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Kurt M. Bryan

Rose-Hulman Institute of Technology, bryan@rose-hulman.edu

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ELEMENTARY INVERSION OF THE LAPLACE TRANSFORM

Kurt Bryan

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Department of Mathematics
Rose-Hulman Institute of Technology

FAX: (812) 877-3198   Phone: (812) 877-8391

Elementary Inversion of the Laplace Transform

Kurt Bryan [1]

Abstract. This paper provides an elementary derivation of a very simple "closed-form" inversion formula for the Laplace Transform.

Key words. Laplace Transform, Inverse Laplace Transform

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1 Introduction

The Laplace transform is a powerful tool in applied mathematics and engineering. Virtually every beginning course in differential equations at the undergraduate level introduces this technique for solving linear differential equations. The Laplace transform is indispensable in certain areas of control theory.

Given a function $f(t)$ defined for $0 \leq t < \infty$, the Laplace transform $F(s)$ is defined as

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt, \tag{1}$$

at least for those s for which the integral converges. In practice when one uses the Laplace transform to, for example, solve a differential equation, one has to at some point invert the Laplace transform by finding the function $f(t)$ which corresponds to some specified $F(s)$. The usual technique is to manipulate $F(s)$ algebraically (e.g., with partial fraction decompositions, shifting theorems, etc.) until one can "guess" the function $f(t)$ to which $F(s)$ corresponds.

But the alert student may ask some difficult questions:

1. Could two different functions $f_1(t)$ and $f_2(t)$ have the same Laplace transform?

2. Is there a constructive or explicit procedure for determining $f(t)$ from $F(s)$?

Under reasonable restrictions on the functions involved, the answer to both questions is "yes", but the standard techniques used to show this—complex analysis, residue computations, and/or Fourier's integral inversion theorem—are generally beyond students in an introductory differential equations course.

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[1] Department of Mathematics, Rose-Hulman Institute of Technology, Terre Haute IN 47840

The purpose of this paper is to show that one can answer both questions above—and give an "explicit" (albeit not necessarily practical) inversion procedure for the Laplace transform—using only very elementary analysis. The main result, Theorem 2.1, was actually proved by E. Post in 1930 [2]. The author of this note "rediscovered" the result while trying to prove results on the non-negativity of a function $f(t)$ from knowledge of its Laplace transform (see Theorem 2.3 below.) The result does not seem to be well-known, and does not appear in most standard texts on the Laplace Transform—indeed, this author first discovered quite by accident that the result was known while browsing through [3] (see section 1.1).

The actual proof of the inversion formula uses only the elementary properties of the Laplace transform as taught in almost all differential equations courses. Indeed, if one is willing to accept the "graphically" obvious fact stated in Lemma 3.1 below, then the proof of the inversion formula becomes an easy one page exercise using only standard tools from any introductory DE course.

2   Inverting the Laplace Transform

Let $f(t)$ be a continuous function on the interval $[0, \infty)$ which is of exponential order, that is, for some $b \in \mathbb{R}$

$$\sup_{t>0} \frac{|f(t)|}{e^{bt}} < \infty. \tag{2}$$

In this case the Laplace transform (1) exists for all $s > b$ and is in fact infinitely differentiable with respect to s for $s > b$; see [1], section 19. The following theorem shows that $f(t)$ can be uniquely recovered from $F(s)$.

Theorem 2.1 *A function f which is continuous on $[0, \infty)$ and satisfies the growth condition (2) can be recovered from $F(s)$ as*

$$f(t) = \lim_{k \to \infty} \frac{(-1)^k}{k!} \left(\frac{k}{t}\right)^{k+1} F^{(k)}\left(\frac{k}{t}\right)$$

for $t > 0$, where $F^{(k)}$ denotes the kth derivative of F.

Since Theorem 2.1 yields $f(t)$ in terms of $F(s)$, we immediately obtain a version of Lerch's Theorem.

Theorem 2.2 *Let $f(t)$ and $g(t)$ be continuous functions defined for $t \geq 0$ which satisfy (2), with Laplace transforms $F(s)$ and $G(s)$, respectively. If for some constant $c > 0$ we have $F(s) = G(s)$ for all $s > c$ then $f(t) = g(t)$ for all $t > 0$.*

Of course, the main difficulty in using Theorem 2.1 for actually computing the inverse Laplace transform is that repeated symbolic differentiation of F may yield rather unwieldy expressions. However, one can apply the theorem in a few simple cases.

Example 1: Let $f(t) = e^{-at}$. The Laplace transform is easily found to be $F(s) = \frac{1}{s+a}$. It is also simple to verify that $F^{(k)}(s) = k!(-1)^k(s+a)^{-k-1}$. Inserting this expression for $F^{(k)}$ into Theorem 2.1 gives

$$
\begin{aligned}
f(t) &= \lim_{k\to\infty} \frac{k^{k+1}}{t^{k+1}}\left(a + \frac{k}{t}\right)^{-k-1} \\
&= \lim_{k\to\infty} \left(1 + \frac{at}{k}\right)^{-k-1}.
\end{aligned}
$$

This last limit is easy to evaluate—take the natural log of both sides and write the resulting indeterminate form as $-\frac{\ln(1+at/k)}{1/(k+1)}$. L'Hopital's rule reveals that the indeterminate form approaches $-at$. The continuity of the natural logarithm immediately shows that $\ln(f(t)) = -at$, so $f(t) = e^{-at}$.

Example 2: Let $f(t) = t^n$, with $n \geq 0$. In this case we have $F(s) = n!s^{-1-n}$. One finds that

$$
F^{(k)}(s) = (-1)^k(n+k)!s^{-n-k-1}.
$$

Inserting this into Theorem 2.1 yields

$$
f(t) = t^n \lim_{k\to\infty} \frac{k^{k+1}(n+k)!}{k!t^{k+1}}\left(\frac{k}{t}\right)^{-n-k-1}.
$$

One can use Stirling's formula, in the form $\lim_{k\to\infty} \frac{k!}{\sqrt{2\pi k}}k^k e^{-k} = 1$, to obtain, after a bit of simplification,

$$
f(t) = t^n e^{-n} \lim_{k\to\infty} \sqrt{1 + \frac{n}{k}}\left(1 + \frac{n}{k}\right)^k \left(1 + \frac{n}{k}\right)^n.
$$

As $k \to \infty$ one finds that $\sqrt{1 + \frac{n}{k}} \to 1$, $(1 + \frac{n}{k})^k \to e^n$, and $\left(1 + \frac{n}{k}\right)^n \to 1$. As a result the limit yields $f(t) = t^n$.

An Application

Suppose $f(t)$ is a nonnegative function which satisfies the bound (2). Then the integrand in the definition of the Laplace transform (1) is nonnegative and as a result $F(s) \geq 0$ for all $s > b$. Differentiating (1) repeatedly with respect to s yields

$$
F^{(k)}(s) = (-1)^k \int_0^\infty t^k e^{-st}f(t)\, dt.
$$

Again, nonnegativity of the integrand shows that $F^{(k)}(s)$ alternates sign with respect to k. Thus if $f(t)$ is nonnegative then we have $(-1)^k F^{(k)}(s) \geq 0$ for all k and $s > b$.

From Theorem 2.1, however, we see that the converse is also true! If $(-1)^k F^{(k)}(s) \geq 0$ holds for each k then the quantity under the limit on the right side of the expression displayed

3

in Theorem 2.1 is nonnegative. As a result, the limit $f(t)$ is necessarily nonnegative and we have

Theorem 2.3 *A function f satisfying the bound (2) with Laplace transform $F(s)$ is nonnegative if and only if*

$$(-1)^k F^{(k)}(s) \geq 0$$

for all $k \geq 0$ and all $s > b$.

3 The Proof of Theorem 2.1

In preparation for the proof of Theorem 2.1, let us define the sequence of functions $g_k(t)$ for $t \geq 0$ and $k \in \mathbf{Z}^+$ as

$$g_k(t) = \frac{k^{k+1}}{k!} t^k e^{-kt}. \tag{3}$$

It is an easy exercise in integration by parts to verify that

$$\int_0^\infty g_k(t)\, dt = 1 \tag{4}$$

for all $k > 0$. Also, g_k has a unique maximum at $t = 1$, with $g_k(1) = \frac{k^{k+1}e^{-k}}{k!}$. Finally, we note that since $k! > k^k e^{-k}$ for all positive integers k (this is easy to prove; see [4], chapter 5, exercises 13 and 14) we have the inequality

$$g_k(t) \leq k t^k e^{k(1-t)} \tag{5}$$

for all $t \geq 0$ and positive k.

Below we plot $g_k(t)$ for several values of k.

Function g for k=1,2,4,20
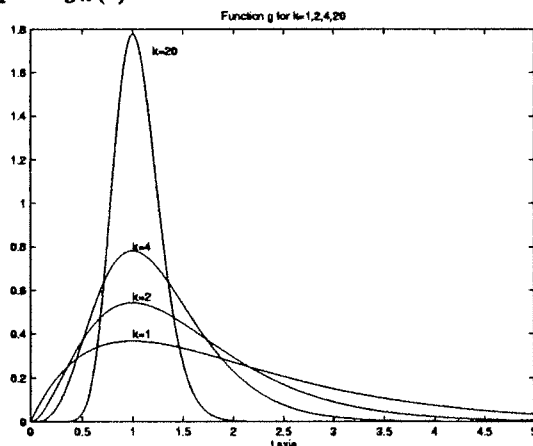
It is apparent that as k increases, $g_k(t)$ has a sharper peak and is "localized" near $t = 1$. Put another way, $g_k(t)$ approximates the delta function $\delta(t - 1)$. The following lemma quantifies this statement.

4

Lemma 3.1 *For any continuous function $\phi(t)$ defined for $t \geq 0$ which satisfies the bound (2) we have*

$$\lim_{k \to \infty} \int_0^\infty g_k(t)\phi(t)\,dt = \phi(1).$$

We defer the proof of Lemma 3.1 to section 4. With Lemma 3.1 in hand, however, it is relatively easy to prove Theorem 2.1.

Proof of Theorem 2.1: Given a function $\phi(t)$, let us denote its Laplace transform as a function of s by $\mathcal{L}(\phi)(s)$, with $F(s) = \mathcal{L}(f)(s)$. We recall some elementary properties of the Laplace Transform (see [1] for proofs):

1. $\dfrac{d^k F(s)}{ds^k} = \mathcal{L}((-1)^k t^k f(t))(s)$

2. $F(s + c) = \mathcal{L}(e^{-ct} f(t))(s)$

3. $\dfrac{1}{c} F(s/c) = \mathcal{L}(f(ct))(s)$

For $t_0 > 0$ define the function $\tilde{f}(t) = f(t_0 t)$ and note that by property (3) above we have

$$\mathcal{L}(\tilde{f})(s) = \mathcal{L}(f(t_0 t))(s) = \frac{1}{t_0} F(s/t_0).$$

It immediately follows that for all sufficiently large s

$$\frac{d^k}{ds^k}\left(\mathcal{L}(\tilde{f})\right)(s) = \frac{1}{t_0^{k+1}} F^{(k)}(s/t_0). \tag{6}$$

Let $\phi(t) = e^{-st}\tilde{f}(t)$. If f satisfies the growth condition (2) then so does ϕ (with b replaced by $b - s$.) From Lemma 3.1 and the definition of $g_k(t)$ we have

$$
\begin{aligned}
\phi(1) = e^{-s}\tilde{f}(1) = e^{-s}f(t_0) &= \lim_{k \to \infty} \frac{k^{k+1}}{k!} \int_0^\infty e^{-st} t^k e^{-kt} \tilde{f}(t)\,dt \\
&= \lim_{k \to \infty} \frac{k^{k+1}}{k!} \mathcal{L}(t^k e^{-kt}\tilde{f}(t))(s).
\end{aligned}
\tag{7}
$$

Using the above properties (1) and (2) of the Laplace Transform, equation (6), and the definition of $\tilde{f}$ we have

$$
\begin{aligned}
\mathcal{L}(t^k e^{-kt}\tilde{f}(t))(s) &= (-1)^k \frac{d^k}{ds^k}(\mathcal{L}(e^{-kt}\tilde{f}(t)))(s) \\
&= (-1)^k \frac{d^k}{ds^k}(\mathcal{L}(\tilde{f}))(s + k) \\
&= (-1)^k \frac{1}{t_0^{k+1}} \frac{d^k}{ds^k}(\mathcal{L}(f))\left(\frac{s+k}{t_0}\right) \\
&= (-1)^k \frac{1}{t_0^{k+1}} F^{(k)}\left(\frac{s+k}{t_0}\right).
\end{aligned}
\tag{8}
$$

5

Equations (7) and (8), with $f(t_0) = e^s \phi(1)$, now yield

$$f(t_0) = e^s \lim_{k \to \infty} \frac{(-1)^k}{k!} \left(\frac{k}{t_0} \right)^{k+1} F^{(k)} \left(\frac{s+k}{t_0} \right) \tag{9}$$

for any s. The statement in Theorem 2.1 is actually just the special case $s = 0$. ■

4   The Proof of Lemma 3.1

We first establish the following technical lemma.

Lemma 4.1 *For any δ with $0 < \delta < 1$ and any real b we have*

$$\lim_{k \to \infty} \int_0^{1-\delta} g_k(t) e^{bt} \, dt = 0, \tag{10}$$

$$\lim_{k \to \infty} \int_{1+\delta}^{\infty} g_k(t) e^{bt} \, dt = 0. \tag{11}$$

Proof: To prove (10), first define the function

$$h(t) = t e^{1-t}.$$

The function $h(t)$ is positive for $t > 0$, has a maximum value of 1 at $t = 1$, and $h(t) < 1$ for all $t \neq 1$. Also, it is easy to check that $h'(t) > 0$ for all $t \in (0, 1)$, so h is strictly increasing on $(0, 1)$. It follows that

$$h(t) \leq h(1 - \delta) < 1$$

for all t with $0 \leq t \leq 1 - \delta$. Note that by inequality (5) we have $g_k(t) \leq k(h(t))^k$. Let $m = \max(1, e^{b(1+\delta)})$. It follows that

$$\begin{aligned}
0 < \int_0^{1-\delta} g_k(t) e^{bt} \, dt &\leq k \int_0^{1-\delta} (h(t))^k e^{bt} \, dt \\
&< mk \int_0^{1-\delta} (h(1-\delta))^k \, dt \\
&= mk(1-\delta)(h(1-\delta))^k.
\end{aligned} \tag{12}$$

Since $0 < h(1 - \delta) < 1$, it is clear that the term on the right in (12) approaches zero as $k \to \infty$, and so equation (10) must hold.

To prove equation (11), again let $h(t) = t e^{1-t}$. For any a with $0 < a < 1$ and all $t > \frac{1}{1-a}$ we have the inequality

$$h(t) < \frac{1}{1-a} e^{-at}. \tag{13}$$

This can easily be shown by examining the function $\frac{h(t)}{e^{-at}}$. This function has a unique maximum for t in $(1, \infty)$ at $t = \frac{1}{1-a}$, with maximum value $\frac{1}{1-a}$, which immediately implies the inequality (13).

6

From $g_k(t) \le k(h(t))^k$ and inequality (13) we obtain

$$g_k(t)e^{bt} < k\left(\frac{1}{1-a}\right)^k e^{(b-ka)t} \tag{14}$$

for all $t > \frac{1}{1-a}$. Let $a_0 = \frac{\delta}{1+\delta}$. Choosing $a < a_0$ gives $0 < a < 1$ and, by simple algebra, $\frac{1}{1-a} < 1 + \delta$. By inequality (14)

$$
\begin{aligned}
\int_{1+\delta}^{\infty} g_k(t)e^{bt}\,dt & < k\left(\frac{1}{1-a}\right)^k \int_{1+\delta}^{\infty} e^{(b-ka)t}\,dt \\
& = k\left(\frac{1}{1-a}\right)^k \left(\frac{e^{(b-ka)(1+\delta)}}{ka-b}\right) \\
& = \frac{e^{b(1+\delta)}}{a}\left(\frac{e^{-a(1+\delta)}}{1-a}\right)^k.
\end{aligned}
\tag{15}
$$

provided of course that k is large enough so that the integrals converge.

Claim: For any $\delta > 0$ we have, for all sufficiently small positive a, that

$$\frac{e^{-a(1+\delta)}}{1-a} < 1.$$

Proof: By simple algebra the inequality above is equivalent to

$$1 + \delta > \frac{-\ln(1-a)}{a}. \tag{16}$$

It's easy to check that the function $\phi(a) = \frac{-\ln(1-a)}{a}$ satisfies $\lim_{a\to 0}\phi(a) = 1$, $\phi'(a) > 0$ for $0 < a < 1$, and $\lim_{a\to 1}\phi(a) = \infty$. It follows that for $\delta > 0$ the equation

$$\phi(a) = 1 + \delta$$

has a unique solution $a = a_1$. If a is chosen so that $a < a_1$ then inequality (16) will hold, and this proves the claim.

To finish the proof of Lemma 4.1 we choose $a < \min(a_0, a_1)$, and let $d = \frac{e^{-a(1+\delta)}}{1-a} < 1$. Inequality (15) yields

$$0 < \int_{1+\delta}^{\infty} g_k(t)e^{bt}\,dt < \frac{e^{b(1+\delta)}}{a}d^k$$

and since $0 < d < 1$ the right hand side clearly tends to zero as $k \to \infty$, which proves equation (11) and finishes the proof of Lemma 4.1.∎
An immediate consequence of Lemma 4.1 and equation (4) is that

$$\lim_{k\to\infty} \int_{1-\delta}^{1+\delta} g_k(t)\,dt = 1. \tag{17}$$

7

We can now prove Lemma 3.1.

Proof of Lemma 3.1: Since ϕ is continuous we may, for any $\epsilon > 0$, choose some δ so that $|\phi(t) - \phi(1)| < \epsilon$ for all t with $1 - \delta < t < 1 + \delta$. We then have

$$\int_0^\infty g_k(t)\phi(t)\,dt - \int_{1-\delta}^{1+\delta} g_k(t)\phi(1)\,dt = \int_{1-\delta}^{1+\delta} g_k(t)(\phi(t) - \phi(1))\,dt$$

$$+ \int_0^{1-\delta} g_k(t)\phi(t)\,dt + \int_{1+\delta}^\infty g_k(t)\phi(t)\,dt$$

so that

$$\left| \int_0^\infty g_k(t)\phi(t)\,dt - \int_{1-\delta}^{1+\delta} g_k(t)\phi(1)\,dt \right| \leq \epsilon \int_{1-\delta}^{1+\delta} g_k(t)\,dt$$

$$+ \left| \int_0^{1-\delta} g_k(t)\phi(t)\,dt \right| + \left| \int_{1+\delta}^\infty g_k(t)\phi(t)\,dt \right|.$$

From Lemma 4.1 and the fact that ϕ is of exponential order we find that the last two integrals on the right above approach zero as $k \to \infty$, and since ϵ is arbitrary we conclude that

$$\lim_{k\to\infty} \int_0^\infty g_k(t)\phi(t)\,dt = \lim_{k\to\infty} \int_{1-\delta}^{1+\delta} g_k(t)\phi(1)\,dt = \phi(1)$$

by equation (17), which proves the claim. ∎

5 Conclusion

As noted, Theorem 2.1 does not provide a very practical means for inverting the Laplace Transform in general, since repeated differentiation of $F(s)$ generally leads to unwieldy expressions. Nonetheless, it is quite interesting to use a computer algebra system such as Maple or Mathematica to approximate the inversion formula (say with $k = 50$) applied to $F(s)$, and compare the result to the original $f(t)$. Although convergence of the inversion to f with respect to k is slow, the graphs clearly show that the procedure works.

It should also be noted that the inversion formula in Theorem 2.1 can be applied to functions with jump discontinuities. Specifically, let f be continuous from the left and right at a point x, but have a jump discontinuity at x, with $f(x^+)$ and $f(x^-)$ the limits from the right and left, respectively. One can then show that the inversion formula recovers $(f(x^+) + f(x^-))/2$.

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