116	0.55987	0.75340	0.31327	0.19020	0.07908
5	1	4.00000	10.00000	3.97532	0.60247	0.64199	0.24446	0.16149	0.06149

μ (cars/hr)	W_s (hrs)	W_s (min)
6	.3736	22.4
7	.287	17.16
8	.23	13.80
9	.19	11.40
10	.16	9.60

Desired service rate = 10 cars/hr
Thus, the service time must be reduced from $\frac{60}{6} = 10$ minutes to $\frac{60}{10} = 6$ minutes, a 40% reduction

2

m = number of parking spaces
An arriving car will not find a space if there are $m+1$ cars in the system. Thus, find m such that $p_{m+1} \leq .01$
TORA input = $(4, 6, 1, m+1, \infty)$

m	$N = m+1$	p_N
4	5	.04812
5	6	.0311
6	7	.0203
7	8	.01335
8	9	.009

Select the number of parking spaces $m \geq 8$

Continued...

3

m = number of seats.
The $N = m+1$, and

$\lambda_{eff} = \lambda p_N = 5 p_N$ customers/hr

TORA input = $(6, 5, 1, N, \infty)$

Title: 17.6d-3
Comparative Analysis

Scenario	c	Lambda	Mu	L'da eff	p0	Ls	Lq	Ws	Wq
1	1	6.00000	5.00000	3.62637	0.27473	1.12088	0.39560	0.30909	0.10909
2	1	6.00000	5.00000	4.06855	0.18629	1.72578	0.91207	0.42418	0.22418
3	1	6.00000	5.00000	4.32810	0.13438	2.35949	1.49387	0.54516	0.34516
4	1	6.00000	5.00000	4.49647	0.10071	3.02117	2.12188	0.67190	0.47190
5	1	6.00000	5.00000	4.61288	0.07742	3.70984	2.78726	0.80423	0.60423

m	$N = m+1$	λ_{eff} (customers/hr)
1	2	3.63
2	3	4.07

Use two seats or less

4

$\lambda = 10$ generators per hour
$\mu = \frac{60}{15} = 4$ generators per hour
$N = 7 + 1 = 8$

Title: 17.6d-4
Scenario 1-- (M/M/1):(GD/8/Infinity)

Lambda = 10.00000 Mu = 4.00000
Lambda eff = 3.99843 Rho/c = 2.50000

Ls = 7.33569 Lq = 6.33609
Ws = 1.83464 Wq = 1.58464

n	Probability, pn	Cumulative, Pn	n	Probability, pn	Cumulative, Pn
0	0.00039	0.00039	5	0.03841	0.06375
1	0.00098	0.00138	6	0.09603	0.15978
2	0.00246	0.00383	7	0.24006	0.39984
3	0.00615	0.00998	8	0.60016	1.00000
4	0.01536	0.02534			

(a) $p_8 \cong .6$

(b) $L_q = 6.34$ generators

(c) Let C = belt capacity. Thus, $N = C + 1$. The assembly department is kept in operation so long as at least one empty space remains on the belt; that is,

$$P\{\text{empty space on belt}\} = p_0 + p_1 + \cdots + p_C$$

$$= \frac{1-\rho}{1-\rho^{C+2}} \sum_{n=0}^{C} \rho^n$$

$$= \frac{1-\rho}{1-\rho^{C+2}} \cdot \frac{1-\rho^{C+1}}{1-\rho}$$

$$= \frac{1-\rho^{C+1}}{1-\rho^{C+2}}$$

Continued...

$$\lim_{c\to\infty} \frac{1-\rho^{c+1}}{1-\rho^{c+2}} = \lim_{c\to\infty} \frac{-(c+1)\rho^{c}}{-(c+2)\rho^{c+1}}$$

$$= \lim_{c\to\infty} \frac{c+1}{(c+2)\rho}$$

$$= \lim_{c\to\infty} \left(\frac{1+1/c}{1+2/c}\right)\frac{1}{\rho}$$

$$= \frac{1}{\rho}$$

In the present example, $\rho = 10/4$ and $1/\rho = .4$. Thus,

$$\lim_{c\to\infty} (p_0+p_1+\cdots+p_c) = 1/\rho = .4$$

This result means that regardless of how large the belt is, the probability of finding an empty space cannot exceed .4. Thus, achieving a 95% utilization for the assembly dept. is impossible.

The result makes sense because the arrival rate λ (=10/hr) is $2\frac{1}{2}$ times larger than the service rate (= 4). The only way we can accomplish the desired result is to reduce λ and/or increase μ.

5

(a) $p_{50} \simeq .00002$

(b) $P\{\text{wish is not fulfilled}\}$

$= P\{\text{48 or more in resturant}\}$

$= p_{48}+p_{49}+p_{50}$

$= 1-(p_0+p_1+\cdots+p_{47})$

$= 1-.99993$

$= .00007$

continued...

Title: 17.6d-5
Scenario 1-- (M/M/1):(GD/50/infinity)

TORA input = $(10, 12, 1, 50, \infty)$

Lambda = 10.00000 Mu = 12.00000
Lambda eff = 9.99982 Rho/c = 0.83333

Ls = 4.99533 Lq = 4.16201
Ws = 0.49954 Wq = 0.41621

n	Probability, pn	Cumulative, Pn	n	Probability, pn	Cumulative, Pn
0	0.16668	0.16668	26	0.00146	0.99281
1	0.13890	0.30558	27	0.00121	0.99402
2	0.11575	0.42133	28	0.00101	0.99504
3	0.09646	0.51779	29	0.00084	0.99588
4	0.08038	0.59818	30	0.00070	0.99658
5	0.06699	0.66516	31	0.00059	0.99717
6	0.05582	0.72098	32	0.00049	0.99765
7	0.04652	0.76750	33	0.00041	0.99806
8	0.03876	0.80627	34	0.00034	0.99840
9	0.03230	0.83857	35	0.00028	0.99868
10	0.02692	0.86549	36	0.00024	0.99892
11	0.02243	0.88792	37	0.00020	0.99911
12	0.01869	0.90662	38	0.00016	0.99928
13	0.01558	0.92220	39	0.00014	0.99941
14	0.01298	0.93518	40	0.00011	0.99952
15	0.01082	0.94600	41	0.00009	0.99962
16	0.00902	0.95501	42	0.00008	0.99970
17	0.00751	0.96253	43	0.00007	0.99976
18	0.00626	0.96879	44	0.00005	0.99982
19	0.00522	0.97401	45	0.00005	0.99986
20	0.00435	0.97835	46	0.00004	0.99990
21	0.00362	0.98198	47	0.00003	0.99993
22	0.00302	0.98500	48	0.00003	0.99996
23	0.00252	0.98751	49	0.00002	0.99998
24	0.00210	0.98961	50	0.00002	1.00000
25	0.00175	0.99136			

6

TORA input = $(20, 7.5, 1, 15, \infty)$

Title: 17.6d-6
Scenario 1-- (M/M/1):(GD/15/infinity)

Lambda = 20.00000 Mu = 7.50000
Lambda eff = 7.50000 Rho/c = 2.66667

Ls = 14.40000 Lq = 13.40000
Ws = 1.92000 Wq = 1.78667

n	Probability, pn	Cumulative, Pn	n	Probability, pn	Cumulative, Pn
0	0.00000	0.00000	8	0.00065	0.00104
1	0.00000	0.00000	9	0.00174	0.00278
2	0.00000	0.00000	10	0.00463	0.00742
3	0.00000	0.00001	11	0.01236	0.01978
4	0.00001	0.00002	12	0.03296	0.05273
5	0.00003	0.00005	13	0.08789	0.14062
6	0.00009	0.00015	14	0.23438	0.37500
7	0.00024	0.00039	15	0.62500	1.00000

(a) $p_0 \cong 0.$

(b) $p_{n\le 14} = p_0+\cdots+p_{14} = .375$

(c) $W_s = 1.92$ hours

7

(a) $p_{n\le 4} = p_0+p_1+\cdots+p_4$

$= .962$

(b) $\lambda_{lost} = \lambda p_5$

$= 5\times.038 = .19$ cust./hr

(c) $L_s = 0\times.399+1\times.249+2\times.156$
$+3\times.097+4\times.061$
$+5\times.038$
$= 1.286$

continued...

(d) $W_q = W_s - \frac{1}{\mu}$

$\lambda_{eff} = 5(1-.038) = 4.81$ cust/hr

$W_s = \frac{L_s}{\lambda_{eff}}$

$= \frac{1.286}{4.81}$

$= .2675$ hour

$W_q = .2675 - 1/8$

$= .1424$ hour

8

$$p_n = \frac{(1-\rho)\rho^n}{1-\rho^{N+1}}$$

$$\lim_{\rho\to 1} p_n = \lim_{\rho\to 1} \frac{\rho^n - \rho^{n+1}}{1-\rho^{N+1}}$$

$$= \lim_{\rho\to 1} \frac{n\rho^{n-1} - (n+1)\rho^n}{-(N+1)\rho^N}$$

$$= \frac{1}{N+1}$$

Thus,

$$L_s = \sum_{n=0}^{N} np_n$$

$$= \frac{1}{N+1}\sum_{n=0}^{N} n$$

$$= \frac{N(N+1)}{2(N+1)} = \frac{N}{2}$$

9

$$W_s = W_q + \frac{1}{\mu}$$

$$\lambda_{eff} W_s = \lambda_{eff} W_q + \frac{\lambda_{eff}}{\mu}$$

Thus,

$$L_s = L_q + \frac{\lambda_{eff}}{\mu}$$

or

$$\lambda_{eff} = \mu(L_s - L_q)$$

1

TORA input = (8, 5, 2, ∞, ∞)

Title: 17.6e-1
Scenario 1– (M/M/2):(GD/infinity/infinity)

Lambda = 8.00000 Mu = 5.00000
Lambda eff = 8.00000 Rho/c = 0.80000

Ls = 4.44444 Lq = 2.84444
Ws = 0.55556 Wq = 0.35556

n	Probability, pn	Cumulative, Pn	n	Probability, pn	Cumulative, Pn
0	0.11111	0.11111	23	0.00131	0.99475
1	0.17778	0.28889	24	0.00105	0.99580
2	0.14222	0.43111	25	0.00084	0.99664
3	0.11378	0.54489	26	0.00067	0.99731
4	0.09102	0.63591	27	0.00054	0.99785
5	0.07282	0.70873	28	0.00043	0.99828
6	0.05825	0.76698	29	0.00034	0.99862
7	0.04660	0.81359	30	0.00028	0.99890
8	0.03728	0.85087	31	0.00022	0.99912
9	0.02983	0.88070	32	0.00018	0.99930
10	0.02386	0.90456	33	0.00014	0.99944
11	0.01909	0.92365	34	0.00011	0.99955
12	0.01527	0.93892	35	0.00009	0.99964
13	0.01222	0.95113	36	0.00007	0.99971
14	0.00977	0.96091	37	0.00006	0.99977
15	0.00782	0.96873	38	0.00005	0.99982
16	0.00625	0.97498	39	0.00004	0.99985
17	0.00500	0.97998	40	0.00003	0.99988
18	0.00400	0.98399	41	0.00002	0.99991
19	0.00320	0.98719	42	0.00002	0.99992
20	0.00256	0.98975	43	0.00002	0.99994
21	0.00205	0.99180	44	0.00001	0.99995
22	0.00164	0.99344			

TORA input = (16, 5, 4, ∞, ∞)

Title: 17.6e-1
Scenario 2– (M/M/4):(GD/infinity/infinity)

Lambda = 16.00000 Mu = 5.00000
Lambda eff = 16.00000 Rho/c = 0.80000

Ls = 5.58573 Lq = 2.38573
Ws = 0.34911 Wq = 0.14911

n	Probability, pn	Cumulative, Pn	n	Probability, pn	Cumulative, Pn
0	0.02730	0.02730	24	0.00138	0.99450
1	0.08737	0.11467	25	0.00110	0.99560
2	0.13979	0.25446	26	0.00088	0.99648
3	0.14911	0.40357	27	0.00070	0.99718
4	0.11929	0.52285	28	0.00056	0.99775
5	0.09543	0.61828	29	0.00045	0.99820
6	0.07634	0.69463	30	0.00036	0.99856
7	0.06107	0.75570	31	0.00029	0.99885
8	0.04886	0.80456	32	0.00023	0.99908
9	0.03909	0.84365	33	0.00018	0.99926
10	0.03127	0.87492	34	0.00015	0.99941
11	0.02502	0.89994	35	0.00012	0.99953
12	0.02001	0.91995	36	0.00009	0.99962
13	0.01601	0.93596	37	0.00008	0.99970
14	0.01281	0.94877	38	0.00006	0.99976
15	0.01025	0.95901	39	0.00005	0.99981
16	0.00820	0.96721	40	0.00004	0.99985
17	0.00656	0.97377	41	0.00003	0.99988
18	0.00525	0.97901	42	0.00002	0.99990
19	0.00420	0.98321	43	0.00002	0.99992
20	0.00336	0.98657	44	0.00002	0.99994
21	0.00269	0.98926	45	0.00001	0.99995
22	0.00215	0.99140	46	0.00001	0.99996
23	0.00172	0.99312			

(a) $C = 2$:

$P\{\text{all servers are busy}\} = (P_{n\geq 2})^2 = (1-.29)^2 = .504$

$C = 4$:

$P\{\text{all servers are busy}\} = 1 - P_{n\leq 3} = 1 - .404 = .596$

C = 4 yields a higher probability that all servers are busy.

continued...

(b)

Title: 17.6e-1
Comparative Analysis

Scenario	c	Lambda	Mu	L'da eff	p0	Ls	Lq	Ws	Wq
1	4	16.00000	5.00000	16.00000	0.02730	5.58573	2.38573	0.34911	0.14911
2	5	16.00000	5.00000	16.00000	0.03715	3.71299	0.51299	0.23206	0.03206
3	6	16.00000	5.00000	16.00000	0.03977	3.34526	0.14526	0.20908	0.00908

For $C = 5$, $W_q = .032$ hour $\cong 2$ min

$C = 4$, $W_q = .149$ hour $\simeq 9$ min

Select $C \geq 5$

2

$C = 2$: $\lambda = 8$ calls/hr

$\mu = \frac{60}{14.5} = 4.1379$ calls/hr

$C = 4$: $\lambda = 16$ calls/hr

$\mu = 4.1379$ calls per hour

utilization $= \lambda/\mu c = .967$

Title: 6e-2
Comparative Analysis

Scenario	c	Lambda	Mu	L'da eff	p0	Ls	Lq	Ws	Wq
1	2	8.00000	4.13790	8.00000	0.01695	29.49805	27.56471	3.68726	3.44559
2	4	16.00000	4.13790	16.00000	0.00332	30.75857	26.89187	1.92241	1.68074

$W_q = \begin{cases} 3.446 \text{ hours for } C = 2 \\ 1.681 \text{ hours for } C = 4 \end{cases}$

Consolidation reduces the waiting time by more than 51%.

3

(a) $\lambda = \frac{60}{5} = 12$ per hour

$\mu = 10$ per hour

$c > \frac{\lambda}{\mu} = 1.2 \Rightarrow c \geq 2$

(b) $\lambda = \frac{60}{2} = 30$ per hour

$\mu = \frac{60}{6} = 10$ per hour

$c > \frac{\lambda}{\mu} = \frac{30}{10} = 3 \Rightarrow c \geq 4$

(c) $\lambda = 30$ per hour, $\mu = 40$ per hr.

$c > \frac{30}{40} = .75 \Rightarrow c \geq 1$

4

$\lambda = 45$ customers/hr

$\mu = \frac{60}{5} = 12$ customers/hr

$c > \frac{45}{12}$ or $c \geq 4$

Desired $W_q \leq 30$ seconds $= .0083$ hr

Title: 6e-4
Comparative Analysis

Scenario	c	Lambda	Mu	L'da eff	p0	Ls	Lq	Ws	Wq
1	4	45.00000	12.00000	45.00000	0.00656	16.72545	12.97545	0.37168	0.28834
2	5	45.00000	12.00000	45.00000	0.01868	5.13537	1.38537	0.11412	0.03079
3	6	45.00000	12.00000	45.00000	0.02208	4.12903	0.37903	0.09176	0.00842
4	7	45.00000	12.00000	45.00000	0.02309	3.86873	0.11873	0.08597	0.00264

Select $C \geq 7$.

5

TORA input: (20, 12, 3, ∞, ∞)

Title: 17.6e-5
Scenario 1-- (M/M/3):(GD/infinity/infinity)

Lambda =	20.00000	Mu =	12.00000
Lambda eff =	20.00000	Rho/c =	0.55556
Ls =	2.04137	Lq =	0.37470
Ws =	0.10207	Wq =	0.01874

n	Probability, pn	Cumulative, Pn	n	Probability, pn	Cumulative, Pn
0	0.17266	0.17266	10	0.00218	0.99728
1	0.28777	0.46043	11	0.00121	0.99849
2	0.23981	0.70024	12	0.00067	0.99916
3	0.13323	0.83347	13	0.00037	0.99953
4	0.07401	0.90748	14	0.00021	0.99974
5	0.04112	0.94860	15	0.00012	0.99986
6	0.02284	0.97144	16	0.00006	0.99992
7	0.01269	0.98414	17	0.00004	0.99996
8	0.00705	0.99119	18	0.00002	0.99998
9	0.00392	0.99510	19	0.00001	0.99999

m = size of waiting room.

$p_0 + p_1 + \cdots + p_{m+2} \geq .999 \Rightarrow m \geq 10$

6

$C = 2$, $\lambda_{windows} = .8 \times \frac{60}{3} = 16$ /hr

$\mu = \frac{60}{5} = 12$ per hour

Title: 6e-6
Scenario 1-- (M/M/2):(GD/infinity/infinity)

Lambda =	16.00000	Mu =	12.00000
Lambda eff =	16.00000	Rho/c =	0.66667
Ls =	2.40000	Lq =	1.06667
Ws =	0.15000	Wq =	0.06667

n	Probability, pn	Cumulative, Pn	n	Probability, pn	Cumulative, Pn
0	0.20000	0.20000	14	0.00137	0.99726
1	0.26667	0.46667	15	0.00091	0.99817
2	0.17778	0.64444	16	0.00061	0.99878
3	0.11852	0.76296	17	0.00041	0.99919
4	0.07901	0.84198	18	0.00027	0.99946
5	0.05267	0.89465	19	0.00018	0.99964
6	0.03512	0.92977	20	0.00012	0.99976
7	0.02341	0.95318	21	0.00008	0.99984
8	0.01561	0.96879	22	0.00005	0.99989
9	0.01040	0.97919	23	0.00004	0.99993
10	0.00694	0.98613	24	0.00002	0.99995
11	0.00462	0.99075	25	0.00002	0.99997
12	0.00308	0.99383	26	0.00001	0.99998
13	0.00206	0.99589			

(a) $P_{n \geq 2} = 1 - (P_0 + P_1)$

$= 1 - .46667$

$= .5333$

(b) $P_0 = .2$

(c) $L_q = 1.067$

(d) NO, because $\lambda > \mu$. The minimum number of windows should $\geq \frac{\lambda}{\mu} = \frac{16}{12} = 1.33$

Number of windows ≥ 2

7

$\lambda = 25 \times \frac{60}{15} = 100$ jobs/hour

$\mu = \frac{60}{2} = 30$ jobs/hour, $C = 4$.

Title: 6e-7
Scenario 1-- (M/M/4):(GD/infinity/infinity)

Lambda =	100.00000	Mu =	30.00000
Lambda eff =	100.00000	Rho/c =	0.83333
Ls =	6.62194	Lq =	3.28861
Ws =	0.06622	Wq =	0.03289

n	Probability, pn	Cumulative, Pn	n	Probability, pn	Cumulative, Pn
0	0.02131	0.02131	28	0.00138	0.99311
1	0.07103	0.09234	29	0.00115	0.99425
2	0.11839	0.21073	30	0.00096	0.99521
3	0.13154	0.34228	31	0.00080	0.99601
4	0.10962	0.45190	32	0.00066	0.99668
5	0.09135	0.54325	33	0.00055	0.99723
6	0.07613	0.61937	34	0.00046	0.99769
7	0.06344	0.68281	35	0.00038	0.99808
8	0.05286	0.73568	36	0.00032	0.99840
9	0.04405	0.77973	37	0.00027	0.99866
10	0.03671	0.81644	38	0.00022	0.99889
11	0.03059	0.84703	39	0.00019	0.99907
12	0.02549	0.87253	40	0.00015	0.99923
13	0.02125	0.89377	41	0.00013	0.99936
14	0.01770	0.91148	42	0.00011	0.99946
15	0.01475	0.92623	43	0.00009	0.99955
16	0.01229	0.93853	44	0.00007	0.99963
17	0.01025	0.94877	45	0.00006	0.99969
18	0.00854	0.95731	46	0.00005	0.99974
19	0.00711	0.96443	47	0.00004	0.99978
20	0.00593	0.97035	48	0.00004	0.99982
21	0.00494	0.97530	49	0.00003	0.99985
22	0.00412	0.97941	50	0.00002	0.99988
23	0.00343	0.98284	51	0.00002	0.99990
24	0.00286	0.98570	52	0.00002	0.99991
25	0.00238	0.98809	53	0.00001	0.99993
26	0.00199	0.99007	54	0.00001	0.99994
27	0.00165	0.99173	55	0.00001	0.99995

(a) $P_{n \geq 4} = 1 - CP_3$

$= 1 - .34228 = .65772$

(b) $W_s = .06622$ hour

(c) $L_q = 3.29$ jobs

(d) $p_0 = .021 \Rightarrow 2.1\%$ idleness

(e) Av. # of idle computers $= 4 - (L_s - L_q)$

$= 4 - (6.62 - 3.29) = .67$

8

$\lambda = 15 + 10 + 20 = 45$ customers/hour

$\mu = \frac{60}{6} = 10$ customers/hour

$C > 45/10 = 4.5 \Rightarrow C \geq 5$

Title: 6e-8
Comparative Analysis

Scenario	c	Lambda	Mu	L'da eff	p0	Ls	Lq	Ws	Wq
1	5	45.00000	10.00000	45.00000	0.00496	11.36244	6.86244	0.25250	0.15250
2	6	45.00000	10.00000	45.00000	0.00914	5.76496	1.26496	0.12811	0.02811
3	7	45.00000	10.00000	45.00000	0.01046	4.89100	0.39100	0.10869	0.00869

(a) $W_s \leq 15/60 = .25$ hour $\Rightarrow C \geq 6$

(b) $\%\text{ idle} = \frac{C - (L_s - L_q)}{C} \times 100$

C	L_s	L_q	$C-(L_s-L_q)$	% idle
5	11.362	6.862	.5	10%
6	5.765	1.265	1.5	25%

Select $C = 5$

(c)

C	5	6	7
P_0	.00496	.00914	.01046

Select $C \leq 6$

9

1. Limited space inside a bank or a grocery store
2. Multiple queues appear to offer more corteous service.

10

For c parallel servers:

$$L_q = \frac{\rho}{c-\rho}, \text{ provided } \frac{\rho}{c} \to 1$$

Thus,

$$W_{q_c} = \frac{1}{\lambda_c}\frac{\rho}{c-\rho} = \frac{1}{(c\mu-\lambda_c)}$$

For a single server

$$W_{q_1} = \frac{\lambda_1}{\mu(\mu-\lambda_1)}$$

Because $\lambda_c = c\lambda_1$, we have

$$\frac{W_{q_c}}{W_{q_1}} = \left(\frac{\frac{1}{c(\mu-\lambda_1)}}{\frac{\lambda_1}{\mu(\mu-\lambda_1)}}\right) = \frac{1}{c\left(\frac{\lambda_1}{\mu}\right)}$$

$$= \frac{1}{c\left(\frac{\lambda_c/\mu}{c}\right)}$$

$$= \frac{1}{c(\rho/c)}$$

$$\lim_{\frac{\rho}{c}\to 1} \frac{W_{q_c}}{W_{q_1}} = \frac{1}{c}$$

11

Determination of p_0 involves the finite series sum

$$\sum_{n=c}^{\infty}\left(\frac{\rho}{c}\right)^{n-c} = \sum_{j=0}^{\infty}\left(\frac{\lambda}{\mu c}\right)^{j}$$

The series will diverge if $\lambda \geq \mu c$. The condition requires that customers be serviced at a rate faster than the rate at which they arrive at the facility. Else, the queue will build up to infinity.

12

$$L_q = \sum_{n=c}^{\infty}(n-c)p_n$$

$$= \sum_{n=c}^{\infty} n p_n - c\sum_{n=c}^{\infty} p_n + \sum_{n=0}^{c-1} n p_n - \sum_{n=0}^{c-1} n p_n + c\sum_{n=0}^{c-1} p_n - c\sum_{n=0}^{c-1} p_n$$

$$= \sum_{n=0}^{\infty} n p_n - c\sum_{n=0}^{\infty} p_n + \sum_{n=0}^{c-1}(c-n)p_n$$

$$= L_s - c + (\text{number of idle servers})$$

$$= L_s - \bar{c}$$

Now, by definition

$$L_s = L_q + \frac{\lambda_{eff}}{\mu}$$

It follows that $\bar{c} = \frac{\lambda_{eff}}{\mu}$

13

$$p_n = \begin{cases} \frac{\lambda^n}{n!\,\mu^n}p_0, & n \leq c \\ \frac{\lambda^n}{c!\,c^{n-c}\mu^n}p_0, & n \geq c \end{cases}$$

for $c = 1$,

$$p_n = \begin{cases} \frac{\lambda}{\mu}p_0 & n = 1 \\ \left(\frac{\lambda}{\mu}\right)^n p_0 & n \geq 1 \end{cases}$$

Thus,

$$p_n = \left(\frac{\lambda}{\mu}\right)^n p_0, \quad n = 1, 2, \ldots$$

14

$$L_q = p_0\frac{1}{c!}\sum_{n=c+1}^{\infty}(n-c)\frac{(\lambda/\mu)^n}{c^{n-c}}$$

$$= p_0\frac{(\lambda/\mu)^c}{c!}\sum_{n=c+1}^{\infty}(n-c)\left(\frac{\lambda}{\mu c}\right)^{n-c}$$

$$= p_0\frac{(\lambda/\mu)^c}{c!}\sum_{j=1}^{\infty} j\left(\frac{\lambda}{\mu c}\right)^{j}$$

$$= p_0\frac{(\lambda/\mu)^c}{c!}\frac{\lambda}{\mu c}\frac{d}{d\left(\frac{\lambda}{\mu c}\right)}\sum_{j=0}^{\infty}\left(\frac{\lambda}{\mu c}\right)^{j}$$

$$= p_0\frac{(\lambda/\mu)^c}{c!}\left\{\frac{\lambda/\mu c}{(1-\lambda/\mu c)^2}\right\}$$

$$= p_c\frac{\rho/c}{(1-\rho/c)^2} = \frac{\rho}{(c-\rho)^2}p_c$$

15

(a) $P\{\text{a customer is waiting}\}$

$$= P\{\text{at least } c+1 \text{ in system}\}$$

$$= \sum_{n=c+1}^{\infty} p_n$$

$$= \sum_{n=c}^{\infty} p_n - p_c$$

$$= p_0 \frac{\rho^c}{c!} \frac{1}{1-\frac{\rho}{c}} - p_c$$

$$= p_c \left\{ \frac{1}{1-\frac{\rho}{c}} - 1 \right\}$$

$$= p_c \left(\frac{\rho}{c-\rho} \right)$$

(b) Expected number in queue given the queue is not empty

$$= \sum_{i=c+1}^{\infty} (i-c) \frac{p_i}{\sum_{j=c+1}^{\infty} p_j}$$

$$= \frac{L_q}{\sum_{j=c+1}^{\infty} p_j} = \frac{L_q}{p_c\left(\frac{\rho}{c-\rho}\right)}$$

Now, $L_q = \frac{p_0}{c!} \sum_{n=c+1}^{\infty} (n-c) \frac{\rho^n}{c^{n-c}}$

$$= p_0 \frac{\rho^c}{c!} \sum_{j=1}^{\infty} j \left(\frac{\rho}{c}\right)^j$$

$$= p_0 \frac{\rho^c}{c!} \left(\frac{\rho/c}{(1-\rho/c)^2} \right), \quad \frac{\rho}{c} < 1$$

$$= p_c \left\{ \frac{c\rho}{(c-\rho)^2} \right\}, \quad \frac{\rho}{c} < 1$$

Substitution for L_q yield the desired result.

(c) Exp. waiting time for those who must wait = Exp. waiting time given there are c in the system.

$$= \frac{1}{\lambda} \sum_{i=c+1}^{\infty} (i-c) \frac{p_i}{\sum_{n=0}^{\infty} p_n}$$

$$= \frac{L_q/\lambda}{p_c/(1-\rho/c)} = \frac{1}{\mu(c-\rho)}$$

16

First convert the c-channel case into an equivalent single channel. Let the customer just arriving be the j^{th} in queue. Because there are c channels in parallel, the service time, t, of each of the other $j-1$ customers and the (one) customer in service are determined as follows: Let $t_1, t_2, \ldots, t_c$ be the actual service times in the c channels. Then,

$$P\{t > T\} = P\{\min_{1\le i\le c} t_i > T\}$$

$$= \left(e^{-\mu T}\right)^c = e^{-\mu c T}$$

This is true because if $\min_i > T$, then every t_i must be $> T$.

Now,

$$F_t(T) = 1 - P\{t > T\}$$

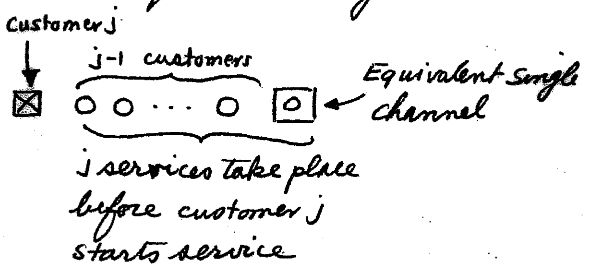
$$= 1 - e^{-\mu c T}, \quad T > 0$$

Thus,

$$f(T) = \frac{\partial F_t(T)}{\partial T} = \mu c e^{-\mu c T}, \quad T > 0$$

which is exponential with mean $\frac{1}{\mu c}$.

The c channels can be converted into an equivalent single channel as



Before customer j starts service, j other customers each with a service time T must be processed first.

Continued...

The assumption here is that all c channels are busy. If there are any idle servers, arriving customer j will have zero waiting time in queue and the special case is treated separately.

Let τ be the waiting time in queue given there are j other customer yet to be serviced. Then

$$\tau = T_1' + T_2 + \cdots + T_j$$

where $T_1', T_2, \ldots, T_j$ are exponential with mean $1/\mu c$. T_1' represents the remaining service time for the customer already in service. The lack of memory property indicate that T_1' is also exponential with mean $1/\mu c$. Thus,

$$W_q(\tau|j) = \frac{\mu c(\mu c\tau)^{j-1} e^{-\mu c\tau}}{(j-1)!}, \quad \tau > 0$$

Let $W_q(\tau)$ be the absolute pdf, then

$$w_q(\tau) = \sum_{j=1}^{\infty} W_q(\tau|j)\, q_j$$

where

$$q_j = \begin{cases} \sum_{k=0}^{c-1} p_k, & j = 0 \\ p_{c+j-1}, & j > 0 \end{cases}$$

Hence, for $\tau > 0$

$$W_q(\tau) = \sum_{j=1}^{\infty} \frac{\mu c(\mu c\tau)^{j-1} e^{-\mu c\tau}}{(j-1)!} \frac{\rho^{c+j-1}}{c!\, c^{j-1}} p_0$$

$$= \frac{\rho^c \mu c\, e^{-\mu c\tau}}{c!} p_0 \sum_{j=0}^{\infty} \frac{(\rho\mu c\tau/c)^j}{j!}$$

$$= \frac{\rho^c \mu c\, e^{-\mu c\tau}}{c!} p_0\, e^{\lambda\tau}$$

$$= \frac{\rho^c \mu\, e^{-\mu(c-\rho)\tau}}{(c-1)!} p_0$$

continued...

For $\tau = 0$, the corresponding probability is $\sum_{k=0}^{c-1} p_k$, or

$$1 - \sum_{k=c}^{\infty} p_k = 1 - \sum_{j=0}^{\infty} p_{c+j}$$

$$= 1 - \sum_{j=0}^{\infty} \frac{\rho^{c+j}}{c!\, c^j} p_0$$

$$= 1 - \frac{\rho^c}{c!}\left(\frac{p_0}{1-\frac{\rho}{c}}\right)$$

$$= 1 - \left\{\frac{\rho^c p_0}{(c-1)!(c-\rho)}\right.$$

Hence,

$$W_q(\tau) = \begin{cases} 1 - \dfrac{\rho^c p_0}{(c-1)!(c-\rho)}, & \tau = 0 \\[2ex] \dfrac{\mu\rho^c e^{-\mu(c-\rho)\tau}}{(c-1)!} p_0, & \tau > 0 \end{cases}$$

17

$$P\{\tau > y\} = \int_y^{\infty} w_q(\tau)\, d\tau$$

$$= \frac{c\mu\rho^c p_0}{c!} \int_y^{\infty} e^{-(c\mu-\lambda)\tau} d\tau$$

$$= \frac{\rho^c c\mu}{c!(c\mu-\lambda)} e^{-(c\mu-\lambda)y} p_0$$

$$= \frac{\rho^c p_0}{c!\left(1-\frac{\rho}{c}\right)} e^{-(c\mu-\lambda)y}$$

$$= P\{\tau > 0\}\, e^{-(c\mu-\lambda)y}$$

where $P\{\tau > 0\} = 1 - P\{\tau = 0\}$

18

From Problem 16, the waiting time in the system is computed as

$$T = T_1' + T_2' + \dots + T_j' + t_j$$

where

$t_j =$ actual service time for customer j.

t_j is exponential with mean $1/\mu$

Thus, T is the convolution of the waiting time in queue and the actual service time of customer j. This means that $w(T)$ is the convolution of $w_q(\tau)$ and $g(t)$; that is,

$$w(T) = w_q(\tau) * g(t)$$

where

$$g(t) = \mu e^{-\mu t}, \quad t > 0$$

$$w(T) = w_q(0) g(T) + \int_{0+}^{T} w_q(\tau) g(T-\tau) d\tau$$

$$= \left(1 - \frac{\rho^c p_0}{(c-1)!(c-\rho)}\right) \mu e^{-\mu T} + p_0 \int_{0+}^{T} \frac{\mu \rho^c e^{-\mu(c-\rho)\tau}}{(c-1)!} \mu e^{-\mu(T-\tau)} d\tau$$

$$= \left(1 - \frac{\rho^c p_0}{(c-1)!(c-\rho)}\right) \mu e^{-\mu T} + \frac{\mu \rho^c e^{-\mu T}}{(c-1)!(c-1-\rho)} p_0 \left\{1 - e^{-\mu(c-1-\rho)T}\right\}$$

$$= \mu e^{-\mu T} - \frac{\rho^c p_0 \mu e^{-\mu T}}{(c-1)!(c-1-\rho)} \frac{(c-\rho-1)}{(c-\rho)} + \frac{\mu \rho^c e^{-\mu T} p_0}{(c-1)!(c-1-\rho)} - \frac{\mu \rho^c e^{-\mu T} e^{-\mu(c-1-\rho)T}}{(c-1)!(c-1-\rho)}$$

Continued...

$$= \mu e^{-\mu T} + \frac{\rho^c p_0 \mu e^{-\mu T}}{(c-1)!(c-1-\rho)} \left\{\frac{1}{c-\rho} - e^{-\mu(c-1-\rho)T}\right\}$$

$$T > 0$$

1

(a) $c-(L_s-L_q)=4-(4.24-1.54)$
$= 1.3$ cabs

(b) $p_q = .04468$

(c) Title: 6f-1
Comparative Analysis

Scenario	c	Lambda	Mu	L'da eff	p0	Ls	Lq	Ws	Wq
1	4	16.00000	5.00000	15.42815	0.03121	4.23984	1.15421	0.27481	0.07481
2	4	16.00000	5.00000	15.25869	0.03236	4.02634	0.97460	0.26387	0.06387
3	4	16.00000	5.00000	15.02834	0.03393	3.78470	0.77903	0.25184	0.05184
4	4	16.00000	5.00000	14.70690	0.03613	3.51216	0.57078	0.23881	0.03881
5	4	16.00000	5.00000	14.24151	0.03931	3.20550	0.35719	0.22508	0.02508

m = length of waiting list
$N = m+4$

m	N	W_q(hr)	W_q(min)
6	10	.075	4.5
5	9	.064	3.83
4	8	.052	3.12
3	7	.039	2.33
2	6	.025	1.5

Select $m \le 3$

2

$c=2$, $\lambda = 20/hr$, $N=5$
$\mu = 60/6 = 10/hr$

(a) $p_5 = .1818$ or 18.18%

(b) $p_1 = .1818$ or 18.18%

(c) $\%$ utilization $= 100\left(\frac{L_s-L_q}{c}\right)$

$$= \frac{2.727-1.091}{2}\times 100$$

$$= 81.8\%$$

(d) Probability $= p_2+p_3+p_4 = .54546$

(e) $p_N \le .1$

N	5	... 8	9	10
p_N	.1818	.1176	.1053	.0952

$N \ge 10$ spaces (including the pumps)

continued...

(f) $p_0 \le .05$

N	5	... 8	9	10
p_N	.0909	.0588	.0526	.0476

$N \ge 10$

3

$\lambda = 60/10 = 6/hr$
$\mu = 60/30 = 2/hr$, $N=18$

(a) # idle mechanics
$= c-(L_s-L_q)$
$= 3-(9.54-6.71) = .17$

(b) $p_{18} = .0559$
$\lambda_{lost} = .0559 \times 6 = .3354$ job/hr
lost jobs in 10 hrs $= 3.354$ jobs

(c) $p_{n\le 17} = p_0+p_1+\cdots+p_{17}$
$= .9441$

(d) $p_{n\le 2} = p_0+p_1+p_2 = .10559$

(e) $L_q = 6.7081$ mower

(f) $\frac{L_s-L_q}{c} = \frac{9.54-6.71}{3} = .944$

4

$N=40$, $c=30$, $\lambda = 20/hr$
$\mu = 60/60 = 1/hr$

(a) $p_{40} = .00014$

(b) $p_{30}+p_{31}+\cdots+p_{39} = p_{n\le 39} - p_{n\le 29}$
$= .99986 - .97533$
$= .02453$

(c) $p_{29} = .01248$

(d) $L_s - L_q = 20.043 - .046 \approx$ **20** spaces

(e) $L_q = .046$

continued...

(f) If there are 30 cars or more in the lot, the student will not make it to class. Thus,

$P\{\text{not finding a parking space}\}$

$= P_{30} + P_{31} + \cdots + P_{40} = 1 - P_{n \le 29}$

$= 1 - .97533 = .02467$

No. of students who cannot park during an 8-hr period $= 20 \times .02467 \times 8$

$\cong 4$ students

5

$$1 = p_0 \left\{ \sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^c}{c!} \sum_{n=c}^{N} \left(\frac{\rho}{c}\right)^{n-c} \right\}$$

$$= p_0 \left\{ \sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^c}{c!} \frac{1-(\rho/c)^{N-c+1}}{(1-\rho/c)} \right.$$

$$p_0 = \left\{ \sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^c}{c!} \left(\frac{1-(\rho/c)^{N-c+1}}{1-\rho/c} \right) \right.$$

6

$$\bar{c} = L_s - L_q$$

$$= \lambda_{eff}(W_s - W_q)$$

$$= \lambda_{eff}\left(\frac{1}{\mu}\right)$$

7

$$1 = \frac{p_0}{c!} \sum_{n=c}^{N} \frac{\rho^n}{c^{n-c}} + p_0 \sum_{n=0}^{c-1} \frac{\rho^n}{n!}$$

$$= \frac{p_0 \rho^c}{c!} \sum_{n=0}^{N-c} \left(\frac{\rho}{c}\right)^n + p_0 \sum_{n=0}^{c-1} \frac{\rho^n}{n!}$$

$$= \frac{p_0 \rho^c}{c!} (N-c+1) + p_0 \sum_{n=0}^{c-1} \frac{\rho^n}{n!}$$

Thus,

$$p_0 = \left\{ \sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^c}{c!}(N-c+1) \right\}^{-1}$$

$$L_q = \sum_{n=c}^{N} (n-c) p_n$$

$$= \sum_{j=0}^{N-c} j\, p_{j+c}$$

$$= \frac{\rho}{c!} \frac{\rho}{c} \sum_{j=0}^{N-c} j \left(\frac{\rho}{c}\right)^{j-1} p_0$$

continued...

$$= \frac{\rho^c}{c!} \sum_{j=0}^{N-c} j\, p_0 \quad \left(\text{because } \frac{\rho}{c} = 1\right)$$

$$= \frac{\rho^c}{c!} \frac{(N-c)(N-c+1)}{2} p_0$$

$$= \frac{\rho^c (N-c)(N-c+1)}{2c!} p_0$$

8

$$\lambda_n = \begin{cases} \lambda, & n = 0, 1, 2, \ldots, c-1 \\ 0, & n = c \end{cases}$$

$$\mu_n = n\mu, \quad n = 0, 1, \ldots, c$$

Thus,

$$p_n = \frac{\rho^n}{n!} p_0, \quad n = 0, 1, 2, \ldots, c$$

$$\sum_{n=0}^{c} p_n = \sum_{n=0}^{c} \frac{\rho^n}{n!} p_0 = 1$$

$$p_0 = \left\{ \sum_{n=0}^{c} \frac{\rho^n}{n!} \right\}^{-1}$$

Set 15.6g

1

(a) $p_0 = 0$

(b) $p_{n \geq 10} = 1 - p_{n \leq 9} = .1$

(c) $p_{n \leq 40} - p_{n \leq 29} = .7771 - .13787$

$= .63923$

(d) $L_s = 36$

Net annual equity

$= \$1000 \times 36 \{.1(1-.3) + .9(1+.15)\}$

$= \$39,780$

2

$\lambda = \frac{100}{8} = 12.5/hr$

$\mu = \frac{60}{30} = 2/hr$

(a) $L_s = 6.25 \simeq 7$ seats

(b) $p_{n \geq 8} = 1 - (p_0 + p_1 + \cdots + p_7)$

$= 1 - .7089 = .2911$

(c) $p_0 = .00193$

3

$\rho = .1$

c	Lambda	Mu	L'da eff	p0	Ls	Lq	Ws	Wq
2	1.00000	10.00000	1.00000	0.90476	0.10025	0.00025	0.10025	0.00025
4	1.00000	10.00000	1.00000	0.90484	0.10000	0.00000	0.10000	0.00000
10	1.00000	10.00000	1.00000	0.90484	0.10000	0.00000	0.10000	0.00000
20	1.00000	10.00000	1.00000	0.90484	0.10000	0.00000	0.10000	0.00000
50	1.00000	10.00000	1.00000	0.90484	0.10000	0.00000	0.10000	0.00000
9999	1.00000	10.00000	1.00000	0.90484	0.10000	0.00000	0.10000	0.00000

4

$\rho = 9$

c	λ	μ	λ_{eff}	p_0	L_s	L_q	W_s	W_q
10	9.00000	1.00000	9.00000	0.00007	15.01858	6.01858	1.66873	0.66873
15	9.00000	1.00000	9.00000	0.00012	9.07235	0.07235	1.00804	0.00804
25	9.00000	1.00000	9.00000	0.00012	9.00000	0.00000	1.00000	0.00000
50	9.00000	1.00000	9.00000	0.00012	9.00000	0.00000	1.00000	0.00000
9999	9.00000	1.00000	9.00000	0.00012	9.00000	0.00000	1.00000	0.00000

1. For very small ρ, $(M/M/\infty):(GD/\infty/\infty)$ provides reliable estimates for $(M/M/c):(GD/\infty/\infty)$.
2. For large ρ, $(M/M/\infty)$ gives reliable estimates only if c is large

1

(a) $R=1$: $\lambda_{eff} = \lambda(22-L_s)$

$= .5(22 - 12.004)$

$= 4.998$

$R=4$: $\lambda_{eff} = .5(22-2.1) = 9.95$

(b) No. of idle repair persons

$= 4-(L_s - L_q)$

$= 4-(2.1-.11) = 2.01$

(c) $p_0 = .10779$

(d) $R=3$:

$P\{2 \text{ or } 3 \text{ are idle}\} = p_0 + p_1$

$= .34492$

Title: 6h-1
Scenario 3-- (M/M/3):(GD/22/22)

Lambda =	0.50000	Mu =	5.00000
Lambda eff =	9.76696	Rho/c =	0.03333
Ls =	2.46596	Lq =	0.51257
Ws =	0.25248	Wq =	0.05248

n	Probability, pn	Cumulative, Pn	n	Probability, pn	Cumulative, Pn
0	0.10779	0.10779	8	0.00953	0.99244
1	0.23713	0.34492	9	0.00445	0.99689
2	0.24899	0.59390	10	0.00193	0.99881
3	0.16599	0.75990	11	0.00077	0.99959
4	0.10513	0.86502	12	0.00028	0.99987
5	0.06308	0.92810	13	0.00009	0.99996
6	0.03574	0.96384	14	0.00003	0.99999
7	0.01906	0.98291			

2

Productivity of repair persons

$$= \frac{\text{Av. \# busy repair persons}}{R}$$

$$= \frac{L_s - L_q}{R}$$

R	Repair prod.	Shop prod.
1	100%	45.44%
2	88.2%	80.15%
3	65.1%	88.7%
4	49.7%	90.45%

$R=2$ yield 80.15% shop productivity and also maintain repair productivity at 88.2%

3

Increasing R, in effect, increases the number of machines that remain operative, and hence the chance of additional breakdowns. Stated differently, if all machines remain broken, there will be no new calls for repair service, and $\lambda_{eff} = 0$

4

$\lambda = \frac{60}{45} = 1.33$ machines/hr

$\mu = \frac{60}{8} = 7.5$ machines/hr

$R=1$, $K=5$

Title: 6h-4
Scenario 1-- (M/M/1):(GD/5/5)

Lambda =	1.33333	Mu =	7.50000
Lambda eff =	4.99939	Rho/c =	0.17778
Ls =	1.25045	Lq =	0.58386
Ws =	0.25012	Wq =	0.11679

n	Probability, pn	Cumulative, Pn	n	Probability, pn	Cumulative, Pn
0	0.33341	0.33341	3	0.11240	0.95293
1	0.29637	0.62978	4	0.03996	0.99290
2	0.21075	0.84053	5	0.00710	1.00000

(a) $L_s = 1.25$ machines

(b) $p_0 = .33341$

(c) $W_s = .25$ hour

5

$\lambda = 60/45 = 1.33/\text{hr}$

$\mu = 60/20 = 3/\text{hr}$

$R=4$, $K=4$

Title: 6h-5
Scenario 1-- (M/M/4):(GD/4/4)

Lambda =	1.33333	Mu =	3.00000
Lambda eff =	3.69230	Rho/c =	0.11111
Ls =	1.23077	Lq =	0.00000
Ws =	0.33333	Wq =	0.00000

n	Probability, pn	Cumulative, Pn	n	Probability, pn	Cumulative, Pn
0	0.22972	0.22972	3	0.08067	0.99104
1	0.40839	0.63811	4	0.00896	1.00000
2	0.27226	0.91037			

(a) $L_s = 1.23$ worker

(b) $p_0 = .22922$

6

$$\lambda = \frac{60}{30} = 2 \text{ calls/hr/baby}$$

$$\mu = \frac{60}{120} = .5/\text{hr}$$

$$R = 5, \quad K = 5$$

Title: 6h-6
Scenario 1-- (M/M/5):(GD/5/5)

Lambda =	2.00000	Mu =	0.50000
Lambda eff =	2.00000	Rho/c =	0.80000
Ls =	4.00000	Lq =	0.00000
Ws =	2.00000	Wq =	0.00000

n	Probability, pn	Cumulative, Pn	n	Probability, pn	Cumulative, Pn
0	0.00032	0.00032	3	0.20480	0.26272
1	0.00640	0.00672	4	0.40960	0.67232
2	0.05120	0.05792	5	0.32768	1.00000

(a) No. "awake" babies

$$= 5 - L_s = 5 - 4 = 1 \text{ baby}$$

(b) $p_5 = .32768$

(c) $p_{n\le 2} = p_0 + p_1 + p_2 = .05792$

7

$$p_n = \begin{cases} \frac{K\lambda}{\mu}\frac{(K-1)\lambda}{2\mu}\cdots\frac{(K-n)\lambda}{n\mu}p_0, & 0 \le n \le R \\ \frac{K\lambda}{\mu}\frac{(K-1)\lambda}{2\mu}\cdots\frac{(K-R)\lambda}{R\mu}\cdots\frac{K-n}{R\mu}p_0, & R \le n \le K \end{cases}$$

Thus,

$$p_n = \begin{cases} \frac{K(K-1)\cdots(K-n)}{1\times 2\times\cdots\times n}\left(\frac{\lambda}{\mu}\right)^n p_0, & 0 \le n \le R \\ \frac{C_n^k\, n!}{R!\, R^{n-R}}\left(\frac{\lambda}{\mu}\right)^n p_0, & R \le n \le K \end{cases}$$

$$= \begin{cases} C_n^K \rho^n p_0, & 0 \le n \le R \\ C_n^k \frac{n!\,\rho^n}{R!\, R^{n-R}} p_0, & R \le n \le K \end{cases}$$

8

$$\bar{R} = L_s - L_q$$

$$= \lambda_{eff}(W_s - W_q)$$

$$= \lambda_{eff}\left(\frac{1}{\mu}\right)$$

hence $\lambda_{eff} = \mu \bar{R}$

9

$$p_n = \begin{cases} C_n^k \rho^n n!\, p_0, & n = 0, 1 \\ C_n^k n!\, \rho^n p_0, & n = 1, 2, \ldots, K \end{cases}$$

$$= \frac{K!}{(K-n)!}\rho^n p_0, \quad n = 0, 1, 2, \ldots, K$$

$$L_s = \sum_{n=0}^{K} n p_n = p_0 K! \sum_{n=0}^{K} \frac{n\rho^n}{(K-n)!}$$

$$= K - \left(\frac{1 - p_0}{\rho}\right)$$

1

$$\%\text{idle} = \frac{1-(L_s-L_q)}{1} \times 100$$
$$= [1-(L_s-L_q)] \times 100$$
$$= (1-1.333+.667) \times 100$$
$$= 33.3\%$$

2

(a) $E\{t\} = 14$ min

$Var\{t\} = \frac{(20-8)^2}{12} = 12 \text{ min}^2$

$\lambda = 4/hr = .0667/min$

$L_s = 7.867$ cars

$W_s = 118$ min $= 1.967$ hours

$L_q = 6.933$ cars

$W_q = 104$ min $= 1.733$ hours

(b) $E\{t\} = 12$ min

$Var\{t\} = 9 \text{ min}^2$

$\lambda = .0667/min$

$L_s = 2.5$ cars

$W_s = 37.5$ min $= .625$ hour

$L_q = 1.7$ cars

$W_q = 25.5$ min $= .425$ hr

(c) $E\{t\} = 4 \times .2 + 8 \times .6 + 15 \times .2 = 8.6$ min

$Var\{t\} = (4-8.6)^2(.2) + (8-8.6)^2(.6) + (15-8.6)^2(.2) = 12.64 \text{ min}^2$

$L_s = 1.0244$ cars

$W_s = 15.3657$ min $= .256$ hr

$L_q = .451$ car

$W_q = 6.765$ min $= .113$ hr

3

$\lambda = .3$ job/day

Service time distribution:

$f(t) = .5, \quad 2 \le t \le 4$ days

$E\{t\} = 3$ days

$Var\{t\} = \frac{4}{12} = .333 \text{ days}^2$

(a) $L_q = 4.2$ homes

(b) $W_s = 17$ days

(c) $E\{t\} = 1.5$, $Var\{t\} = \frac{1}{12} = .0833$

$L_q = .191$ home

$W_s = 2.14$ days

4

$\lambda = \frac{30}{8 \times 60} = .0625$ prescr./min

$E\{t\} = 12+3 = 15$ min

$Var\{t\} = 9 + \frac{(4-2)^2}{12} = 9.333 \text{ min}^2$

(a) $p_0 = .0625$

(b) $L_q = 7.3$ prescriptions

(c) $W_s = 132.17$ min $= 2.2$ hours

5

$\lambda = 1/45$ /min $= .0222$ /min

$E\{t\} = 28 + 4.5 = 32.5$ min

$Var\{t\} = \frac{(6-3)^2}{12} = .75$

(a) $L_q = .9395$ item

(b) $p_0 = .278$

(c) $W_s = 74.78$ min

6

$$L_s = \lambda E\{t\} + \frac{\lambda^2(E^2(t) + Var\{t\})}{2(1-\lambda E\{t\})}$$
$$= \lambda E\{t\} + \frac{(\lambda E\{t\})^2}{2(1-\lambda E\{t\})}$$
$$= \rho + \frac{\rho^2}{2(1-\rho)}$$

7

$$L_s = \frac{m\lambda}{\mu} + \frac{\lambda^2\left(\frac{m^2}{\mu^2} + \frac{n}{\mu^2}\right)}{2\left(1-\frac{m\lambda}{\mu}\right)}$$

$$= m\rho + \frac{m^2\rho^2 + m\rho^2}{2(1-m\rho)}$$

$$= m\rho + \frac{m(m+1)\rho^2}{2(1-m\rho)}$$

8

$E\{t\} = \frac{1}{\mu}$, $Var\{t\} = \frac{1}{\mu^2}$

$$L_s = \frac{\lambda}{\mu} + \frac{\lambda^2\left(\frac{1}{\mu^2} + \frac{1}{\mu^2}\right)}{2(1-\lambda/\mu)}$$

$$= \rho + \frac{\rho^2}{1-\rho}$$

$$= \frac{\rho}{1-\rho}$$

9

(a) Because each server receives every c^{th} customer and the interarrival time at the channel is exponential with mean $1/\lambda$, the interarrival time at each server is the convolution of c exponential distributions each with mean $\frac{1}{\mu}$. This means that the interarrival time is gamma with mean c/λ and variance c/λ^2.

(b) The interarrival time at the i^{th} server is exponential with mean $\frac{1}{\alpha_i \lambda}$. This means that the arrivals at server i is Poisson with mean $\alpha_i \lambda$, $i=1,2,\ldots,c$

1

(a) $\mu_2 = \frac{24}{\left(\frac{1000}{36}\right)\frac{1}{60}} = 5.184$ jobs/day

$\mu_3 = \frac{24}{\left(\frac{1000}{50}\right)\times\frac{1}{60}} = 7.2$ jobs/day

$\mu_4 = \frac{24}{\left(\frac{1000}{66}\right)\times\frac{1}{60}} = 9.5$ jobs/day

(b) $ETC_i = 24\,C_{1i} + 80\,Lq_i$

i	λ_i	μ_i	Lq_i	C_{1i}	ETC_i
1	4	4.32	11.57	\$15	\$1285.60
2	4	5.18	2.62	20	689.60
3	4	7.20	.69	24	631.20
4	4	9.50	.31	27	672.80

Select model 3.

2

$\lambda = 3/hr$

$\mu_1 = 5/hr$, $C_1 = \$15$

$\mu_2 = 8/hr$, $C_2 = \$20$

Cost/Broken machine = \$50/hr

(M/M/1):(GD/10/10):

$\lambda = 3,\ \mu = 5 \Rightarrow L_{s_1} = 8.33$

(M/M/1):(GD/10/10):

$\lambda = 3,\ \mu = 8 \Rightarrow L_{s_2} = 7.33$

$TC_1 = 50L_{s_1} + 15 = 50\times 8.33 + 15 = \$431.50/hr$

$TC_2 = 50L_{s_2} + 20 = 50\times 7.33 + 20 = \$386.50/hr$

Hire second repair person.

3

$\lambda = 10/hr = .167/min$

Scanner A:

Service time distribution:

$f_A(t) = \frac{1}{\left(\frac{35}{10}\right)-\left(\frac{25}{10}\right)} = 1,\ 2.5 \le t \le 3.5$

Continued...

$E_A\{t\} = 3$ min

$Var_A\{t\} = \frac{1}{12}$ min^2

Scanner B:

$f_B(t) = \frac{1}{\frac{35}{15}-\frac{25}{15}} = 1.5,\quad 5/3 \le t \le 7/3$

$E_B\{t\} = 2$ min

$Var_B\{t\} = \frac{(2/3)^2}{12} = 1/27$ min^2

From Excel file PKFormula.xls,

$L_{s_A} = .755$ customer

$L_{s_B} = .419$ customer

$TC_A = .2L_{s_A} + C_A$

$= \left(.2\times .755 + \frac{\$10}{10\times 60}\right)\times 60 = \$10.06/hr$

$TC_B = .2L_{sB} + C_B$

$= \left(.2\times .419 + \frac{\$15}{10\times 60}\right)\times 60 = \$6.53/hr$

Select scanner B

4

(a) μ = number of filled orders/hr

λ = number of requested orders/hr

C_1 = cost/unit increase in production rate

C_2 = cost of waiting/unit waiting time/cust.

$TC(\mu)$ = Total cost/unit waiting time given μ

$= C_1\mu + C_2 L_s$

$= C_1\mu + C_2\frac{\lambda}{\mu-\lambda}$

(b) $\frac{\partial TC(\mu)}{\partial\mu} = C_1 - C_2\frac{\lambda}{(\mu-\lambda)^2} = 0$

$\mu = \lambda + \sqrt{\frac{C_2}{C_1}\lambda}$

(c) $\lambda = 3$, $C_1 = .1\times 500 = \$50$, $C_2 = \$100$

$\mu = 3 + \sqrt{\frac{100}{50}\times 3} = 5.45$ orders/hr

Optimum production rate

$= 500\times 5.45 \simeq 2725$ pieces/hr

5

$\lambda = 80$ jobs/wk

$C_1 = \$250$/wk $C_2 = \$500$/job/wk

$\mu = \lambda + \sqrt{\frac{C_2}{C_1}\lambda}$

$= 80 + \sqrt{\frac{500}{250} \times 80} = 92.65$ jobs/wk

6

$\lambda = 25$ groups/hr

Model A: $\mu_A = 26$/hr, $N = 20$

Operating cost $C_A = \$12000$/month

From TORA: $p_{20} = .03128$

$L_q = 7.65$ groups

Cost/hr = operating cost/hr + waiting cost/hr + cost of lost customers/hr

$= \frac{C_A}{30 \times 10} + 10L_q + \lambda p_N \times 15$

$= \frac{12000}{30 \times 10} + 10 \times 7.65 + 25 \times .03128 \times 15$

$= \$128.23$/hr

Model B: $\mu_B = 29$/hr, $N = 30$

$C_B = \$16000$/month

From TORA: $p_{30} = .0016$

$L_q = 5.07$ groups

Cost/hr $= \frac{\$16000}{30 \times 10} + 10 \times 5.07 + 25 \times .0016 \times 15$

$= \$104.63$

Select model B

7

Let

C_3 = cost/unit time/additional capacity unit.

The cost model in Problem 6 is modified by adding the term $C_3 N$ to the cost equation.

8

p_0 is the probability of running out of stock. Thus,

Cost of lost sales per hour $= C_1 \lambda p_0$

$E\{\text{cost}\}$/unit time

$= E\{\text{lost sales cost}\}$/unit time $+ E\{\text{holding cost}\}$/unit time

$= C_1 \lambda p_0 + C_2 L_s$

For $(M/M/1):(GD/\infty/\infty)$

$p_0 = (1-\rho)$

$L_s = \frac{\rho}{1-\rho}$

Thus,

$E\{\text{cost}\}/\text{unit time} = C_1\lambda(1-\rho) + C_2\frac{\rho}{1-\rho}$

$\frac{\partial E\{\text{cost}\}}{\partial \rho} = -C_1\lambda + \frac{C_2}{(1-\rho)^2} = 0$

Thus,

$\rho = 1 \pm \sqrt{\frac{C_1\lambda}{C_2}}$

Under steady state, ρ must be less than 1. Thus,

$\rho = 1 - \sqrt{\frac{C_1\lambda}{C_2}}$

The solution requires $\sqrt{\frac{C_1\lambda}{C_2}} < 1$ in order for ρ not to assume a negative value. Note that $\rho = \frac{\lambda}{\mu}$, where λ is a constant. This means that μ is the actual optimization variable.

1

$C_1 = \$20,\ C_2 = \$45,$

$\lambda = 17.5/hr,\ \mu = 10/hr$

Title: 17.9b-1 (M/M/c)(GD/infinity/infinity)
Comparative Analysis

Scenario	c	Lambda	Mu	L'da eff	p0	Ls	Lq	Ws	Wq
1	2	17.50000	10.00000	17.50000	0.06667	7.46667	5.71667	0.42667	0.32667
2	3	17.50000	10.00000	17.50000	0.15564	2.21712	0.46712	0.12669	0.02669
3	4	17.50000	10.00000	17.50000	0.17038	1.84206	0.09206	0.10526	0.00526
4	5	17.50000	10.00000	17.50000	0.17314	1.76962	0.01962	0.10112	0.00112

$ETC(c) = 20c + 45L_s$

c	$L_s(c)$	$ETC(c)$
2	7.467	$20\times2+45\times7.467 = \$376.02$
→ 3	2.217	$20\times3+45\times2.217 = \$159.77$
4	1.842	$20\times4+45\times1.842 = \$162.89$
5	1.770	$20\times5+45\times1.770 = \$179.65$

Use three clerks

2

$\text{Cost/hr} = C_1 L_s + C_2 c$

$C_1 = \$30,\quad C_2 = \$18$

$(M/M/c):(GD/10/10):\ \lambda = 1/20 = 0.05/hr$

$\mu = 1/3 = 0.333/hr$

Title: 9b-2
Comparative Analysis

Scenario	c	Lambda	Mu	L'da eff	p0	Ls	Lq	Ws	Wq
1	2	0.05000	0.33300	0.41603	0.21439	1.67942	0.43010	4.03683	1.03383
2	3	0.05000	0.33300	0.43187	0.24269	1.36246	0.06554	3.15476	0.15175

$(\text{Cost/hr for } c=2) = 30\times1.68+18\times2 = \$86.40$

$(\text{Cost/hr for } c=3) = 30\times1.36+18\times3 = \$94.80$

(a) No, because the cost is higher

(b) Schedule loss/breakdown $= C_1 W_s$

$c=2:\ W_s = 4.037$ hours

Schedule loss $= 30\times4.037 = \$121.11$

$c=3:\ W_s = 3.155$ hours

Schedule loss $= 30\times3.155 = \$94.65$

3

The problem is similar to the machine repair model. The executives are the "machines" and the WATS line is the "server"

arrival rate/executive = 2 calls/day

$\text{service rate} = \dfrac{480}{6}$

$= 80$ calls/day

Continued...

TORA input:

$R=1:\ (2, 80, 1, 100, 100)$

$R=2:\ (2, 80, 2, 100, 100)$

Title: 9b-3
Comparative Analysis

Scenario	c	Lambda	Mu	L'da eff	p0	Ls	Lq	Ws	Wq
1	1	2.00000	80.00000	80.00000	0.00000	59.99881	58.99881	0.74999	0.73749
2	2	2.00000	80.00000	159.28000	0.00200	20.35920	18.36820	0.12782	0.11532

(a) <u>No WATS</u>:

Cost/month = (2 calls/8 hrs/exec) × (100 exec) × (6 min/call) × (50 ¢/min) × (200 hrs/month)

= \$15,000/month

<u>One WATS Line</u>: $L_q = 59$

Cost/month = Cost of WATS line + $C_1 L_q$

$= \$2000/\text{month} + 59\left(\frac{1¢}{100}\times60\times200\right)$

$= \$9080$

Savings = 15,000 − 9080

= \$5920/month

(b) <u>Two WATS lines</u>: $L_q = 18.4$

Cost/month = 2 × 2000 + $18.4\left(\frac{1¢}{100}\times200\times60\right)$

= \$6200

Additional savings

= 9080 − 6200 = \$2880

Lease a second WATS line

4

Rate of breakdown/machine, λ

$= \frac{57.8}{8 \times 20} = .36125/\text{hr}$

$\mu = \frac{60}{6} = 10/\text{hr}$

TORA model: $(M/M/3):(GD/20/20)$

W_s = lost time per breakdown

λ = number of breakdowns/hr/mach

lost time per mach/hr $= \lambda W_s$

From TORA, $W_s = .10118$ hr

Lost revenue/machine/hr

$= 25 \times (.36125 \times .10118) \times \$2$

$= \$1.83$

Lost revenue for all machines

$= 20 \times 1.83 = \$36.60$

Cost of 3 repairpersons/hr

$= 3 \times 20 = \$60.$

5

$TC(c) = cC_1 + C_2 L_s(c)$

$TC(c-1) = (c-1)C_1 + C_2 L_s(c-1)$

$TC(c+1) = (c+1)C_1 + C_2 L_s(c+1)$

$TC(c-1) - TC(c)$

$= -C_1 + C_2 \{L_s(c-1) - L_s(c)\}$

$TC(c+1) - TC(c)$

$= C_1 - C_2 \{L_s(c) - L_s(c+1)\}$

At a minimum point, we must have

$TC(c-1) \geq TC(c)$

$TC(c+1) \geq TC(c)$

Thus,

$L_s(c-1) - L_s(c) \geq \frac{C_1}{C_2}$

$L_s(c) - L_s(c+1) \leq \frac{C_1}{C_2}$

continued...

or

$L_s(c) - L_s(c+1) \leq \frac{C_1}{C_2} \leq L_s(c-1) - L_s(c)$

$\frac{C_1}{C_2} = \frac{12}{50} = .24$

c	$L_s(c)$	$L_s(c) - L_s(c+1)$
2	7.467	–
3	2.217	5.25
4	1.842	.375
		$\leftarrow \frac{C_1}{C_2} = .24$
5	1.764	.078

$c^* = 4$

1

$\lambda = 1/7 = .1428$ breakdown/hr

$\mu = .25$ repair per hour

TORA model: (M/M/R):(GD/10/10)

Comparative Analysis

Scenario	c	Lambda	Mu	L'da eff	p0	Ls	Lq	Ws	Wq
1	1	0.14280	0.25000	0.25000	0.00001	8.24932	7.24934	32.99772	28.99772
2	1	0.14280	0.25000	0.25000	0.00001	8.24932	7.24934	32.99772	28.99772
3	3	0.14280	0.25000	0.71071	0.00540	5.02302	2.18017	7.06758	3.06758
4	4	0.14280	0.25000	0.83618	0.00899	4.14443	0.79972	4.95641	0.95641
5	5	0.14280	0.25000	0.88773	0.01043	3.78339	0.23247	4.26187	0.26187
6	6	0.14280	0.25000	0.90407	0.01081	3.66899	0.05272	4.05831	0.05831
7	7	0.14280	0.25000	0.90807	0.01089	3.64096	0.00867	4.00955	0.00955
8	8	0.14280	0.25000	0.90878	0.01091	3.63602	0.00091	4.00100	0.00100

(a) From TORA's output

$L_s < 4 \Rightarrow R \geq 5$

(b) From TORA's output

$W_q < 1 \Rightarrow R \geq 4$

2

$C_1 = \$12$

c	L_s
2	7.467
3	2.217
4	1.842

$$2.217 - 1.842 \leq \frac{12}{C_2} \leq 7.467 - 2.217$$

$$.375 \leq \frac{12}{C_2} \leq 5.25$$

or

$$\$2.29 \leq C_2 \leq \$32$$