

DYNAMIC EPISTEMIC LOGIC

# Study Notes

Filip Rehburg<sup>1</sup>

filip.rehburg@student.uva.nl

<sup>1</sup> University of Amsterdam, Amsterdam, The Netherlands

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**Instructor:** Alexandru Baltag (TheAlexandruBaltag@gmail.com)

**TA:** Giuseppe Manes (giuseppe.manes@student.uva.nl)

Do not distribute, please send this link: [https://github.com/frehburg/mol\\_DEL\\_notes](https://github.com/frehburg/mol_DEL_notes)

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Lecture	Status
Introduction: Motivation, Main Themes, Puzzles : Section 1.1	✓
Main Themes, Puzzles, and Paradoxes Continued : Section 1.2	✓
Single-Agent Epistemic-Doxastic Logics: Kripke Models : Section 1.3	✓
Multi-agent Models and Public Announcement Logic (PAL) : Section 2.1	WIP
PAL Continued : Section 2.2	X
Does this one even exist?? : Section 2.3	X
"Learnability" and "Knowability" : Section 3.1	X
The problem of belief revision : Section 3.3	X
: Section 3.3	∅
: Section 4.2	∅
: Section 4.3	∅
: Section 5.2	∅
: Section 5.3	∅
: Section 6.2	∅
: Section 6.3	∅
: Section 7.2	∅
: Section 7.3	∅
: Section 8.2	∅
Tutorial 1 : Section 3.2	X
: Section 4.1	∅
: Section 5.1	∅
: Section 6.1	∅
: Section 7.1	∅
: Section 8.1	∅

## Prompt for generating summaries

Create a summary of the attached slides including the most important intuition, all mathematical formulas, relevant examples, and theorems, but no proofs. Pay special attention to the provided examples, their continuations and modifications.

Be concise and technical using expert vocabulary. Explain in a suitable manner for a master of Logic student familiar with the relevant background but unfamiliar with the discussed material as of yet. Write the summary in typst. The slides are attached. Only focus on content and leave out organizational information about the course. I am pasting all of this into my typst document where each lecture is a level two heading e.g. == Lecture 1, so subchapters have to be at the correct level, at least three e.g. === Core Intuitions and Definitions.

Important: Wrap the generated typst syntax summary in “““ to make it copyable

### Notable features of typst syntax:

1. if there is more than one letter in a name in typst math block then it needs to be wrapped in “”.
2. to make text bold, wrap it in singular stars and to make it italic wrap it in underscores
3. If you are more used to different typesetting languages, typst always uses () as parentheses and only uses {} for set notation

### Style guide:

1. do not include and or [cite: x] in your output
2. I have defined custom functions to represent definitions, theorems (“theorem”), proofs (“proof”), examples (“example”), intuitions [only use this for informal introductions] (“intuition”), warnings to watch out (“attention”), questions (“question”), calls to recall something learned before (“remember”), note something carefully (“note”), and an info (“info”).
  - To define a new concept, call `#def("Name of Concept")[Definition body]`
  - For all others call `#callout(title: "Title", style: "style-name")[Box body]
  - Each box generates a tag #label("def-concept-name-hyphenated"). Refer to any concept you reference back to always @def-concept-name-hyphenated
  - use my custom definitions for common operators such as the set of propositional letters or epistemic operators:

**TODO:** For some reason bullet point markers after 1 have no symbols

**TODO:** fix that def title cannot be a [content block]

# Week 1

## Session 1-1 (Lecture): Introduction: Motivation, Main Themes, Puzzles

### Motto of Dynamic Epistemic Logic

*“The wise sees action and knowledge as one. They see truly.” - Bhagavad Gita*

#### A Core Intuitions and Definitions

##### ☰ Example 1: Multi-Agent Systems

1. **Computation:** a network of communicating computers (e.g., the internet)
2. **Games:** players in a game (e.g., chess or poker)
3. **AI:** a team of robots exploring their environment and interacting with each other
4. **Cryptographic Communication:** agents (“principals”) using a cryptographic protocol to communicate in private
5. **Economics:** transactions in a market
6. **Society:** social activities
7. **Politics:** diplomacy, war
8. **Science:** a community of scientists, engaged in creating theories, making observations and performing experiments to test their theories

**Def 1 (Properties of Multi-Agent Systems):**

- *dynamic:* Agents perform *actions* which change the system (via interaction)
- *informational:* Agents acquire, store, process, and exchange *information* about each other and the environment

- *Evolving knowledge:* The knowledge an agent has may *change* in time, due to their or other players' actions.
- Certain actions increase information.
- *General rule:* players try to minimize their uncertainty and increase their knowledge.

**Def 2 (Knowledge):** Truthful information.

**Def 3 (Justified Belief):** Information that is plausible, well-justified, probable, but possibly false.

**Def 4 (Belief Revision):** A sustained, dynamic, self-correcting, truth-tracking action. Non-monotonic. True knowledge can only be recovered by effort. Made more difficult by deceit.

### ② Question

Is knowledge a form of belief, or is knowledge more fundamental than belief?

**Def 5 (Uncertainty):** A corollary of imperfect knowledge or “imperfect information”.

**Def 6** (*Game of imperfect information*): A game where some moves are hidden, preventing players from knowing everything that is going on; they only have a partial view of the situation.

- An agent may be *uncertain* () about the real situation at a given time: they cannot *distinguish* between possible outcomes.

*Wrong Beliefs:* Agents...

- ... may be induced (even with malicious intent e.g., cheating) to acquire false “certainty” in their drive for more knowledge.
- ... causing them to “know” things that are not true (e.g., due to bluffing in poker).
- Wrong beliefs are indistinguishable from true beliefs for an agent once they have become “certainty” (they really think they “know”).

**Def 7** (*Strategic Ignorance*): It can be advantageous not to know (or pretend not to).

## B Distributed, Nested, and Common Knowledge

**Def 8** (*Distributed Knowledge*): Potential/virtual knowledge that is not reducible to one individual.

Knowledge that is not necessarily held by any individual agent prior to communication, but is known when multiple agents pool their distinct information.

### ☰ Example 2: Distributed Knowledge: Business dealings

- $A$  knows  $B$  made a deal with either  $C$  or  $E$  (exclusively).
- $B$  actually made a deal with  $E$ , so  $C$  knows  $B$  did **not** go make a deal with them.
- Neither  $A$  nor  $C$  individually know  $B$  made a deal with  $E$  before communicating.
- If  $A$  and  $C$  communicate (pool their knowledge), they deduce the truth. The fact is *distributed knowledge* among them.

**Def 9** (*Nested Knowledge*): Knowledge about the knowledge of others, leading to potential infinite regress or deep epistemic reasoning (e.g., “how can you know that I do not know?”).

**Def 10** (*Introspection*): An agent’s capability (or lack thereof) to reason about their own epistemic state.

- **Known knowns:** things we know we know.
- **Known unknowns:** things we know we do not know.
- **Unknown unknowns:** things we do not know that we do not know.

**Def 11** (*Common Knowledge*): A condition where an entire group knows a fact, everybody knows that everybody knows it, and everybody knows that everybody knows that everybody knows it, ad infinitum.

### ☰ Example 3: Common Knowledge vs. 'Everybody Knows'

- Suppose everybody knows the road rules (e.g., red means “stop”) and respects them.
- **Question:** Is this enough to drive safely? **No**.
- **Reasoning:** Merely knowing the rule is insufficient if you lack the certainty that **others** know the rules and will abide by them.
- **Resolution:** Safe driving requires the rules to be *Common Knowledge* (Def 11).

## Session 1-2 (Lecture): Main Themes, Puzzles, and Paradoxes Continued

### A Epistemic Puzzles and Paradoxes

#### ☰ Example 4: Puzzle 0: The Coordinated Attack

Two army divisions (A and B) must attack simultaneously to win. They communicate via messengers over a channel where messages might be captured.

- A sends “attack at dawn” and B receives it.
- B must acknowledge receipt, but A does not know if the acknowledgment will arrive.
- A must acknowledge the acknowledgment, ad infinitum.

**Result:** No finite sequence of successful message deliveries can achieve coordination.

#### ⦿ Remember: Fixpoints and Byzantine Generals

**Def 12 (Fixpoint):**  $x$  is a fixpoint iff  $f : X \rightarrow X; x = f(x)$ .

In the case of Puzzle 0:

$$C\Box\varphi \equiv K_A C\Box\varphi \wedge K_B C\Box\varphi \quad (1)$$

Where  $K_X$  is the knowledge operator of agent  $X$ ,  $C\Box$  is the common knowledge operator,  $\varphi$  is the message about the attack time.

#### ⓘ Intuition: Coordinated Attack Intuition

Achieving *Common Knowledge* (Def 11) over an unreliable communication channel is logically impossible in a finite number of steps. Unbounded nested knowledge (Def 9) does not equate to true common knowledge.

### ☰ Example 5: Puzzle 1: To Learn is to Falsify

*A* sends an email to her lover *C*: “*B* doesn’t know about us.”

*B* secretly intercepts and reads it.

**Result:** The proposition was true right before reading, but the act of learning the message immediately falsifies it (a dynamic variant of Moore’s Paradox).

#### ∅ Note: Instantaneous truth value change

**Paradox:** usually learning  $\varphi$  means believing phi  $\Box\varphi$ , but here reading  $\varphi$  leads to not believing  $\varphi$ :  $\Box\neg\varphi$ .

**Less paradoxical with dynamic thinking:** The truth value of the statement changes instantaneously when *B* reads and accepts it.

### ⚠ Attention: Non-standard Belief Revision

Standard belief-revision postulates (e.g., AGM) fail for complex learning actions where the informational payload refers directly to the epistemic state of the receiver.

### ☰ Example 6: Puzzle 2 & 3: Self-Fulfilling and Self-Enabling Falsehoods

- **Self-Fulfilling:** *A* falsely believes *B* knows about her affair and sends a warning message. *B* intercepts it and thereby learns of the affair. Communicating a false belief makes it true.  
“*B* doesn’t know about us.”
- **Self-Enabling:** *C* (wanting to seduce faithful *A*) forges a message to himself from *A* saying *B* knows they are having an affair. *B* reads it and divorces *A*. *A*, on the rebound, starts an affair with *C*. The transmission of a falsehood causally enables its own validation.

## B The Muddy Children and Epistemic Updates

### ☰ Example 7: Puzzle 4: Muddy Children

4 perfect logicians (children), exactly 3 have dirty faces. They see others but not themselves.

- Father publicly announces: “At least one of you is dirty.”
- Father iteratively asks: “Do you know if you are dirty or not?”
- Children answer publicly and simultaneously based strictly on their knowledge without guessing.

**Result:** For 2 rounds, they answer in the negative. In the 3rd round, all 3 dirty children confidently state they are dirty. In the 4th round, the clean child deduces they are clean.

## ⓘ Socratic Questioning

Discovering answers by asking questions of students. (Wikipedia)

## ⓘ Intuition: Muddy Children

1. *What's the point of the father's first announcement ("At least one of you is dirty")?*

The initial announcement transforms distributed implicit knowledge into public *Common Knowledge* (Def 11).

2. *What's the point of the father's repeated questions?*

The iterated Socratic questioning acts as sequential epistemic updates: public statements of ignorance incrementally eliminate possible worlds in the Kripke model until the true state is uniquely isolated.

## Ⓜ Example 8: Modifications of Muddy Children

- **The Amazon Island:** Isomorphic to Muddy Children. A law mandates wives to execute their cheating husbands at noon once discovered. Queen announces at least one cheater exists and if somebody's husband is cheating, all other wives know it. With 17 cheaters, for 16 days nothing happens, and all 17 are shot on day 17.
- **The Dangers of Mercy:** Wives of the 17 cheaters secretly decide to spare them, while others believe strict obedience to the law is common knowledge. No shots are fired on day 17. On day 18, all faithful husbands are erroneously shot by their wives, who logically deduce (from flawed public premises) that their husbands must be cheating.

## Ⓜ Example 9: Puzzle 5: Sneaky Children

Children are incentivized for speed and punished for errors. After round 1, two dirty children cheat by secretly confirming to each other they are dirty, thus answering "I know" prematurely in round 2.

- **Honest Children Always Suffer:** The 3rd dirty child logically deduces it must be clean, answers incorrectly in round 3, and is punished.
- **Clean Children Always Go Crazy:** The 4th (clean) child faces a strict contradiction. If it blindly applies monotonic updates via classical logic, it undergoes logical explosion (believing everything).

## C Paradoxes of Induction and Probability

### ☰ Example 10: Puzzle 6: Surprised Children

Teacher announces an exam next week, but the date will be a surprise (students won't even know the night before).

- **Paradoxical Argumentation:** Students apply backward induction. It cannot be Friday (they'd know Thursday night). By elimination, it cannot be any day. They deduce the announcement is false.
- **Result:** They dismiss the announcement. The exam occurs (e.g., Tuesday) and is indeed a complete surprise.

### ☰ Example 11: Puzzle 7: The Lottery Paradox

A fair lottery with 1,000,000 tickets.

- Probability of ticket  $x$  winning is 0.000001.
- It is rational to hold the belief that ticket  $x$  will lose.
- This reasoning applies symmetrically to all tickets.
- Yet, the agent knows one ticket will win.

**Result:** The conjunction of highly probable rational beliefs yields a strict logical **inconsistency**.

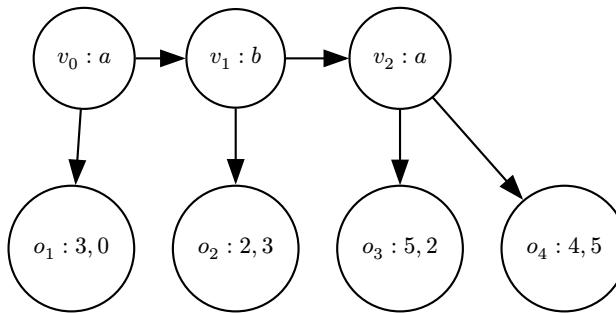
### ☰ Example 12: Puzzle 7 Modification: The Infinite Lottery

An infinite lottery over arbitrary natural numbers. The probability of any given ticket winning is exactly 0. The agent is mathematically correct to believe a specific ticket will not win, yet one must win. Any finite subset of beliefs is consistent, but the infinite global set is inconsistent.

## D Backward Induction and Social Epistemology

### ☰ Example 13: Puzzle 8: The Centipede Game

A sequential game with alternating moves by  $a$  and  $b$ , deciding between stopping the game or continuing:



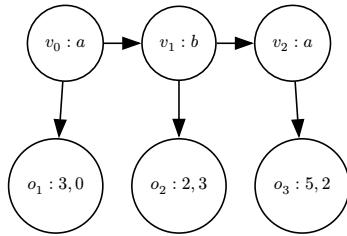
In the leaves ("outcomes"  $o_j$ ) the first number is  $a$ 's payoff, the second number is  $b$ 's payoff.

- $v_0 : a$  stops for  $o_1(3, 0)$  or continues to  $v_1$
- $v_1 : b$  stops for  $o_2(2, 3)$  or continues to  $v_2$
- $v_2 : a$  stops for  $o_3(5, 2)$  or continues to  $v_4(4, 5)$

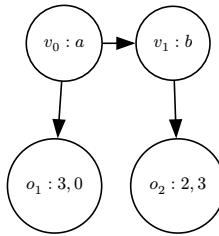
### The Backwards Induction (BI) Method

- Iteratively eliminate the *obviously* “bad” moves
- Proceeding backwards from the leaves

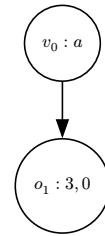
#### Elimination Step 1



#### Elimination Step 2



#### Elimination Step 3



- BI outcome:**  $o_1 : 3, 0$
- Why not another outcome?*: Strikes many as irrational

#### ① Intuition: The BI Paradox and Rational Pessimism

- Aumann's Argument:** Assuming *Common Knowledge* (Def 11) of Rationality (CKR), backward induction dictates *A* chooses  $o_3$  at  $v_2$ , so *B* chooses  $o_2$  at  $v_1$ , so *A* chooses  $o_1$  at  $v_0$ . The game terminates immediately at a suboptimal Pareto outcome.
- Counterargument:** If *B* reaches  $v_1$ , he observes *A* violating CKR (she didn't stop at  $v_0$ ). If *B* adopts **Rational Pessimism**—assuming *A* is irrational and will thus choose  $o_4$  at  $v_2$ —he should continue. If *A* anticipates this belief revision, her initial deviation becomes strictly rational. The epistemic foundation of backward induction contradicts its own counterfactuals.

## E Social Epistemology

Group dynamics often deviate from ideal individualized epistemic logic due to the recursive nature of social evidence.

**Def 13 (Pluralistic Ignorance):** A situation where the group collectively knows or acts upon less information than the individuals possess privately. Often observed in totalitarian regimes where public behavior contradicts private beliefs.

#### ☰ Example 14: Puzzle 9: Wisdom vs. Madness of the Crowds

- Wisdom of the Crowds:** Distributed group knowledge often empirically exceeds the most expert individual (e.g., aggregating independent estimates).
- Madness of the Crowds:** Systems can fail systematically due to cascading social epistemology.

#### ① Intuition: Information Cascades

An information cascade occurs when agents base their decisions on the observable behavior of prior agents rather than their own private evidence, leading to a breakdown of *epistemic democracy* (the wisdom of crowds).

### Example 15: The Black and White Urn Problem

**Setup:** One urn is in a room. It is either Urn B (2/3 black marbles) or Urn W (2/3 white marbles). Agents enter one by one, draw a marble, replace it, and publicly record their guess of the urn on a blackboard.

- The Cascade:**
1. Voter 1 draws Black and guesses Urn B.
  2. Voter 2 draws Black and guesses Urn B.
  3. Voter 3 draws White. However, the public evidence (two B votes) combined with their private evidence (one W draw) yields an aggregate evidence of (B, B, W). The rational epistemic choice is still to guess Urn B.

**Result:** From Voter 3 onwards, everyone will vote Urn B regardless of their private draw. If the first two voters happened to draw the minority color (probability  $\frac{1}{9}$ ), the entire crowd of  $n$  voters will lock into the wrong conclusion.

### Example 16: Biological and Geopolitical Cascades

- **Army Ant Circular Mill:** If an army ant loses the pheromone trail, it is biologically programmed to follow the ant directly in front of it. This simple rule works locally but can result in a massive recursive loop (a death spiral up to 400m in diameter) where the ants walk in a circle until they die.
- **The Men Who Stare at Goats (Cold War):** A French newspaper published a fabricated story about US military research into psychic weapons. Soviet intelligence read this, assumed it was a cover-up, and initiated their own psychic research program. US intelligence discovered the Soviet program and, assuming the Soviets were onto a real threat, started their own actual research program, sparking a 30-year arms race built on an initial cascade of false information.

## Session 1-3 (Lecture): Single-Agent Epistemic-Doxastic Logics: Kripke Models

### A Syntax and Core Definitions

Single-agent epistemic-doxastic logic expands standard propositional logic to formally capture an agent's knowledge and beliefs.

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid B\varphi \quad (2)$$

where  $p \in \text{Prop}$ .

**Def 14 (Single-Agent, pointed Epistemic-Doxastic Model):** Is a tuple  $S = (S, S_0, \|\cdot\|, s_*)$ , where

- $S$ : A set of *ontic* states defining the agent's *epistemic state* (epistemically possible).
- $S_0$ : A non-empty subset  $S_0 \subseteq S$ , called the *sphere of beliefs* or the agent's *doxastic state*.
- $\|\cdot\| : \text{Prop} \rightarrow \mathcal{P}(S)$ : A *valuation* map assigning atomic propositions to sets of states.
- $s_* \in S$ : The designated "actual world" representing the real state of the world.

*Sphere-based:* represents beliefs as nested layers of possible worlds, ranking worlds by their plausibility

## B Semantics

### ⓘ Intuition: Interpretation

- **Epistemic state:** state of the agent's knowledge: they believe  $s_*$  is among  $S$ , but cannot distinguish between  $s_i, s_j \in S; i \neq j$ .
- **Doxastic state:** the agent believes  $s_* \in S_0$

### ⓘ Notation: Truth

We write the following if  $\varphi$  is *true* in world  $w$ . When the model  $\mathbf{S}$  is fixed, we skip the subscript.

$$w \models_{\mathbf{S}} \varphi \quad (3)$$

### ⓘ Note: Atomic logical connectives

We interpret negation  $\neg$  and conjunction  $\wedge$  as atomic logical connectives, but disjunction  $\vee$ , the conditional  $\rightarrow$ , and the biconditional  $\leftrightarrow$  as compound connectives.

**Def 15 (Truth in an Interpretation):** A sentence  $\varphi$  is true in a model  $\mathbf{S}$  under the valuation map  $\|\cdot\|_{\mathbf{S}}$  if

- $\varphi = p; p \in \text{Prop}: w \models p$  iff  $w \in \|\varphi\|$ ,
- $\varphi = \neg\psi: w \models \neg\psi$  iff  $w \not\models \psi$ ,
- $\varphi = \psi \wedge \chi: w \models \psi \wedge \chi$  iff  $w \models \psi$  and  $w \models \chi$ ,
- $\varphi = K\varphi: w \models K\varphi$  iff  $\forall s \in S, s \models \varphi$ ,
- $\varphi = B\varphi: w \models B\varphi$  iff  $\forall s \in S_0, s \models \varphi$ .

**Def 16 (Validity):** A sentence  $\varphi$  is **valid** in a model  $\mathbf{S}$  if it is true at every state  $w \in \mathbf{S}$ .

**Def 17 (Satisfiability):** A sentence  $\varphi$  is **satisfiable** in a model  $\mathbf{S}$  if it is true at some state  $w \in \mathbf{S}$ .

### ⓘ Note: Semantics of Knowledge and Belief

The universal quantifier over the domain of possibilities is interpreted as knowledge or belief.

- **Knowledge** ( $K\varphi$ ): Truth in all epistemically possible worlds.
- **Belief** ( $B\varphi$ ): Truth in all doxastically possible worlds within the sphere of beliefs.

## C Learning and Mistaken Updates

Learning corresponds to world elimination. An update with a sentence  $\varphi$  is the operation of deleting all non- $\varphi$  possibilities from the model.

### Example 17: The Concealed Coin and Mistaken Updates

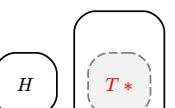
**Base Scenario:** A coin is on the table; the agent does not know if it is Heads ( $H$ ) or Tails ( $T$ ). 

**Standard Update:** The agent looks and sees  $H$ . The  $T$  world is eliminated, and only the  $H$  epistemic possibility survives. 

#### Note: Update as World Elimination

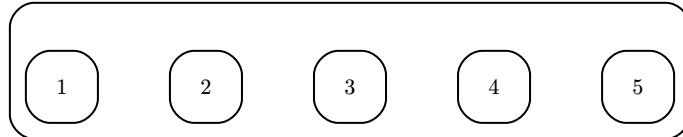
In general, updating corresponds to world elimination: an update with a sentence  $\varphi$  is simply the operation of deleting all the non- $\varphi$  possibilities.

**Mistaken Update:** The agent mistakenly believes they saw  $H$ . If we eliminate  $T$ , the actual world  $s_*$  is no longer in the agent's model, making it impossible to evaluate objective truth. 

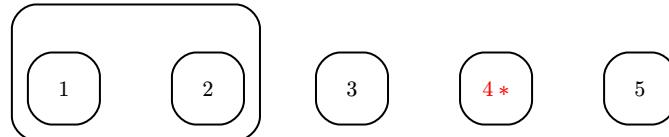
**Resolution (Third-Person Models):** We maintain an objective perspective where the real possibility always remains in the global model  $S$ , even if the agent believes it to be impossible. The sphere of beliefs  $S_0$  ( $\square$ ) is restricted to  $T$ , meaning the agent believes  $H$ , but their belief is false because  $s_* \in S/S_0$ . 

### Example 18: Continuation of Puzzle 6: Surprised Children

Situation before the teacher's announcement (we don't know  $s_*$ : no star in the figure):



A student believes (for some reason) the exam will be on Monday or Tuesday, but it is on Thursday ( $s_* = 4$ ):



## D Kripke Semantics for Epistemic-Doxastic Logic

Sphere models can be generalized using Kripke semantics to allow for varying strengths of knowledge and belief.

### ⦿ Remember: Kripke Model

**Def 18 (Kripke Model):** A Kripke model is a tuple  $S = (S, \{R_i\}_{i \in I}, \|\cdot\|, s_*)$  with set of states  $S$ , accessibility relations  $R_i$ , valuation  $\|\cdot\|$ , and actual state  $s_*$ .

**Def 19 (Epistemic-Doxastic Kripke Model):** To model knowledge K and belief B, this becomes  $(S, \sim, \rightarrow, \|\cdot\|, s_*)$ , where  $\sim$  is the epistemic relation (for K) and  $\rightarrow$  is the doxastic relation (for B).

For atomic sentences and for Boolean connectives, we use the same semantics (and notations) as on epistemic-doxastic models.

**Def 20 (Kripke modalities):** For every sentence  $\varphi$ , we can define a new sentence using the *universal Kripke modality*  $[R_i]$  by universally quantifying over  $R_i$  accessible worlds. The dual *existential Kripke modality*  $\langle R_i \rangle$  is given by

$$\langle R_i \rangle \varphi := \neg [R_i] \neg \varphi. \quad (4)$$

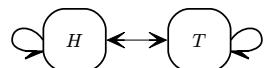
### ◻ Notation: Kripke modalities: subscript

If  $R$  is unique, we abbreviate  $[R_i]\varphi$  as  $\Box\varphi$ , and  $\langle R_i \rangle \varphi$  as  $\Diamond\varphi$ .

**Def 21 (Truth in an interpretation continued: Kripke modalities):** We continue Def 15 by adding vi.  $\varphi = [R_i]\varphi$ :  $w \models [R_i]\varphi$  iff  $v \models \varphi \forall v : wR_i v$ .

### Ξ Example 19: Example 17 contd.: Coins and Knowledge

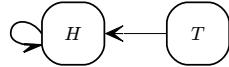
The agent's knowledge in the concealed coin scenario can be represented as:



- The arrows represent the **epistemic relation**  $\sim$ , capturing the agent's uncertainty about the state of the world.
- An arrow from state  $s$  to state  $t$  means that if  $s_* = s$ , the agent could not distinguish between  $s$  and  $t$ .

### Example 20: Example 17 contd.: Coins and Belief

The agent's belief after the mistaken update can be represented as:



- The arrows represent the **epistemic relation**  $\sim$ , capturing the agent's uncertainty about the state of the world.
- An arrow from state  $s$  to state  $t$  means that if  $s_* = s$ , the agent could not distinguish between  $s$  and  $t$ .

### Remember: Named axioms in Modal Logic

Certain axioms have set names in Modal Logic:

- **(K)** Basic Modal Logic
- **(T)** Reflexivity  $\Box\varphi \rightarrow \varphi$
- **(4)** Transitivity  $\Box\varphi \rightarrow \Box\Box\varphi$
- **(5)** Euclideaness  $\Diamond\varphi \rightarrow \Box\Diamond\varphi$
- **(D)** Seriality  $\Box\varphi \rightarrow \Diamond\varphi$
- **Weak Epistemic Model (S4)** = **(K)** + **(T)** + **(4)**: No negative introspection (only reflex. & trans.)
- **Epistemic Model: (S5)** = **(K)** + **(T)** + **(5)**  
Note: An **(S5)**-model is one where the accessibility relations are equivalence relations: reflexive, transitive, symmetric (with the other two properties, Symmetry  $\equiv$  Euclidean)
- **Doxastic Model: (KD45)** = **(K)** + **(D)** + **(4)** + **(5)**  
Note: Doxastic Models are not symmetric, but serial ( $\forall s : \exists t : s \rightarrow t$ ), transitive, Euclidean

### Theorem 1: Axioms and Relational Properties

A Kripke model satisfying all the below conditions on the relations  $\sim$  and  $\rightarrow$  is called an **epistemic-doxastic Kripke model**.

Validities for Knowledge (Equivalence relation  $\sim$ , giving an S5 model):

- i. **Veracity** ( $K\varphi \Rightarrow \varphi$ ):  $\sim$  is reflexive.
- ii. **Positive Introspection** ( $K\varphi \Rightarrow KK\varphi$ ):  $\sim$  is transitive (4).
- iii. **Negative Introspection** ( $\neg K\varphi \Rightarrow K\neg K\varphi$ ):  $\sim$  is Euclidean (and symmetric) (5).

Validities for Belief (KD45 model properties for  $\rightarrow$ ):

- iv. **Consistency** ( $\neg B(\varphi \wedge \neg\varphi)$ ):  $\rightarrow$  is serial.
- v. **Positive Introspection** ( $B\varphi \Rightarrow BB\varphi$ ):  $\rightarrow$  is transitive.
- vi. **Negative Introspection** ( $\neg B\varphi \Rightarrow B\neg B\varphi$ ):  $\rightarrow$  is Euclidean.

KB Interaction Properties:

- vii. **Knowledge implies Belief** ( $K\varphi \Rightarrow B\varphi$ ): If  $s \rightarrow t$  then  $s \sim t$ .
- viii. **Strong Positive Introspection** ( $B\varphi \Rightarrow KB\varphi$ ): If  $s \sim t$  and  $t \rightarrow w$  then  $s \rightarrow w$ .
- ix. **Strong Negative Introspection** ( $\neg B\varphi \Rightarrow K\neg B\varphi$ ): If  $s \sim t$  and  $t \rightarrow w$  then  $s \rightarrow w$ .

### Observations:

1. Epistemic-doxastic Kripke models are equivalent to Simple Epistemic-Doxastic Models(Def 14)
2. The epistemic relation is completely determined by the doxastic relation.

## Sound and Complete Proof System for single agent epistemic-doxastic logic:

- Axioms:

From above: i. - iv., vii. - ix.

All propositional tautologies

Modus Ponens: from  $\varphi$  and  $(\varphi \rightarrow \psi)$  infer  $\psi$

Necessitation: from  $\varphi$  infer  $K\varphi$  and  $B\varphi$

Kripke's axioms for K and B:

$$K\varphi \rightarrow \psi \rightarrow K\varphi \rightarrow K\psi$$

$$B\varphi \rightarrow \psi \rightarrow B\varphi \rightarrow B\psi$$

### Generalization

- It is convenient to have a more general semantics where the above do not hold

Introspection is not universally accepted

People may believe they know things they don't actually know

There might be “crazy” agents with inconsistent beliefs

### Theorem 2: Equivalence of Models

Every epistemic-doxastic sphere model  $S = (S, S_0, \|\cdot\|, s_*)$  is completely equivalent to an epistemic-doxastic Kripke model  $S' = (S, \sim, \rightarrow, \|\cdot\|, s_*)$  that satisfies the same sentences at  $s_*$ .

### Attention: Logical Omniscience

Any Kripke modality validates axiom K ( $K(\varphi \Rightarrow \psi) \Rightarrow (K\varphi \Rightarrow K\psi)$ ) and the Necessitation rule (if  $\varphi$  is valid,  $K\varphi$  is valid). Consequently, Kripke semantics models “ideal reasoners” with unlimited inference powers who know/believe all logical entailments, failing to capture bounded rationality.

## Week 2

### Session 2-1 (Lecture): Multi-agent Models and Public Announcement Logic (PAL)

#### A Multi-Agent Kripke Models & Modalities

**Def 22** (*Multi-Agent Kripke Model*): A multi-agent Kripke model is a tuple

$$S = \left( S, \left\{ \xrightarrow[a]{\cdot} \right\}_{a \in \mathcal{A}}, \|\cdot\| \right) \quad (5)$$

where  $\mathcal{A}$  is a set of labels representing the names of epistemic agents.

**Def 23** (*Epistemic Modalities*): For every sentence  $\varphi$ , we define  $\Box_a \varphi$  by universally quantifying over  $\xrightarrow[a]{\cdot}$ -accessible worlds:  $s \models_S \Box_a \varphi \Leftrightarrow t \models_S \varphi$  for all  $t$  such that  $s \xrightarrow[a]{\cdot} t$ . This is interpreted as knowledge, denoted  $K_a \varphi$ , or belief, denoted  $B_a \varphi$ . Its existential dual  $\Diamond_a \varphi := \neg \Box_a \neg \varphi$  denotes epistemic possibility.

#### ☰ Example 21: The Concealed Coin

Two players  $a, b$ , along with a referee  $c$  play a game. The referee covers a fair coin so nobody knows the outcome. Using concatenated arrows, we can express iterated knowledge. For instance,  $b$  knows that  $a$  does not know the outcome but knows it is Heads ( $H$ ) or Tails ( $T$ ):  $w \models \Box_b (\neg \Box_a H \wedge \neg \Box_a T) \wedge \Box_b \Box_a (H \vee T)$ .

#### B Common Knowledge

**Def 24** (*Common Knowledge (Group)*): Common knowledge within a group  $G \subseteq \mathcal{A}$ , denoted  $C \Box_G \varphi$ , is evaluated by quantifying over all worlds accessible by any finite concatenation of arrows within  $G$ :  $s \models_S C \Box_G \varphi \Leftrightarrow t \models_S \varphi$  for every  $t$  plus every finite chain  $s = s_0 \xrightarrow[a_1]{\cdot} s_1 \xrightarrow[a_2]{\cdot} \dots \xrightarrow[a_n]{\cdot} s_n = t$  with  $a_1, \dots, a_n \in G$ . It is equivalent to the Kripke modality for the reflexive-transitive closure of the union of all epistemic relations:  $\left[ \left( \cup_{a \in G} \xrightarrow[a]{\cdot} \right)^* \right]$ .

#### ⓘ Intuition: Common Knowledge as Infinite Conjunction

Let  $E_G \varphi := \wedge_{a \in G} \Box_a \varphi$  (“everybody in  $G$  knows  $\varphi$ ”). Def 24 is semantically equivalent to the infinite conjunction:  $\varphi \wedge E_G \varphi \wedge E_G E_G \varphi \wedge \dots$

#### ☒ Theorem 3: Validities for Common Modalities

- **Fixed-Point Axiom** (Mix):  $C \Box_G \varphi \Rightarrow (\varphi \wedge E_G C \Box_G \varphi)$
- **Induction Axiom**:  $C \Box_G \varphi \Rightarrow E_G \varphi \Rightarrow (\varphi \Rightarrow C \Box_G \varphi)$

## C Distributed Knowledge

**Def 25** (*Distributed Knowledge (Group)*): Distributed knowledge within a group  $G$ , denoted  $D\Box_G \varphi$  or  $Dk_G \varphi$ , is obtained by quantifying over worlds simultaneously accessible by all arrows for agents in  $G$  (i.e., the intersection of epistemic relations  $\cap_{a \in G} \xrightarrow{a}$ ):  $s \models_S D\Box_G \varphi \Leftrightarrow t \models_S \varphi$  for every  $t$  such that  $s \xrightarrow{a} t$  holds for all  $a \in G$ .

### ⓘ Intuition: Epistemic Potential

Def 25 captures the implicit knowledge of the group: what the agents in  $G$  could come to know if they pooled all their private knowledge.

### Ⓛ Example 22: Two Muddy Children

Two children (1 & 2) have dirty foreheads ( $d_1, d_2$ ). Each sees the other but not themselves. In the real world  $w = (d_1, d_2)$ , neither knows both are dirty, but it is distributed knowledge:  $w \models \neg K_1 d_1 \wedge d_2 \wedge \neg K_2 d_1 \wedge d_2 \wedge Dk(d_1 \wedge d_2)$ .

### ⓘ Theorem 4: Validities for Distributed Knowledge

- $K_a \varphi \Rightarrow Dk \varphi$
- $(K_a \varphi \wedge K_b \psi) \Rightarrow Dk(\varphi \wedge \psi)$

## D Dynamics & Public Announcements

**Def 26** (*Public Announcement Logic*): A public announcement  $!\varphi$  is a joint update that deletes all non- $\varphi$  worlds from a model. The model transformer maps  $S$  to  $S^{!}\varphi = (S_\varphi, \xrightarrow{\varphi}, \|\cdot\|_\varphi)$ , where  $S_\varphi = \|\varphi\|_S$ , while the relations as well as valuations are restricted to  $S_\varphi$ . The dynamic modality is evaluated as:  $s \models_S [!]\varphi \psi \Leftrightarrow t \models_{S^{!}\varphi} \psi$  for all  $t \in S^{!}\varphi$  such that  $s \xrightarrow{!}\varphi t$ .

### ⓘ Theorem 5: PAL Reduction Axioms

PAL (Def 26) allows translating dynamic formulas to basic modal logic via reduction axioms:

- **Atomic Permanence:**  $[!]\varphi p \Leftrightarrow (\varphi \Rightarrow p)$
- **Announcement-Negation:**  $[!]\varphi \neg \psi \Leftrightarrow (\varphi \Rightarrow \neg[!]\varphi \psi)$
- **Announcement-Conjunction:**  $[!]\varphi (\psi_1 \wedge \psi_2) \Leftrightarrow ([!]\varphi \psi_1 \wedge [!]\varphi \psi_2)$
- **Announcement-Knowledge:**  $[!]\varphi \Box_a \psi \Leftrightarrow (\varphi \Rightarrow \Box_a [!]\varphi \psi)$

### ⓘ Expressivity & Succinctness

PAL has the exact same expressivity as basic modal logic because dynamic modalities can be eliminated using the reduction axioms. However, PAL is exponentially more succinct.

## E Moore Sentences & Paradoxes

**Def 27 (Moore Sentences):** Sentences that become false after being truthfully announced. For a Moore sentence  $\varphi$ , we have  $[\!\varphi]\neg\varphi$ . Therefore, they become known to be false after being announced:  $[\!\varphi]K_a\neg\varphi$ .

### ☰ Example 23: Moore Sentences in Muddy Children

“You are dirty but you do not know it” is a Moore sentence for child 1:  $d_1 \wedge \neg K_1 d_1$ . This is initially true in the muddy children model. However, after it is publicly announced, child 1 learns they are dirty. The second conjunct becomes false, making the entire sentence false.

*Continuation:* In the grand finale of the muddy children puzzle, children repeatedly announce their ignorance:  $!(\wedge_i (\neg K_i d_i \wedge \neg K_i \neg d_i))$ . Each round deletes worlds where a child would have known their state, converting distributed knowledge into common knowledge until the dirty children deduce they are dirty.

### ☒ Theorem 6: Closure under Composition

Performing two successive public announcements is equivalent to a single, more complex announcement:  $[\!\varphi][\!\psi]\theta \Leftrightarrow [!(\varphi \wedge [\!\varphi]\psi)]\theta$ .

**TODO:** read and improve

**Session 2-2 (Lecture): PAL Continued**

**Session 2-3 (Lecture): Does this one even exist??**

## **Week 3**

**Session 3-1 (Lecture): "Learnability" and "Knowability"**

**Session 3-2 (Lecture): Tutorial 1**

**Session 3-3 (Lecture): The problem of belief revision**

## **Week 4**

**Session 4-1 (Lecture):**

**Session 4-2 (Lecture):**

**Session 4-3 (Lecture):**

## **Week 5**

**Session 5-1 (Lecture):**

**Session 5-2 (Lecture):**

**Session 5-3 (Lecture):**

## **Week 6**

**Session 6-1 (Lecture):**

**Session 6-2 (Lecture):**

**Session 6-3 (Lecture):**

## **Week 7**

**Session 7-1 (Lecture):**

**Session 7-2 (Lecture):**

**Session 7-3 (Lecture):**

## **Week 8**

**Session 8-1 (Lecture):**

**Session 8-2 (Lecture):**

**Session 8-3 (Lecture):**

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