Nonlinear Differential Equations with Deviating Arguments and Polynomial Approximation

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Abstract. A simple computational approach will be developed for a very broad family of differential equations with deviating arguments in their vector fields. Results will be given for problems with delays which are linear in time and state independent, and also have analytic initial data and nonlinear differential equations which are retarded, neutral or advanced. A simple modification of the approach is applicable to so called stiff equations. In addition, an explicit a priori error estimate that does not require derivatives of the vector field is presented.

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1 Introduction

PSM, or the Parker-Sochacki method, was presented in [9] which has an explicit a priori error estimate free of derivatives of the vector field, see [10]. This approach combines polynomial vector fields via auxiliary variables with Picard iteration, which preserves the polynomial structure and can be shown to be a suitably defined Taylor expansion of the solution, computed without needing derivatives of the vector field. A combination of PSM and method of steps was given in [8], while the combination used in [4] is capable of a more efficient numerical simulator, which works on a wide variety of problems with deviating arguments, including so called stiff problems. The hope is that the universality of the approach along with its simplicity will be attractive not only to mathematicians, but to scientists and engineers as well.

The class of problems under consideration, a discussion of notation and a sketch of the existence and uniqueness of solutions are given in Section 2. An example is presented in Section 3 in the case of finding a polynomnial vector field equivalent to the original one via auxiliary variables. This will include a generalization for the case of stiff problems. This approach will be formalized in Section 4, which includes a discussion of some computational points that arise from this choice and an error analysis to determine stability with respect to initial data errors. The results of some numerical experiments will be given in Section 5.

2 DDEs and Convergence of Picard Iteration

The family of differential equations with deviating arguments that are considered is

$$\begin{cases} D^{L_1}u(t) &= f\left(t, D_{\mathbf{L}_1}u(t), D_{\mathbf{L}_2}u(\Delta(t))\right), & t > t_0 \\ u(t) &= \Psi(t), & t \in [a, t_0] \end{cases} \tag{2.1}$$

with L_1, $L_2 \in \mathbb{N}$, $t_0 \in \mathbb{R}$ and D being the differential operator with the subscripted version indicating an ordered list of derivatives of u. The entries of the subscript specify which

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derivatives, for example, $D_{[0,2,4]}u(t)$ would indicate f has a $u(t)$, $u''(t)$ and $u''''(t)$ dependence. For convenience, take $\mathbf{L}_1 = [0, \ldots, L_1]$ and let f absorb unused derivatives. Similarly, take $\mathbf{L}_2 = [0, \ldots, L_2]$, with $L_2 \in \mathbb{N}$ independent of L_1.

Following [2] and [3], if $L_1 > L_2$ the equation is considered to be *retarded*, if $L_1 = L_2$, then the equation is considered to be *neutral*, while if $L_1 < L_2$ the equation is considered to be *advanced*. In practice, one would specify a particular $\mathbf{L}_1 \subseteq [0, \ldots, L_1]$ and $\mathbf{L}_2 \subseteq [0, \ldots, L_2]$ to use. It is interesting to note that the approach developed here does not need to be altered if the equation is retarded, neutral or advanced.

The data Ψ given in the interval $[a, t_0]$, which will be referred to as the *initial history*, is taken to be analytic for convenience. The delay Δ as a function of t is assumed to be continuous and strictly increasing with $\Delta(t) < t$, to ensure $\Delta(t)$ represents a true delay. From these assumptions, the inverse Δ^{-1} exists. For convenience, the delay will be given by $\Delta(t) = \sigma t - \tau$ with some $0 < \sigma \leq 1$ and $\tau \in \mathbb{R}^+$ since $\Delta(t) < t$ for all $t \in [0, \infty)$. A state dependent delay is possible but an order of magnitude more difficult, so for convenience, only the state independent case will be considered.

The standard change of variables converts (2.1) to a first order system, so that the Picard iteration can applied to

$$\begin{cases} D\mathbf{u}(t) &= \mathbf{f}\left(t, \mathbf{u}(t), \mathbf{u}(\Delta(t))\right), \quad t > t_0 \\ \mathbf{u}(t) &= \mathbf{\Psi}(t), \quad t \in [a, t_0] \end{cases} \tag{2.2}$$

where $1 \leq l \leq \max L \equiv \{L_1, L_2\}$ and $\mathbf{u} = \left(u^l\right)_{l=0}^{L}$ is such that $u^l = D^l u$ and $\mathbf{\Psi} = \left(\Psi^l\right)_{l=0}^{L}$, hence, $\Psi^l = D^l \Psi$. For constant Ψ, note that $D^l \Psi = 0$ for $l > 0$. Thus, the approximation method may also be applied to systems of higher order DDEs of the form (2.1).

Let $\mathbf{f}$ in (2.2) be continuous with respect to t and Lipschitz continuous in the remaining variables on $[t_0, T^*] \times [-U, U]^{2L}$, with U, $T^* < \infty$. Such vector fields are called *admissible*. Denote by C_1^l the Lipschitz constant of admissible $\mathbf{f}$ with respect to component $u^l(t)$ of $\mathbf{u}(t)$ and denote by C_2^l the Lipschitz constant of admissible $\mathbf{f}$ with respect to component $u^l(t)$ of $\mathbf{u}(\Delta(t))$. Define $C_f \equiv \sum_{l=0}^{L} C_1^l + C_2^l$ and $M_f \equiv \max_l \max_{[t_0, T^*]} \mathbf{f}$. Existence and uniqueness can be extended to (2.2) via the method of successive approximations. For brevity, only a sketch of the proof is given, which covers the important bound and skips lightly over standard details.

Proposition 2.1. *For admissible vector fields* $\mathbf{f}$*, let* $T \equiv \min\left\{T^*, UM_f^{-1}\right\}$ *and let* $\mathbf{\Psi}$ *in (2.2) be analytic and such that* $M_\Psi \equiv \max_l \max_{[a,T^*]} |\mathbf{\Psi}(t) - \mathbf{\Psi}(t_0)| < \infty$ *and for* $t > t_0$*, let* $\mathbf{u}_0(t) = \mathbf{\Psi}(t)$*. For each* $k \geq 0$ *define*

$$\mathbf{u}_{k+1}(t) = \mathbf{u}_k(\tau_0) + \int_{t_0}^{t} \mathbf{f}\left(s, \mathbf{u}_k(s), \mathbf{u}_k(\Delta(s))\right)\, ds \tag{2.3}$$

then $\mathbf{u}_k(t)$ *converges uniformly to some* $\mathbf{u}_*(t)$*, which solves (2.2) uniquely over* $[t_0, T]$*.*

Proof. (sketch) Analyticity of Ψ, the FTOC and induction yield a well defined $\mathbf{u}_k$ for all $k \geq 0$. Note that $t \in [t_0, T] \implies \Delta(t) \in [a, \Delta(T)] \subset [a, T)$, so

$$|\mathbf{u}_k(\Delta(t)) - \mathbf{u}_{k-1}(\Delta(t))| \leq \max_{t \in [a,T)}$$

Combining that with the Lipschitz codition on $\mathbf{f}$ yields

$$|\mathbf{f}\left(\mathbf{u}_k(\Delta(t))\right) - \mathbf{f}\left(\mathbf{u}_{k-1}(\Delta(t))\right)| \leq \max_{t\in[a,T)} \leq C_f M_k$$

where $M_k(t) \equiv \max_{s\in[t_0,t]} |\mathbf{u}_k(s) - \mathbf{u}_{k-1}(s)|$. It is straightforward to show

$$|u_1^l - u_0^l|(t) = \left|\Psi^l(t_0) + \int_{t_0}^t f_0^l(s)\,\mathrm{d}s - \Psi^l(t)\right| \leq M_\Psi + M_f(t-t_0)$$

so both $|u_1^l - u_0^l|(t)$ and $|u_1^l - u_0^l|(\Delta(t))$ are bound by $M_\Psi + M_f(t-t_0)$ for every $1 \leq l \leq L$. So $|u_{k+1}^l - u_k^l|(t) \leq M_\Psi \frac{(t-t_0)^k}{k!} + M_f \frac{(t-t_0)^{k+1}}{k+1!}$ holds when $k = 0$. Now consider

$$|u_{k+2}^l - u_{k+1}^l|(t) \leq \int_{t_0}^t |f_{k+1}^l - f_k^l|(s)\,\mathrm{d}s \leq \int_{t_0}^t C_f^{k+1}\left(M_\Psi \frac{(t-t_0)^k}{k!} + M_f \frac{(t-t_0)^{k+1}}{k+1!}\right)\mathrm{d}s \tag{2.4}$$

and subsequently, one has

$$|u_{k+2}^l - u_{k+1}^l|(t) \leq C_f^{k+1}\left(M_\Psi \frac{(t-t_0)^{k+1}}{k+1!} + M_f \frac{(t-t_0)^{k+2}}{k+2!}\right)$$

and so $|u_{k+1}^l - u_k^l|(t) \leq M_\Psi \frac{(t-t_0)^k}{k!} + M_f \frac{(t-t_0)^{k+1}}{k+1!}$ holds for $k \geq 0$ by induction. It is straightforward to demonstrate (see [**?**] for example) to show existence, uniqueness and continuity of the solution to (2.2) by summing and using the Weierstrass M-test to obtain uniform convergence. □

3 Auxiliary Variables for DDEs

The method of steps approach to DDEs, see for example [5], allows the approach of Parker and Sochacki for ODEs, see [9], to be introduced to (2.2). Known results will be summarized and cited. In addition, intuition for the PSM procedure on ODEs will be given along the way for readers unfamiliar with this approach. In particular, auxiliary variables must be determined to achieve a polynomial, autonomous vector field that begins at time zero.

Example. Let $u'(t) = \cos(u(t)) + u(t-\tau) + u'(t-\tau)$ with $t > 0$. It can be shown that this DDE is equivalent to the quadratic system

$$\begin{cases} u' &= v + R + S \\ v' &= -w(v + R + S) \\ w' &= v(v + R + S) \end{cases}$$

if one introduces $v = \cos(u)$ and $w = \sin(u)$ along with $R(t) = u(t-\tau)$ and $S(t) = u'(t-\tau)$, so that $u' = v + R + S$, $v' = -wu' = -w(v + R + S)$ and $w' = vu' = v(v + R + S)$.

Remarks.

1. The auxiliary variables R, S are comprised of polynomial preserving transformations: delay by τ and 'take derivative'. This and the explicit nature of successive approximations allows the update of R via $R' = S$ to be bypassed with a function evaluation of u's evolution. This could not have been done for v since cos does not preserve the polynomial structure. Hence R' and S' equations are not necessary additions because they are not computationally cheaper than the change of variable.

2. If a term in the original vector field is not polynomial and has a delayed argument, then its initial history will be required. If the initial history for the components were not constant, then the history for the term will not be a polynomial. Using an appropriate Maclaurin polynomial, or a more general curve fitting, the results here will show that the error can be controlled if the error term can be made arbitrarily small.

3. Vector fields which cannot be transformed into polynomial ones exist, but they are hard to come by, see [7] for an example. Such a vector field does not have a *projectively polynomial* solution, i.e. a solution which is a component in an equivalent polynomial sytem.

In conjunction with the conversion to a polynomial vector field, [9], a standard change of variable will accommodate autonomy and the evolution also needs to begin at $t = 0$. Define $\underline{t} \equiv t - t_0$ and $\mathbf{u}(t) = \mathbf{u}(\underline{t} + t_0) = \underline{\mathbf{u}}(\underline{t})$. Since recovering the original variables is simply to shift by t_0, relabel both $\underline{t}$ and $\underline{\mathbf{u}}$ as t and $\mathbf{u}$. Due to the derivative being invariant with respect to the translation in time, (2.2) becomes

$$\begin{cases} D\mathbf{u}(t) &= \mathbf{f}(\mathbf{u}(t), \mathbf{u}(\Delta(t))), & t \in (0, T) \\ \mathbf{u}(t) &= \mathbf{\Psi}(t), & t \in [a - t_0, 0] \end{cases} \tag{3.1}$$

where $\mathbf{u}$ now includes not only derivatives of u but the auxiliary variables necessary to make the vector field be polynomial and autonomous.

PSM for initial value problems, see [7], [9] and [10], will produce one term in the Maclaurin series for $\mathbf{u}$ in (3.1) for each Picard iteration performed. In particular, any polynomial vector field ca be made into a quadratic one, hence the integration in Picard's is reduced to convolving appropriate terms, adding one to the exponent and dividing by that exponent. In addition, only the next power of the expansion needs to becomputed, so the entire vector field is not needed in each integration. the idea extends to PDEs as well.

The method of steps will now be applied to (3.1). Define $\tau_0 \equiv 0$ and $\tau_{-1} \equiv a - t_0 = \Delta(\tau_0)$ so that a is determined by Δ via $a = t_0 + \Delta(\tau_0)$. Suppose now that $\mathbf{u}$ is known over $[\tau_{m-1}, \tau_m]$. By standard theory, see e.g. [1], if the vector field is Lipschitz continuous, then the unique solution that ensues would be valid over $[\tau_m, \tau_{m+1}]$, where τ_{m+1} is the time which delays to τ_m, the edge of the current information: $\tau_{m+1} = \Delta^{-1}(\tau_m)$. Denote this unique solution by $\mathbf{u}_m(t) \equiv \mathbf{u}(t),\ t \in [\tau_m, \tau_{m+1}]$, where the time intervals are referred to as *τ-blocks*. Define $\mathcal{T} \equiv \{\tau_m \mid \Delta(\tau_m) = \tau_{m-1}, m \in \mathbb{Z}_0^+\}$.

Noticing that $\Delta : [\tau_m, \tau_{m+1}] \to [\tau_{m-1}, \tau_m]$, consider composition, denoted by

$$\Delta_m(t) \equiv \Delta \circ \cdots \circ \Delta(t) \tag{3.2}$$

with $\Delta_1(t) = \Delta(t)$ and $m > 0$. Define $\Delta_0(t) \equiv t = t - \tau_0$. It follows that $\Delta_m(\tau_m) = \tau_0$ for all $m \geq -1$. The linear in time and state independent delay

$$\Delta(t) = \sigma t + \tau_{-1} \tag{3.3}$$

with $0 < \sigma \leq 1$ and *lag* $\tau_{-1} < 0$, includes the constant lag case: $\Delta(t) = t - \tau$ for some fixed $\tau > 0$. In this case, one has $\tau_m = m\tau$ for all $m \geq 0$, while $\Delta_m(t)$ reduces to $t - m\tau$ and τ_m reduces to $m\tau$.

A change of variables will be invoked as the problem enters a new τ-block. For $t \in [\tau_m, \tau_{m+1}]$ define $\underline{t} \equiv \Delta_m(t)$ so that $\underline{t} \in [\tau_0, \tau_1]$ and let $\mathbf{u}_m(t) = \mathbf{u}(\Delta_m^{-1}(\underline{t})) \equiv \mathbf{U}_m(\underline{t})$ so that $\mathbf{U}_m = \mathbf{u} \circ \Delta_m^{-1}$. Now (3.1) can be written, for each m such that $\tau_m < T$

$$\begin{cases} D\mathbf{U}_m(\underline{t}) &= \mathbf{f}_\Delta\left(\mathbf{U}_m(\underline{t}), \mathbf{U}_{m-1}((\underline{t}))\right), \quad \underline{t} \in (\tau_0, \tau_1] \\ \mathbf{U}_m(\underline{t}) &= \mathbf{U}_{m-1}(\underline{t}), \quad \underline{t} \in [\tau_{-1}, \tau_0] \end{cases} \tag{3.4}$$

where $\mathbf{f}_\sigma = (\sigma_m^{-1}) \cdot \mathbf{f}$.

The important part here, especially for the computation, is that the $\mathbf{u}_{m-1}(\Delta(t))$ term in the vector field of (2.2) when $t \in [\tau_m, \tau_{m+1}]$ appears to lose its delayed argument, but really it has been absorbed by the local variable $\underline{t}$:

$$\mathbf{u}_{m-1}(\Delta(t)) = \mathbf{U}_{m-1}(\Delta_{m-1}(\Delta(t))) = \mathbf{U}_{m-1}(\Delta_m(t)) = \mathbf{U}_{m-1}(\underline{t})$$

when $t \in [\tau_m, \tau_{m+1}]$. This allows for rapid computation of successive $\mathbf{U}_m$'s since the previously computed approximation can be used in its current form as the delay term in the vector field and one only needs to run a normal PSM problem over each τ-block. Note that the solution to (2.2) can be recovered via

$$\mathbf{u}_m(t) = \mathbf{U}_m(\Delta_m(t))$$

and the solution to (2.2) is recovered upon shifting time further by t_0.

Since (3.4) implies all problems exist computationally over a fixed interval $[\tau_0, \tau_1]$, then for a given m, $\mathbf{U}_{m-1}(\underline{t})$, with $\underline{t} \in [\tau_{-1}, \tau_0]$ will be referred to as the *initial history* for the $[\tau_m, \tau_{m+1}]$ problem.

3.1 Stiff Problems

To accommodate stiff problems, it becomes necessary to subdivide τ-blocks to allow lower order polynomials a chance to approximate decaying exponentials. To this end, break all τ-blocks into N *segments* of length δ, so that $N\delta = \tau$. Let i be an index for the segment endpoints, i.e. $i = 0, \ldots, N$. Then in an arbitrary segment in an arbitrary τ-block, let $\mathbf{U}_{mi}(\bar{t}) = \mathbf{U}_m(\bar{t})$ for $\bar{t} \in [i, i+1]\delta$. Then (3.4) becomes

$$\begin{cases} D\mathbf{U}_{mi}(\bar{t}) &= \mathbf{f}\left(\mathbf{U}_{mi}(\bar{t}), \mathbf{U}_{m-1,i}(\bar{t})\right), \quad \bar{t} \in (i, i+1]\delta \\ \mathbf{U}_{mi}(\bar{t}) &= \mathbf{U}_{m-1,i}(\bar{t}), \quad \bar{t} \in [i-1, i]\delta \end{cases}$$

To accommodate PSM, one more change of variable will be invoked, resetting the time in each segment to begin at zero: let $\underline{t} = \bar{t} - i\delta$ with $\underline{\mathbf{U}}_{mi}(\underline{t}) = \mathbf{U}_{mi}(\bar{t})$, which is independent of m, or which τ-block is under consideration. Hence, the previous IVP becomes

$$\begin{cases} D\underline{\mathbf{U}}_{mi}(\underline{t}) &= \mathbf{f}\left(\underline{\mathbf{U}}_{mi}(\underline{t}), \underline{\mathbf{U}}_{m-1,i}(\underline{t})\right), \quad \underline{t} \in (0, \delta] \\ \underline{\mathbf{U}}_{mi}(\underline{t}) &= \underline{\mathbf{U}}_{m-1,i}(\underline{t}), \quad \underline{t} \in [-\delta, 0] \end{cases} \tag{3.5}$$

and upon completion of the computation, the original approximation can be found from

$$\mathbf{u}_m(t) = \underline{\mathbf{U}}_{mi}(\Delta_m(t) - i\delta))$$

For the constant lag case, $\Delta_m(t) - i\delta = t - (m\tau + i\delta)$. Note that (3.5) contains (3.4) if $N = 1$. A small benefit to stiff problems occurs because of the small segment lengths. When the initial history is not constant, a smaller order can be used to achieve the same accuracy when compared to having to expand over the whole τ-block.

4 qPSM Procedure and Error Analysis

A procedure will now be outlined by specifying how PSM and method of steps interact. In [8], PSM was applied globally, i.e. across τ-blocks. Here, PSM is applied to a single τ-block and then repeated in subsequent τ-blocks. While solving only locally, i.e. over $[\tau_m, \tau_{m+1}]$, this version avoids some messier computations arising from nonlinear interactions between delay terms and non-delayed terms.

Consider (3.4) initially for convenience, and apply PSM to this IVP over $t \in [\tau_0, \tau_1]$, i.e. with $m = 0$, to generate the kth order Maclaurin expansion to $\mathbf{u}_0(t)$. To link PSM with the method of steps, use this Maclaurin expansion as the initial history for (3.4) over $[\tau_1, \tau_2]$ and apply PSM again. Repeat this until the desired τ-block $[\tau_m, \tau_{m+1}]$ is reached, using the previously calculated apprximation as the initial history. Conditions so that this approach is stable with respect to initial history errors will be developed after some notation.

The PSM approximation to $\mathbf{U}_m(\underline{t})$ from (3.4), denoted $\mathbf{U}_m^k(\underline{t})$, is computed via

$$\mathbf{U}_m^{k+1}(\underline{t}) = \mathbf{U}_m^k(\tau_0) + \int_{\tau_0}^{\underline{t}} \mathbf{f}(\mathbf{U}_m^k(s), \mathbf{U}_{m-1}^k(s))\, \mathrm{d}s \tag{4.1}$$

for $k \in Z^+$ and $k^* = k_{m-1}^*$ is the largest iteration achieved by PSM when it was applied over $[\tau_{m-1}, \tau_m]$. Since k^* is finite, an additional error is introduced through the initial history and propagated.

Now consider an idealized system: (3.4) and the previously calculated initial data was error free, so that the only error that would be incurred is the PSM error for the current τ-block. Denote the solution as $\mathbf{v}_m(t) = \mathbf{V}_m(\underline{t})$, where $\mathbf{V}_m(\underline{t})$ solves

$$\begin{cases} D\mathbf{V}_m(\underline{t}) &= \mathbf{f}\left(\mathbf{V}_m(\underline{t}), \mathbf{V}_{m-1,i}(\underline{t})\right), \quad \underline{t} \in (\tau_0, \tau_1] \\ \mathbf{V}_m(\underline{t}) &= \mathbf{V}_{m-1}(\underline{t}), \quad \underline{t} \in [\tau_{-1}, \tau_0]\delta \end{cases} \tag{4.2}$$

Denote by $\mathbf{V}_m^k$ the PSM approximation to (4.2) over $[\tau_m, \tau_{m+1}]$. Note that $\mathbf{U}_m^k \to \mathbf{V}_m^k$ as $k_{m-1}^* \to \infty$.

To prove convergence of qPSM approach, it is necessary to show that it is stable with respect to initial data error, since after $[\tau_0, \tau_1]$, the PSM error incurred with be transferred to $[\tau_1, \tau_2]$ through both an error in the initial height of the approximation as well as error in the delayed argument. To this end, consider the error between the qPSM approximation and the solution to the IVP

$$|\mathbf{U}_m^k - \mathbf{U}|(\underline{t}) \le |\mathbf{U}_m^k - \mathbf{V}_m^k|(\underline{t}) + |\mathbf{V}_m^k - \mathbf{U}|(\underline{t})$$

The second term on the right hand side of the inequality is just the PSM error, since the initial data is error free. This error is quantified in [10]. Hence, it remains to look at the first term. Let $E_m^k \equiv \max_{[\tau_0,\tau_1]} |\mathbf{U}_m^k - \mathbf{V}_m^k|(\underline{t})$ and for convenience, assume k_m^* is constant for all m. Then invoking (4.1) along with (4.1) with $\mathbf{U}$ replaced by $\mathbf{V}$, using the triangle inequality and Lipschitz condition, there is

$$E_{km} \quad \le \quad E_{m-1}^{k^*}(1 + C_2\tau_1) + C_1\tau_1 E_m^{k-1} \tag{4.3}$$

Noting that $k^* = k_{m-1}^*$ will not change as this inequality is iterated, there is

$$\begin{aligned} &\le \quad E_{m-1}^{k^*}(1 + C_2\tau_1) + C_1\tau_1(E_{m-1}^{k^*}(1 + C_2\tau_1) + C_1\tau_1 E_m^{k-2}) \\ &\le \quad \sum_{i=0}^{k-1}(1 + C_2\tau_1)(C_1\tau_1)^i E_{m-1}^{k^*} + (C_1\tau_1)^k E_m^0 \end{aligned} \tag{4.4}$$

By construction, $E_m^0 = \max_{[\tau_m,\tau_{m+1}]} |\mathbf{u}_m^0 - \mathbf{v}_m^0|(t) = |\mathbf{u}_{m-1}^{k^*} - \mathbf{v}_{m-1}^{k^*}|(\tau_m) \leq E_{m-1}^{k^*}$, so that the inequality may be updated to, when $k = k_m^*$,

$$\begin{aligned} E_m^{k^*} &\leq (1 + C_2\tau_1)\left(1 + \sum_{i=1}^{k^*}(C_1\tau_1)^i\right) E_{m-1}^{k^*} \\ &\leq \left((1 + C_2\tau_1)\left(1 + \sum_{i=1}^{k^*}(C_1\tau_1)^i\right)\right)^m E_0^{k^*} \equiv M_{k^*m}^{f\tau} E_0^{k^*} \end{aligned}$$

Computing conditions imply k^* and m are both finite. It follows that $M_{k^*m}^{f\tau}$ is finite and so there exists an $E_0^{k^*}$ small enough so that $E_m^{k^*} \leq \epsilon$ for any $\epsilon > 0$. If $k^* \to \infty$ then dependng on whether $C_1\tau_1 \leq 1$ or $c_1\tau > 1$, the approach converges as before. Otherwise, the initial history error $E_0^{k^*}$ must go to zero faster than the sum of the $C_1\tau_1$ terms in order to converge.

In the case of $N > 1$ segements, the multiplier $M_{k^*m}^{f\tau}$ now has an exponent mN rather than m and the previous analysis can be applied with minor modifications. Computationally, with N segments, then N different sets of initial histories must be stored and they are referenced in the same order.

5 Numerical Experiments

Plots for approximations to two numerical experiments are presented. The first experiment, dubbed the *circuit problem*, can be found in [6] and this system is a three component linear system, which is both neutral as well as *moderately* stiff (eigenvalue ratio is $3.1135 \cdot 10^5$). The approximation uses a 30th degree polynomial and segemnt lengths of 0.01, with $\tau = 1$. Cubic approximations to the initial history are used, which are comprised of sine functions. The overall view of the three components is given in Figure 1, along with a close up of the approximations near $t = 0$ and $t = 1$. The neutral term is responsible for the discontinuity of the first derivative at $t = \tau_m$ for $m > 0$.

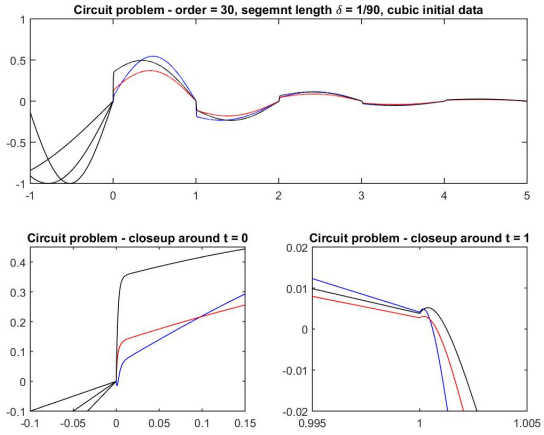

Figure 1: Circuit problem - stiffness ratio is $3.1135 \cdot 10^5$

The second experiment involves the Kuramoto model using $N = 12$ oscillators with a constant lag time delay, see [11], where the notation used in that paper is followed. This problem in terms of auxiliary variables becomes a $2N^2 + n = 300$ component system. The top subplot in Figure 2 is an instance of synchronization; the phase differences for oscillator

i and oscillator $i+1$ for $i = 1, \ldots, 11$ are shown, and note all phase differences are smaller than 10^{-4} (mod 2π). The bottom subplot shows an instance of incoherence; the oscillators don't synchronize.

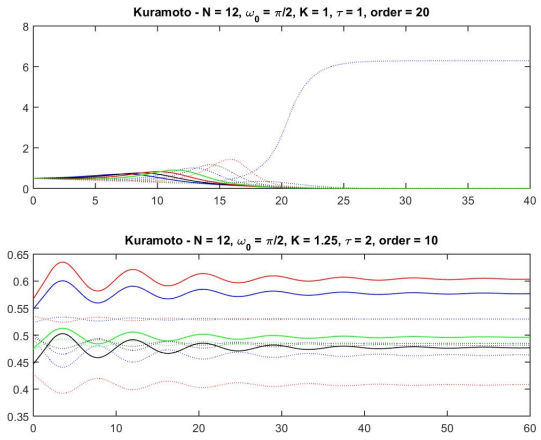

Figure 2: Kuramoto - 12 oscillators, $\omega_0 = \frac{1}{2}\pi$, synchronization and incoherence

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