ML Homework Assignment 5

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Exercise 1

By question 1 in Tutorial 10: A normal modal logic Λ has the finite model property iff it is sound and complete with respect to $\mathrm{FinFr}(\Lambda)$.

(a) Show that S4.2 has the FMP

Recall that **S4.2** is the logic defined by the transitivity and directness axioms:

$$\begin{aligned} \mathbf{S4.2} &= \boldsymbol{K} + (4) + (.2) \\ \mathbf{S4.2} &= \boldsymbol{K} + (\Diamond\Diamond p \to \Diamond p) + (\Diamond\Box p \to \Box\Diamond p) \end{aligned} \tag{1}$$

By definition $\mathrm{FinFr}(\mathbf{KMU}) \Vdash \mathbf{KMU}$.

Assume there is some formula φ s.t. $\underset{\mathbf{S4.2}}{\nvdash} \varphi$. To show the contraposition of completeness w.r.t. $\mathrm{FinFr}(\mathbf{S4.2})$, we will show $\exists \mathbb{F} \in \mathrm{FinFr}(\mathbf{S4.2}) : \mathbb{F} \not\Vdash \varphi$.

Because **S4.2** is complete, there must be a frame $\mathbb{F}$ which refutes φ in $\mathrm{Fr}(\mathbf{S4.2})$: $\mathbb{F} \not\Vdash \varphi$. Thus, there must be some valuation V and point $r \in \mathbb{F}$, we write $\mathbb{M} = (\mathbb{F}, V)$, s.t. $\mathbb{M}, r \not\Vdash \varphi$. To show that there is a finite model, we will first take a generated submodel, and then apply a filtration. For this we choose the Lemmon filtration.

Let $\mathbb{M}'$ be the the generated submodel of $\mathbb{M}$ with r as its root node (where as mentioned above r is a point such that $\mathbb{M}, r \vDash \neg\varphi$). It is clear that $\mathbb{M}', r \vDash \neg\varphi$, and also that $\mathbb{M}'$ is still transitive.

Let $\Sigma = \mathrm{Sub}(\{\varphi\})$. Let $\mathbb{M}^{\mathrm{t}}$ be the model resulting from applying the Lemmon filtration to $\mathbb{M}'$. It is clear that $\mathbb{M}^{\mathrm{t}}, [r] \vDash \neg\varphi$ (filtration theorem), and $\mathbb{M}^{\mathrm{t}}$ is finite.

To show: $\mathbb{M}^{\mathrm{t}} \in \mathrm{FinFr}(\mathbf{S4.2})$. Since $\mathbb{M}^{\mathrm{t}}$ is finite, what rests is to show that the axioms of the logic are valid in $\mathbb{M}^{\mathrm{t}}$.

(4) Since we know that the Lemmon filtration preserves transitivity, clearly the (4) axiom's validity is preserved in $\mathbb{M}^{\mathrm{t}}$.

(.2) We let $[w]$ some point in $\mathbb{M}^t$, assume that $[w] \vDash \Diamond\Box p$. We show that $[w] \vDash \Box\Diamond p$.

We know that there exists a point $[v]$ such that $[w]R^t[v]$, and $[v] \vDash \Box p$. Now take some successor $[a]$ of $[w]$. We show that $[a] \vDash \Diamond p$.

We examine points v, a in $\mathbb{M}'$ such that $v \in [v], a \in [a]$. By the fact that $\mathbb{M}'$ is a rooted and transitive model, we know that $rR'v$ and $rR'a$.

By applying the Sahlqvist algorithm, we get the standard translation of the axiom as:

$$\forall x \forall s (xRs \to \forall a (xRa \to \exists c (aRc \to sRc))) \tag{2}$$

We know that $\mathbb{M}'$ has this frame property, and since we have $rR'v$ and $rR'a$, we know there exists some point c such that $vR'c$ and $aR'c$. By the nature of filtrations, we can infer that $[v]R^t[c]$, and $[a]R^t[c]$.

By $[v]R^t[c]$, and $[v] \vDash \Box p$, we have that $[c] \vDash p$. Since also $[a] \vDash [c]$, we have $[a] \vDash \Diamond p$ as desired.

This shows that if a point $[w]$ in $\mathbb{M}^{\mathrm{t}}$ satisfies $\Diamond\Box p$, then it satisfies $\Box\Diamond p$. We conclude that all the axioms of **S4.2** hold on the finite frame $\mathbb{M}^{\mathrm{t}}$, and so $\mathbb{M}^{\mathrm{t}} \in \mathrm{FinFr}(\mathbf{S4.2})$.

This proves completeness of **S4.2** with respect to FinFr(**S4.2**).

We conclude that **S4.2** is sound and complete with respect to FinFr(**S4.2**) and thus has the FMP.

(b) Show that KMU has the FMP

$$\mathbf{KMU} = \boldsymbol{K} + (\Diamond\Box p \to \Diamond p) \tag{3}$$

By definition $\mathrm{FinFr}(\mathbf{KMU}) \Vdash \mathbf{KMU}$.

Assume there is some formula φ s.t. $\underset{\mathbf{KMU}}{\nvdash} \varphi$. To show the contraposition of completeness w.r.t. FinFr(**KMU**), we will show $\exists \mathbb{F} \in \mathrm{FinFr}(\mathbf{KMU}) : \mathbb{F} \not\Vdash \varphi$.

Because **KMU** is complete, there must be a frame $\mathbb{F}$ which refutes φ in Fr(**KMU**): $\mathbb{F} \not\Vdash \varphi$. Thus, there must be some valuation V and point $x \in \mathbb{F}$, we write $\mathbb{M} = (\mathbb{F}, V)$, s.t. $\mathbb{M}, x \not\Vdash \varphi$. To show that there is a finite frame, we will apply filtration to the discussed model. For this we choose the smallest filtration $\mathbb{R}^{\mathrm{s}}$.

Define $\Sigma = \mathrm{Sub}(\{\varphi\})$. Let $\mathbb{M}^s$ be the model resulting from applying the smallest filtration to $\mathbb{M}$. Note that by the filtration theorem, we still have $\mathbb{M}^t, [x] \vDash \neg\varphi$.

To show: $\mathbb{M}^s \in \mathrm{Fr}(\mathbf{KMU})$. We show all axioms of **KMU** are preserved in filtered frame.

$(\Diamond\Box p \to \Diamond p)$ Take some world $[w] \in \mathbb{M}^s$. Suppose $\mathbb{M}^s, [w] \Vdash \Diamond\Box p$. Thus there must be some $[v] \in \mathbb{M}^s$ s.t. $[w]R^s[v]$ and $\mathbb{M}^s, [v] \Vdash \Box p$.

The standard translation of $\Diamond\Box p \to \Diamond p$ is:

$$\forall x \forall s (xRs \to \exists a (xRa \wedge sRa)) \tag{4}$$

Thus there must be representatives $w, v \in \mathbb{M}$ s.t. wRv. Since the unfiltered model satisfies the frame condition, there is $a \in \mathbb{M}$ s.t. wRa and vRa. By the definition of the smallest filtration, there is the Σ-equivalence class $[a] \in \mathbb{M}^s$ s.t. $[w]R^s[a]$ and $[v]R^s[a]$. Since $\mathbb{M}^s, [v] \Vdash \Box p$, we get $\mathbb{M}^s, [a] \Vdash p$ and thus $\mathbb{M}^s, [w] \Vdash \Diamond p$. Since w was chosen arbitrarily, we can generalize $\mathbb{M}^s \Vdash \Diamond\Box p \to \Diamond p$.

Since $(\Diamond\Box p \to \Diamond p)$ holds in $\mathbb{M}^s$, we conclude $\mathbb{F}^s \in \mathrm{Fr}(\mathbf{KMU})$, since $\mathbb{F}^s$ is finite $\mathbb{F}^s \in \mathrm{FinFr}(\mathbf{KMU})$.

This proves that **S4.2** is complete with respect to FinFr(**KMU**).

Since we have that **S4.2** is sound and complete wrt FinFr(**KMU**), we conclude that it has the FMP.

$\Box$

(c) Deduce that they are decidable

Axiomatizations:

- $\Sigma_{\mathbf{S4.2}} = \{(4), (.2)\}$, $\mathbf{S4.2} = \boldsymbol{K}\boldsymbol{\Sigma}_{\mathbf{S4.2}}$
- $\Sigma_{\mathbf{KMU}} = \{(\Diamond\Box p \to \Diamond p)\}$, $\mathbf{KMU} = \boldsymbol{K}\boldsymbol{\Sigma}_{\mathbf{KMU}}$

It is clear that $\Sigma_{\mathbf{S4.2}}$ and $\Sigma_{\mathbf{KMU}}$ are both finite, thus, by definition 6.10 in the modal logic book, both **S4.2** and **KMU** are finitely axiomatizable.

By Harrop's theorem (Thm. 4.12 in the study notes) since both **KMU** and **S4.2** have the FMP and are finitely axiomatizable, they are decidable.

Exercise 2

(a) Show that $\mathfrak{g} \vDash \Box(p \leftrightarrow \Box p) \to \Box p$.

Let $w \in W$ and V some admissible valuation. We show that $\mathfrak{g}, V, w \vDash \Box(p \leftrightarrow \Box p) \to \Box p$. We assume $\mathfrak{g}, V, w \vDash \Box(p \leftrightarrow \Box p)$ and distinguish two cases:

- Case 1: $w \in \mathbb{N}$.

By definition of R we know that wRv for any $v \in \mathbb{N}$ such that $v < w$. If $w = 0$ then it is clear that $\mathfrak{g}, V, w \vDash \Box p$ as desired, so we continue with the case $w \neq 0$. Let $w' < w$. We use induction to show that $g, V, w' \vDash p$.

Base: $w' = 0$. In this case we have $w' \vDash \Box p$, and by wRw' and $w \vDash \Box(p \leftrightarrow \Box p)$ we have that $w' \vDash p$.

IH: assume $\mathfrak{g}, V, k \vDash p$ for $k < n - 1$.

Induction step $w' = k + 1$. Take some successor w'' of w'. By definition of R we have $w'' < w'$. By the inductive hypothesis we know that $w'' \vDash p$. It follows that $w' \vDash \Box p$, and again by wRw' and $w \vDash \Box(p \leftrightarrow \Box p)$ we have that $w' \vDash p$.

We conclude that every successor of w (every $w' < w$) satisfies p, and thus $w \vDash \Box p$ as desired.

- Case 2: $w = \omega + n$ for some $n \in \mathbb{N}$.

We first note that wRv for any $v \in \mathbb{N}$. It is easy to see that the induction constructed above can be extended to induction on $\mathbb{N}$, since in this case $w \vDash \Box(p \leftrightarrow \Box p)$ implies $p \leftrightarrow \Box p$ holds at any $v \in \mathbb{N}$. We thus focus our attention to successors of w of the form $\omega + m$.

We know that every $V(p)$ is either finite or cofinite, and since we have shown that $w \vDash \Box(p \leftrightarrow \Box p)$ implies that $v \vDash p$ for any $v \in \mathbb{N}$, we know that in this case $V(p)$ is cofinite. This allows us to conclude the existence of some $n \in \mathbb{N}$ with $n > m$ such that $u \geq \omega + n \Rightarrow u \vDash p$. We now show by induction on ℓ that also $\omega + n - \ell \vDash p$ for $1 \leq \ell \leq n - m$.

Base: $\ell = 1$

We assume that $\omega + n \vDash p$, and since $n > m$ we know that $\omega + n \vDash p \leftrightarrow \Box p$, and thus $\omega + n \vDash \Box p$. By definition of R we have that $(\omega + n)R(\omega + n - 1) = (\omega + n - \ell)$. It follows that $\omega + n - \ell \vDash p$.

IH: $\omega + n - k \vDash p$ for $k < n - m - 1$

Induction step: $\ell = k + 1$

We know by the induction hypothesis that $\omega + n - k \vDash p$. Since we have that $k < k + 1 < n - m$ we know that $\omega + m < \omega + n - (k + 1) < \omega + n - k$ we know that both $(\omega + m)R(\omega + n - k)$ and $(\omega + m)R(\omega + n - (k + 1))$. We can thus conclude that $\omega + m - k \vDash \Box p$. By definition of R we know that $(\omega + n - k)R(\omega + n - (k + 1))$. It follows that $\omega + n - (k + 1) \vDash p$.

We have thus shown that $\omega + t \vDash p$ for $t \geq m$. In particular, since we have $\omega + m \vDash p$, and by definition of R we have $(\omega + m)R(\omega + m)$, we see that $\omega + m \vDash \Box p$ as desired.

We conclude that $\mathfrak{g} \vDash \Box(p \leftrightarrow \Box p) \to \Box p$

(b) Show that GL* is Kripke incomplete

To show that $\mathbf{GL}^*$ is incomplete (with respect to the class of frames $\mathrm{Fr}(\mathbf{GL}^*) = \mathrm{Fr}(\mathbf{GL})$) is suffices to find a formula which is valid on this class of frames, but which is not a theorem of $\mathbf{GL}^*$. We propose the formula $\varphi := \Box(\Box p \to p) \to \Box p$. Since this formula is an axiom of $\mathbf{GL}$, we know that it is valid on $\mathrm{Fr}(\mathbf{GL}) = \mathrm{Fr}(\mathbf{GL}^*)$.

Suppose that $\varphi \in \mathbf{GL}^*$. We showed in part (a) that $\mathfrak{g} \vDash \Box(\Box p \leftrightarrow p) \to \Box p$. This shows that $\mathbf{GL}^*$ is sound with respect to $\mathfrak{g}$ (page 193 of the modal logic book says that axiom validity is enough for soundness). Since we assumed that $\varphi \in \mathbf{GL}^*$, it follows that $\mathfrak{g} \vDash \varphi$. We now show that this is not the case.

Let V be a valuation with $V(p) = W - \{\omega\}$. Let $\mathbb{M} = (W, R, V)$, we show that $\mathbb{M}, \omega \nvDash \Box(\Box p \to p) \to \Box p$. We show that $\mathbb{M}, \omega \vDash \Box(\Box p \to p)$. Take some successor v of ω. If $v \neq \omega$ then $v \vDash p$, so $v \vDash \Box p \to p$. If $v = \omega$, then we have $v \nvDash \Box p$, so $v \vDash \Box p \to p$. It thus holds that $\mathbb{M}, \omega \vDash \Box(\Box p \to p)$. However since $\omega R \omega$, we do not have $\omega \vDash \Box p$. It follows that $\omega \nvDash \Box(\Box p \to p) \to \Box p$. This formula is thus not valid on $\mathbb{M}$.

This is a contradiction, and thus it follows that $\varphi \not\in \mathbf{GL}^*$. Since we have found a formula which is valid on $\mathrm{Fr}(\mathbf{GL}^*)$, but which is not a theorem of $\mathbf{GL}^*$, we have shown that $\mathbf{GL}^*$ is kripke incomplete as desired.