UNIT 16 TRIGONOMETRY AND ITS APPLICATION

Structure

16.1 Introduction

16.2 Objectives

16.3 Trigonometric Ratios

16.3.1 Definitions of Trigonometric Ratios

16.3.2 Trigonometric Ratios of some Specific Angles

16.3.3 Trigonometric Ratios of Complementary Angles

16.4 Trigonometric Identities

16.5 Height and Distance

16.6 Let Us Sum Up

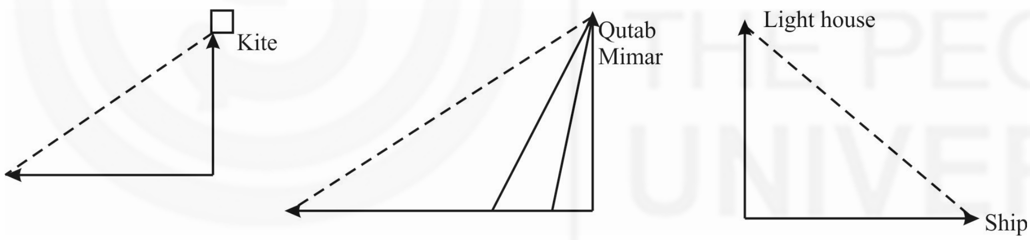
16.7 Unit End Activities

16.8 Answers to Check your Progress

16.9 References and Suggested Readings

16.1 INTRODUCTION

Have you ever thought how we can find the height at which the kite is flying or the height of QutabMinar or the distance of a ship from a light house.



In all the situations given above, the distance or height can be found by using some mathematical technique which come under a branch of mathematics called trigonometry.

The word Trigonometry is derived from the Greek words 'tri' meaning three, 'gon' meaning sides and 'metron' meaning measure. These three words together mean triangle measurement. **Thus trigonometry is the study of relationships between the sides and angles in a triangle**. Trigonometric ratios of angles which always have a unique value for any given angle form the basic tool for the study of these relationships. So in this unit we will study about trigonometric ratio and identities. We will also study about how to apply this knowledge to solve problems on heights and distances. Knowledge of trigonometry is useful in many situations such as navigation of ships or movements of aeroplanes, rockets, astronomical sciences, engineering surveys etc. Thus we will also discuss how trigonometry can be used in our real life situations.

16.2 OBJECTIVES

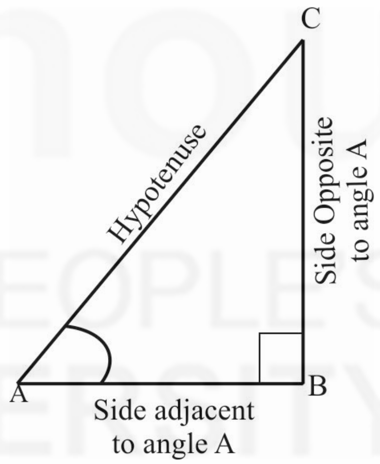
After going through this unit, you will be able to:-

- make the students understand the importance of Trigonometry;
- demonstrate to the students how the concepts of similarity form the basis of trigonometric ratios;
- develop among students the skill of manipulating trigonometric ratios and appreciate their relationship;
- help the students in using Pythagoras thereom to solve problems involving trigonometric ratios;
- help the students in finding the trigonometric rations of complementary angles and applying them;
- develop among students the skill of proving trigonometric identities; and
- develop problem solving skills as required to solve problems of height and distance;

16.3 TRIGONOMETRIC RATIOS

16.3.1 Definitions of Trigonometric Ratios

Ask student to consider a right triangle ABC right angled at B as shown in the figure.



They know, that side opposite to the right angle is always called the hypotenuse.

Ask students to consider $\angle A$ or in brief angle A.

It is an acute angle.

Now, for the remaining two sides AB and BC they will see that BC is the <u>side opposite to Angle A</u>, which is perpendicular to point B and AB is the side of the Angle A (Angle A is formed by two sides - AB and the hypotenuse). They can say that side AB as the <u>side adjacent to Angle A.</u>

Students we have already studied the concept of ratio. You can now define certain ratios involving the sides of a right triangle and call them Trigonometric Ratios. These trigonometric ratios express the relationship between the angle and the lengths of its sides. Trigonometric ratios of angle A are defined as:

$$\text{sine of } \angle A = \frac{\text{side opposite to angle A}}{\text{hypotenuse}} = \frac{BC}{AC}$$

$$\text{cosine of } \angle A = \frac{\text{side adjacent of angle A}}{\text{hypotenuse}} = \frac{AB}{AC}$$

$$\text{tangent of } \angle A = \frac{\text{side opposite to angle A}}{\text{side adjacent to angle A}} = \frac{BC}{AB}$$

In abbreviated form, these ratios are sin A, cos A, tan A. The ratios cosecant A, secant A and cotangent A are the reciprocals of ratios sin A, cos A and tan A respectively and are written as cosec A, sec A and cot A.

$$\text{cosec A} = \frac{1}{\sin A} = \frac{\text{hypotenuse}}{\text{side opposite to angle A}} = \frac{AC}{BC}$$

$$\sec A = \frac{1}{\cos A} = \frac{\text{hypotenuse}}{\text{side adjacent to angle A}} = \frac{AC}{AB}$$

$$\cot A = \frac{1}{\tan A} = \frac{\text{side adjacent to angle A}}{\text{side opposite to angle A}} = \frac{AB}{BC}$$

Students can observe that

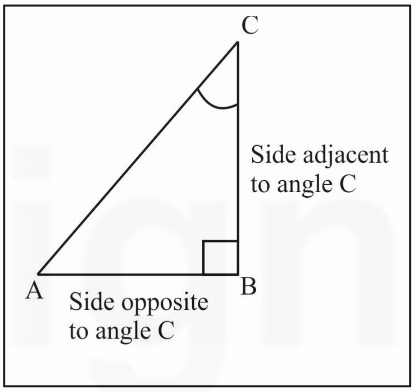
$$\tan A = \frac{BC}{AB} = \frac{BC/AC}{AB/AC} = \frac{\sin A}{\cos A} \text{ and}$$

$$\cot A = \frac{1}{\tan A} = \frac{\cos A}{\sin A}$$

Now if you define the trigonometric ratios for angle C in the same right triangle, you will see that AC remains the hypotenuse as the right angle B is fixed.

Now in place of angle A, you can take angle C. So the side opposite to angle C is AB and side adjacent to angle C is BC.

Ask your student to write the trigonometricratios for angle C yourself.



Note:-

1 Note that sinA is not the product of sin and A, sin separated from A has no meaning. Sin is always of some angle. The same follows for other Trigonometric ratios also.

2 For the sake of convenience, we may write $\sin^2 A$, $\cos^2 A$, etc. in place of $(\sin A)^2$, $(\cos A)^2$ etc. respectively. But we should not write cosec A, which is the reciprocal of sin A as cosec A= $\sin^{-1}$ A. We can however write cosec A = $(\sin A)^{-1}$

i. e. $(\sin A)^{-1} \neq \sin^{-1} A$.

3 Greek letter θ(theta) is also sometimes used to denote angle.

4 The trigonometric ratio of an angle is always a real number and so it does not have any unit.

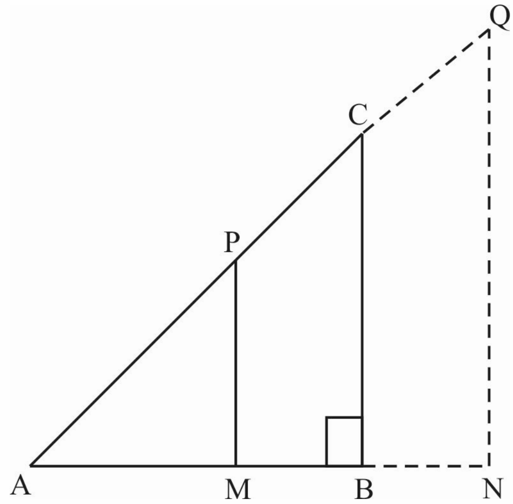
5 The word ‘trigonometric ratios’ is sometimes written briefly as t-ratios.

6 In this unit, you shall always assume that

$0^\circ \leq \theta \leq 90^\circ$

Ask your students that if, they keep the angle A same but change the lengths of the sides, will the value of trigonometric ratios of that angle change?

For this, ask them to consider again a right Δ ABC right angled at B.



Take a point P on the hypotenuse AC.

Draw PM $\perp$ AB

Now we want to find whether trigonometric ratios of angle A differ in

$\triangle$AMP and $\triangle$ABC.

Student already know that in ΔABC, $\sin A = \dfrac{BC}{AC}$ _____ (1)

and they can see that in ΔAMP, $\sin A = \dfrac{PM}{AP}$ _____ (2)

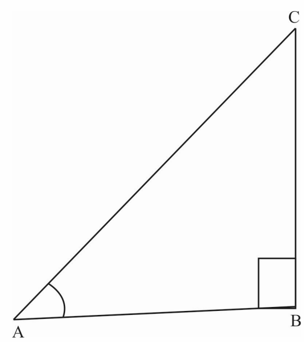
Now, $\Delta AMP \sim \Delta ABC$ (AA criterion of similarity).

So, $\dfrac{MP}{AP} = \dfrac{BC}{AC}$ __________(3)

So, from (1), (2) and (3) students can conclude that

$\sin A$ in $\Delta ABC = \sin A$ in ΔAMP

In the same way, you can check that if students take any point Q on extended AC and $QN \perp AB$ extended, the value of $\sin A$ remains the same in Δ AQN also. Similarly, we can show for other trigonometric ratios, so from the above you can conclude that values of the trigonometric ratios of an angle do not vary with the lengths of the sides of the triangle, if the angle remains the same.



When you know any one trigonometric ratio supposesinA in the figure, you also know the lengths of two sides BC and AC.

If yourstudents are able to find the third side they can find the value of all the other trigonometric ratios.

Ask students, do they remember Pythagoras thereom for a right Δ ABC ?

Student, may reply $AC^2 = AB^2 + BC^2$

From this, they can find side $AB = \sqrt{AC^2 - BC^2}$

So, now, students know the length of all the three sides AB, BC and AC and so, they can obtain all other remaining five trigonometric ratios. Also note that as hypotenuse is the longest side in a right triangle, the value of sinA and cosA is always less than 1 (or in particular equal to 1). The following examples will make the idea clear.

Example-1:

In ΔPQR , right angled at Q, if PQ = 20 units and PR= 29 units, find sinP, cosR, cotR.

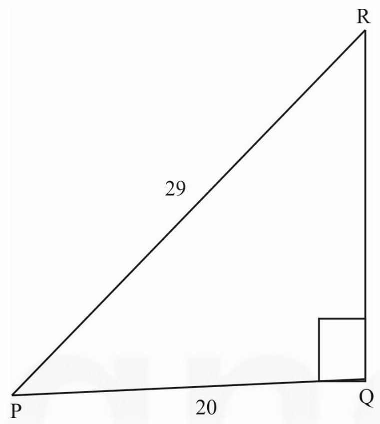
Solution:-

By Pythagoras theorem students will have

$$PR^2 = PQ^2 + QR^2$$

$$29^2 = 20^2 + QR^2$$

$$\text{or } QR^2 = 29^2 - 20^2 = 841 - 400 = 441$$



$$\text{So, } QR = 21 \text{ units}$$

$$\text{Therefore, } \sin P = \frac{QR}{PR} = \frac{21}{29}$$

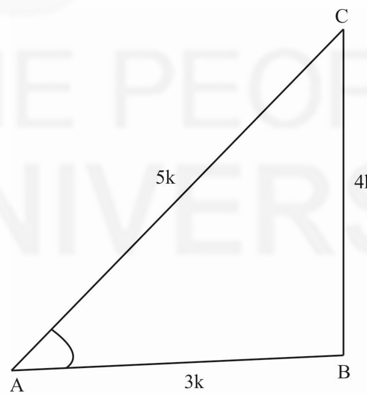
$$\cos R = \frac{QR}{PR} = \frac{21}{29}$$

$$\cot R = \frac{QR}{PQ} = \frac{21}{20}$$

Example 2:

Given $\cos A = \frac{3}{5}$, Find other t-ratios of angle

<u>Solution:</u>



Take a right ΔABC (Figure)

$$\cos A = \frac{3}{5} = \frac{AB}{AC}$$

Note that $\frac{AB}{AC} = \frac{3}{5}$ does not necessarily mean AB=3 units and AC=5 units. In general, this means AB = 3k units and AC = 5k units, where K is some constant.

By Pythagoras thereom, $AC^2=AB^2+BC^2$

$$\text{So,} (5k)^2=(3k)^2+BC^2$$

$$\text{Or, } BC^2 = (5k)^2 - (3k)^2 = 25k^2 - 9k^2 = 16k^2$$

$$\text{So, } \quad BC=4k$$

$$\text{Therefore, } \sin A = \frac{BC}{AC} = \frac{4k}{5k} = \frac{4}{5}$$

$$\text{cosec } A = \frac{1}{\sin A} = \frac{5}{4}$$

$$\sec A = \frac{1}{\cos A} = \frac{5}{3}$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{4}{3}$$

$$\cot A = \frac{1}{\tan A} = \frac{3}{4}$$

Example 3:

In right ΔABC, right angled at A, in which AB=5 units, BC=13 units and $\angle ABC = \theta$, determine the value of $\sin^2\theta - \cos^2\theta$.

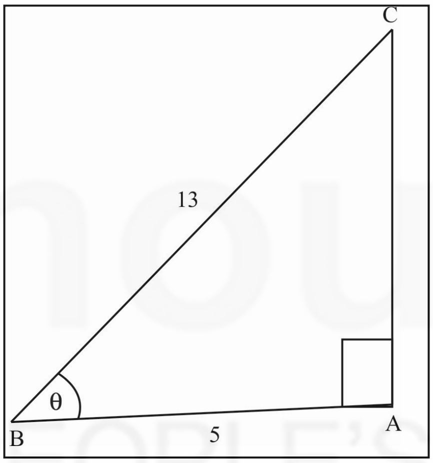
Solution:-

In ΔABC,

$BC^2 = AB^2 + AC^2$

So, $AC^2 = BC^2 - AB^2 = 13^2 - 5^2 = 169 - 25 = 144$

So, $AC = 12$ units



Therefore, $\sin\theta = \frac{AC}{BC} = \frac{12}{13}$

$$\cos\theta = \frac{AB}{BC} = \frac{5}{13}$$

so, $\sin^2\theta - \cos^2\theta = \left(\frac{12}{13}\right)^2 - \left(\frac{5}{13}\right)^2$

$$= \frac{144 - 25}{169}$$

$$= \frac{119}{169}$$

Check Your Progress

Notes: a) Write your answers in the space given below.

b) Compare your answers with those given at the end of the Unit .

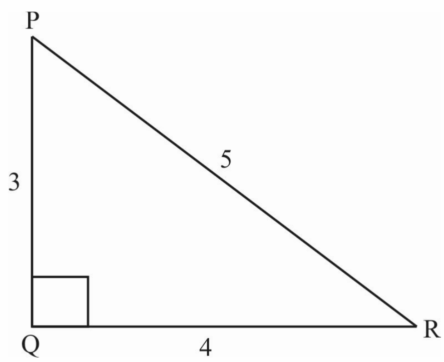
1) From the figure, write

sin P =

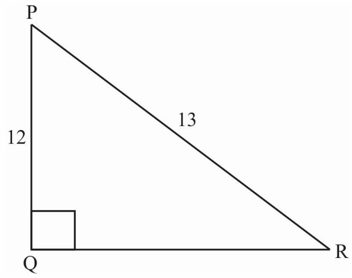
cosP =

tanR =

cosecR =



2) From the figure, find tanP–cotR



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3) Given $\sec\theta = \frac{13}{12}$, find the value of $\frac{1-\tan^2\theta}{1+\tan^2\theta}$

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16.3.2 Trigonometric Ratios of some Specific Angles

In this section we will find the values of Trigonometric Ratios of $0^o, 30^o, 45^o, 60^o$ and 90^o.

Trigonometry ratio of 45^o could be introduced by the following way.

Ask students to consider Δ ABC, right angled at B, if $\angle A = 45^o$ what can you say about $\angle C$? Students may respond that $\angle C$ will also be 45^o. The teacher can give measurent of each side to be 'a'. so

AB= BC =a (say)

To find the value of all t-ratios of A, you will have to find the third side, Using Pythogorasthereom we have

$AC^2 = AB^2 + BC^2$

$AC^2 = a^2 + a^2 = 2a^2$

So, $AC = a\sqrt{2}$

Ask students by using the definitions of t-ratios, workout the different trigonometric ratios for angle 45^o

They may bring out that

$$\sin 45^\circ = \frac{BC}{AC} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\cos 45^\circ = \frac{AB}{AC} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

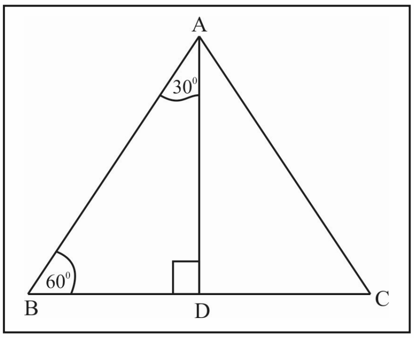
$$\tan 45^\circ = \frac{BC}{AB} = \frac{a}{a} = 1$$

Also, $\operatorname{cosec} 45° = \dfrac{1}{\sin 45°} = \sqrt{2}$

$\sec 45° = \sqrt{2}$

$\cot 45° = 1$

T-Ratios of 30° and 60°



You can take an equilateral Δ ABC

So, $\angle A = \angle B = \angle C = 60°$

Draw, $AD \perp BC$

Now $\Delta ABD \cong \Delta ACD$ (by RHS)

$$\left.\begin{array}{l}\text{So, } BD = CD \\ \text{and } \angle BAD = \angle CAD\end{array}\right\}\text{(cpct.)}$$

You know that to find the value of all the t-ratios, you must know all the three sides of the triangle.

In Δ ABD, let AB = 2a

As BD= CD So $BD = \frac{1}{2}\ BC = \frac{1}{2} \times 2a = a$

By Pythagoras theorem,

$$AD^2 = AB^2 - BD^2$$
$$= (2a)^2 - a^2 = 3a^2$$
$$\text{so, } AD = a\sqrt{3}$$

Therefore, in Δ ABD, AB = 2a, BD =a and $AD = a\sqrt{3}$

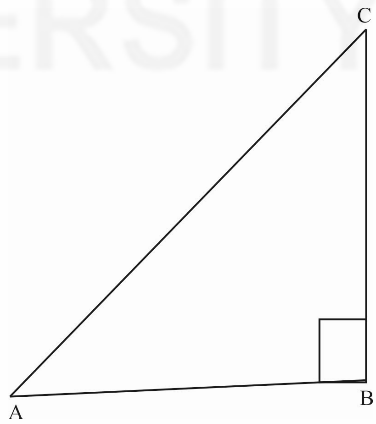
Now you as well as your students can find all the t-ratios for angle 60° and $\angle BAD = 30°$

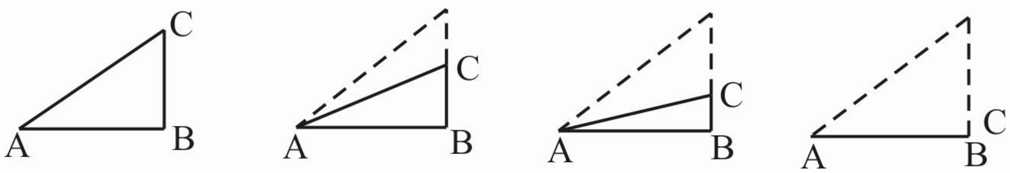
Ask student to all find the t –ratios yourself.

T-Ratios of 0° and 90°

In the figure, in right ΔABC,

If $\angle A$ is made smaller and smaller, the length of the side BC goes on decreasing and point C gets closer and closer to point B and finally when $\angle A$ becomes very close to 0°, BC gets very close to 0 and AC becomes almost the same as AB.





So, $\sin A = \frac{BC}{AC}$ is very close to 0.

$\cos A = \frac{AB}{AC}$ is very close to 1.

So, you define

$\sin 0^o = 0$

$\cos 0^o = 1$

Using these, you have

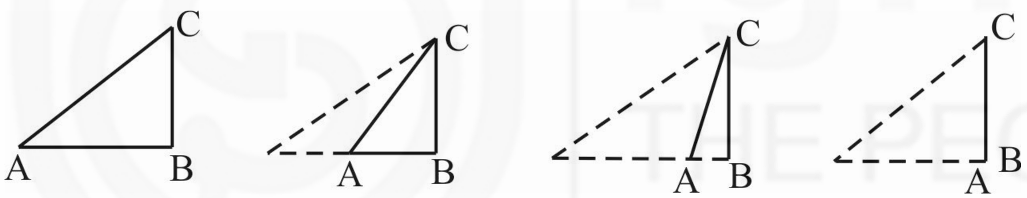
$$\tan 0^0 = \frac{\sin 0^0}{\cos 0^0} = 0$$

$\cot 0^0 = \frac{1}{\tan 0^0} = \infty$ which is not defined as division by 0 is not defined.

$$\sec 0^0 = \frac{1}{\cos 0^0} = 1$$

$\operatorname{cosec} 0^0 = \frac{1}{\sin 0^0} = \infty$ which again is not defined.

Similarly, when $\angle A$ is made larger and larger in ΔABC till it becomes 90^o, point A gets closer to point B and side AC almost coincides with side BC.



Thus, you have

$\sin 90^o = 1$ and $\cos 90^o = 0$

Ask student to find the other Trigonometric ratios of 90^o.

Note:

It should be noted that there exist values of all the trigonometric ratios for angles other than these angles also. The values can be obtained from trigonometric tables which your student will study in higher classes.

Values of all t- ratios of 0^o, 30^o, 45^o, 60^o and 90^o.

$\angle A$	0^o	30^o	45^o	60^o	90^o
sin A	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos A	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan A	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined

cosec A	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
sec A	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined
cot A	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

From the above table, students can observe that as $\angle A$ increases from 0^o to 90^o, sinA increases from 0 to 1 and cosA decreases from 1 to 0 and also that

$$0 \le \sin A \le 1$$
$$1 \le \cos A \le 0$$

Example 4:-

Evaluate $\cot^2 45^o - \sec^2 60^o + \sin^2 60^o + \cos^2 90^o$

Solution: Given expression

$$= 1^2 - 2^2 + \left(\frac{\sqrt{3}}{2}\right)^2 + 0^2$$

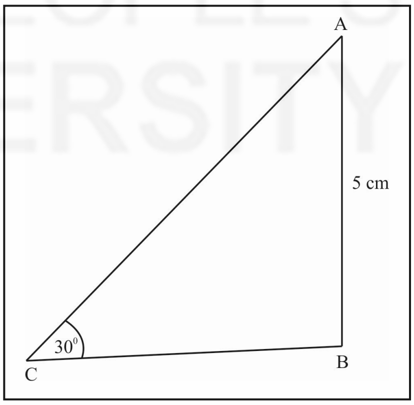
$$= 1 - 4 + \frac{3}{4}$$

$$= -\frac{9}{4}$$

In previous section, you have found all the t-ratios when any two side of the right triangle are given. Now if, your students are given one side of the triangle and one angle of the right triangle, then they can find the other two sides of the triangle. Let us see an example.

Example 5:-

In Δ ABC, right angled at B, if AB= 5cm, $\angle C = 30^o$, determine the sides BC and AC.



Solution:-

You are given AB and we want to find BC. So we will choose that-ratio, which involves these two sides. Here, it is tanC (or cotC)

$$\tan 30° = \frac{AB}{BC}$$

Ask student, as they know that $\tan 30° = \frac{1}{\sqrt{3}}$

So, $\frac{1}{\sqrt{3}} = \frac{5}{BC}$

Or , $BC = 5\sqrt{3}$ cm

To determine AC, you have

$$\sin C = \frac{AB}{AC}$$

$$\sin 30° = \frac{5}{AC}$$

i.e., $\frac{1}{2} = \frac{5}{AC}$

Or, AC = 10 cm

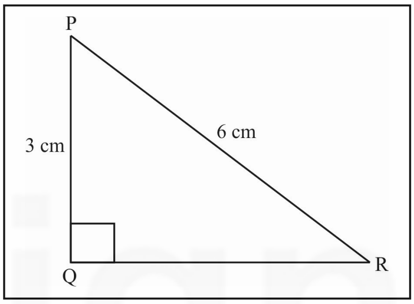
To determine AC you could have used cosC, secC or Pythagoras theorem also.

Now, if your students are given two sides of right triangle they can find its angles also. Let us an example.

Example 6:-

In right Δ PQR, right angled at Q, PQ = 3 cm and PR = 6cm, determine $\angle$P, $\angle$R.

Solution:



You choose that t-ratio which involves the two given sides.

So, $\sin R = \frac{PQ}{PR} = \frac{3}{6} = \frac{1}{2}$

So, $\angle$ R = 30°

and so, $\angle$ P = 60°

From the above, you must have noticed that if one of the sides and any other part either an acute angle or any side of a right triangle are given, you can find the remaining sides and angles of the triangle.

Check Your Progress

Notes: a) Write your answers in the space given below.

b) Compare your answers with those given at the end of the Unit .

4. Evaluate $4\cot^2 45° - \sec^2 60° + \sin^2 60° + \cos^2 90°$

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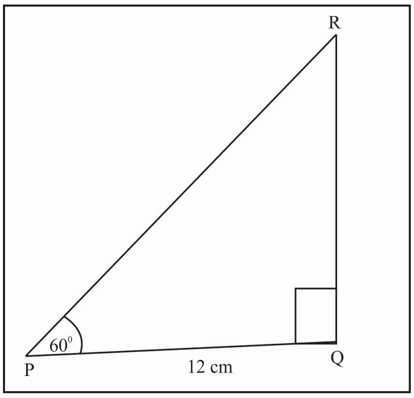
5. In Δ PQR, right angled at Q,

if PQ = 12cm, $\angle$P = 60°,

determine PR and QR.



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6. In Δ ABC right angled at B, if AB = 3cm, AC = 2 $\sqrt{3}$ cm, find $\angle$A and $\angle$C.

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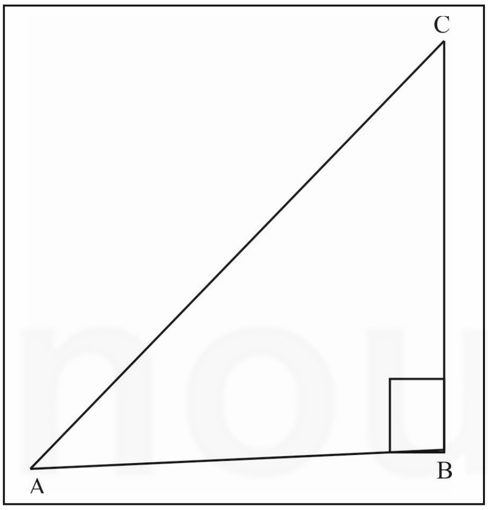
16.3.3 Trigonometric Ratios of Complementary Angles

Ask your student to recall that two angles are complementary angles, if their sum is 90^o.



In right Δ ABC right angled at B,

$\angle A + \angle C = 90^o$.

For convenience, you can write

$A + C = 90^o$

i.e. $C = 90° - A$

Now, $\sin A = \frac{BC}{AC}$, $\cos A = \frac{AB}{AC}$

$\sin C = \frac{AB}{AC}$, $\cos C = \frac{BC}{AC}$

You can see that

$\sin A = \cos C$ and $\cos A = \sin C$

i.e. $\sin A = \cos(90° - A)$, $\cos A = \sin(90° - A)$

Similarly, you can show other t-ratios.

So, for angles lying between 0^o and 90^o, you have

$\sin(90° - A) = \cos A$

$\cos(90° - A) = \sin A$

$\tan(90° - A) = \cot A$

$\operatorname{cosec}(90° - A) = \sec A$

$\sec(90° - A) = \operatorname{cosec} A$

$\cot(90° - A) = \tan A$

so, you have

$\sin 30^o = \cos 60^o$

$\cos 30^o = \sin 60^o$

$\tan 30^o = \cot 60^o$

$\operatorname{cosec} 30^o = \sec 60^o$

$\sec 30^o = \operatorname{cosec} 60^o$

$\cot 30^o = \tan 60^o$

Angle 45^o is its own complement. Hence

$\sin 45^o = \cos 45^o$

$\tan 45^o = \cot 45^o$

$\sec 45^o = \mathrm{cosec}\, 45^o$

Example 7

Evaluate $\cos 48^o - \sin 42^o$

Solution

$\cos 48^o - \sin 42^o = \cos 48^o - \sin(90^o - 48^o)$

$= \cos 48^o - \cos 48^o$

$= 0$

Example 8

If $\sec 4A = \mathrm{cosec}\,(A - 20^o)$, where 4A is an acute angle, find A.

Solution

since, $\sec 4A = \mathrm{cosec}\,(90^o - 4A)$

So, $\mathrm{cosec}\,(90^o - 4A) = \mathrm{cosec}\,(A - 20^o)$

As $90^o - 4A$ and $A - 20^o$, both are acute angles, you have

$90^o - 4A = A - 20^o$

Or $5A = 110^o$

Or $A = 22^o$

Check Your Progress

Notes: a) Write your answers in the space given below.

b) Compare your answers with those given at the end of the Unit .

7) Evaluate $\dfrac{\sin 18°}{\cos 72°}$

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8) Prove that

$$\frac{\cos 20°}{\sin 70°} + \frac{\cos\theta}{\sin(90° - \theta)} = 2$$

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9) If $\sin 3A = \cos(A - 26°)$, where 3A is an acute angle, find the value of A.

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16.4 TRIGONOMETRIC IDENTITIES

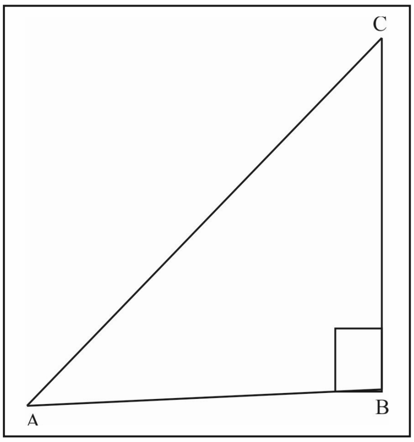
Can you recall what is an identity ?

An equation which is true for all the values of the variables is called an identity.

A trigonometric identity is an equation, which involves Trigonometric ratios and is true for all the values of the angles involved.

Consider Δ ABC right angled at B

$AB^2 + BC^2 = AC^2$ ____ (1)



Dividing each term of (1) by AC^2

You will have

$$\frac{AB^2}{AC^2} + \frac{BC^2}{AC^2} = 1$$

$$\text{i.e.} \left(\frac{AB}{AC}\right)^2 + \left(\frac{BC}{AC}\right)^2 = 1$$

$$\text{or } (\cos A)^2 + (\sin A)^2 = 1$$

$$\text{or } \cos^2 A + \sin^2 A = 1$$

This is true for all A such that $0^o \le A \le 90^o$,

So, this is a Trigonometric Identity.

In (i) above, dividing by AB^2, you will have

$$\left(\frac{AB}{AB}\right)^2 + \left(\frac{BC}{AB}\right)^2 = \left(\frac{AC}{AB}\right)^2$$

i.e. $1 + \tan^2 A = \sec^2 A$ ________ (2)

As tanA and secA are not defined for A $=90^o$, so (2) is true for all A such that $0^o \le A < 90^o$.

Similarly, dividing (i) by BC^2 , you can get

$$\left(\frac{AB}{BC}\right)^2 + \left(\frac{BC}{BC}\right)^2 = \left(\frac{AC}{BC}\right)^2$$

i.e. $\cot^2 A+1 = \text{cosec}^2 A$ ________ (3)

(3) is true for all A such that $0^o < A \le 90^o$.

Thus, you will have

$\sin^2 A+\cos^2 A = 1$

$\sec^2 A=1+\tan^2 A$

$\text{cosec}^2 A=1+\cot^2 A$

These three relations are all identities and are called fundamental identities. Each of these identities can be obtained from the other.

If your students know one t-ratio, they can determine other t-ratios using these identities.

Suppose you are given sinA. From the identity $\sin^2A+\cos^2A=1$ you canfind cosA.AlsotanA will be obtained from the relations $\tan A = \dfrac{\sin A}{\cos A}$

So, now you know sinA, cosA and tanA and the other t-ratio are reciprocals of these.

Example 9

Express $\operatorname{cosec}\theta$ in term of $\cos\theta$.

Solution:

Since, $\sin^2\theta + \cos^2\theta = 1$

So, $\sin^2\theta = 1 - \cos^2\theta$

$$\text{or,}\quad \frac{1}{\operatorname{cosec}^2\theta} = 1-\cos^2\theta$$

$$\text{or,}\quad \operatorname{cosec}^2\theta = \frac{1}{1-\cos^2\theta}$$

$$\text{or,}\quad \operatorname{cosec}\theta = \pm\frac{1}{\sqrt{1-\cos^2\theta}}$$

As θ is an acute angle, you have

$$\operatorname{cosec}\theta = \frac{1}{\sqrt{1-\cos^2\theta}}$$

Example 10

Prove that $(\operatorname{cosec}\theta - \cot\theta)^2 = \dfrac{1-\cos\theta}{1+\cos\theta}$

Solution:

$$\text{L.H.S.} = \left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right)^2$$

$$= \left(\frac{1-\cos\theta}{\sin\theta}\right)^2 = \frac{(1-\cos\theta)^2}{\sin^2\theta}$$

$$= \frac{(1-\cos\theta)^2}{1-\cos^2\theta}$$

$$= \frac{(1-\cos\theta)^2}{(1-\cos\theta)(1+\cos\theta)} = \frac{1-\cos\theta}{1+\cos\theta} = \text{R.H.S.}$$

Check Your Progress

Notes: a) Write your answers in the space given below.

b) Compare your answers with those given at the end of the Unit .

10) Express secA in terms of sinA.

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11) Simplify $\dfrac{1+\tan^2 A}{1+\cot^2 A}$

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12) Prove that

$\tan^4 A + \tan^2 A = \sec^4 A - \sec^2 A$

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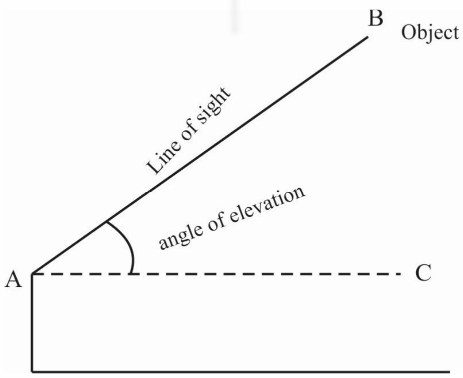
16.5 HEIGHT AND DISTANCE

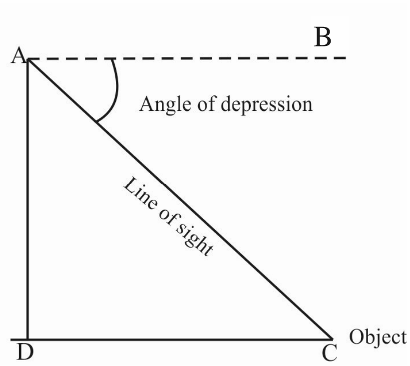
In the earlier section you have studied about trigonometric ratios. In this section, you will study how trigonometry is used in finding the heights and distances of various objects without actually measuring them.

Whenever, an engineer faces problem in determining the width of a river or height of a tower etc., which may not be easily possible to measure with a measuring tape, she/he imagines a big right triangle. Knowing a side and an angle by using any surveying instrument she/he can use the knowledge of trigonometric ratios to calculate the unknown side i.e. width of the river or height of the tower.

Let us consider an observer viewing a certain object



(i)



(ii)

The line drawn from the eye of the observer to the point in the object being viewed is called the line of sight. The angle made by the line of sight with the horizontal is called the angle of elevation or angle of depression depending upon the object viewed is above the horizontal line or below the horizontal line.

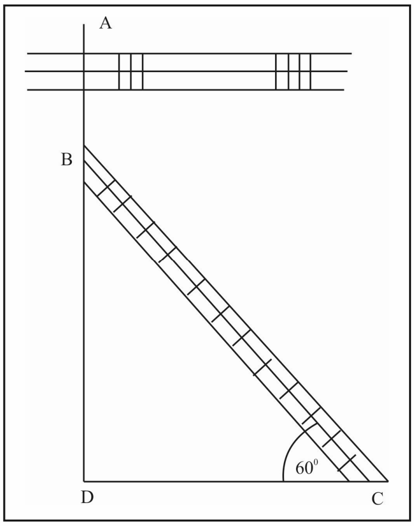
If the object being viewed is above the horizontal line, the angle $\angle BAC$ is the angle of elevation as in fig (i).

If the observer is standing on a balcony and viewed the object C on the road the $\angle BAC$ is the angle of depression as in the figure (ii). Note here $\angle CAD$ is not the angle of depression.

Now let us solve problems.

Example 11

An Electrician has to repair an electric fault on a pole height 5 m. He needs to reach a point 1.3m below the top of the pole to undertake the repair work what should be the length of the ladderthat he should use which, when inclinedat an angle of 60^{o} to the horizontal, would enable his to reach the required position?Also, how far from the foot of the pole should he place the foot of the ladder? (take $\sqrt{3}=1.73$)



Solution:-

The electrician has to reach to point B on the pole AD

So, BD = AD – AB = (5 - 1.3) m = 3.7 m

Here, BC represents the ladder in the right ΔBCD and we wants to find its length.

Which t-ratio should be used?

It should be Sin60^{o}

So, $\frac{BD}{BC} = \sin 60^{2}$ or $\frac{3.7}{BC} = \frac{\sqrt{3}}{2}$

$$\text{or,} \quad BC = \frac{3.7 \times 2}{\sqrt{3}} = 4.28\text{m}(\text{approx})$$

i.e. length of the ladder should be 4.28m.

We also want to find how far from the foot of the pole should be the foot of the ladder.

$$\frac{DC}{BC} = \cot 60^{0} = \frac{1}{\sqrt{3}}$$

$$\text{Or,} \quad \frac{DC}{3.7} = \frac{1}{\sqrt{3}}$$

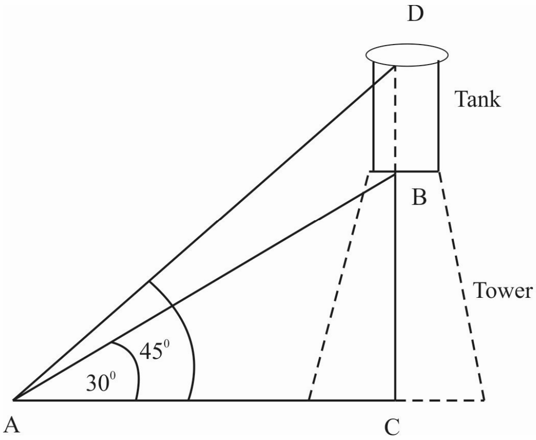
$$\text{So,} \quad DC = \frac{3.7}{\sqrt{3}} = 2.14\text{m}(\text{approx})$$

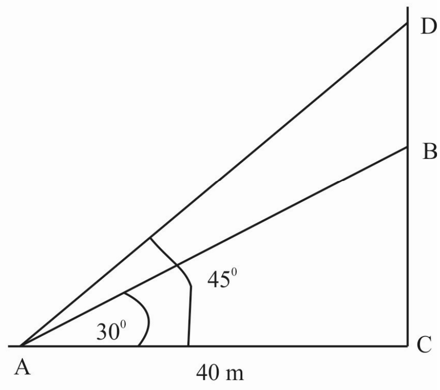
Therefore, she should place the foot of the ladder at a distance of 2.14m from the pole.

Example 12

From a point on the ground 40 m away from the foot of a tower, the angle of elevation of the top of the tower is 30^{o}. The angle of the elevation to the top of a water tank on the top of the tower is 45°. Find the height of the tower and the depth of the tank.

Solution





In right Δ ACB, we have

$$\frac{BC}{AC} = \tan 30^0 = \frac{1}{\sqrt{3}}$$

$$\text{or} \quad \frac{BC}{40} = \frac{1}{\sqrt{3}}$$

$$\text{or} \quad BC = \frac{40}{\sqrt{3}} = \frac{40\sqrt{3}}{3} = 23.1\text{m approx}$$

Thus, height of tower is 23.1m (approx).

To find the depth BD of the water tank you require the length CD

In Δ ACD , you have

$$\frac{CD}{AC} = 1 \ \text{or} \ \frac{CD}{40} = 1 \ \text{or,} \ CD = 40\text{m}$$

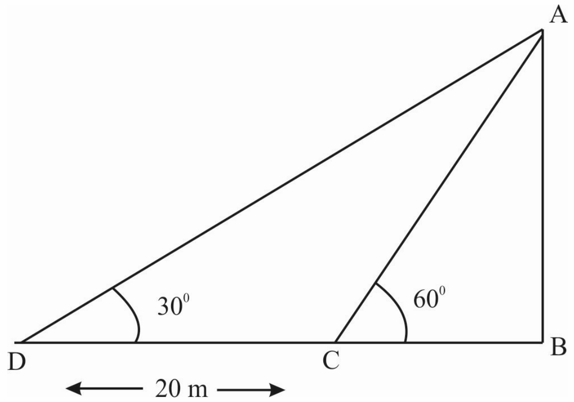
Hence, depth of the water tank = $BD \Rightarrow CD - CB \Rightarrow (40 - 23.1)\text{m} \Rightarrow 16.9\text{m}$ (approx.)

Example 13

A tree stands vertically on the bank of a river. From a point on the other bank directly opposite the tree, the angle of the elevation of the top of the tree is 60°.

From a point 20m behind this point on the same bank, the angle of elevation of the top of the tree is 30°. Find the height of the tree and the width of the river.

Solution



Let AB be the height of the tree and let C and D be the two points on the other bank opposite to the tree so that BC measures the width of the river, you want to find AB and BC.

In Δ ABC

$$\frac{AB}{BC} = \tan 60^\circ$$

Or $\quad AB = BC\sqrt{3}$ ----------------(i)

Now in Δ ABD

$$\frac{AB}{BD} = \tan 30^\circ = \frac{1}{\sqrt{3}}$$

i.e. $\dfrac{AB}{(20+BC)} = \dfrac{1}{\sqrt{3}}$

or $\quad AB = \dfrac{20+BC}{\sqrt{3}}$ ----(ii)

From (i) & (ii) you will have

$$BC\sqrt{3} = \frac{20+BC}{\sqrt{3}}$$

or $\quad 3BC = 20 + BC$

or, $\quad 2BC = 20$

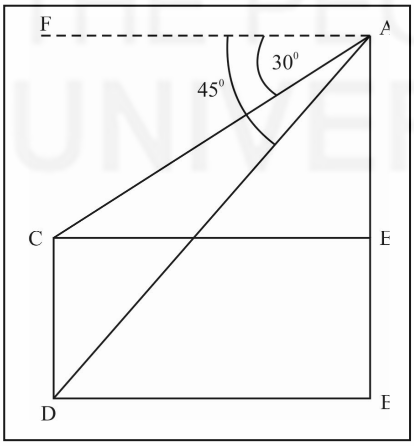
or, $\quad BC = 10$

So, by (i) $AB = 10\sqrt{3}$ =17.3m (aprox.)

Hence, height of the tree is 17.3 m approx. and width of the river is 10m.

Example 14

The angels of depression of the top and bottom of an 8m tall building from the top of a multistoried building are 30° and 45° respectively. Find the height of the multi-storied building and the distance between the two buildings.



Solution

In the figure, AB denotes the multi-storied building and CD the 8m tall building.

You want to find out AB and BD.

$\left.\begin{array}{l}\angle FAC = \angle ACE \\ \text{and} \\ \angle FAD = \angle ADB\end{array}\right\}$ Alternative angles

So, $\angle ACE = 30^\circ$ and $\angle ADB = 45^\circ$

Now in right Δ ACE ,

$$\frac{AE}{CE} = \tan 30^\circ = \frac{1}{\sqrt{3}}$$

So, $\quad CE = AE\sqrt{3}$ --------- (1)

In right Δ ADB,

$$\frac{AB}{BD} = \tan 45° = 1$$

So, BD=AB -----------(2)

Also BD=CE -----------(3)

So, by (1), (2) & (3) you have,

CE= $AE\sqrt{3}$ = BD = AB = AE + BE = AE +8

so, $AE\sqrt{3} = AE + 8$

i.e, $AE\left(\sqrt{3}-1\right) = 8$

$$\text{so, } AE = \frac{8}{\sqrt{3}-1} = \frac{8\left(\sqrt{3}+1\right)}{\left(\sqrt{3}-1\right)\left(\sqrt{3}+1\right)} = \frac{8\left(\sqrt{3}+1\right)}{3-1} = 4\left(\sqrt{3}+1\right)$$

So, height of the multi –storied building

= AB =AE + BE = $4(\sqrt{3}+1) + 8 = 4(\sqrt{3}+3)$ m

Distance between two buildings = BD = AB = $4(\sqrt{3}+3)$ m

Check Your Progress

Notes: a) Write your answers in the space given below.

b) Compare your answers with those given at the end of the Unit .

13) The length of the shadow of a man is equal to the height of the man. The angle of depression is?

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14) From a point 20m away from the foot of the tower the angle of elevation of the top of the tower is 30^{o}. The height of the tower is:

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15) A circus artist is climbing from the ground along a rope stretched from the top of vertical pole and tied at the ground. The height of the pole is 12m and the angle made by the rope with ground level is 30^{o}. Calculate the distance covered by the artist in climbing to the top of the pole.

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16) From a point on a bridge across a river, the angles of depression of the banks on opposite sides of the river are 30^o and 45^o respectively. If the bridge is at a height of 3m from the banks, find the width of the river.

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16.6 LET US SUM UP

In this unit, we have defined six Trigonometric ratios. These ratios are unique for any given angle and form the basic building blocks in the study of trigonometry. T-ratios of some specific angles and the t-ratios of complementary angles are derived. Three fundamental identities using Pythagoras theorem and inter relationship of t-ratios has been used to solve problems on trigonometric identities.

An important use of trigonometry has been illustrated by solving problems on height and distances. You would have realized that for solving height and distance problems, it is very necessary to draw a diagram as per the given information. Thus an elementary treatment of Trigonometry has been provided in this unit since it lays a foundation for further study of subject.

16.7 UNIT END ACTIVITIES

1) In Δ PQR, Q=90^o and $\sin R = \frac{3}{5}$, write the value of cosP.

2) Given that 16cotA = 12, find the value of $\frac{\sin A + \cos A}{\sin A - \cos A}$

3) Simplify

$\tan^2 60^o + 2\cos^2 45^o + 3(\sec^2 30^o + \cos^2 90^o)$

4) Evaluate

$$2\ \frac{\cos 58^\circ}{\sin 32^\circ} - \sqrt{3}\ \frac{\cos 38^\circ \operatorname{cosec} 52^\circ}{\tan 15^\circ \tan 60^\circ \tan 75^\circ}$$

5) If $\sin 2\theta = \cos(\theta - 36^\circ)$, 2θ and $\theta - 36^\circ$ are acute angles, then find the value of θ.

6) Prove that:-

$$\frac{\cos A}{1-\tan A} + \frac{\cos A}{1-\cot A} = \cos A, \quad A \neq 45^\circ$$

7) From a point P on the ground the angle of elevation of the top of a 10m tall building is 30^o. A flag is hoisted at the top of the building and the angle of elevation of the top of the flagstaff from P is 45^o. Find the length of the flagstaff and the distance of the building from the point P. ($\sqrt{3}$ = 1.732)

8) The angle of elevation of the top of a building from the foot of the tower is 30^o and the angle of elevation of the top of the tower from the foot of the building is 60^o. If tower is 50m high, find the height of the building.

9) As observed from the top of a 75m high lighthouse from the sea level, the angles of depression of two ships are 30^o and 45^o. If one ship is exactly behind the other on the same side of the lighthouse, find the distance between two ships.

16.8 ANSWERS TO CHECK YOUR PROGRESS

1) $\frac{4}{5}, \frac{3}{5}, \frac{3}{4}, \frac{5}{3}$

2) 0

3) $\frac{119}{169}$

4) $\frac{3}{4}$

5) 24cm, $12\sqrt{3}$cm

6) 30^o, 60^o

7) 1

9) 29^o

10) $\frac{1}{\sqrt{1-\sin^2 A}}$

11) $\tan^2 A$

13) 45^o

14) $20/\sqrt{3}$

15) 24m

16) $3(\sqrt{3}+1)$m

16.9 REFERENCES AND SUGGESTED READINGS

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