* Logic and Proofs *

* Preposition :-

A preposition is a declarative statement which is either true or false but not both.

e.g.

1) Delhi is a capital of India. (True) → Prepositio
2) Sun rises in a west (False) → preposition.
3) $2 + 1 = 3$ (True) → preposition.
4) $2 + 2 = 5$ (False) → Preposition.

1) what time it is? → Not preposition.
2) Read it carefully → Not preposition.
3) $x + 1 = 5$ → Not preposition.
4) $x + Y = Z$ → Not preposition.
5) What an astonishing day! (Exclamatory Sentences)
6) You were amazing!

No. of columns $= 2^n$ .... n = no. of prepositio

* Negation :-

Let P be a preposition then its Negation is denoted by $\bar{P}$ / $\sim p$. It is defined as :

It is not a case that ______ P ______

e.g.

p : It is a parrot.

$\sim p$: It is not a parrot

$\sim p$: It is not a case that It is a parrot.

Conjuction, Disjunction,
Exclusive OR

Example 2: Let r: 2 divides 4
s: 2 divides 6.

Then $r \wedge s$ is the statement:

$r \wedge s$: 2 divides 4 and 2 divides 6. or 2 divides both 4 and 6.

As r is T & s is T, it follows that $r \wedge s$ is T.

Exampale 3: Let p: 5 is an integer
q: 5 is not an odd integer.

~~p is T~~ ; ~~q is F~~ here $p \wedge q$: 5 is an integer and 5 is not an odd integer.

As p is T
q is F
$\therefore$ $p \wedge q$ is F.

Also; $p \wedge q$: 5 is an integer but 5 is not an odd integer
We Can write

Truth table :

p	~p
T	F
F	T

$\sim(\sim p) = p$

$\sim(\sim(\sim(\sim p))) = \sim p$

Conjuction :-

Let p and q be two preposition then the conjuction of p & q is denoted by $p \wedge q$ and it is true when both statements are true otherwise false

p : He is teacher.
q : He is single.

$p \wedge q$: He is teacher and single

Truth table :-

p	q	$p \wedge q$	$q \wedge p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

$p \wedge q = q \wedge p$

→ $\wedge$ operator is cummutative

Def. 3: Disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

☆ For given two statements p and q, we can form the statement "p or q" by putting word "or" between the statements such that the ~~s~~ p or q is true if at least one of the statements p or q is true.

Truth table:

Example 1: p: 2 is an integer
q: 3 is greater than 5.

then $p \vee q$: 2 is an integer or 3 is greater than 5.

Here, p is T & q is F ;
it follows that $p \vee q$ is true.

p	q	$p \vee q$
T	F	T

Example 2:

p: 2^2+3^3 is an even integer
q: 2^2+3^3 is an odd integer

Then $p \vee q$: 2^2+3^3 is an even integer or 2^2+3^3 is an odd integer.

OR we can also write

$p \vee q$: Either 2^2+3^3 is an even integer or 2^2+3^3 is an odd integer

OR

$p \vee q$: 2^2+3^3 is an even integer or an odd integer.

Now $2^2+3^3 = 4+27 = 31$ is an odd integer

p	q	$p \vee q$
F	T	T

(Inclusive OR)

Disjunction of two preposition :-

Let p and q be two preposition then disjunction of p and q is denoted by $p \vee q$ and it is false if both the statement are false.

p: He is a teacher

q: He is a singer

$p \vee q$: Either he is teacher or a singer.

Truth table :-

p	q	$p \vee q$	$q \vee p$	
T	T	T	T	
T	F	T	T	
F	T	T	T	
F	F	F	F	

$p \vee q = q \vee p$ → cummutative

$p \vee (q \vee r) = (p \vee q) \vee r$ → Assosiative

disjunction → Inclusive or

Exclusive OR: $\oplus$ (XOR): Either p or q is true [not both], then $p \oplus q$ is true

Example: p: Student who have taken calculus can take this class.

q: Student who have taken Computer Science can take this class.

$p \vee q$: student who have taken calculus or Computer Science can take this class.

$p \oplus q$: Student who have taken calculus or computer Science, but not both can take this class.

$\wedge$ Both T ✓

$\vee$ atleast 1 T ✓

$\oplus$ Exactly 1 T ✓

(Conjunction) $\wedge$ = Both true ✓

disjunction $\vee$ = ~~Both false~~

Exactly one T ✓

$\oplus$ = atleast one

★ Conditional Statements: → Converse
→ Contra-positive
→ Inverse
→ Bi-conditional

★ Let p and q be two propositions. Then "if p, then q" is a statement called an implication, or a condition, written as $p \to q$.
The conditional statement $p \to q$ is false when p is true and q is false; otherwise it is considered true.

★ In the implication $p \to q$, p is called the hypothesis (or antecedent or premise) & q is called the conclusion (or consequence)

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Inverse

q	p	$q \to p$
T	T	T
F	~~F~~ T	~~F~~ T
T	F	F
F	F	~~F~~ T

- **Exclusive or (XOR) :-** Let p and q be propositions. The exclusive or of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.

Truth table :

p	q	$p \oplus q$	$q \oplus p$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	F	F

$p \oplus q = q \oplus p$ .... cummutative

$q \oplus (p \oplus R) = (q \oplus p) \oplus R$ .... Assosiative.

*** Conditional statements :-**

Let p and q be two prepositions then the conditional of p and q is denoted by $p \rightarrow q$ and it is false if p is true and q is false otherwise true.

$p \rightarrow$ Hypothesis

$q \rightarrow$ conclusion.

e.g.

p : You are born in Delhi.

q : You are born in India.

$p \rightarrow q$: If you are born in Delhi then you are born in India.

* If it is Cold then I will wear a jacket.

* If ABC is a triangle then $\angle A + \angle B + \angle C = 180^\circ$

* If my program has no syntax error then it will compile

*

* Let p and q be two statements

(i) The statement $q \to p$ is called the <u>converse</u> of the implication $p \Rightarrow q$.

(ii) The statement $\sim p \to \sim q$ is called the <u>Inverse</u> of the conditional statement $p \to q$.

(iii) The statement $\sim q \to \sim p$ is called the Contrapositive " " $p \to q$.

<u>Example</u>: Consider the statement "If today is Sunday, then I will go for a walk".

Let p & q be the foll. statements:

p: Today is Sunday

q: I will go for a walk.

$p \Rightarrow q$

Converse $(q \to p)$: If I will go for a walk, then today is Sunday.

Inverse $(\sim p \to \sim q)$: If Today is not Sunday, then I will not go for a walk.

Contrapositive $(\sim q \to \sim p)$: If I will not go for a walk, then today is not Sunday.

p	q	$p \to q$	$q \to p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

$p \to q = q \to p$

Converse, Inverse and contrapositive :-

$p \to q$

converse : $q \to p$

Inverse : $(\sim p \to \sim q)$

contra(+ve): $(\sim q \to \sim p)$

Que- What are the contra(+ve), converse and Inverse of the following conditional statement

The home team wins whenever its raining.

p: It is raining

q: The home team wins.

converse $(q \to p)$: If the home team wins then it is raining.

Inverse $(\sim p \to \sim q)$: If it is not raining then home team does not win.

contra(+ve) $(\sim q \to \sim p)$: If the home team does not win then it is not raining.

Biconditional: Let p and q be two statements, then "p if and only if q" written as $p \Leftrightarrow q$, is called biconditional statement of p and q.

Example: ☆ You can get this shirt if and only if you pay for it.
☆ My program will compile if and only if it has no syntax errors.

For two statements (propositions) p & q

$\sim p$: negation p ; $p \wedge q$: Conjuction ; $p \vee q$: disjunction (inclusive or) ;
$p \oplus q$: Exclusive or ; $p \rightarrow q$: conditional ; $p \Leftrightarrow q$: biconditional

The symbols $\sim, \wedge, \vee, \oplus, \rightarrow, \Leftrightarrow$ are called logical connectives.

Precedence of Logical operators

Operator	Precedence
$\sim$	1
$\wedge$	2
$\vee$, $\oplus$	3
$\rightarrow$	4
$\leftrightarrow$	5

Eg1) $\sim p \wedge q$ — $\sim(p \wedge q)$ ✗ ; $(\sim p) \wedge q$ ✓

Eg2) $p \vee q \rightarrow r$ — $p \vee (q \rightarrow r)$ ✗ ; $(p \vee q) \rightarrow r$ ✓

Eg3 $p \wedge q \vee r$ — $(p \wedge q) \vee r$ ✓ ; $p \wedge (q \vee r)$ ✗

☆ Some other ways to express $p \Leftrightarrow q$

- "p is necessary and sufficient for q"
- If p then q, and conversely"
- p iff q"

Truth table :

p	q	$p \to q$	$q \to p$	$\sim p$	$\sim q$	$\sim p \to \sim q$	$\sim q \to \sim p$
T	T	T	T	F	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

$p \to q = \sim q \to \sim p$ (Imp)

$q \to p = \sim p \to \sim q$

* Demorgan's law :-

1) $\sim(p \vee q) = \sim p \wedge \sim q$

2) $\sim(p \wedge q) = \sim p \vee \sim q$

* Biconditional :- or Biimplication

Let P and q be two preposition then biconditional of p and q are denoted by $p \leftrightarrow q$ and it is true If both P and q have same truth value otherwise it is false.

e.g.

p : you can take a flight

q : you buy a ticket.

$p \leftrightarrow q$: you can take a flight if and only if you buy a ticket.

Propositional Equivalences

* **Tautology**: A compound proposition that is always true, no matter what the truth value of the propositions that occur in it.

* **Contradiction**: A compound proposition that is always false is called a contradiction.

* **Contingency**: A compound proposition that is neither a tautology nor a contradiction is called a contingency.

Example 1:

disjunction (atleast one TV) — conjuction (both True ✓)

p	$\sim p$	$p \vee \sim p$	$p \wedge \sim p$
T	F	T	F
F	T	T	F

here
As $p \vee \sim p$ is always true, so it is a tautology.
& $p \wedge \sim p$ is always false, so it is a contradiction.

Example 2: $\sim p \wedge q \rightarrow (\sim(q \rightarrow p))$

p	q	$\sim p$	$\sim p \wedge q$	$q \rightarrow p$	$\sim(q \rightarrow p)$	$\sim p \wedge q \rightarrow \sim(q \rightarrow p) = A$
T	T	F	F	T	F	T
T	F	F	F	T	F	T
F	T	T	T	F	T	T
F	F	T	F	T	F	T

$\therefore$ A is tautology

gical Equivalences: Compound propositions that have the same truth values in all possible cases e called logical equivalent.

e The compound propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology.

nd it is denoted by $\boxed{p \equiv q}$.

To check whether p & q are logically equivalent or not, check if $p \leftrightarrow q$ has truth value only 'T'.

Show that $\sim(p \vee q)$ and $\sim p \wedge \sim q$ are logically equivalent.

q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$p \vee q$	$\sim(p \vee q)$
T	F	F	F	T	F
F	F	T	F	T	F
T	T	F	F	T	F
F	T	T	T	F	T

$\therefore \sim(p \vee q) \equiv (\sim p \wedge \sim q)$

Show that $p \rightarrow q$ & $\sim p \vee q$ are logically equivalent

p	q	$\sim p$	$p \rightarrow q$	$\sim p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

$\therefore p \rightarrow q \cong \sim p \vee q$

Laws of Logical Equivalence

Identity Laws	$p \wedge T \equiv p$ $p \vee F \equiv p$
Domination Laws	$p \vee T \equiv T$ $p \wedge F \equiv F$
Idempotent Laws	$p \vee p \equiv p$ $p \wedge p \equiv p$
Double Negation Law	$\sim(\sim p) \equiv p$
Commutative Laws	$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$
Associative Laws	$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Distributive Laws	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
De Morgan's Laws	$\sim(p \vee q) \equiv \sim p \wedge \sim q$ $\sim(p \wedge q) \equiv \sim p \vee \sim q$
Absorption Laws	$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$
Negation Laws	$p \vee \sim p \equiv T$ $p \wedge \sim p \equiv F$

Truth table :

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

$$p \leftrightarrow q = (p \leftrightarrow q) \wedge (q \rightarrow p)$$

Propositional Equivalences

Toutology → always true statement.

$p \vee T \equiv T$

$p \wedge T \equiv p$

$p \vee F \equiv p$

$p \wedge F \equiv F$

p	T	$p \wedge T$
T	T	T
F	T	F

$= p$

$p \vee \sim p \equiv T$

$p \wedge \sim p \equiv F$

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (q \wedge r)$

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (q \vee r)$

Important Laws

$p \rightarrow q \equiv \sim p \vee q$

Predicate refers to the property that the subject of the statement can have.

* x is greater than 3

x ↙ Variable

is greater than 3 ↳ Predicate ~~⟹ is propert~~

⟹ is greater than 3 is the value that x can have.

Eg 1

$P(x): x > 3$

P → Propositional function

$P(x)$ → value of Propositional function P at x

* When 'x' is variable, we cannot decide whether $P(x)$ is proposition or not.

As we assign some value to 'x', we get proposition.

Eg: At $x = 4$; $P(4): 4 > 3$

⟹ It is true

* At $x = 2$; $P(2): 2 > 3$

⟹ It is false proposition

Example 2:

Let $Q(x, y)$ denote the statement "$x = y + 3$", what are the truth values of the propositions $Q(1, 2)$ & $Q(3, 0)$?

Sol: $Q(x, y): x = y + 3$

Now $Q(1, 2): 1 = 2 + 3$

⟹ $1 = 5$ which is false.

& $Q(3, 0): 3 = 0 + 3$

: $3 = 3$

which is true.

Similarly, we can let $R(x, y, z)$ denote the statement "$x + y = z$". when values will be assigned to x, y, z, this statement has a truth val.

Eg: what are truth value of proposition $R(1, 2, 3)$ & $R(0, 0, 1)$ if

$R(x, y, z): x + y = z$

Sol: Now $R(1, 2, 3): 1 + 2 = 3$

: $3 = 3$

∴ Its truth value is true

$R(0, 0, 1): 0 + 0 = 1$

: $0 \neq 1$

∴ Its truth value is false

Predicates & Quantifiers

Statements involving Variables, such as "$x > 3$," "$x = y+3$", "$x + y = z$"

*** Predicates :-**

Statements involving variables are not called preposition. OR Predicate is a sentence depending on variables which become a proposition upon substituting values in a domain.

$P(x) = x > 3$ ..... $x \to$ subject
or $> 3 \to$ predicate.

$P(x)$: Computer x is functioning properly.

but when we asign a particular value to x then it becomes preposition.

$P(x) : x > 3$

$P(4) = 4 > 3$ its truth value is true.

$P(2) : 2 > 3$ its truth value is false.

e. Let $Q(x, y) : x = y + 3$ what are the truth values of $Q(1, 2)$ and $Q(3, 0)$

→

$Q(x, y) : x = y + 3$

$Q(1, 2) : 1 \neq 5$

Its truth value is false.

$Q(3, 0) : 3 = 3$

Its truth value is true.

* In general, a statement involving the n variables $x_1, x_2, \ldots, x_n$ can be denoted by $P(x_1, x_2, \ldots, x_n)$.

A statement of the form $P(x_1, x_2, \ldots, x_n)$ is the value of the propositional function P at n-tuple $(x_1, x_2, \ldots, x_n)$, & P is also called an n-place predicate or a ~~n-array~~ n-ary predicate

Universal. $\forall x P(x)$

Eg: In section A12 of 70 students

$P(x)$: x is Present in class

So, $P(x_1)$: x_1 is Present in class → It is true
$P(x_3)$: x_3 " " → T

∀ student in A12 $P(x)$ is True

Let ~~RX~~ x_4 is absent
$P(x_4)$: x_4 is Present in class → F

Remark: ☆ Generally, an implicit assumption is made that all domains of discourse for quantifiers are non-empty.

Note: if the domain is empty, then $\forall x P(x)$ is true for any propositional function $P(x)$; because there are no elements 'x' in the domain for which $P(x)$ is false.

$\forall x P(x)$ is F, if ∃ some x s.t. $P(x)$ is false

☆ If the domain is empty, then $\exists x Q(x)$ is false for $Q(x)$, because when the domain is empty, there can be no element 'x' in the domain for which $Q(x)$ is true.

$\exists x Q(x)$ is F, if it is false for every value of x

To create a proposition from a propositional function, the quantification is used. Quantification express the extent to which

* **Quantifiers :** a predicate is true over a range of elements.

Quantifiers are of two types.
1) Universal quantifiers.
2) Existential quantifiers.

* In English, the words all, some, many, none and few are used in quantification.

1) **Universal quantifier :** "The universal quantification of $P(x)$ is the statement "$P(x)$ for all values of x in the domain""

The notation of $\forall x\, P(x)$ denotes the universal quantification of $P(x)$. Here $\forall$ is called the universal quantifier. We read $\forall x\, P(x)$ as "for all x $P(x)$" and "for every x $P(x)$". An element for which $P(x)$ is false is called a counterexample of $\forall x\, P(x)$.

Eg1) Let $P(x)$ be the statement "$x+1 > x$". What is the truth value of quantification $\forall x P(x)$, where the domain consist of all real numbers?

$P(x)$: $x + 1 > x$ $\quad x \in R$ $\rightarrow$ always true.

$\forall x\, P(x)$ is true.

Eg2

Let $Q(x)$: $x < 2$ what is the truth value of $\forall x\, P(x)$

$\rightarrow$ $Q(x)$: $x < 2$

$Q(3)$: $3 < 2$

The truth value of $\forall x\, P(x)$ is false.

Eg3

Let $Q(x)$: $x^2 > 0$ where $x \in R$ what is the truth value of $\forall x\, Q(x)$

$\rightarrow$ $Q(x)$: $x^2 > 0$

$Q(0)$: $0^2 > 0$

$0 > 0$ (false)

The truth value of $\forall x\, Q(x)$ is false.

Que- Let $g(x): x^2 \geq x$ what is the truth value of $\forall x\, g(x)$

1) $x \in R$ 2) $x \in Z$

→ $g(x): x^2 \geq x$

where $x \in R$

$g(1/2): \left(\frac{1}{4}\right) \geq \left(\frac{1}{2}\right)$

which is false

∴ Truth value of $\forall x\, g(x)$ is false.

$g(x): x^2 \geq x$

where $x \in Z$

$\forall x\, g(x)$ is true where $x \in Z$

* When all the elements in the domain can be listed as $x_1, x_2 \cdots, x_n$ it follows that $\forall x\, P(x)$ it is same as saying that:

$$\forall x\, P(x): P(x_1) \wedge P(x_2) \wedge P(x_3) \cdots \wedge P(x_n)$$

Que- What is the truth value $\forall x\, P(x)$ where $P(x): x^2 < 10$ and domain consist of positive integer not exceeding 4?

→ Domain: 1, 2, 3, 4

$\forall x\, P(x) \equiv P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge (P(x_4)$

$P(1) = 1 < 10$ (T)

$P(2) = 4 < 10$ (T)

$P(3) = 9 < 10$ (T)

$P(4) = 16 < 10$ (F)

Existential Quantifier

Eg: What is the truth value for $\exists x\, P(x)$; $\{1,2,3,4\}$
where $P(x): x^2 > 10$ in the domain

Sol:

Now $P(1): 1^2 > 10$
$1 > 10$ (F)

$P(2): 2^2 > 10$
$4 > 10$ (F)

$P(3): 3^2 > 10$
$9 > 10$ (F)

$P(4): 4^2 > 10$
$16 > 10$ (T)

$\therefore \exists\, x = 4$ for which $P(x)$ is true.
$\therefore$ truth value for $\exists x\, P(x)$ is true.

So; if $P(1) \vee P(2) \vee P(3) \vee P(4)$ is true (if any of $P(x)$ is true)

*So, in existensial quantifier, we just need to find one value for which $P(x)$ is true.

Table: for Quantifiers

Statement	When True?	When False?
$\forall x\, P(x)$	$P(x)$ is true for every x.	there is an x for which $P(x)$ is false
$\exists x\, P(x)$	There is an x for which $P(x)$ is true	$P(x)$ is false for every x.

Eg: Determine the truth value, for following statements having real domain

a) $\exists x (x^3 = -1)$ as $\exists\, x = -1$ such that $(-1)^3 = -1$ $\therefore \exists x (x^3 = -1)$ is T

b) $\exists x (x^4 < x^2)$ let $x = \frac{1}{2}$ $\left(\left(\frac{1}{2}\right)^4 < \left(\frac{1}{2}\right)^2\right) = \frac{1}{16} < \frac{1}{4}$

(c) $\forall x ((-x)^2 = x^2)$ As $\forall x$, this result (is) hold. $\therefore$ it is T

d) $\forall x (2x > x)$ if $x = -1$; then $-2 \not> 2$ $\therefore$ it is False statement

$\therefore \forall x P(x) \equiv T \wedge T \wedge T \wedge F$

$\equiv F$

$\therefore$ Truth value of $\forall x P(x)$ is false.

The Existential quantification of $P(x)$ is the Proposition "There exist an element x in the domain such that $P(x)$." We use the notation: $\exists x P(x)$ for the existential quantification of $P(x)$.

2) Existential Quantifier:-

If we can find one such x for which $P(x)$ is true then $\exists x P(x)$ it truth value is true.

$\exists x P(x)$ Here '$\exists$' is called the existential quantifier.

If we cannot find any x for which $P(x)$ is true then the truth value of $\exists x P(x)$ is false.

$P(x): x > 3$ what is the truth value of $\exists x P(x)$?

$P(4): 4 > 3$ (True)

$\therefore$ The truth value of $\exists x P(x)$ is true.

$Q(x): x = x+1$ what is the truth value of $\exists x P(x)$

$Q(2): 2 = 3$ (false)

$Q(-2): -2 = -1$ (false)

$\therefore$ Truth value of $\exists x P(x)$ is false.

when all element in the domain can be listed as $x_1, x_2, x_3 \cdots x_n$ then this. If any value true the truth value is true.

$\exists x P(x) = P(x_1) \vee P(x_2) \vee P(x_3) \cdots \vee P(x_n)$

Some Terminologies

Theorem: Formally, a theorem is a statement that can be shown to be true.
Proved

Propositions: Less important theorems sometimes are called propositions.

Proof: A proof is a valid argument that establishes the truth of a theorem.

Axioms or Postulates: The statements used in proof can include axioms, which are statements we assume to be true.

Lemma: A less important theorem that is helpful in the proof of other results is called a lemma.
Complicated proofs are usually easier to understand when they are proved using a series of lemma, where each lemma is proved individually.

Corollary: A corollary is a theorem that can be established directly from a theorem, that ~~can b~~ has been proved.

Conjecture: A conjecture is a statement that is being proposed to be a true statement, usually on the basis of some partial evidence, a heuristic argument, or the intuition of an expert. When a proof of a conjecture is found, the conjecture becomes a theorem.

Different Proof methods

Proof: A proof is a valid argument that establishes ~~the truth~~ of a mathematical statement.

* **Direct Proof method :-** For conditional statement $p \to q$, a direct proof shows, if p is true and then using a set of valid arguments, show that q is true.

Que- Give a direct proof of a theorem if n is odd integer and n^2 is odd integer.

→ p: n is odd integer.

q: n^2 is odd integer.

$p \to q$: If n is odd integer then n^2 is odd integer.

As n is odd integer

$n = 2k+1 \qquad k \in Z$

Squaring both sides.

$n^2 = (2k+1)^2$

$= 4k^2+1+4k$

$= 4k^2+4k+1 \qquad = 2(2k^2+2k)+1$

$= 4(k^2+k)+1 \qquad = 2k_1+1 \qquad \{k_1 = 2k^2+2k, \; k \in \mathbb{Z} \Rightarrow k_1 \in \mathbb{Z}\}$

$= 4k(k+1)+1$

$= 2(2k(k+1)+1$

put $2k(k+1) = b$ we get

$n^2 = 2b+1$

hence n^2 is odd integer.

Que- Prove with the help of direct method if m and n are Perfect squares then $m \times n$ is also a perfect square.

→ p: m and n are perfect squares

q: $m \times n$ is a perfect square.

$p \to q$: If m and n are perfect squares then $m \times n$ is also a perfect square.

As m & n are perfect squares.

∃ integers a and b such that $m = a^2$, $n = b^2$

$mn = a^2b^2$ as $a \in Z$ and $b \in Z$

$mn = (ab)^2$ $\therefore ab \in Z$

$\therefore$ If m and n are perfect squares then mn is also a perfect square.

Que- Prove that sum of two rational numbers is a rational number

$\rightarrow$ p: a and b are rational numbers

q: a+b is a rational number.

$p \rightarrow q$: If a and b are rational numbers then a+b is rational number.

As a and b are rational numbers.

$a = \frac{p}{q}$, $b = \frac{r}{s}$ $p, q, r, s \in Z$ and $q \neq 0, s \neq 0$

$a+b = \frac{p}{q} + \frac{r}{s}$ as $p, q, r, s \in Z$

$= \frac{ps + qr}{qs} \in Q$ $\therefore pq, ps, qr, qs \in Z$

$\therefore ps + qr \in Z$

$q \neq 0, s \neq 0$

hence proved $\therefore qs \neq 0$

If a & b are rational numbers then a+b is also a rational number.

This is similar for subtraction & multiplication but for division $\frac{a}{b}$ is not rational.

* ## Proof by Contraposition: (or Indirect Proof)

* An extremely useful type of indirect proof is know as proof by contraposition.
* Proofs by Contraposition make use of the fact that the Conditional statement $p \to q$ can be proved by showing that its contrapositive, ~~$(\sim q \to \sim q)$~~ $(\sim q \to \sim p)$ is true.

$\Rightarrow$ So, In proof by Contraposition of $p \to q$, we take $\sim q$ as a premise, & using axioms, definition & previously proven theorems, together with the rules of inference, we show that $\sim p$ must follow.

So, Here we use $\boxed{p \to q \equiv \sim q \to \sim p}$

Thm: Prove that if n is an integer & $3n+2$ is odd, then n is odd.
First we attempt it by direct proof.

Sol:
Let p = $3n+2$ is odd integer, when n is an integer
q = n is an odd integer.

Suppose p is true
i.e $3n+2$ is an odd integer for some integer n.

$\therefore$ $3n+2 = 2k+1$ for some integer k
$\Rightarrow 3n+1 = 2k$
$\Rightarrow 3n = 2k-1$ or $n = \frac{2k-1}{3}$ for some integer k.

Now If $k=1$, $n = \frac{1}{3}$, which is not an odd integer.
If $k=2$, $n = 1$, which is odd.
So, there does not seem to be any direct way to conclude that n is odd.

Qu. * **Proof by contrapositive method :-**

When it is not possible to apply direct method then we apply contrapositive method.

Que- Prove that If n is an integer and $3n+2$ is odd then n is odd.

→ p : n is an integer and $3n+2$ is odd.

q : n is odd.

$p \to q$: If n is an integer and $3n+2$ is odd then n is odd.

→

As $3n+2$ is odd

$3n+2 = 2k+1 \qquad k \in Z$

$3n = 2k+1-2$

$n = \dfrac{2k-1}{3} \notin Z$ for $k=1$

∴ by using contrapositive method.

$\sim q$: n is not odd i.e. n is even.

$\sim p$: n is integer and $3n+2$ is even.

$\sim q \to \sim p$: If n is even then $3n+2$ is even.

As n is even

$n = 2k \qquad k \in Z$

$3n+2 = 3(2k)+2 \qquad$ as $k \in Z$

$= (3k+1)2 \qquad \therefore 3k+1 \in Z$

put $3k+1 = b$

$\therefore 3n+2 = 2b$

∴ $3n+2$ is even

hence, $\sim q \to \sim p \equiv p \to q$ — hence proved.

that if $n=ab$, whe...

Sol. Assume $p: n = ab$, where a & b are +ve integer

$q: a \leq \sqrt{n}$ or $b \leq \sqrt{n}$

☆ To Prove: $p \to q$

We will prove $\sim q \to \sim p$

Now

$q: a \leq \sqrt{n} \vee b \leq \sqrt{n}$

$\sim q: \sim(a \leq \sqrt{n} \vee b \leq \sqrt{n}) \Rightarrow \sim(a \leq \sqrt{n}) \wedge \sim(b \leq \sqrt{n})$ {D Morgan's Law}

$\Rightarrow a > \sqrt{n} \wedge b > \sqrt{n}$

ie $a > \sqrt{n}$ and $b > \sqrt{n}$

$\Rightarrow ab > n$ {Multiplying above equations}

$\Rightarrow ab \neq n$

$\sim p$ $\quad\quad \therefore \sim q \to \sim p \equiv p \to q$

$\therefore$ If $n = ab$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$

Vacuous Proof:

~~The~~ conditional statement $p \to q$ is true, when p is false, whatever value of q have.

whether 'q' is true or false, if p is false

$p \to q$ is true

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

So, if we can show that ~~$p \to q$~~ p is false, then we have a proof, called a vacuous proof, of the conditional statement $p \to q$.

Trivial Proof: By showing that q is true, it follows that $p \to q$ must also be true. A proof of $p \to q$ that uses the fact that q is true is called a trivial proof.

Ques Let $P(n)$ be "If a and b are positive integers with $a \geq b$, then $a^n \geq b^n$" where the domain consist of all nonnegative integers. show that $P(0)$ is true.

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If n is an integer and $3n+2$ is odd the n is odd.

Vacuous Proof :- depends on p (hypothesis)

eg

show that the preposition $P(0)$ is true, wher $P(n)$:
If $n>1$ then $n^2>n$" domain $\in Z$.

by truth table

$P(n)$: If $n>1$ then $n^2>n$
$P(0)$: If $0>1$ then $0>0$
F

Hypothesis is false, conclusion is false
$\therefore$ by thuth table
$\therefore$ $P(0)$ is True. ✓

Trivial Proof :- depends on q statement (conclusion)
Solve the following problem $P(n)$: If $a \geq b$ then $a^n \geq b^n$

$P(n)$: If $a \geq b$ then $a^n \geq b^n$
$P(0)$: If $a \geq b$ then $a^0 \geq b^0$
If $a \geq b$ then $1 \geq 1$
$\hookrightarrow$ True.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

The truth value of $P(0)$ is true.

Q: Give a proof by contradiction of theorem "If $3n+2$ is odd, then n is odd".

Solution: Given statement r: If $3n+2$ is odd, then n is odd.

Proof Contradiction: Assume $\sim r$, & Contradiction

Here; p_1: $3n+2$ is odd.
p_2: n is odd.

$$p_1 \to p_2 \equiv \sim p_1 \vee p_2$$

& $\Rightarrow$ $$\sim(p_1 \to p_2) \equiv \sim(\sim p_1 \vee p_2)$$
$$\equiv p_1 \wedge \sim p_2$$

$\therefore \sim r$: $3n+2$ is odd and n is even.

~~Sol~~ So,
Consider, $3n+2$ is odd and n is even.
As n is even
$\therefore n = 2t, \ t \in \mathbb{Z}$
So, $3n+2 = 3(2t)+2 = 6t+6$
$= 2(3t+1)$
$= 2k$; where $k = 3t+1 \in \mathbb{Z}$

$\Rightarrow 3n+2$ is an even number.

$\Rightarrow\Leftarrow$

So, we have a contradiction.
$\therefore$ ~~If~~ Our assumption is wrong.
$\Rightarrow$ If $3n+2$ is odd, then n is odd.

Proof by Contradiction: In a proof by contradiction, we assume that the conclusion is not true and then arrive at a Contradiction.

* contradiction proof :-

Prove that $\sqrt{2}$ is irrational number :

→ suppose that $\sqrt{2}$ is rational number

$\therefore \sqrt{2} = \frac{a}{b}$ $a, b \in Z$, $b \neq 0$ $(a, b) = 1$

i.e there is no common factor between a and b.

$a = \sqrt{2} \cdot b$

squaring on both sides

$a^2 = 2b^2$ $a^2 = 2 \cdot b^2$ … (1)

$2 | a^2 \Rightarrow 2 | a$ ⇒ a^2 is an even number

⇒ a is also an even number

$\cancel{\frac{a}{2} = k \; (k \in Z)}$

$\therefore a = 2k$ $(k \in \mathbb{Z})$ … (2)

using (2) in equⁿ (1)

$(2k)^2 = 2b^2$

$4k^2 = 2b^2$

$2k^2 = b^2$ ⇒ b^2 is an even number

$2 | b^2 \Rightarrow 2 | b$ ⇒ b is also an even number

$2 \times (a, b) \Rightarrow 2 \setminus 1$ ⇒ a & b both have common factor 2.

This is not possible, $\sqrt{2}$ is not rational number $\therefore \sqrt{2}$ is irrational number.

que- Prove that $5+3\sqrt{2}$ is irrational number.

→ If possible suppose that $5+3\sqrt{2}$ is rational

$$5+3\sqrt{2} = \frac{p}{q} \qquad p,q \in Z \,,\; q \neq 0 \,,\; (p,q) = 1$$

$$3\sqrt{2} = \left(\frac{p}{q} - 5\right)$$

$$\sqrt{2} = \frac{1}{3}\left(\frac{p}{q} - 5\right)$$

LHS = Irrational

RHS = Rational

Irrational $\neq$ Rational

our assumption is wrong

$\therefore$ $5+3\sqrt{2}$ is irrational number.

* Mistakes in the Proofs:-

que - What is wrong with the famous proof that $1 = 2$

→ let $a = b$

$a^2 = ab$ (multiply both sides by a)

$a^2 - b^2 = ab - b^2$ (subtract both sides by b^2)

$\cancel{(a-b)}(a+b) = b\cancel{(a-b)}$

$b+b = b$

$2b = b$

$2 = 1$

There is mistake in step no. 5

⋆ Logical Equivalence involving conditional Statements

- $p \to q \equiv \sim p \vee q$
- $p \to q \equiv \sim q \to \sim p$
- $p \vee q \equiv \sim p \to q$ → (Solve)
- $p \wedge q \equiv \sim(p \to \sim q)$ → (Solve)
- $\sim(p \to q) \equiv p \wedge \sim q$ (Solve)
- $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
- $(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
- $(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
- $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

⋆ Logical equivalence involving Biconditional

- $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
- $p \leftrightarrow q \equiv \sim p \leftrightarrow \sim q$
- $p \leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$
- $\sim(p \leftrightarrow q) \equiv p \leftrightarrow \sim q$

⋆ Some examples

Eg1 Using logical equivalences; show that $\sim(p \to q)$ and $p \wedge \sim q$ are equivalent.

Sol: Given $\sim(p \to q) \equiv \sim(\sim p \vee q)$

$\equiv \sim(\sim p) \wedge \sim q$ (Using De-Morgan's Law)

$\equiv p \wedge \sim q$

Eg2 Show that $\sim(p \vee (\sim p \wedge q)) \equiv \sim p \wedge \sim q$

Sol: Given: $\sim(p \vee (\sim p \wedge q)) \equiv \sim p \wedge \sim(\sim p \wedge q)$ (D-M's Law)

$\equiv \sim p \wedge (\sim(\sim p) \vee \sim q)$

$\equiv \sim p \wedge [p \vee \sim q] \equiv [\sim p \wedge p] \vee [\sim p \wedge \sim q]$

$\equiv F \vee [\sim p \wedge \sim q] \equiv \sim p \wedge \sim q$

Eg3 By Logical equivalences show that $(p\wedge q)\to(p\vee q)$ is a tautology.

Sol: $(p\wedge q)\to(p\vee q) \equiv \sim(p\wedge q)\vee(p\vee q)$ $\{\because p\to q \equiv \sim p\vee q\}$

$\equiv (\sim p\vee\sim q)\vee(p\vee q)$

$\equiv (\sim p\vee p)\vee(\sim q\vee q)$ $\left\{\begin{array}{l}(a+b)+(c+d)\\ =(a+c)+(b+d)\end{array}\right.$

$\equiv T\vee T$

$\equiv T$