Introduction to Data Structures and Algorithms

Pham Quang Dung and Do Phan Thuan

Computer Science Department, SoICT,
Hanoi University of Science and Technology.

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First example

Find the largest weight subsequence of a given sequence of numbers

- Given a sequence $s = \langle a_1, \ldots, a_n \rangle$
- a subsequence is $s(i, j) = \langle a_i, \ldots, a_j \rangle$, $1 \leq i \leq j \leq n$
- weight $w(s(i, j)) =$

$$\sum_{k=i}^{j} a_k$$

- Problem: find the subsequence having largest weight
- http://www.spoj.com/problems/MAXSUMSU/

Example

- sequence: -2, 11, -4, 13, -5, 2
- The largest weight subsequence is 11, -4, 13 having weight 20

How many subsequences?

- Number of couple (i, j) where $1 \le i \le j \le n$
- $\binom{n}{2} + n$
- This give a direct algorithm

Direct algorithm — $\mathcal{O}(n^3)$

- Scan all possible subsequences $\binom{n}{2} + n = \frac{n^2 + n}{2}$
- Compute and keep the largest weight subsequence

```java
public long algo1(int[] a){
  int n = a.length;
  long max = a[0];
  for(int i = 0; i < n; i++){
    for(int j = i; j < n; j++){
      int s = 0;
      for(int k = i; k <= j; k++)
        s = s + a[k];
      max = max < s ? s : max;
    }
  }
  return max;
}
```

Direct algorithm
Faster algorithm — $\mathcal{O}(n^2)$

- Observation: $\sum_{k=i}^{j} a[k] = a[j] + \sum_{k=i}^{j-1} a[k]$

```
public long algo2(int[] a){
  int n = a.length;
  long max = a[0];
  for(int i = 0; i < n; i++){
    int s = 0;
    for(int j = i; j < n; j++){
      s = s + a[j];
      max = max < s ? s : max;
    }
  }
  return max;
}
```

Divide And Conquer

- Divide the sequence into 2 subsequences at the middle $s = s_1 :: s_2$
- The largest subsequence might
  - be in s_1 or
  - be in s_2 or
  - start at some position of s_1 and end at some position of s_2
- Java code:

```java
private long maxSeq(int i, int j){
  if(i == j) return a[i];
  int m = (i+j)/2;
  long ml = maxSeq(i,m);
  long mr = maxSeq(m+1,j);
  long maxL = maxLeft(i,m);
  long maxR = maxRight(m+1,j);
  long maxLR = maxL + maxR;
  long max = ml > mr ? ml : mr;
  max = max > maxLR ? max : maxLR;
  return max;
}
public long algo3(int[] a){
  int n = a.length;
  return maxSeq(0,n-1);
}
```

Divide and Conquer — $\mathcal{O}(n \log n)$

```java
private long maxLeft(int i, int j){
  long maxL = a[j];
  int s = 0;
  for(int k = j; k >= i; k--){
    s += a[k];
    maxL = maxL > s ? maxL : s;
  }
  return maxL;
}
private long maxRight(int i, int j){
  long maxR = a[i];
  int s = 0;
  for(int k = i; k <= j; k++){
    s += a[k];
    maxR = maxR > s ? maxR : s;
  }
  return maxR;
}
```

Dynamic programming

General principle

- Division: divide the initial problem into smaller similar problems (subproblems)
- Storing solutions to subproblems: store the solution to subproblems into memory
- Aggregation: establish the solution to the initial problem by aggregating solutions to subproblems stored in the memory

Dynamic programming

Largest subsequence

- Division:
  - Let s_i be the weight of the largest subsequence of $a_1, \ldots, a_i$ ending at a_i
- Aggregation:
  - $s_1 = a_1$
  - $s_i = \max\{s_{i-1} + a_i, a_i\}, \forall i = 2, \ldots, n$
  - Solution to the original problem is $\max\{s_1, \ldots, s_n\}$
- Number of basic operations is n (best algorithm)

Dynamic programming — $\mathcal{O}(n)$

```java
public long algo4(int[] a){
  int n = a.length;
  long max = a[0];
  int[] s = new int[n];
  s[0] = a[0];
  max = s[0];
  for(int i = 1; i < n; i++){
    if(s[i-1] > 0) s[i] = s[i-1] + a[i];
    else s[i] = a[i];
    max = max > s[i] ? max : s[i];
  }
  return max;
}
```

Analyzing algorithms

- Resources (memory, bandwithd, CPU, etc.) required by the algorithms
- Most concern is the computational time
- Input size: number of items in the input
- Running time: measured in term of the number of primitive operations performed

Analyzing algorithms

- algo1: $T(n) = \frac{n^3}{6} + \frac{n^2}{2} + \frac{n}{3}$
- algo2: $T(n) = \frac{n^2}{2} + \frac{n}{2}$
- algo3:
  - Count the number of addition ("+") operation $T(n)$

$$T(n) = \begin{cases} 0 & \text{if} \quad n = 1 \\ T(\frac{n}{2}) + T(\frac{n}{2}) + cn & \text{if} \quad n > 1 \end{cases}$$

  - By induction: $T(n) = n \; c \; logn$
- algo4: $T(n) = c \; n$

Analyzing algorithms

- Worst-case running time: the longest running time for any input of size n
- Best-case running time: the shortest running time for any input of size n
- Average-case running time: probabilistic analysis (make assumption of a distribution of the input) yields expected running time

Order of growth

- Consider only leading term of the function
- Ignore constant coefficient
- Example
  - $an^3 + bn^2 + cn + d = \Theta(n^3)$

Asotic notations

- Given a fucntion $g(n)$, we denote:
  - $\Theta(g(n)) = \{f(n) : \exists c_1, c_2, n_0 \text{ s.t. } 0 \le c_1 g(n) \le f(n) \le c_2 g(n), \forall n \ge n_0\}$
  - $\mathcal{O}(g(n)) = \{f(n) : \exists c, n_0 > 0 \text{ s.t. } f(n) \le c g(n), \forall n \ge n_0\}$
  - $\Omega(g(n)) = \{f(n) : \exists c, n_0 > 0 \text{ s.t. } c g(n) \le f(n), \forall n \ge n_0\}$
- Examples
  - $10n^2 - 3n = \Theta(n^2)$
  - $10n^2 - 3n = \mathcal{O}(n^3)$
  - $10n^2 - 3n = \Omega(n)$

Analysis of algorithms

Experiments studies

- Write a program implementing the algorithm
- Execute the program on a machine with different input sizes
- Measure the actual execution times
- Plot the results

Analysis of algorithms

Shortcomings of experiments studies

- Need to implement the algorithm, sometime difficult
- Results may not indicate the running time of other input not experimented
- To compare two algorithms, it is required to use the same hardware and software environments.

Analysis of algorithms

Asymptotic algorithm analysis

- Use high-level description of the algorithm (pseudo code)
- Determine the running time of an algorithm as a function of the input size
- Express this function with asymptotic notations

Analysis of algorithms

- Sequential structure: P and Q are two segments of the algorithm (the sequence $P; Q$)
    - $\text{Time}(P; Q) = \text{Time}(P) + \text{Time}(Q)$ or
    - $\text{Time}(P; Q) = \Theta(max(\text{Time}(P), \text{Time}(Q)))$
- **for** loop: **for** $i = 1$ to m **do** $P(i)$
    - $t(i)$ is the time complexity of $P(i)$
    - time complexity of the **for** loop is $\sum_{i=1}^{m} t(i)$

Analysis of algorithms

while (**repeat**) loop

- Specify a function of variables of the loop such that this function reduces during the loop
- To evaluate the running time, we analyze how the function reduces during the loop

Analysis of algorithms

Example: binary search
Function BinarySearch($T[1..n], x$)
begin
    $i \leftarrow 1; j \leftarrow n$;
    while $i < j$ **do**
        $k \leftarrow (i + j)/2$;
        case
            $x < T[k]$: $j \leftarrow k - 1$;
            $x = T[k]$: $i \leftarrow k; j \leftarrow k$; exit;
            $x > T[k]$: $i \leftarrow k + 1$;
        endcase
    endwhile
end

Analysis of algorithms

Example: binary search

Denote

- $d = j - i + 1$ (number of elements of the array to be investigated)
- i^*, j^*, d^* respectively the values of i, j, d after a loop

We have

- If $x < T[k]$ then $i^* = i$, $j^* = (i+j)/2 - 1$, $d^* = j^* - i^* + 1 \leq d/2$
- If $x > T[k]$ then $j^* = j$, $i^* = (i+j)/2 + 1$, $d^* = j^* - i^* + 1 \leq d/2$
- If $x = T[k]$ then $d^* = 1$

Hence, the number of iterations of the loop is $\lceil log n \rceil$

Master theorem

$T(n) = aT(n/b) + cn^k$ with $a \geq 1, b > 1, c > 0$ are constant

- If $a > b^k$, then $T(n) = \Theta(n^{log_b a})$
- If $a = b^k$, then $T(n) = \Theta(n^k log n)$
- If $a < b^k$, then $T(n) = \Theta(n^k)$

Example

- $T(n) = 3T(n/4) + cn^2 \Rightarrow T(n) = \Theta(n^2)$
- $T(n) = 2T(n/2) + n^{0.5} \Rightarrow T(n) = \Theta(n)$
- $T(n) = 16T(n/4) + n \Rightarrow T(n) = \Theta(n^2)$
- $T(n) = T(3n/7) + 1 \Rightarrow T(n) = \Theta(log n)$

Sorting

- Put elements of a list in a certain order
- Designing efficient sorting algorithms is very important for other algorithms (search, merge, etc.)
- Each object is associated with a key and sorting algorithms work on these keys.
- Two basic operations that used mostly by sorting algorithms
  - Swap(a, b): swap the values of variables a and b
  - Compare(a, b): return
    - ★ true if a is before b in the considered order
    - ★ false, otherwise.
- Without loss of generality, suppose we need to sort a list of numbers in nondecreasing order

- A sorting algorithm is called **in-place** if the size of additional memory required by the algorithm is $\mathcal{O}(1)$ (which does not depend on the size of the input array)
- A sorting algorithm is called **stable** if it maintains the relative order of elements with equal keys
- A sorting algorithm uses only comparison for deciding the order between two elements is called **Comparison-based sorting algorithm**

Insertion Sort

- At iteration k, put the kth element of the original list in the right order of the sorted list of the first k elements $(\forall k = 1, \ldots, n)$
- Result: after k^{th} iteration, we have a sorted list of the first k^{th} elements of the original list

```
void insertion_sort(int a[], int n){
  int k;
  for(k = 2; k <= n; k++){
    int last = a[k];
    int j = k;
    while(j > 1 && a[j-1] > last){
      a[j] = a[j-1];
      j--;
    }
    a[j] = last;
  }
}
```

Selection Sort

- Put the smallest element of the original list in the first position
- Put the second smallest element of the original list in the second position
- Put the third smallest element of the original list in the third position
- ...

```c
void selection_sort(int a[], int n){
  for(int k = 1; k <= n; k++){
    int min = k;
    for(int i = k+1; i <= n; i++)
      if(a[min] > a[i])
        min = i;
    swap(a[k],a[min]);
  }
}
```

Bubble sort

- Pass from the beginning of the list: compare and swap two adjacent elements if they are not in the right order
- Repeat the pass until no swaps are needed

```c
void bubble_sort(int a[], int n){
  int swapped;
  do{
    swapped = 0;
    for(int i = 1; i < n; i++)
    if(a[i] > a[i+1]){
      swap(a[i],a[i+1]);
      swapped = 1;
    }
  }while(swapped == 1);
}
```

Merge sort

Divide-and-conquer

- Divide the original list of n elements into two lists of $n/2$ elements
- Recursively merge sort these two lists
- Merge the two sorted lists

Merge sort

```
void merge(int a[], int L, int M, int R){
  // merge two sorted list a[L..M] and a[M+1..R]
  int i = L;// first position of the first list a[L..M]
  int j = M+1;// first position of the second list a[M+1..R]
  for(int k = L; k <= R;k++){
    if(i > M){// the first list is all scanned
      TA[k] = a[j]; j++;
    }else if(j > R){// the second list is all scanned
      TA[k] = a[i]; i++;
    }else{
      if(a[i] < a[j]){
        TA[k] = a[i]; i++;
      }else{
        TA[k] = a[j]; j++;
      }
    }
  }
  for(int k = L; k <= R; k++)
    a[k] = TA[k];
}
```

Merge sort

```
void merge_sort(int a[], int L, int R){
  if(L < R){
    int M = (L+R)/2;
    merge_sort(a,L,M);
    merge_sort(a,M+1,R);
    merge(a,L,M,R);
  }
}
```

Quick sort

- Pick an element, called a **pivot**, from the original list
- Rearrange the list so that:
    - All elements less than **pivot** come before the **pivot**
    - All elements greater or equal to to **pivot** come after **pivot**
- Here, **pivot** is in the right position in the final sorted list (it is fixed)
- Recursively sort the sub-list before **pivot** and the sub-list after **pivot**

Quick sort

```
void quick_sort(int a[], int L, int R){
  if(L < R){
   int index = (L+R)/2;
    index = partition(a,L,R,index);
    if(L < index)
       quick_sort(a,L,index-1);
    if(index < R)
       quick_sort(a,index+1,R);
  }
}
```

Quick sort

```
int partition(int a[], int L, int R, int indexPivot
    int pivot = a[indexPivot];
    swap(a[indexPivot],a[R]);// put the pivot in the
    int storeIndex = L; // store the right position o

    for(int i = L; i <= R-1; i++){
        if(a[i] < pivot){
            swap(a[storeIndex],a[i]);
            storeIndex++;
        }
    }
    swap(a[storeIndex],a[R]); // put the pivot in the

    return storeIndex;
}
```

Heap sort

Sort a list $A[1..N]$ in nondecreasing order

1. Build a heap out of $A[1..N]$
2. Remove the largest element and put it in the Nth position of the list
3. Reconstruct the heap out of $A[1..N-1]$
4. Remove the largest element and put it in the $N-1^{th}$ position of the list
5. ...

Heap sort - Heap structure

- Shape property: Complete binary tree with level L
- Heap property: each node is greater than or equal to each of its children (max-heap)
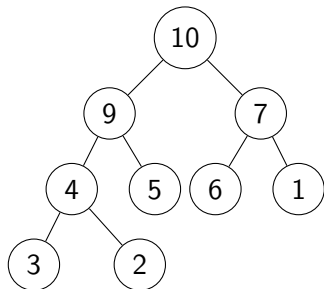
1	2	3	4	5	6	7	8	9
10	9	7	4	5	6	1	3	2

Heap sort

- Heap corresponding to a list $A[1..N]$
  - Root of the tree is $A[1]$
  - Left child of node $A[i]$ is $A[2*i]$
  - Right child of node $A[i]$ is $A[2*i+1]$
  - Height is $logN+1$
- Operations
  - Build-Max-Heap: construct a heap from the original list
  - Max-Heapify: repair the following binary tree so that it becomes Max-Heap
    - A tree with root $A[i]$
    - $A[i] < max(A[2*i], A[2*i+1])$: heap property is not hold
    - Subtrees rooted at $A[2*i]$ and $A[2*i+1]$ are Max-Heap

Heap sort

```
void heapify(int a[], int i, int n){
  // array to be heapified is a[i..n]
  int L = 2*i;
  int R = 2*i+1;
  int max = i;
  if(L <= n && a[L] > a[i])
    max = L;
  if(R <= n && a[R] > a[max])
    max = R;
  if(max != i){
    swap(a[i],a[max]);
    heapify(a,max,n);
  }
}
```

Heap sort

```
void buildHeap(int a[], int n){
    // array is a[1..n]
  for(int i = n/2; i >= 1; i--){
    heapify(a,i,n);
  }
}

void heap_Sort(int a[], int n){
    // array  is a[1..n]
  buildHeap(a,n);
  for(int i = n; i > 1; i--){
    swap(a[1],a[i]);
    heapify(a,1,i-1);
  }
}
```

Comb sort

Comb sort is a relatively simple and quick sorting algorithm originally designed by Dobosiewicz in 1980. Later it was rediscovered by Lacey and Box in 1991. Comb sort improves on bubble sort.

- Bubble sort: any two comparing elements has a gap of 1;
- Comb sort: the gap can be much more than 1 and goes down (for each iteration of outer loop) in steps of shrink factor: [input size / shrink factor, input size / shrink factor^2, input size / shrink factor^3, . . . , 1];
- The shrink factor has a great effect on the efficiency of comb sort. The value 1.3 has been suggested as an ideal shrink factor by the authors of the original article after empirical testing on over 200,000 random lists. A value too small slows the algorithm down by making unnecessarily many comparisons, whereas a value too large fails to effectively deal with turtles.
- The final stage of the sort is thus equivalent to a bubble sort, but by this time most turtles have been dealt with, so a bubble sort will be efficient.

Comb sort

```
1   void combsort(int a[], int n){
2     gap = n // Initialize gap size
3     shrink = 1.3; // Set the gap shrink factor
4     bool sorted = false;
5
6     while ((gap > 1) || (sorted == false)) {
7       // Update the gap value for a next comb
8       gap = floor(gap / shrink);
9       if (gap < 1)  gap = 1;
10      int i = 0;
11      if (gap == 1)  sorted = true;
12
13      // A single "comb" over the input list
14      while ( i + gap < n) { // See Shell sort for a similar idea
15        if (a[i] > a[i+gap]) {
16          swap(a[i], a[i+gap]);
17          if (gap == 1) sorted = false;
18            //If this assignment never happens within the loop,
19            //then there have been no swaps and the list is sorted.
20        }
21        i := i + 1
22      }
23    }
24  }
```

Data structures

- List
- Stack
- Queue

List

- Collection of objects which are arranged in a linear order
- Array
  - Continuous allocation
  - Accessing elements via indices
- Linked List
  - Elements are not necessarily allocated continuously
  - User pointer to link an element with its successor
  - Accessing elements via pointers

Stacks

- An ordered list in which all insertions and deletions are made at one end (called top)
- Principle: the last element inserted into the stack must be the first one to be removed (**Last-In-First-Out**)
- Operations
  - Push(x, S): push an element x into the stack S
  - Pop(S): remove an element from the stack S, and return this element
  - Top(S): return the element at the top of the stack S
  - Empty(S): return true if the stack S is empty

Queues

- An ordered list in which the insertions are made at one end (called tail) and the deletions are made at the other end (called head)
- Principle: the first element inserted into the queue must be the first one to be removed (**First-In-First-Out**)
- Applications: items do not have to be processed immediately but they have to be processed in FIFO order
  - Data packets are stored in a queue before being transmitted over the internet
  - Data is transferred asynchronously between two processes: IO buffered, piples, etc.
  - Printer queues, keystroke queues (as we type at the keyboard), etc.

Queues

- Operations
  - ▶ Enqueue(x, Q): push an element x into the queue Q
  - ▶ Dequeue(Q): remove an element from the queue Q, and return this element
  - ▶ Head(Q): return the element at the head of the queue Q
  - ▶ Tail(Q): return the element at the tail of the queue Q
  - ▶ Empty(Q): return true if the queue Q is empty

Java Libraries

- List
  - ArrayList (dynamic array): get(int index), size(), remove(int index), add(int index, Object o), indexOf(Object o)
  - LinkedList (doubly linked list): remove, poll, element, peek, add, offer, size
- Stack
  - push, pop, size
- Queue
  - LinkedList
  - remove, poll, element, peek, add, offer, size
- Set
  - Collection of items
  - Methods: add, size, contains
- Map
  - Map an object (key) to another object (value)
  - Methods: put, get, keySet

Examples

```java
package week2;

import java.util.ArrayList;

public class ExampleArrayList {

  public ExampleArrayList(){
    ArrayList<Integer> L = new ArrayList();
    for(int i = 1; i <= 10; i++)
      L.add(i);
    for(int i = 0; i < L.size(); i++){
      int item = L.get(i);
      System.out.print(item + " ");
    }
    System.out.println("size of L is " + L.size());
  }
  public static void main(String[] args) {
    ExampleArrayList EAL = new ExampleArrayList();
  }
}
```

Examples

```
package week2;
import java.util.HashSet;
public class ExampleSet {

  public void test(){
    HashSet<Integer> S = new HashSet();
    for(int i = 1; i <= 10; i ++)
      S.add(i);
    for(int i: S){
      System.out.print(i + " ");
    }
    System.out.println("S.size() = " + S.size());
System.out.println(S.contains(20));
  }
  public static void main(String[] args) {
    ExampleSet ES = new ExampleSet();
    ES.test();
  }
}
```

```java
package week2;
import java.util.LinkedList;
import java.util.Queue;
public class ExampleQueue {
  public static void main(String[] args) {
    Queue Q = new LinkedList();
    /*
     * Q.element(): return the head of the queue without removing it.
     * If Q is empty, then raise exception
     * Q.peek(): return the head of the queue without removing it.
     * If Q is empty, then return null
     * Q.remove(): remove and return the head of the queue.
     * If Q is empty, then raise exception
     * Q.poll(): remove and return the head of the queue.
     * If Q is empty, then return null
     * Q.add(e): add an element to the tail of Q.
     * If no space available, then raise exception
     * Q.offer(e): add an element to the tail of Q. If no space availa
     * then return false. Otherwise, return true
     */
    for(int i = 1; i <= 10; i++)  Q.offer(i);
    while(Q.size() > 0){
      int x = (int)Q.remove();
      System.out.println("Remove " + x + ", head of Q is " + Q.peek()
    }
```

Examples

```java
package week2;

import java.util.*;
public class ExampleStack {
  public ExampleStack(){
    /*
     * S.push(e): push an element to the stack
     * S.pop: remove the element at the top of the stack and return it
     */
    Stack S = new Stack();
    for(int i = 1; i <= 10; i++)
      S.push(i + "000");
    while(S.size() > 0){
      String x = (String)S.pop();
      System.out.println(x);
    }
  }
  public static void main(String[] args) {
    ExampleStack S = new ExampleStack();
  }
}
```

Examples

```
package week2;

import java.util.HashMap;
public class ExampleHashMap {

  public ExampleHashMap(){
    HashMap<String, Integer> m = new HashMap<String, Integer>();
    m.put("abc",1);
    m.put("def", 1000);
    m.put("xyz", 100000);
    for(String k: m.keySet()){
      System.out.println("key = " + k + " map to " + m.get(k));
    }
  }
  public static void main(String[] args) {
    ExampleHashMap EHM = new ExampleHashMap();
  }
}
```

Examples

```java
package week2;
import java.util.Scanner;
import java.util.HashMap;
import java.io.File;
public class CountWords {
  public CountWords(String filename){
    HashMap<String, Integer> count = new HashMap<String, Integer>();
    try{
      Scanner in = new Scanner(new File(filename));
      while(in.hasNext()){
        String s = in.next();
        if(count.get(s) == null)
        count.put(s, 0);
        count.put(s, count.get(s) + 1);
      }
      for(String w: count.keySet())
        System.out.println("Word " + w + " appears " + count.get(w) +
      in.close();
    }catch(Exception ex){
      ex.printStackTrace();
    }
  }
  public static void main(String[] args) {
    CountWords  CW = new CountWords("data\\week2\\CountWords.txt");
  }
}
```

Checking parentheses expression

```java
private boolean match(char c, char cc){
  if(c == '(' && cc == ')') return true;
  if(c == '[' && cc == ']') return true;
  if(c == '{' && cc == '}') return true;
  return false;
}
public boolean check(String expr){
  Stack S = new Stack();
  for(int i = 0; i < expr.length(); i++){
    char c = expr.charAt(i);
    if(c == '(' || c == '{'|| c == '[')
      S.push(c);
    else{
      if(S.size() == 0) return false;
      char cc = (char)S.pop();
      if(!match(cc,c)) return false;
    }
  }
  return S.size() == 0;
}
```

Water Jug Problem

There are two jugs, a *a*-gallon one and a *b*-gallon one (*a*, *b* are positive integer). There is a pump with unlimited water. Neither jug has any measuring marking on it. How can you get exactly *c*-gallon jug (*c* is a positive integer)?

Water Jug Problem

- Search problem
- State (x, y): quantity of water in two jugs
- Neighboring states
  - $(x, 0)$
  - $(0, y)$
  - (a, y)
  - (x, b)
  - $(a, x + y - a)$ if $x + y >= a$
  - $(x + y, 0)$ if $x + y < a$
  - $(x + y - b, b)$ if $x + y >= b$
  - $(0, x + y)$ if $x + y < b$
- Final states: (c, y) or (x, c)

Water Jug Problem

Algorithm 1: WaterJug(a, b, c)

$Q \leftarrow \emptyset$;
Enqueue($(a, b), Q$);
$Visited(a, b) \leftarrow$ TRUE;
while Q *is not empty* **do**
    $(x, y) \leftarrow$ Dequeue(Q);
    foreach $(x', y') \in$ *neighboring states of* (x, y) **do**
        if $(x' = c \lor y' = c)$ **then**
            Solution();
            BREAK;
        else
            if *NOT* $Visited(x', y')$ **then**
                Enqueue($(x', y'), Q$);
                $Visited(x', y') \leftarrow$ TRUE;