1. 设 $A = \begin{pmatrix} 1 & 1 \\ 2 & 5 \end{pmatrix}$, $x = \begin{Bmatrix} 2 \\ 3 \end{Bmatrix}$, 试求 $\|A\|_1$, $\|A\|_2$, $\|A\|_\infty$, $\|Ax\|_1$, $\|Ax\|$

$\|Ax\|_\infty$。

$$\|\mathbf{A}\|_1 = \max(1+2,1+5) = 6, \|\mathbf{A}\|_\infty = \max(1+1,2+5) = 7$$

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & 5 \end{pmatrix}, \quad \mathbf{A}^T \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 7 \\ 7 & 29 \end{pmatrix}$$

$$\left| \lambda \mathbf{E} - \mathbf{A}^T \mathbf{A} \right| = \begin{vmatrix} \lambda - 2 & -7 \\ -7 & \lambda - 29 \end{vmatrix} = \lambda^2 - 31\lambda + 9 = 0$$

$$\lambda_{\max} = (31 + \sqrt{31^2 - 4*9})/2, \quad \|\mathbf{A}\|_2 = \sqrt{\lambda_{\max}} \approx 5.54$$

$$\mathbf{Ax} = \begin{pmatrix} 1 & 1 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 19 \end{pmatrix} \quad \|\mathbf{Ax}\|_1 = 24, \quad \|\mathbf{Ax}\|_2 \approx 19.65, \quad \|\mathbf{Ax}\|_\infty = 19$$

2. 设 $A = \begin{pmatrix} 2 & 1 \\ -5 & 4 \end{pmatrix}$，则 A 的谱半径 $\rho(A)$ 和条件数 $\mathrm{Cond}(A)_\infty$ 怎样计算？

$$|\lambda \mathbf{E} - \mathbf{A}| = \begin{vmatrix} \lambda - 2 & -1 \\ 5 & \lambda - 4 \end{vmatrix} = \lambda^2 - 6\lambda + 13 = 0$$

$$\lambda_{\max} \approx 3 + 2i, \quad \rho(\mathbf{A}) = |\lambda_{\max}| \approx 3.61$$

$$\mathbf{A}^{-1} = \begin{pmatrix} 4 & -1 \\ 5 & 2 \end{pmatrix} / 13$$

$$\left\| \mathbf{A}^{-1} \right\|_\infty = 7/13, \quad \left\| \mathbf{A} \right\|_\infty = 9, \quad cond(\mathbf{A})_\infty = \frac{63}{13} \approx 4.85$$

9. 已知方程组 $\begin{pmatrix} 1 & 2 \\ 0.32 & 1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ ，则解此方程组的 Jacobi 迭代法是否收敛？

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0.32 & 1 \end{pmatrix} \qquad L = \begin{pmatrix} 0 & 0 \\ 0.32 & 0 \end{pmatrix}, U = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

$$-\mathbf{D}^{-1}(\mathbf{L}+\mathbf{U}) = -\begin{pmatrix} 0 & 2 \\ 0.32 & 0 \end{pmatrix}$$

$$\lambda^2 - 0.64 = 0 \Rightarrow \lambda = \pm 0.8$$

收敛！

13. 解方程组 $\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$ 的 Jacobi 迭代公式为

$$\begin{cases} x_1^{(k)} = \dfrac{1}{a_{11}}(b_1 - a_{12}x_2^{(k-1)}) \\ \\ x_1^{(k)} = \dfrac{1}{a_{22}}(b_2 - a_{21}x_1^{(k-1)}) \end{cases} \qquad k = 1, 2 \cdots$$

求证:上述迭代公式产生的向量序列 $\{x^{(k)}\}$ 收敛的充要条件是 $r = \left| \dfrac{a_{12}a_{21}}{a_{11}a_{22}} \right| < 1$

$$\mathbf{B} = -\mathbf{D}^{-1}(\mathbf{L} + \mathbf{U}) = - \begin{pmatrix} 0 & \dfrac{-a_{12}}{a_{11}} \\ \dfrac{-a_{21}}{a_{22}} & 0 \end{pmatrix}$$

$$\left| \lambda\mathbf{E} - \mathbf{B} \right| = - \begin{pmatrix} \lambda & \dfrac{a_{12}}{a_{11}} \\ \dfrac{a_{21}}{a_{22}} & \lambda \end{pmatrix} = 0 \qquad\qquad \rho(\mathbf{B}) = \sqrt{\dfrac{a_{12}}{a_{11}}\dfrac{a_{21}}{a_{22}}} < 1$$

14. 已知方程组（1）$\begin{cases} x_1 + 0.4x_2 + 0.4x_3 = 1 \\ 0.4x_1 + x_2 + 0.8x_3 = 2 \\ 0.4x_1 + 0.8x_2 + x_3 = 3 \end{cases}$ 和（2）$\begin{cases} x_1 + 2x_2 - 2x_3 = 1 \\ x_1 + x_2 + x_3 = 1 \\ 2x_1 + 2x_2 + x_3 = 1 \end{cases}$

用 Jacobi 迭代法和 Gauss – Seidel 迭代法解此方程组的收敛性。

(1) $\mathbf{A} = \begin{pmatrix} 1 & 0.4 & 0.4 \\ 0.4 & 1 & 0.8 \\ 0.4 & 0.8 & 1 \end{pmatrix}$ $\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\mathbf{L} = \begin{pmatrix} 0 & 0 & 0 \\ 0.4 & 0 & 0 \\ 0.4 & 0.8 & 0 \end{pmatrix}$ $\mathbf{U} = \begin{pmatrix} 0 & 0.4 & 0.4 \\ 0 & 0 & 0.8 \\ 0 & 0 & 0 \end{pmatrix}$

Jacobi迭代矩阵：

$$\mathbf{B} = -\mathbf{D}^{-1}(\mathbf{L}+\mathbf{U}) = -\begin{pmatrix} 0 & 0.4 & 0.4 \\ 0.4 & 0 & 0.8 \\ 0.4 & 0.8 & 0 \end{pmatrix}$$

$$\lambda^3 + 0.4*0.4*0.8*2 - 0.16\lambda - 0.16\lambda - 0.64\lambda = \lambda^3 - 0.96\lambda + 0.256$$

$$= \lambda(\lambda^2 - 0.64) - 0.32(\lambda - 0.8) = (\lambda - 0.8)(\lambda^2 + 0.8\lambda - 0.32) = 0$$

$$\lambda_1 = 0.8, \quad \lambda_2 \approx 0.2928, \quad \lambda_3 = -1.0928$$

$$\rho(\mathbf{B}) \approx 1.0928 > 1 \quad \text{根据定理4.6，不收敛。}$$

Gauss-Seidel迭代矩阵：

$$\mathbf{G} = -(\mathbf{D}+\mathbf{L})^{-1}\mathbf{U} = -\begin{pmatrix} 1 & 0 & 0 \\ 0.4 & 1 & 0 \\ 0.4 & 0.8 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 0 & 0.4 & 0.4 \\ 0 & 0 & 0.8 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= -\begin{pmatrix} 1 & 0 & 0 \\ -0.4 & 1 & 0 \\ -0.08 & -0.8 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0.4 & 0.4 \\ 0 & 0 & 0.8 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0.4 & 0.4 \\ 0 & -0.16 & 0.64 \\ 0 & -0.032 & -0.672 \end{pmatrix}$$

$$\|\mathbf{B}\|_\infty = 0.8 < 1 \qquad 根据定理\mathbf{4.7}收敛。$$

或求解谱半径来判断：

$$\lambda(\lambda+0.16)(\lambda+0.672)+0.64\times0.032\lambda = \lambda(\lambda^2+0.832\lambda+0.128) = 0$$

$$\lambda_1 = 0, \quad \lambda_2 \approx -0.2037, \quad \lambda_3 = 0.6283$$

$$\rho(\mathbf{B}) \approx 0.6283 < 1 \qquad 根据定理\mathbf{.6}，收敛。$$

$$A = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 2 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 2 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Jacobi 迭代矩阵：

$$B = -D^{-1}(L+U) = -\begin{pmatrix} 0 & 2 & -2 \\ 1 & 0 & 1 \\ 2 & 2 & 0 \end{pmatrix}$$

$$|\lambda E - B| = \begin{vmatrix} \lambda & 2 & -2 \\ 1 & \lambda & 1 \\ 2 & 2 & \lambda \end{vmatrix} = \lambda^3 + 4 - 4 + 4\lambda - 2\lambda - 2\lambda = \lambda^3 = 0$$

$\rho(B) = 0 < 1$，根据定理 4.6，收敛。

Gauss-Seidel 迭代矩阵：

$$\mathbf{G} = -(\mathbf{D}+\mathbf{L})^{-1}\mathbf{U} = -\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 0 & 2 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= -\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 2 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 & -2 \\ 0 & -2 & 3 \\ 0 & 0 & -2 \end{pmatrix}$$

$$|\lambda\mathbf{E}-\mathbf{B}| = \begin{vmatrix} \lambda & 2 & -2 \\ 0 & \lambda-2 & 3 \\ 0 & 0 & \lambda-2 \end{vmatrix} = \lambda(\lambda-2)^2 = 0$$

$\rho(\mathbf{B}) = 2 > 1$，根据定理 4.6，不收敛。

16. 用 Gauss 消去法求解方程组 $\begin{cases} x_1 + x_2 - 4x_4 = 1 \\ -x_1 + x_2 + x_3 + x_4 = -2 \\ x_1 + 3x_2 + 5x_3 - 4x_4 = -4 \\ x_2 + 2x_3 - x_4 = -2 \end{cases}$

$$\begin{pmatrix} 1 & 1 & 0 & -4 \\ -1 & 1 & 1 & 1 \\ 1 & 3 & 5 & -4 \\ 0 & 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -4 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & -4 \\ 0 & 2 & 1 & -3 \\ 0 & 2 & 5 & 0 \\ 0 & 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -5 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & -4 \\ 0 & 2 & 1 & -3 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 3/2 & 1/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -4 \\ -3/2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & -4 \\ 0 & 2 & 1 & -3 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & -5/8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -4 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

19. 给出下列矩阵的 **LU** 分解 $A = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{bmatrix}$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 6 \\ 2 & 1 & 3 \\ 6 & 3 & 1 \end{pmatrix}$$

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & 3 & 1 \end{pmatrix}, \mathbf{U} = \begin{pmatrix} 1 & 2 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

基本要求

- 向量范数和矩阵范数的计算
- 雅克比、高斯-赛德尔迭代矩阵的计算、收敛性判断；
- 高斯消去法、高斯列主元消去法应用；
- 矩阵的LU分解法。