Desiderata and Suggestions: No. 2. The Theory of Groups: Graphical Representation

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DESIDERATA AND SUGGESTIONS.

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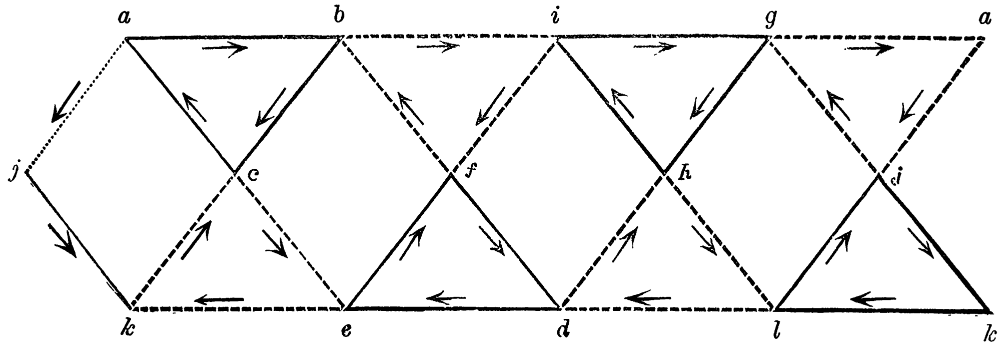
No. 2.—THE THEORY OF GROUPS: GRAPHICAL REPRESENTATION.

In regard to a substitution-group of the order n upon the same number of letters, I omitted to mention the important theorem that every substitution is *regular* (that is, either cyclical or composed of a number of cycles each of them of the same order). Thus in the group of 6 given in No. 1, writing a, b, c, d, e, f in place of $1, \alpha, \beta, \gamma, \delta, \varepsilon$, the substitutions of the group are 1, $ace.bfd$, $aec.bdf$, $ab.cd.ef$, $ad.be.cf$, $af.bc.de$.

Let the letters be represented by points; a change a into b will be represented by a directed line (line with an arrow) joining the two points; and therefore a cycle abc, that is, a into b, b into c, c into a, by the three sides of the trilateral abc, with the three arrows pointing accordingly, and similarly for the cycles $abcd$, &c.: the cycle ab means a into b, b into a, and we have here the line ab with a two headed arrow pointing both ways; such a line may be regarded as a bilateral. A substitution is thus represented by a multilateral or system of multilaterals, each side with its arrow; and in the case of a regular substitution the multilaterals (if more than one) have each of them the same number of sides. To represent two or more substitutions we require different colours, the multilaterals belonging to any one substitution being of the same colour.

In order to represent a group we need to represent only independent substitutions thereof; that is, substitutions such that no one of them can be obtained from the others by compounding them together in any manner. I take as an example a group of the order 12 upon 12 letters, where the number of independent substitutions is $= 2$. See the diagram, wherein the continuous lines represent black lines, and the broken lines, red lines.



The diagram is drawn, in the first instance, with the arrows but without the letters, which are then affixed *at pleasure;* viz: the *form of group* is quite independent of the way in which this is done, though the group itself is of course dependent upon it. The diagram shows two substitutions, each of them of the third order, one represented by the black triangles, and the other by the red triangles. It will be observed that there is *from* each point of the diagram (that is, in the direction of the arrow) one and only one black line, and one and only one red line; hence, a symbol, B, "move along a black line," B^2, move successively along two black lines," BR (read always from right to left), "move first along a red line and then along a black line," has in every case a perfectly definite meaning and determines the path when the initial point is given; any such symbol may be spoken of as a "route."

The diagram has a remarkable property, *in virtue whereof it in fact represents a group.* It may be seen that any route leading from some one point a to itself, leads also from every other point to itself, or say from b to b, from c to c, ... and from l to l. We hence see that a route applied in succession to the whole series of initial points or letters *abcdefghijkl*, gives a new arrangement of these letters, wherein no one of them occupies its original place; a route is thus, in effect, a substitution. Moreover, we may regard as distinct routes, those which lead from a to a, to b, to c, . . . to l, respectively. We have thus 12 substitutions (the first of them, which leaves the arrangement unaltered, being the substitution unity), and these 12 substitutions form a group. I omit the details of the proof; it will be sufficient to give the square obtained by means of the several routes, or substitutions, performed upon the primitive arrangement *abcdefghijkl*, and the cyclical expressions of the

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*l*	1
b	*c*	*a*	*e*	*f*	*d*	*h*	*i*	*g*	*k*	*l*	*j*	*abc* . *def* . *ghi* . *jkl* (= B)
c	*a*	*b*	*f*	*d*	*e*	*i*	*g*	*h*	*l*	*j*	*k*	*acb* . *dfe* . *gih* . *jlk*
d	*l*	*h*	*a*	*g*	*j*	*e*	*c*	*k*	*f*	*i*	*b*	*ad* . *bl* . *ch* . *eg* . *fj* . *ik*
e	*j*	*i*	*b*	*h*	*k*	*f*	*a*	*l*	*d*	*g*	*c*	*aeh* . *bjd* . *cil* . *fkg*
f	*k*	*g*	*c*	*i*	*l*	*d*	*b*	*j*	*e*	*h*	*a*	*afl* . *bkh* . *cgd* . *eij*
g	*f*	*k*	*l*	*c*	*i*	*j*	*d*	*b*	*a*	*e*	*h*	*agj* . *bfi* . *cke* . *dlh*
h	*d*	*l*	*j*	*a*	*g*	*k*	*e*	*c*	*b*	*f*	*i*	*ahe* . *bdj* . *cli* . *fgk*
i	*e*	*j*	*k*	*b*	*h*	*l*	*f*	*a*	*c*	*d*	*g*	*ai* . *be* . *cj* . *dk* . *fh* . *gl*
j	*i*	*e*	*h*	*k*	*b*	*a*	*l*	*f*	*g*	*c*	*d*	*ajg* . *bif* . *cek* . *dhl* (= R).
k	*g*	*f*	*i*	*l*	*c*	*b*	*j*	*d*	*h*	*a*	*e*	*ak* . *bg* . *cf* . *di* . *el* . *hj*
l	*h*	*d*	*g*	*j*	*a*	*c*	*k*	*e*	*i*	*b*	*f*	*alf* . *bhk* . *cdg* . *eji*

substitutions themselves: it will be observed that the substitutions are unity, 3 substitutions of the order (or index) 2, and 8 substitutions of the order (or index) 3.

It may be remarked that the group of 12 is really the group of the 12 positive substitutions upon 4 letters *abcd*, viz., these are 1, *abc*, *acb*, *abd*, *adb*, *acd*, *adc*, *bcd*, *bdc*, *ab.cd*, *ac.bd*, *ad.bc*.

CAMBRIDGE, *16th May, 1878.*