Problem 1. Пусть $(\mathbf{X}_0, \mathbf{X}_1) \sim p_{0,1}$ — некоторое распределение парных данных, а процесс интерполяции в момент времени t получается семплированием из условного распределения $p_{\mathbf{X}_t|\mathbf{X}_0,\mathbf{X}_1}(\boldsymbol{x}|\boldsymbol{x}_0, \boldsymbol{x}_1)$, порождаемого СДУ с векторным полем $f(\boldsymbol{x}|\boldsymbol{x}_0, \boldsymbol{x}_1)$ и коэффициентом диффузии $g(t)$:

$$\mathrm{d}\mathbf{X}_t^{0,1} = f_t(\mathbf{X}_t^{0,1}|\boldsymbol{x}_0, \boldsymbol{x}_1)\mathrm{d}t + g(t)\mathrm{d}\mathbf{W}_t.$$

Покажите, что динамика распределений $p_{\mathbf{X}_t|\mathbf{X}_0}(\boldsymbol{x}|\boldsymbol{x}_0)$ порождается СДУ

$$\mathrm{d}\mathbf{X}_t^{0} = f_t^*(\mathbf{X}_t^{0}|\boldsymbol{x}_0)\mathrm{d}t + g(t)\mathrm{d}\mathbf{W}_t,$$

где

$$f_t^*(\boldsymbol{x}|\boldsymbol{x}_0) = \mathbb{E}\left[f_t(\mathbf{X}_t|\mathbf{X}_0, \mathbf{X}_1)|\mathbf{X}_t = \boldsymbol{x}, \mathbf{X}_0 = \boldsymbol{x}_0\right].$$

Для динамики $p_{\mathbf{X}_t|\mathbf{X}_0,\mathbf{X}_1}(\boldsymbol{x}|\boldsymbol{x}_0, \boldsymbol{x}_1)$ можем записать уравнение Фоккера-Планка:

$$\frac{\partial}{\partial t} p_{\mathbf{X}_t|\mathbf{X}_0,\mathbf{X}_1}(\boldsymbol{x}|\boldsymbol{x}_0, \boldsymbol{x}_1) = -\mathrm{div}\left(p_{\mathbf{X}_t|\mathbf{X}_0,\mathbf{X}_1}(\boldsymbol{x}|\boldsymbol{x}_0, \boldsymbol{x}_1) f_t(\boldsymbol{x}|\boldsymbol{x}_0, \boldsymbol{x}_1)\right) + \frac{g^2(t)}{2}\Delta p_{\mathbf{X}_t|\mathbf{X}_0,\mathbf{X}_1}(\boldsymbol{x}|\boldsymbol{x}_0, \boldsymbol{x}_1)$$

Тогда:

$$\frac{\partial}{\partial t} p_{\mathbf{X}_t|\mathbf{X}_0}(\boldsymbol{x}|\boldsymbol{x}_0) = \frac{\partial}{\partial t}\int_{\mathbb{R}^n} p_{\mathbf{X}_t|\mathbf{X}_0,\mathbf{X}_1}(\boldsymbol{x}|\boldsymbol{x}_0, \boldsymbol{x}_1) p_{\mathbf{X}_1|\mathbf{X}_0}(\boldsymbol{x}_1|\boldsymbol{x}_0)\mathrm{d}\boldsymbol{x}_1 =$$

$$= \int_{\mathbb{R}^n} \frac{\partial}{\partial t} p_{\mathbf{X}_t|\mathbf{X}_0,\mathbf{X}_1}(\boldsymbol{x}|\boldsymbol{x}_0, \boldsymbol{x}_1) p_{\mathbf{X}_1|\mathbf{X}_0}(\boldsymbol{x}_1|\boldsymbol{x}_0)\mathrm{d}\boldsymbol{x}_1 =$$

$$= \int_{\mathbb{R}^n} -\mathrm{div}\left(p_{\mathbf{X}_t|\mathbf{X}_0,\mathbf{X}_1}(\boldsymbol{x}|\boldsymbol{x}_0, \boldsymbol{x}_1) f_t(\boldsymbol{x}|\boldsymbol{x}_0, \boldsymbol{x}_1)\right) p_{\mathbf{X}_1|\mathbf{X}_0}(\boldsymbol{x}_1|\boldsymbol{x}_0)\mathrm{d}\boldsymbol{x}_1 + \int_{\mathbb{R}^n} \frac{g^2(t)}{2}\Delta p_{\mathbf{X}_t|\mathbf{X}_0,\mathbf{X}_1}(\boldsymbol{x}|\boldsymbol{x}_0, \boldsymbol{x}_1) p_{\mathbf{X}_1|\mathbf{X}_0}(\boldsymbol{x}_1|\boldsymbol{x}_0)\mathrm{d}\boldsymbol{x}_1 =$$

$$= -\mathrm{div}\left(p_{\mathbf{X}_t|\mathbf{X}_0}(\boldsymbol{x}|\boldsymbol{x}_0)\int_{\mathbb{R}^n} \frac{p_{\mathbf{X}_t|\mathbf{X}_0,\mathbf{X}_1}(\boldsymbol{x}|\boldsymbol{x}_0, \boldsymbol{x}_1)}{p_{\mathbf{X}_t|\mathbf{X}_0}(\boldsymbol{x}|\boldsymbol{x}_0)} p_{\mathbf{X}_1|\mathbf{X}_0}(\boldsymbol{x}_1|\boldsymbol{x}_0) f_t(\boldsymbol{x}|\boldsymbol{x}_0, \boldsymbol{x}_1)\mathrm{d}\boldsymbol{x}_1\right) +$$

$$+ \frac{g^2(t)}{2}\Delta\int_{\mathbb{R}^n} p_{\mathbf{X}_t|\mathbf{X}_0,\mathbf{X}_1}(\boldsymbol{x}|\boldsymbol{x}_0, \boldsymbol{x}_1) p_{\mathbf{X}_1|\mathbf{X}_0}(\boldsymbol{x}_1|\boldsymbol{x}_0)\mathrm{d}\boldsymbol{x}_1 = -\mathrm{div}\left(p_{\mathbf{X}_t|\mathbf{X}_0}(\boldsymbol{x}|\boldsymbol{x}_0) f_t^*(\boldsymbol{x}|\boldsymbol{x}_0)\right) + \frac{g^2(t)}{2}\Delta p_{\mathbf{X}_t|\mathbf{X}_0}(\boldsymbol{x}|\boldsymbol{x}_0),$$

где

$$f_t^*(\boldsymbol{x}|\boldsymbol{x}_0) = \int_{\mathbb{R}^n} \frac{p_{\mathbf{X}_t|\mathbf{X}_0,\mathbf{X}_1}(\boldsymbol{x}|\boldsymbol{x}_0, \boldsymbol{x}_1)}{p_{\mathbf{X}_t|\mathbf{X}_0}(\boldsymbol{x}|\boldsymbol{x}_0)} p_{\mathbf{X}_1|\mathbf{X}_0}(\boldsymbol{x}_1|\boldsymbol{x}_0) f_t(\boldsymbol{x}|\boldsymbol{x}_0, \boldsymbol{x}_1)\mathrm{d}x_1 =$$

$$= \int_{\mathbb{R}^n} p_{\mathbf{X}_1|\mathbf{X}_0,\mathbf{X}_t}(\boldsymbol{x}_1|\boldsymbol{x}_0, \boldsymbol{x}) f_t(\boldsymbol{x}|\boldsymbol{x}_0, \boldsymbol{x}_1)\mathrm{d}x_1 = \mathbb{E}\left[f_t(\mathbf{X}_t|\mathbf{X}_0, \mathbf{X}_1)\Big|\mathbf{X}_t = \boldsymbol{x}, \mathbf{X}_0 = \boldsymbol{x}_0\right]$$