## * The Principle of Inclusion - Exclusion

It defines that the number of elements in the union of set A and B is the sum of the numbers of elements in the sets minus the number of elements in their intersection.

Let ($|A|$: represents Cardinality of A or no. of elements in A)

then.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

* **Substraction Rule:** If a task can be done in either $n_1$ ways or $n_2$ ways, then the no. of ways to do the task is $n_1 + n_2$ minus the no. of ways to the the task that are common to the two different ways.

**Example 1:** In a discrete mathematics class every student is a major in Computer science or mathematics or both. The number of students having Computer Science as a major (possibly along mathematics) is 25, the number of students having mathematics as a major (possibly along with Computer Science) is 13, and the number of students majoring in both Computer Science and mathematics is 8. How many students are in this class.

**Sol:**

$|A|$: no. of students in the class majoring in C.S

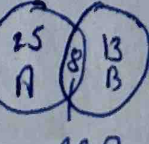
$|B|$: " " " " " Maths

$|A \cap B|$: " " " " in both C.S & maths

$|A \cup B|$: " " " majoring either C.S or math or both

using principle of inclusion - exclusion.

$|A \cup B| = |A| + |B| - |A \cap B| = 25 + 13 - 8 = 30$




$A \cap B$

$\Rightarrow |A \cup B| = 30$

Example 2: How many positive integers not exceeding 1000, are divisible by 7 or 11?

Sol: Let A: the set of positive integers not exceeding 1000, that are divisible by 7.

B: " " " " " that are divisible by 11.

$A \cap B$: " " " " divisible by both 7 and 11

$A \cup B$ " " " " by either 7 or 11

By Principle of inclusion and exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= \left\lfloor \frac{1000}{7} \right\rfloor + \left\lfloor \frac{1000}{11} \right\rfloor - \left\lfloor \frac{1000}{7 \times 11} \right\rfloor$$

$$= \lfloor 142.85 \rfloor + \lfloor 90.90 \rfloor - \lfloor 12.98 \rfloor$$

$$= 142 + 90 - 12$$

$$= 220$$

$\lfloor \; \rfloor$ floor function

floor $= \lfloor x \rfloor = \max\{n \in \mathbb{Z} \mid n \leq x\}$

$\lfloor 8.1 \rfloor = 8$

$\lfloor 8.3 \rfloor = 8$

$\lfloor 8.7 \rfloor = 8$

Extended form of the principle of inclusion-exclusion.

Let $A_1, A_2, \dots, A_n$ be finite sets, then

$$|A_1 \cup A_2 \cup \dots \cup A_n| = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| \dots$$

$$\dots + (-1)^{n+1} |A_1 \cap A_2 \cap \dots \cap A_n|$$

for $n=3$

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

For $n=4$

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4|$$
$$- |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4| + |A_1 \cap A_2 \cap A_3|$$
$$+ |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4| - |A_1 \cap A_2 \cap A_3 \cap A_4|$$

Eg of Subtraction rule,

**Example 3.** Suppose that there are 1807 freshmen at your school. Of these, 453 are taking a course in C.Sc., 567 are taking a course in maths, and 299 are taking course in both Compute Science and mathematics. How many are not taking a course either in C.Sc. or in mathematics?

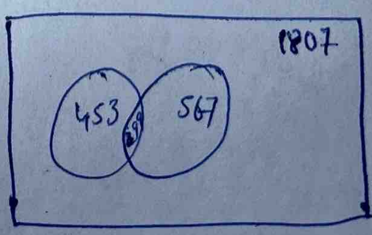
**Sol:** Let $A$: Freshmen taking Computer science.
$B$: " " mathematics

$|A| = 453$, $|B| = 567$, $|A \cap B| = 299$

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$= 453 + 567 - 299 = 721$$

Total freshmen = 1807

no. of freshmen not taking any course either C.S or maths
$= 1807 - 721$
$= 1086$



**Example 4:** A total of 1232 students have taken a Course in spanish, 879 have taken a Course in French, and 114 have taken a Course in Russian. Further, 103 have taken Course in both Spanish and French, 23 have taken Course in both Spanish and Russian, and 14 have taken Courses in both French and Russian. If 2092 students have taken atleast one of spanish, French and Russian, how many students have taken a Course in all three languages

**Sol:** Let $S$: students have taken spanish

$F$: " " French

$S \cap R$: " " " both Spanish & Russian

$S \cap F$: " " " both spanish & French

$F \cap R$: " " " both French & Russian

$S \cap F \cap R$: " " " spanish, French and Russian

$S \cup F \cup R$: " " " atleast one of Spanish, French, and Russian

By principle of inclusion - exclusion

$$|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |S \cap R| - |F \cap R| + |S \cap F \cap R|$$

$$2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |S \cap F \cap R|$$

$$\Rightarrow 2092 = 2225 - 140 + |S \cap F \cap R|$$

$$\Rightarrow 2092 = 2085 + |S \cap F \cap R|$$

$$\Rightarrow |S \cap F \cap R| = 7$$

* **The Product rule:** Suppose that a procedure can be broken down into a sequence of two tasks. If there are $n_1$ ways to do the first task and for each of these ways of doing the first task, there are $n_2$ ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

Eg1. A new company with just two employees, Sachin and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees.

Sol: The Procedure of assigning offices to these two employees consist of assigning an office to Sachin, which can be done in 12 ways, then assigning an office to Patel (different from the office assigned to Sachin), which can be done in 11 ways. ∴ By product rule, there are $12 \cdot 11 = 132$ ways to assign offices to these two employees.

Eg2 There are 32 microcomputers centers each microcomputer has 24 ports. How many different ports to a microcomputer in the center are there?

Sol: $\overset{n_1 \times n_2}{32 \times 24} = 768$ ports

Eg3. The chairs of a class are to be labelled with a letter and a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently

Sol: $\overset{n_1 \times n_2}{26 \times 100} = 2600$

Eg4: How many different bit strings of length seven are there

Sol: $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^7$ $\left[\begin{matrix}\text{bit strings} \\ 0 \quad 1\end{matrix}\right]$

Eg5 How many functions are there from set with m elements to set with n elements.

Sol: $\underbrace{n \cdot n \cdot \cdots \cdot n}_{m \text{ times}} = n^m$

## The Pigeonhole Principle

If 'k' is a positive integer (natural number); and 'k+1' or more objects are placed into 'k' boxes, then there is at least one box containing two or more of the objects.

$$\boxed{\begin{cases} \text{box (Pigeonhole)} \\ \text{objects (Pigeons)} \end{cases}}$$

**Sol:** Proving by Contradiction

Suppose that none of the 'k' boxes contains more than one objects. Then the total numbers of objects would be at most k. This is a contradiction, because there are atleast k+1 objects.

**Corollary:** A function 'f' from a set with 'k+1' or more elements to a set with k elements is not one to one.

**Example 1:** Among any group of 367 people, there must be atleast two with the same birthday because there are only 366 possible birthday (bcs there are only 366 days if we take leap year).

**Example 2:** In any group of 27 english words, there must be atleast two that begin with the same letter because there are 26 letters in the english alphabet.

**Example 3:** How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

**Sol:** Points are $\underbrace{0, 1, 2, 3, \ldots, 99, 100}$ → total 101 points in Count

So, to have equal or same points from 0 to 100 such that atleast two students receive the same score is 102.

## The Generalized Pigeonhole Principle

If 'N' objects are placed into 'k' boxes, then there is atleast one box containing at least $\lceil \frac{N}{k} \rceil$ objects

$\lceil x \rceil \rightarrow$ ceiling function

$Ceil(x) = \lceil x \rceil = \min\{n \in \mathbb{Z} : n \geq x\}$

$\lceil 7.2 \rceil = 8$

$\lceil 7.6 \rceil = 8$

$\lceil 3.4 \rceil = 4$

$\lceil 2.7 \rceil = 3$

Example: Among 100 people, how many (at least) born in the same month.

Sol: 100 people (100 objects)

12 months (12 boxes)

$\therefore$ by general Pigeonhole Principle

$$\left\lceil \frac{100}{12} \right\rceil = \lceil 8.33 \rceil = 9$$

a) 7 b) 8 c) 9 (d) None

Eg2 What is the minimum numbers of students required in a discrete mathematics class to be sure that atleast 6 will receive the same grade, if there are five positive grades A, B, C, D and E.

Sol: Given $\left\lceil \frac{N}{5} \right\rceil = 6$

As $\frac{30}{5} = 6$ $\frac{25}{5} = 5$

(a) 24 (b) 25 (c) 26 (d) 27

also $\lceil \frac{26}{5} \rceil = \lceil 5.2 \rceil = 6$

$\lceil \frac{27}{5} \rceil = \lceil 5.4 \rceil = 6$

$\lceil \frac{28}{5} \rceil = \lceil 5.6 \rceil = 6$

$\lceil \frac{29}{5} \rceil = \lceil 5.8 \rceil = 6$

min value of 'N' for which we are having 6 is 26

if $\lceil \frac{N}{5} \rceil = 6 \Rightarrow \boxed{N = 26}$

Eg3 How many cards must be selected from a standard deck of 52 cards to guarantee that atleast three cards of the same suit are chosen?

Sol: 4 = boxes $\longrightarrow$ pigeonhole

$N \rightarrow$ Cards $\rightarrow$ Pigeons.

By general pigeonhole principle

$\lceil \frac{N}{4} \rceil \geq 3$ ($\because$ at least 3 of the same suit)

Find min 'N' for which it is satisfying

i.e $\lceil \frac{9}{4} \rceil \geq 3$ $\because$ 9 is min integer

$\frac{12}{4} = 3$ & $\frac{8}{4} = 2$

$\lceil \frac{9}{4} \rceil = \lceil 2.25 \rceil = 3$

$\lceil \frac{10}{4} \rceil = \lceil 2.5 \rceil = 3$

So, 9 cards needs to be selected out of 52 cards.