복습

가우스 소거법 → 행렬

$$\begin{cases} 2u+v+w=5 & -① \\ 4u-6v \quad =-2 & -② \\ -2u+7v+2w=9 & -③ \end{cases} \rightarrow \begin{array}{l} 2u+v+w=5 \\ 0-8v-2w=-12 \Leftarrow ②-\boxed{2}\times① \end{array}$$

행렬표현

$$\overset{A}{\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & 2 \\ & & \end{bmatrix}$$

②번식을 ②-2×① 한 과정을 행렬식으로 표현

$$\underset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 2 & 1 & 1 \\ -4 & 6 & 0 \\ -2 & 7 & 2 \end{bmatrix}} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & 2 \\ -2 & 7 & 2 \end{bmatrix}$$

(1번식을 이용해서 2번식 변경)

$$\rightarrow \begin{array}{l} 2u+v+w=5 \\ 0-8v-2w=-12 \\ 0+8v+3w=14 \Leftarrow ③-\boxed{-1}\times① \end{array} \Longrightarrow E_{31}E_{21}A \Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & 2 \\ 0 & 8 & 3 \end{bmatrix}$$

행렬표현

$$\underset{E_{31}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ 0 & 8 & 3 \end{bmatrix}$$

$2u + v + w = 5$ — ①
$0 \; -8v - 2w = -12$ — ②
$0 \; 8v + 3w = 14$ — ③

$\Rightarrow$

$2u + v + w = 5$
$-8v - 2w = -12$
$0 \; w = 2$ $\Rightarrow$ ③ $-$ $\boxed{-1}$ $\times$ ②

행렬로 표현

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & +3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

E_{32} $\qquad$ U

☆ $E_{32}E_{31}E_{21}A \Rightarrow \underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$

즉, $A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_{31} & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & l_{32} & 1 \end{bmatrix} U$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} U = L \cdot U$$

Lower triangular Mat

⊙ Elementary Matrix in G.E

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -l_{21} & & \\ 0 & 0 & 1 \end{bmatrix}$$

E_{21} : ② $-$ $l_{21} \times$ ① $\rightarrow$ ②

$\Rightarrow$ ②′ $=$ ② $+$ $l_{21} \times$ ①

$E_{21}A \Rightarrow$ ②′

$E_{21}A' \Rightarrow A$ ②

$2u + v + w = 5$
② $4u - 6v \quad = -2$
$\Downarrow$
②′ $0 - 8v - 2w = -12$

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\boxed{E_{21}E_{21}^{-1} = I}$

1.5 Triangular Factors
인수

$$A \qquad X \qquad = b$$

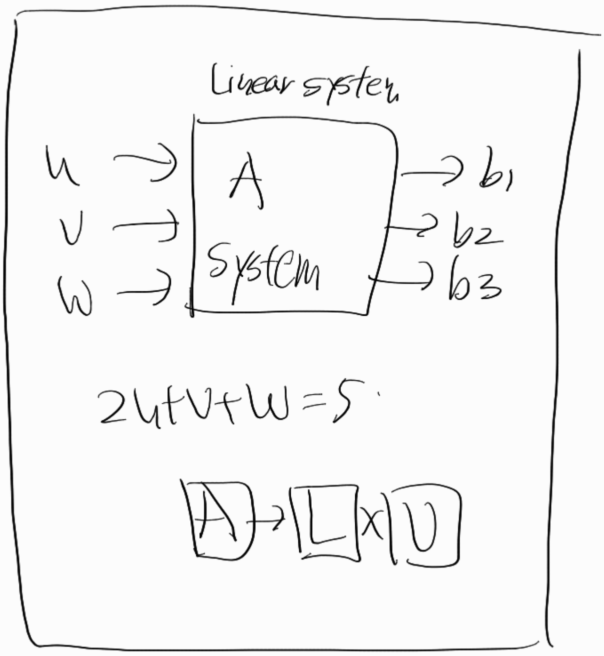
A: 연립방정식 식들을 가지고 구한 A, u v 계수

X: 미지수 모은 벡터

$$A = L\,U$$

lower upper

⇒ LU factorization
(decomposition)



$$Ax = b$$

$$\underline{L^{-1}Ax} = L^{-1}b = c$$

$$U\,x = c$$

upper triang M.

$$A = LU \qquad Lc = b$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

ex)

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix}$$

↳ ①,②,③번식 변형하고,
다시 ②번식에 4를 곱해서 ③번식이랑 빼줌.

ex)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$$

⇓ ② − 3 × ①

$$U = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$\det(A) = 2 = \det(U) \times \det(L)$
$\det(U) = 2$
$\det(L) = 1$

Ex 4)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

$$= L\underset{U}{I}$$

$$U = \begin{bmatrix} d_1 & u_{12} & u_{13} & \cdots & u_{1n} \\ & d_2 & & & \\ & & d_3 & & \\ & \bigcirc & & \ddots & \\ & & & & d_n \end{bmatrix}$$

d = pivot.

0이 아닌 pivot이 나오면 unique한 sol을 갖는 Upper triangle Mt.가 나옴

pivot만 뽑기

$$= \begin{bmatrix} d_1 & & & \\ & d_2 & & \bigcirc \\ \bigcirc & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} 1 & \frac{u_{12}}{d_1} & \frac{u_{13}}{d_1} & \cdots & \frac{u_{1n}}{d_1} \\ & 1 & \frac{u_{23}}{d_2} & \cdots & \\ \bigcirc & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} = DU$$

Diagonal Matrix
(대각 행렬)

Upper.

$$A = LU$$

$$\Rightarrow LDU$$

$$D^n = \begin{bmatrix} d_1^n & & \\ & d_2^n & \bigcirc \\ \bigcirc & & \ddots \\ & & & d_n^n \end{bmatrix}$$

D 거듭제곱시 유용하다.
but, L과 U가 독특한 조건을 갖게됨

⊙ LU factorization is unique!

행렬 A에 대해서 L과 U의 조합이 unique하다.
(LU decomposition이 가능하다면)

⊙ Row Exchange (pivoting)

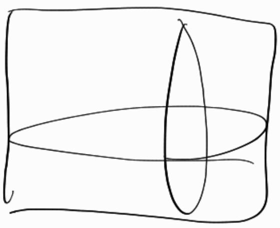
⇒ Permutation. P.

$$\underset{P_{21}}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -6 & 0 \\ 2 & 1 & 1 \\ -2 & 7 & 2 \end{bmatrix}$$

⊙ Permutation Matrix

⇒ has the same rows with I

⇒ They is a single "1" in every row and column



$$P_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P_{32}P_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\neq$$

$$P_{21}P_{32} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

교환법칙 X

행렬의 곱셈 법칙을 따름

$$A \Rightarrow LU \ , \ PA = LU$$

$$P^{-1} = P^{T}$$

$$P_{21} \rightarrow P_{21}^{-1}$$

$$A = P^{T}LU$$