Unit 3

Cartesian product of sets:→

The set of all ordered pairs (a,b) of elements $a \in A$, $b \in B$ is called the Cartesian product of sets A and B, and is denoted by $A \times B$. Thus

$$A \times B = \{(a,b) : a \in A, b \in B\}$$

e.g Let $A = \{1,2\}$ and $B = \{4,5\}$, then

$$A \times B = \{(1,4),(1,5),(2,4),(2,5)\}$$

Binary Relation (or Relation):→

Let A and B be two sets, then the binary relation from A to B is a subset of $A \times B$.

e.g Let $A = \{0,1\}$ and $B = \{a,b,c\}$ then

$$A \times B = \{(0,a),(0,b),(0,c),(1,a),(1,b),(1,c)\}$$

Now $R_1 = \{(0,a),(0,b)\} \subseteq A \times B$

$R_2 = \{(0,a),(0,b),(1,a)\} \subseteq A \times B$

$R_3 = \{(0,a)\} \subseteq A \times B$

Hence, R_1, R_2 and R_3 are relations defined on sets A and B.

Relation on a set A :→ A relation on the set A is a relation from A to A. ie a relation on a set A is a subset of $A \times A$.

Ex:→ Let $A = \{1, 2, 3\}$ be the set. which ordered pairs are in the relation

$$R = \{(a, b) : a \text{ divides } b\}$$

Sol:→ We have $A = \{1, 2, 3\}$, therefore

$A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

Since $R = \{(a,b) : a \text{ divides } b\}$

$\therefore R = \{(1,1), (2,2), (3,3), (1,2), (1,3)\}$

Properties of Relations :

① Reflexive Relation :→ A relation R on a set A is called reflexive if $(a,a) \in R$ for every element $a \in A$.

② Symmetric Relation :→ A relation R is said to be symmetric if $(b,a) \in R$ whenever $(a,b) \in R$, for every $a, b \in R$.

③ Antisymmetric Relation :→ A Relation R on a set A is said to be antisymmetric if $(a,b) \in R$, and $(b,a) \in R$, ~~only if~~ then $a = b$ for every $a, b \in R$.

Transitive Relation :→ A relation R on a set A is said to be transitive if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$ for all $a,b,c \in A$.

Ex:→ Identify which of the following Relations defined on a set $A = \{1,2,3\}$ is reflexive:

$R_1 = \{(1,1),(2,2),(1,3)\}$.

$R_2 = \{(1,1),(2,2),(3,3),(2,3)\}$

$R_3 = \{(1,1),(2,2),(3,3)\}$

$R_4 = \{(1,1),(2,2)\}$

Sol:→ The relations R_2 and R_3 are reflexive because they both all pairs of the form (a,a) i.e $(1,1),(2,2),(3,3)$. R_1 and R_4 are not reflexive because they do not contain all of these ordered pairs.

Ex:→ Identify which of the following relations defined on a set $A = \{1,2,3\}$ is symmetric:

① $R_1 = \{(1,1),(1,2),(2,1),(1,3)\}$

② $R_2 = \{(1,1),(1,3),(3,1),(2,2),(1,2),(2,1)\}$

③ $R_3 = \{(1,1),(2,2),(3,3)\}$

④ $R_4 = \{(2,3),(3,2),(1,2)\}$

Sol:→ The relations R_2 and R_3 are symmetric because they both contain all pairs of the form (b,a) whenever (a,b) belongs to them. But R_1 is not symmetric because $(1,3) \in R_1$ and $(3,1) \notin R_1$. Also, R_4 is not symmetric because $(1,2) \in R_4$ and $(2,1) \notin R_4$.

Ex:→ Identify which of the following relations defined on a set $A = \{1,2,3\}$ is antisymmetric:

$R_1 = \{(1,2), (2,2), (2,1)\}$

$R_2 = \{(1,2), (1,1), (2,2), (3,2)\}$

$R_3 = \{(1,1), (2,2), (3,3), (2,3), (1,3), (1,2)\}$

$R_4 = \{(1,3)\}$

Sol:→ Relations R_2, R_3 and R_4 are all antisymmetric. For each of these relations there is no pair of elements a and b with $a \neq b$ such that (a,b) and (b,a) belong to the relation. R_1 is not antisymmetric because it is impossible that $(1,2) \in R_1$ and $(2,1) \in R_1$ then $1 = 2$.

Remark:→ A relation R is antisymmetric if and only if there are no pairs of distinct elements a and b with $(a,b) \in R$ and $(b,a) \in R$. Hence, the only way to have $(a,b) \in R$ and $(b,a) \in R$ is for a and b to be the same point.

→ Identify which of the following relations defined on a set $A = \{1,2,3\}$ is transitive:

$R_1 = \{(1,2), (1,3)\}$

$R_2 = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

$R_3 = \{(1,2), (2,3), (3,1), (1,3)\}$

Sol:→ Relations R_1 and R_2 are transitive. For each of these relations, we can show that it is transitive by verifying if (a,b) and (b,c) belong to this relation, then (a,c). Relation R_3 is not transitive because $(2,3)$ and $(3,1)$ belongs to R_3 but $(2,1)$ does not belong to R_3.

Remark:→

① Number of relations on set A with n elements $= 2^{n^2}$

② Number of reflexive relations on set A with n elements $= 2^{n(n-1)}$

③ Number of symmetric relations on set A with n elements $= 2^{\frac{n(n+1)}{2}}$

④ No. of antisymmetric relations $= 2^n \, 3^{\frac{n(n-1)}{2}}$

⑤ No. of asymmetric relations $= 3^{\frac{n(n-1)}{2}}$

⑥ No. of irreflexive relations $= 2^{n(n-1)}$.

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(i) If the sets A and B has m and n elements respectively, then number of relations from A to B

$= 2^{m \cdot n}$.

~~Ex: Consider the~~

Ex: Consider the following relations on the set of integers I:

① $R_1 = \{(a,b) : a \le b\}$

② $R_2 = \{(a,b) : a > b\}$

③ $R_3 = \{(a,b) : a = b \text{ or } a = -b\}$

④ $R_4 = \{(a,b) : a = b\}$

⑤ $R_5 = \{(a,b) : a = b+1\}$

⑥ $R_6 = \{(a,b) : a+b \le 3\}$

Sol:→ ① $R_1 = \{(a,b) : a \le b\}$

Since $a \le a$ for all $a \in I$

$\therefore a R_1 a \Rightarrow R_1$ is reflexive

Suppose $a R_1 b$ ie $a \le b$

then it is not necessary $b \le a$

Hence R_1 is not symmetric.

Suppose $a R_1 b$ and $b R_1 c$ ie $(a,b) \in R$ and $(b,c) \in R_1$

then $a \le b$ and $b \le c$

$\Rightarrow a \le c$ ie $a R_1 c \Rightarrow (a,c) \in R_1$

Hence, R_1 is transitive.

2) $R_2 = \{(a,b) : a > b\}$

since $a \not> a$ i.e $a \not R_2 a$. Hence $(a,a) \notin R_2$

$\Rightarrow R_2$ is not reflexive

Suppose $a R_2 b$ i.e $a > b \Rightarrow (a,b) \in R_2$

Then, it is not possible that $b > a$

$\Rightarrow b \not> a$ i.e $(b,a) \notin R_2$

Hence, R_2 is not symmetric.

Suppose $a R_2 b$ and $b R_2 c$, therefore

$(a,b) \in R_2$ and $(b,c) \in R_2$

$\Rightarrow a > b$ and $b > c$

$\Rightarrow a > c$

Hence, $(a,c) \in R_2$ i.e $a R_2 c$.

Therefore, R_2 is transitive.

Remark :→ We use the notation aRb to denote that $(a,b) \in R$ and $a \not R b$ to denote that $(a,b) \notin R$.

Combining Relations :→ Since relations behave like sets only, hence we can find their union, intersection, difference etc.

Ex: Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$.
Let $R_1 = \{(1,1),(2,2),(3,3)\}$ and
$R_2 = \{(1,1),(1,2),(1,3),(1,4)$ be two relations on sets A and B. Find $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$ and $R_2 - R_1$.

Sol:→ We have

$R_1 \cup R_2 = \{(1,1),(2,2),(3,3),(1,2),(1,3),(1,4)\}$

$R_1 \cap R_2 = \{(1,1)\}$.

$R_1 - R_2 = \{(2,2),(3,3)\}$

$R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$

Remark :→ In above example, we get

$A \times B = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3)$
$(2,4),(3,1),(3,2),(3,3),(3,4)\}$.

~~Since clearly $R_1 \subseteq A \times B$~~

Clearly, $R_1 \cup R_2 \subseteq A \times B$, $R_1 \cap R_2 \subseteq A \times B$
$R_1 - R_2 \subseteq A \times B$ and $R_2 - R_1 \subseteq A \times B$.

Hence, union, intersection, difference of two relations are also a relation

Composition of two relations:→

Let R be a relation from a set A to a set B and S a relation from B to a set C. ~~The composite of~~ The composite of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$ and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

Ex:→ What is the composite of the relations R and S, where R is the relation from $\{1,2,3\}$ to $\{1,2,3,4\}$ with $R = \{(1,1), (1,4), (2,3), (3,1), (3,4)\}$ and S is the relation from $\{1,2,3,4\}$ to $\{0,1,2\}$ with

$$S = \{(1,0), (2,0), (3,1), (3,2), (4,1)\}.$$

Sol:→ $S \circ R$ is contructed using all ordered pairs in R and ordered pairs in S, where the second element of the ordered pair in R agrees with the first element of the ordered pair in S. Therefore,

$$S \circ R = \{(1,0), (1,1), (2,1), (2,2), (3,0), (3,1)\}.$$

Definition:→ Let R be a relation on a set A. The powers R^n, $n = 1, 2, 3, \ldots$ are defined recursively by $R^1 = R$ and $R^{n+1} = R^n \circ R$

The definition shows that

$R^2 = R^{1+1} = R^1 \circ R = R \circ R.$

$R^3 = R^{2+1} = R^2 \circ R = (R \circ R) \circ R$ and so on.

Ex:→ Let $R = \{(1,1), (2,1), (3,2), (4,3)\}$. Find the powers of R^n, $n = 2, 3, 4, \ldots$

Sol:→ Since $R^2 = R \circ R$, we find that

$R^2 = \{(1,1), (2,1), (3,1), (4,2)\}$.

Furthermore, since $R^3 = R^2 \circ R$

$= \{(1,1), (2,1), (3,1), (4,1)\}$.

Also, $R^4 = R^3 \circ R$

$= \{(1,1), (2,1), (3,1), (4,1)\} = R^3$

Further, we can verify that

$R^n = R^3$ for $n = 5, 6, 7, \ldots$

Definition:→ If a and b are integers with $a \neq 0$, we say that a divides b if there is an integer c such that $b = ac$.

e.g $8 = 4 \times 2$, therefore 4 divides 8.

Congruent modulo m:→

If a and b are integers and m is a integer, then a is congruent to b modulo m if m divides a−b

i.e $\frac{a-b}{m} = c$, where c is an integer.

We use the notation $a \equiv b \pmod{m}$ to indicate that a is congruent to b modulo m. e.g $6 \equiv 3 \pmod{3}$

M.Imp

Ex:→ (Congruence Modulo m):→

Let m be an integer with $m > 1$. Show that the relation

$$R = \{(a,b) : a \equiv b \pmod{m}\}$$

is an equivalence relation on the set of integers.

Sol:→ Note that $a \equiv b \pmod{m}$ if and only if m divides $a-b$. Note that

$$\frac{a-a}{m} = \frac{0}{m} = 0 \Rightarrow a \equiv a \pmod{m}$$

Hence, Congruent modulo m is reflexive.

Now, suppose $a \equiv b \pmod{m}$. Then,

$a-b$ is divisible by m i.e $\frac{a-b}{m} = K$

or $a-b = Km$, where K is an integer

$\Rightarrow -(a-b) = -(Km)$

$\Rightarrow b-a = (-K)m$ or $\frac{b-a}{m} = -K$

Therefore, $b \equiv a \pmod{m}$. Hence, Congruent modulo m is symmetric.

Next, suppose that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$. Therefore,

$\frac{a-b}{m} = K_1$ and $\frac{b-c}{m} = K_2$

where K_1 & K_2 are integers.

$\Rightarrow a-b = k_1 m$ and $b-c = k_2 m$

$\Rightarrow a-b+b+c = k_1 m + k_2 m$

or $a-c \equiv m(k_1+k_2)$

$\Rightarrow \frac{a-c}{m} = k_1+k_2$, where k_1+k_2 is an Integer.

Hence, $a \equiv c \pmod{m}$. Thus, Congruent modulo m is transitive. It follows that congruence modulo m is an equivalence relation.

Ex:→ Let R be the relation on the set of real numbers such that aRb if and only if a-b is an Integer. Show that R is an equivalence.

Sol:→ We have

$R = \{(a,b) : a-b \text{ is an Integer}\}$

Since $a-a=0$ is an Integer for all integers a. Hence $(a,a) \in R$, R is reflexive.

Now, suppose that aRb. Then, a-b is an Integer say K.

$\therefore a-b = K \Rightarrow -(a-b) = -K$

$\Rightarrow b-a = -K$, where $-K$ is an Integer.

Hence, bRa i.e $(b,a) \in R$. It follows that R is symmetric. If aRb and bRc i.e $(a,b) \in R$ and $(b,c) \in R$, then a-b and b-c are Integers.

Therefore, $a-b=K_1$ and $b-c=K_2$, where K_1 and K_2 are integers.

$\Rightarrow$ $a-b+b-c=K_1+K_2$

or $a-c=K'$, where $K'=K_1+K_2$ is an integer.

$\Rightarrow$ aRc or $(a,c)\in R$.

It follows that R is transitive. Hence, R is an equivalence relation on the set of integers.

Matrix Representation of a Relation

Adjacency Matrix or Zero-one matrix:→

Let R be a relation from a finite set $A=\{a_1, a_2, \ldots, a_m\}$ to a finite set $B=\{b_1, b_2, \ldots, b_n\}$. The relation R can be represented by the matrix $M_R=[m_{ij}]$, called the zero-one matrix, where

$$m_{ij}=\begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R. \end{cases}$$

In other words, the zero-one matrix M_R representing relation R has a '1' as its (i,j) entry when a_i is related to b_j and a '0' in this position if a_i is not related to b_j.

Ex:→ Let $A=\{4,5,6\}$ and $B=\{r,s\}$ be two sets. Let $R=\{(4,r),(5,s),(6,r)\}$ is a relation from A to B. ~~Write~~ Write down the zero-one matrix of R.

Sol:→ The zero-one matrix M_R is given by

$$M_R=\begin{array}{c} \\ 4 \\ 5 \\ 6 \end{array}\begin{array}{c} \begin{array}{cc} r & s \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

:→ Suppose that $A=\{1,2,3\}$ and $B=\{1,2\}$. Let R be the relation from A to B containing (a,b) if $a \in A$, $b \in B$ and $a>b$. Find the zero-one matrix of R.

Sol:→ We have $A=\{1,2,3\}$ and $B=\{1,2\}$.

$\therefore A \times B = \{(1,1),(1,2),(2,1),(2,2),(3,1),(3,2)\}$

Since $R=\{(a,b): a>b\}$, therefore

$R=\{(2,1),(3,1),(3,2)\}$. The matrix for R is

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}_{3\times 2} \end{array}$$

Ex:→ Let $A=\{(a_1,a_2,a_3)\}$ and $B=\{b_1,b_2,b_3,b_4,b_5\}$. Which ordered pairs are in the relation R represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Sol:→ Because the relation R consists of those ordered pairs (a_i,b_j) with $m_{ij}=1$, it follows that

$R=\{(a_1,b_2),(a_2,b_1),(a_2,b_3),(a_2,b_4),$
$(a_3,b_1),(a_3,b_3),(a_3,b_5)\}$

$$M_R = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{ccccc} b_1 & b_2 & b_3 & b_4 & b_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

:→ For the given adjacency matrix

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

define the relation R.

Sol:→ Since the matrix M_R is of the order 3×4 so if R is a relation from the set A to B, then A and B will have 3 and 4 elements respectively.

Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4\}$

Now $(a_i, b_j) \in R$ if and only if $m_{ij} = 1$.

$$\text{Now } M_R = \begin{array}{c|cccc} & b_1 & b_2 & b_3 & b_4 \\ \hline a_1 & 1 & 0 & 0 & 1 \\ a_2 & 0 & 1 & 1 & 0 \\ a_3 & 1 & 0 & 1 & 1 \end{array}$$

$\therefore R = \{(a_1, b_1), (a_1, b_4), (a_2, b_2), (a_2, b_3), (a_3, b_1), (a_3, b_3), (a_3, b_4)\}$

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Remark:→ A square matrix of a relation on a set can be used to determine whether the relation has certain properties. Recall that a relation R on A is reflexive if $(a, a) \in R$ whenever $a \in A$. Thus, R is reflexive if and only if $(a_i, a_i) \in R$ for $i = 1, 2, \ldots, n$. Hence, R is reflexive if and only if $m_{ii} = 1$ for $i = 1, 2, \ldots, n$. In other words, R is reflexive if all the elements of the main diagonal of M_R are equal to 1.

The relation R is symmetric if $(a,b) \in R$ implies that $(b,a) \in R$. Consequently, a relation R on the set $A = \{a_1, a_2, \ldots, a_n\}$ is symmetric if and only if $(a_j, a_i) \in R$ whenever $(a_i, a_j) \in R$. In terms of the enteries of M_R, R is symmetric if and only if $m_{ji} = 1$ whenever $m_{ij} = 1$. This also means $m_{ji} = 0$ whenever $m_{ij} = 0$. Consequently, R is symmetric if and only if $m_{ij} = m_{ji}$, for all pairs of integers i and j with $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. Hence, R is symmetric if and only if $M_R = (M_R)^t$ i.e if M_R is a symmetric matrix.

The relation R is antisymmetric if and only if (a,b) and $(b,a) \in R$ implies that $a = b$. Consequently, the matrix of an antisymmetric relation has the property that if $m_{ij} = 1$ with $i \neq j$, then $m_{ji} = 0$. In other words, either $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$.

Ex:→ Let $A = \{1, 2, 3\}$ and ~~define~~ check whether the following relations are reflexive, symmetric and antisymmetric.

① $R_1 = \{(1,1), (2,2), (1,3), (2,3)\}$

② $R_2 = \{(1,1), (2,2), (3,3), (1,3), (2,3)\}$

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(3) $R_3 = \{(1,2), (2,1), (1,3), (2,3)\}$

(4) $R_4 = \{(1,2), (2,1), (1,3), (3,1)\}$.

(5) $R_5 = \{(1,1), (1,2), (1,3), (3,1), (2,3)\}$.

(6) $R_6 = \{(1,1), (2,2), (3,3), (1,2), (2,3)\}$.

(7) $R_7 = \{(1,1), (2,2), (1,2), (3,1)\}$.

Sol: $\rightarrow$ (1)

$$M_{R_1} = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 \\ 3 & 0 & 0 & 0 \end{array}$$

Since all the diagonals elements of M_{R_1} are not 1, hence R_1 is ~~not~~ not reflexive. Clearly, $a_{13} = 1$ and $a_{31} = 0$, therefore R_1 is not symmetric. Since R_1 is ~~not~~ not symmetric, hence R_1 is antisymmetric.

(2)

$$M_{R_2} = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 \\ 3 & 0 & 0 & 1 \end{array}$$

Since all the diagonals elements are 1, there~~fore~~fore R_2 is reflexive. Also, $a_{13} = 1$ and $a_{31} = 0$, therefore R_2 is not symmetric and hence R_2 is antisymmetric.

③ $M_{R_3} = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 0 & 0 & 0 \end{array}$

Since all diagonals elements are not 1, therefore R_3 is not reflexive. Also, $a_{23}=1$ and ~~[illegible]~~ $a_{32}=0$, therefore R_3 is not symmetric and ~~therefore~~ hence R_3 is antisymmetric.

④ $M_{R_4} = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 \end{array}$

clearly, R_4 is not reflexive as the diagonal elements are not equal to 1. clearly, $a_{12}=a_{21}=1$ and $a_{13}=a_{31}=1$, therefore R_4 is symmetric but not antisymmetric as it contains a symmetric pair.

⑤ $M_{R_5} = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1 & 1 \\ 2 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 \end{array}$

Since all the diagonal elements are not equal to 1, hence R_5 is not Reflexive. Since $a_{12}=1$ ~~[illegible]~~ and $a_{21}=0$, therefore R_5 is not Symmetric. Since $a_{13}=1$ and $a_{31}=1$, hence it contains a symmetric pair, therefore it is not antisymmetric.

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(6)

$$M_{R_6} = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & & & \\ 2 & \multicolumn{3}{c}{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}} \\ 3 & & & \end{array}$$

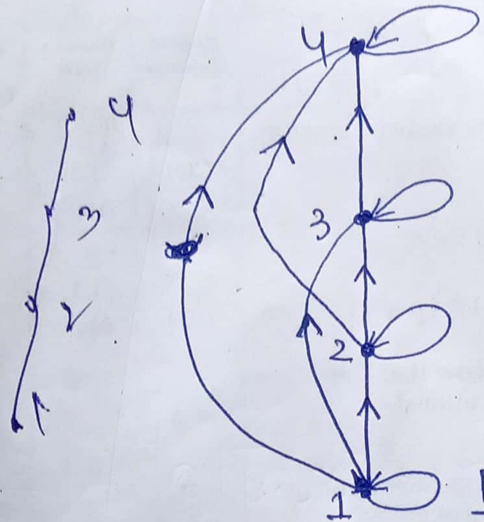
clearly, R_6 is reflexive, symmetric and antisymmetric.

(7)

$$M_{R_7} = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & & & \\ 2 & \multicolumn{3}{c}{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}} \\ 3 & & & \end{array}$$

Since all the diagonal elements are not equal to 1, therefore it is not reflexive. Also, $a_{12} = 1$ but $a_{21} = 0$, therefore it is not antisymmetric. But it is antisymmetric because it does not contain any symmetric pair except the diagonal elements.

Hasse Diagrams:→ Many edges in the ~~directed~~ directed graph for a finite poset do not have to be shown because they must be present. For instance, consider the directed graph ~~graph~~ for the poset $\{(a,b) : a \le b\}$ on the set $A = \{1,2,3,4\}$, shown in Figure A



Here
$R = \{(1,1), (2,2), (3,3), (4,4)$
$(1,2), (1,3), (1,4)$
$(2,3), (2,4), (3,4)\}$

Fig A

Because the relation is a partial ordering, therefore it reflexive and its directed graph has loops at all vertices. Consequently, we do not have to show these loops because they must be present. In ~~Fig~~ Figure B, loops are not shown. Because a partial ordering is transitive, we do not have to show those edges that must be present because of transitivity. For example, in Figure C, the ~~edges~~ edges (1,3), (1,4) and (2,4) are not shown because they must be present.

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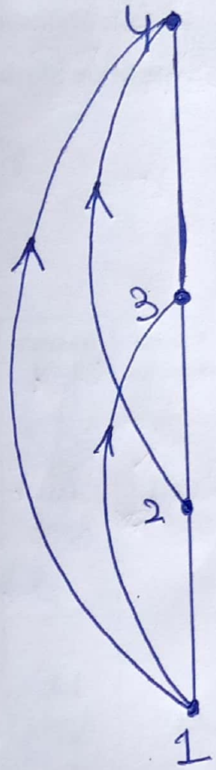
If we assume that all edges are painted upward, we do not have to show the directions of the edges. Figure C does not show directions.



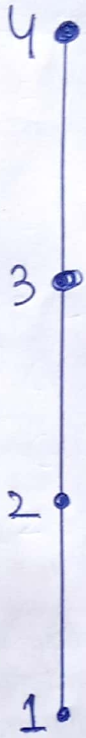


Figure B

Figure C

Contruction of the Hasse Diagram for ($\{1,2,3,4\}, \leq$).

Ex:→ Draw the Hasse diagram representing the partial ordering $\{(a,b): a \text{ divides } b\}$ on $\{1,2,3,4,6\}$

Sol:→ Begin with the directed graph for this partial order, as shown in Figure A. Remove all loops as shown in Fig B. Then, delete all the edges implied by the transitive property. These are (1,4). and (1,6). Arrange all the edges to point upward and delete all arrows on the ~~edges~~ directed edges to obtain the Hasse diagram. The resulting Hasse diagram is shown in Figure C.

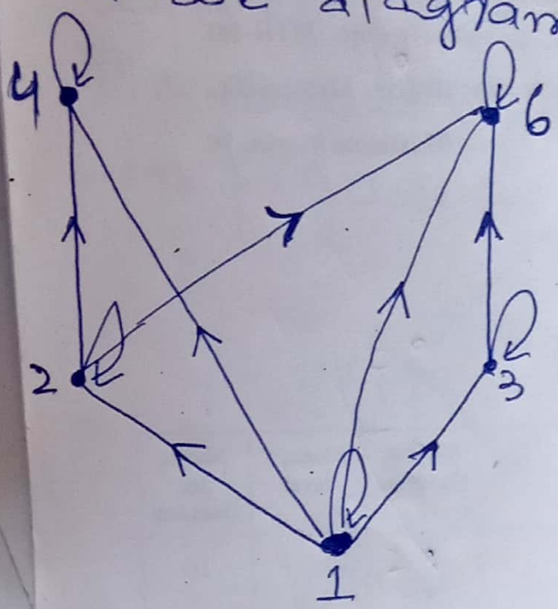


Figure A

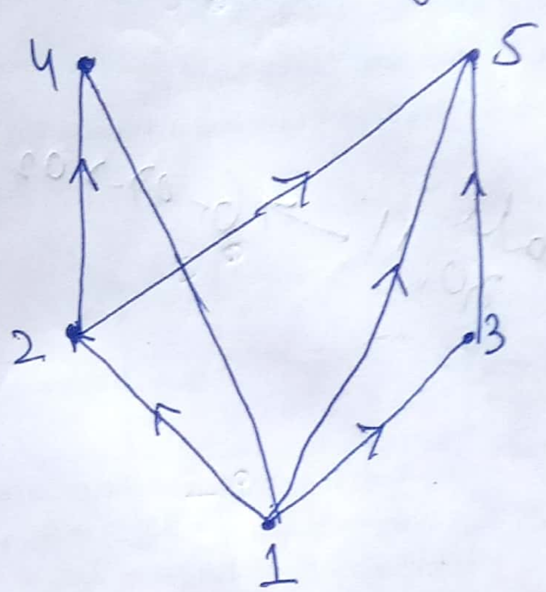
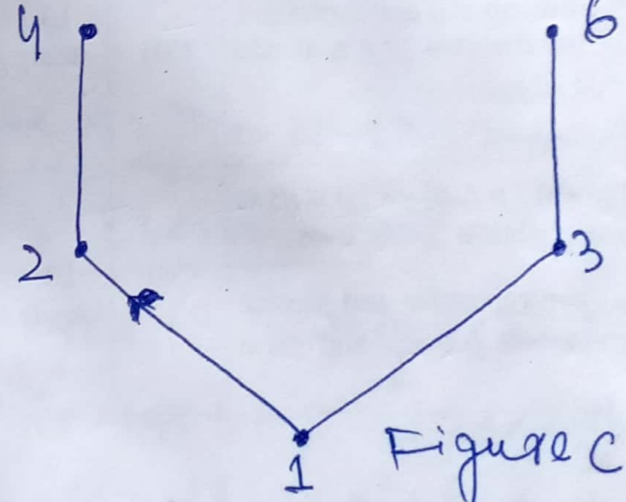


Figure B.



Here $R = \{(1,1),(2,2),(3,3),(4,4),(6,6),(1,2),(1,3),(1,4),(1,6),(2,4),(2,6),(3,6)\}$

Figure C.

Ex:→ Draw the Hasse diagram for the partial ordering $\{(A,B) : A \subseteq B\}$ on the power set $P(S)$, where $S = \{a, b, c\}$.

Sol:→ The Hasse diagram for this partial ~~order~~ ordering is obtained from the directed graph by deleting all the loops and all the edges that occur from transitivity, namely

$(\emptyset, \{a,b\}), (\emptyset, \{a,c\}), (\emptyset, \{b,c\}), (\emptyset, \{a,b,c\})$, $(\{a\}, \{a,b,c\}), (\{b\}, \{a,b,c\})$ and $(\{c\}, \{a,b,c\})$.
Finally all edges point upward and directed arrows are deleted.
The resulting Hasse diagram is illustrated in Figure A.

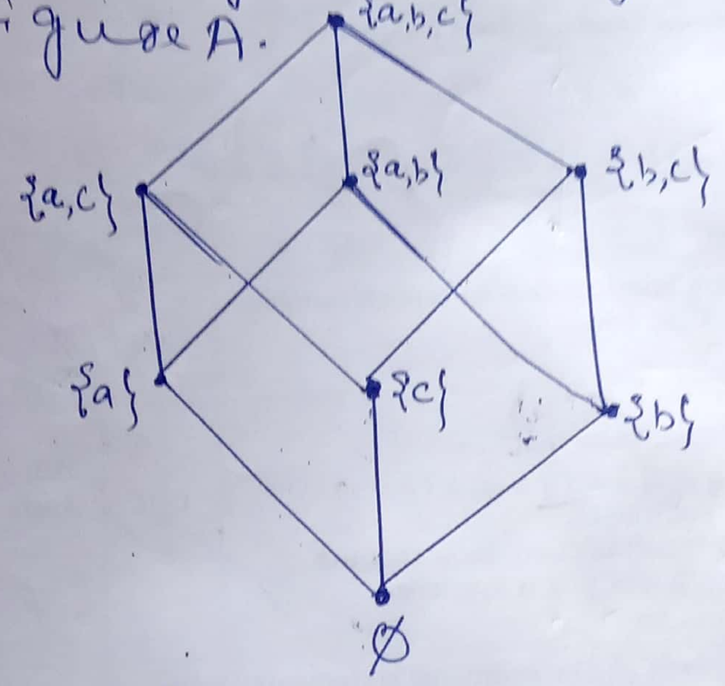


Figure A: The Hasse Diagram of $(P(\{a,b,c\}), \subseteq)$

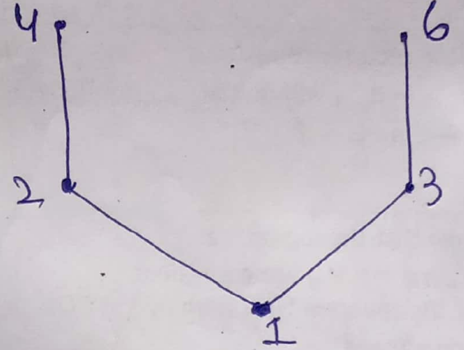
<u>Maximal Element:</u>→ An element 'a' of a poset $(S, \preceq)$ is said to be maximal if there is no $b \in S$ such that $a \preceq b$ i.e. a is not related to any other element in the poset.

<u>Minimal Element:</u>→ An element 'a' of a poset $(S, \preceq)$ is said to be minimal if there is no element $b \in S$ such that $b \preceq a$ i.e. there exists no element in the poset which is related to a.

aximal and minimal elements are asy to spot using a Hasse diagram. They are the top and bottom elements in the the diagram.

Ex:→ Which elements of poset $(\{1,2,3,4,6\}, |)$ are maximal and minimal?

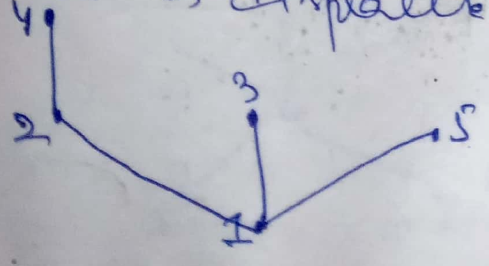
Soln:→ Here $R = \{(1,1), (2,2), (3,3), (1,2), (1,3), (1,4), (1,6), (2,4), (2,6), (3,6)\}$. The Hasse diagram for this poset is displaced as follows



Hence, from Hasse diagram, it follows that the maximal elements are 4 and 6 and minimal element is 1.

Ex:→ Which elements of poset $(\{1,2,3,4,5\}, |)$ are maximal and minimal

Soln:→ Here $R = \{(1,1), (2,2), (3,3), (4,4), (5,5)$ $(1,2)$ $(1,3), (1,4), (1,5), (2,4)\}$. The Hasse diagram for this poset is displaced as follows:



⑥

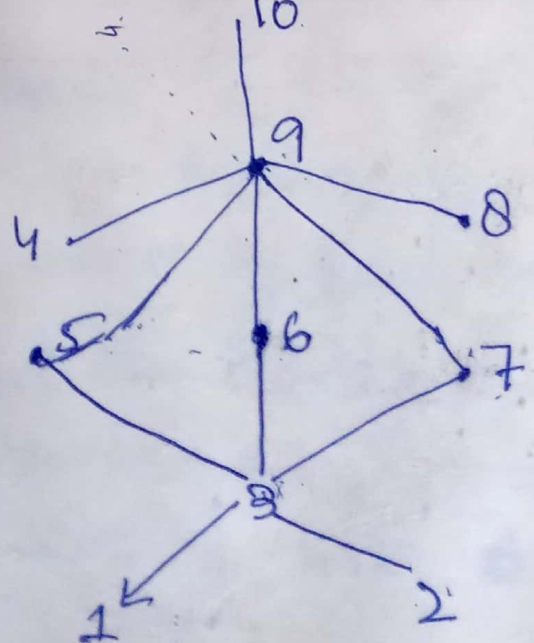
nce, from the Hasse diagram, it follows that the maximal elements are 4, 3 and 5 and minimal element is 1.

<u>Theorem</u>: A finite nonempty poset $(S, \preceq)$ has atleast one maximal element and atleast one minimal element.

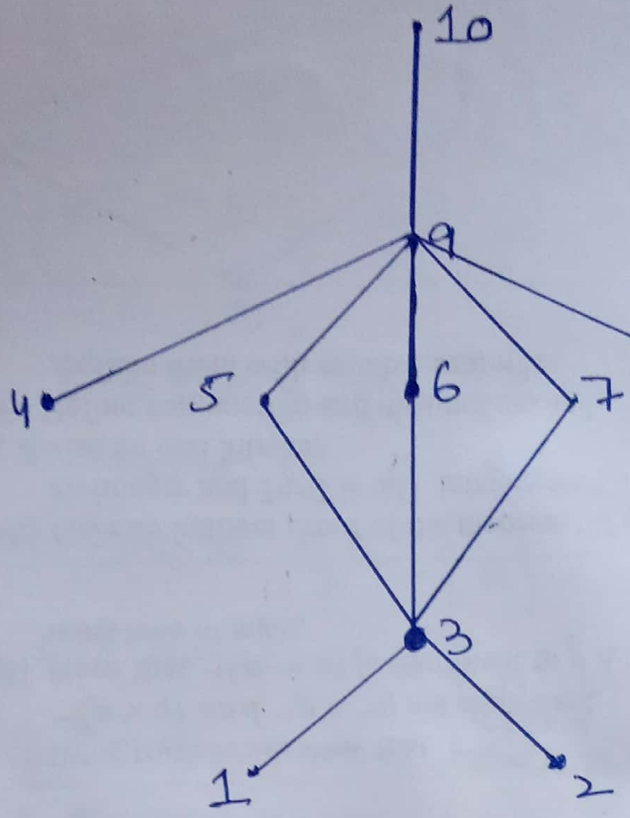
<u>Upper bound</u>:→ Let $(S, \preceq)$ be a poset and A be a subset of S. An element $u \in S$ is an upper bound of A if $a \preceq u$ for all elements $a \in A$

<u>Lower Bound</u>:→ Let $(S, \preceq)$ be a poset and A be a subset of S. An element $l \in S$ is an lower bound of A if $l \preceq a$ for all elements $a \in A$.

Ex:→



Find the lower and upper bounds of the subsets $A=\{5,6,7\}$ and $B=\{5,6,8\}$ in the poset with the Hasse diagram shown below



Sol:→ We have $A=\{5,6,7\}$

Upper bound of $A=\{9,10\}$

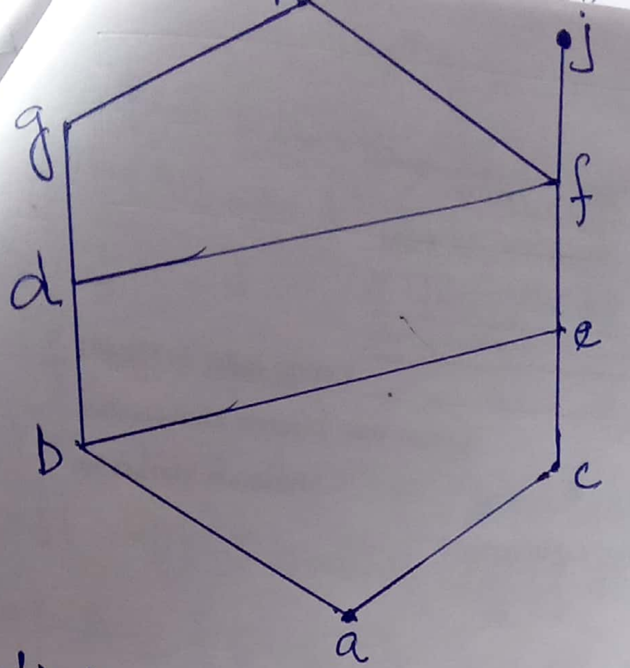
Lower bound of $A=\{1,2,3\}$

Also, we have $B=\{5,6,8\}$

Upper bound of $B=\{9,10\}$

Lower bound of $B=$ does not exist.

Ex2:→ Find the lower and upper bounds of the subsets $A=\{a,b,c\}$, $B=\{j,h\}$ and $C=\{a,c,$ in the poset with the Hasse diagram given below.



Sol:→ We have A= {a,b,c}
Upper bound of A={e,f,j,h}
Lower bound of A={a}
Also, we have B={j,h}
Clearly there is no ~~lower~~ upper bound of B.
Lower bound of B={a,b,c,d,e,f}.
Also, C={a,c,d,f}
Upper bound of C={a,c,d,f}
Lower bound of C={a}.
~~Least Upper~~

Least Upper Bound:→ The element x is called the least upper bound of the subset A if x is an upper bound that is less than every other upper bound of A.

Greatest Lower Bound:→ The element y is called the greatest lower bound of the subset of A if y is an lower bound that is greater than every other lower bound of A.

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The greatest lower bound and
least upper bound of a subset A are denoted
by glb(A) and lub(A), respectively. The least
upper bound and greatest lower bound of a set
A, if existed, are unique.

Ex:→ Find the greatest lower bound and the
least upper bound ~~if they exist, in the poset~~
of $A = \{5, 6, 7\}$ and $B = \{5, 6, 8\}$ in the poset
Shown in Example 1.

Sol:→ We have $A = \{5, 6, 7\}$
upper bound of $A = \{9, 10\}$.
Lower of $A = \{1, 2, 3\}$.
$\therefore$ glb(A) = 3 and lub(A) = 9.

Also, we have $B = \{5, 6, 8\}$
upper of $B = \{9, 10\}$
Lower bound of B does not exist.
Therefore, lub(B) = {9} and glb(B) does
not exist.

14-A
Ex:→ Find the greatest lower bound and the
least upper bound, if they exist, in the poset
shown in Example 2, of the sets
$A = \{b, d, g\}$ and $B = \{c, e, f\}$

tice:→ A partially ordered set in which ery pair of elements has both a least upper bound and a greatest lower bound is called a lattice.

Ex:→ Determine whether the posets represented by each of the Hasse diagrams given below are lattices:

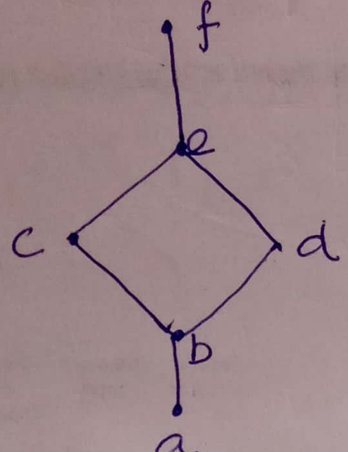


Fig 1

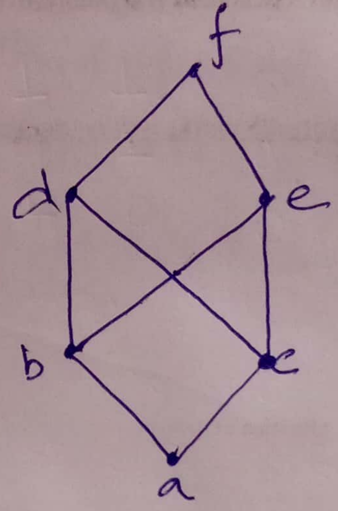
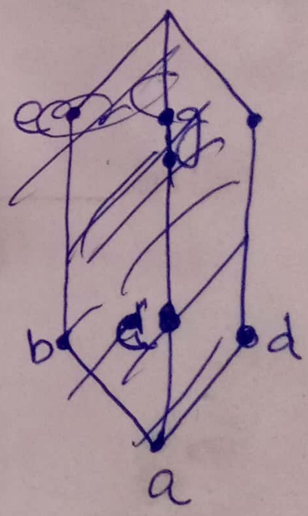


Fig 2



Sol:→ The posets represented by the Hasse diagram in Fig 1 is a lattice because every pair of elements has both a least upper bound and a greatest lower bound.

On the other hand, the poset with the Hasse Diagram shown in Fig 2 is not a lattice because b and c have no least upper bound.

⑥

Is the poset $(Z^+, |)$ a lattice?

Sol:→ Let a and b be two positive Integers. The least upper bound and greatest lower bound of these two Integers are the least Common multiple and the greatest Common divisor of these integers, respectively. It follows that this poset is a lattice.

Ex:→ Determine whether the posets $(\{1,2,3,4,5\}, |)$ and $(\{1,2,4,8,16\}, |)$ are lattices.