



Iran University of  
Science & Technology

**INTERNATIONAL CONFERENCE  
ON COMBINATORICS,  
CRYPTOGRAPHY AND  
COMPUTATION**



*Proceedings of the 2<sup>nd</sup> International Conference on Combinatorics, Cryptography and  
Computation (I4C2017)*

**Solving Some Ordinary Differential Equations in Mechanical  
Engineering using Runge Kutta and Heun's methods**

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**ABSTRACT**

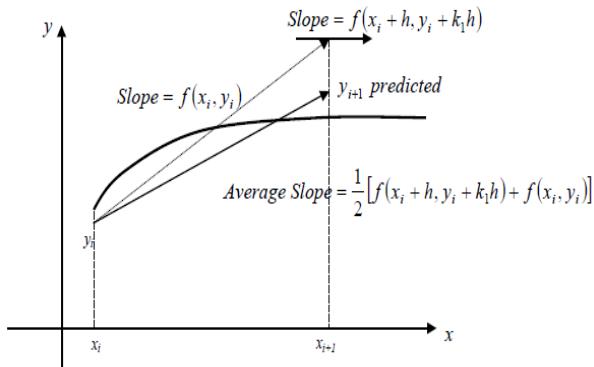
In this paper, we apply a general formula of Runge-Kutta methods in order 2 and 4 to solve two problems in mechanical engineering. At first, we will study Runge-Kutta types; then we will discuss about the problems. Results will be gathered in order to be compared. In the comparison we will discuss about the accuracy of each method.

**KEYWORDS:** Differential equations, Runge-Kutta types, Heun's method, fluids mechanic

**1 INTRODUCTION**

Differential equations have wide applications in various engineering and science disciplines. In general, modeling of the variation of a physical quantity, such as temperature, pressure, displacement, velocity, stress, strain, current, voltage, or concentration of a pollutant, with the change of time or location, or both would result in differential equations. Similarly, studying the variation of some physical quantities on other physical quantities would also lead to differential equations. In fact, many engineering subjects, such as mechanical vibration or structural dynamics, heat transfer, or theory of electric circuits are founded on the theory of differential equations. It is practically important for engineers to be able to model physical problems using mathematical equations and then solve these equations so that the behaviour of the systems concerned can be

studied. The theory of differential equations has become an essential tool economic analysis, particularly since the computer has become commonly available. It would be difficult to comprehend the contemporary literature on economics if one does not understand the basic concepts (such as bifurcations and chaos) and the results of the modern theory of differential equations [1,2,3,4].



## 2 NUMERICAL METHODS

The Runge-Kutta algorithm is used for solving the numerical solution of the ordinary differential equation  $y'(x) = f(x, y)$  with the initial condition  $y(\kappa) = w_0$ .

As already noted, Runge-Kutta method can be used in order 2 and 4. We can study each method below:

### 2.1 Runge-Kutta 2nd order

Only first order ordinary differential equations can be solved by using the Runge-Kutta 2nd order method. There are several methods for 2<sup>nd</sup> order such as Heun, midpoint and Ralston.

The general form of Runge-Kutta 2<sup>nd</sup> order is defined as:

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h$$

Where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$$

$$a_1 + a_2 = 1$$

$$a_2 p_1 = \frac{1}{2}$$

$$a_2 q_{11} = \frac{1}{2}$$

The coefficients  $a_1, a_2, p_1$  and  $q_{11}$  are different for each method:

In Heun's method  $a_2 = \frac{1}{2}$  is chosen, giving:

$$a_1 = \frac{1}{2}$$

$$p_1 = 1$$

$$q_{11} = 1$$

Resulting in:

$$y_{i+1} = y_i + \left( \frac{1}{2}k_1 + \frac{1}{2}k_2 \right)h$$

Where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + h, y_i + k_1 h)$$

In midpoint method  $a_2 = 1$  is chosen, giving:

$$a_1 = 0$$

$$p_1 = \frac{1}{2}$$

$$q_{11} = \frac{1}{2}$$

Resulting in:

$$y_{i+1} = y_i + k_2 h$$

Where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

And in Ralston method  $a_2 = \frac{2}{3}$  is chosen, giving:

$$a_1 = \frac{1}{3}$$

$$p_1 = \frac{3}{4}$$

$$q_{11} = \frac{3}{4}$$

Resulting in:

$$y_{i+1} = y_i + \left( \frac{1}{3}k_1 + \frac{2}{3}k_2 \right)h$$

Where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h\right)$$

## 2.2 Runge-Kutta 4th order

In about 1900, Runge and Kutta developed the following classical fourth order Runge-Kutta iterative method which has the accumulative error in order of  $O(h^4)$ . Since then, many mathematicians have tried to develop more Runge-Kutta like methods in a variety of directions. In 1969, England developed another fourth order for Runge-Kutta method.

The general form of Runge-Kutta 4<sup>th</sup> order is defined as:

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + (4-t)k_2 + tk_3 + k_4)h$$

Where

$$k_1 = f(x_k, y_k)$$

$$k_2 = f\left(x_k + \frac{h}{2}, y_k + \frac{1}{2}k_1 h\right)$$

$$k_3 = f\left(x_k + \frac{h}{2}, y_k + \left(\frac{1}{2} - \frac{1}{t}\right)k_1 h + \frac{1}{t}k_2 h\right)$$

$$k_4 = f\left(x_k + h, y_k + \left(1 - \frac{t}{2}\right)k_2 h + \frac{t}{2}k_3 h\right)$$

$t$  is a free parameter which:

If  $t=2$  is chosen, then upper equation becomes the classical Runge-Kutta method, and if  $t=4$  is chosen, then the equation becomes the England's Runge-Kutta method.

In the first problem we will discuss about a fluids mechanic problem, which is resulted in a first order ODE using Bernoulli's equation. Assume we have a large reservoir of water with 5-meter height; an exit tap is assembled at the bottom of the tank. We are to the time when the reservoir gets empty.

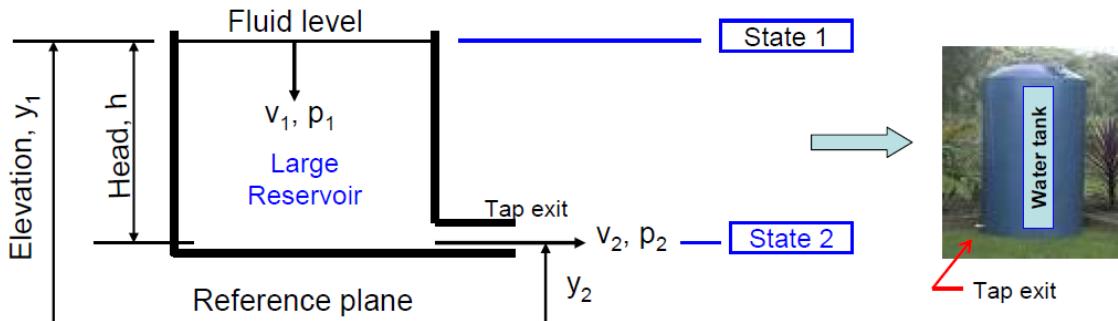


Figure 1: N

From the Bernoulli's equation we have:

$$\frac{v_1^2 - v_2^2}{2g} + \frac{p_1 - p_2}{\rho g} + (y_1 - y_2) = 0$$

As the difference of elevations between states 1 and 2 is not too large, we can have:  $P_1 \approx P_2$ .

Further, because it is a large reservoir, we realize that  $v_1 \ll v_2$  or  $v_1 \approx 0$ .

Thus, the equation is reduced to the form:

$$-\frac{v_2^2}{2g} + 0 + h = 0 \quad \text{with } h = y_1 - y_2$$

From which, we may express the exit velocity of the liquid at the tap to be:  $v_2 = \sqrt{2gh}$  the volume of water leaving the tap is:

$$\Delta V_{exit} = A v(t) \Delta t = \left( \frac{\pi d^2}{4} \right) \sqrt{2g h(t)} \Delta t$$

Where A is the cross section of the tap and d is its diameter and  $v(t) = \sqrt{2gh}$ .

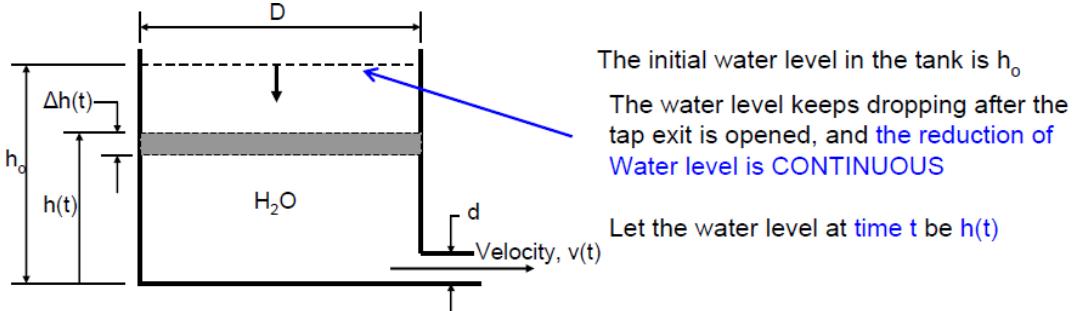


Figure 2: N

The volume of water loss in the tank is:

$$\Delta V_{tank} = -\frac{\pi D^2}{4} \Delta h(t)$$

Where  $D$  is the tank's diameter.

The volume of the water leaving the tap is equal to the water loss in the tank, thus we have:

$$\frac{\pi d^2}{4} \sqrt{2g h(t)} \Delta t = -\frac{\pi D^2}{4} \Delta h(t)$$

By re-arranging the above:

$$\frac{\Delta h(t)}{\Delta t} = -[h(t)]^{1/2} \left( \frac{d^2}{D^2} \right) \sqrt{2g}$$

If the processing of the draining is indeed continuous, i.e.,  $\Delta t \rightarrow 0$ , we will have the above equation expressed in the differential rather than difference form as follows:

$$\frac{dh(t)}{dt} = -\sqrt{2g} \left( \frac{d^2}{D^2} \right) \sqrt{h(t)}$$

With initial condition  $h(0) = 5m$ .

### 3 RESULTS

Obviously we can see that the more the step sizes are reduced, the more results are accurate. Also when  $h$  approaches zero, both 2<sup>nd</sup> and 4<sup>th</sup> order results are close to each other. The running time of the methods do not have sensible difference and they are about 3 seconds; but we can guess that if dimension of our problem is larger than what we saw here, Runge-Kutta 2<sup>nd</sup> order will operate faster than the 4<sup>th</sup> order. In the following figure we can see each method's error from the exact answer of problem:

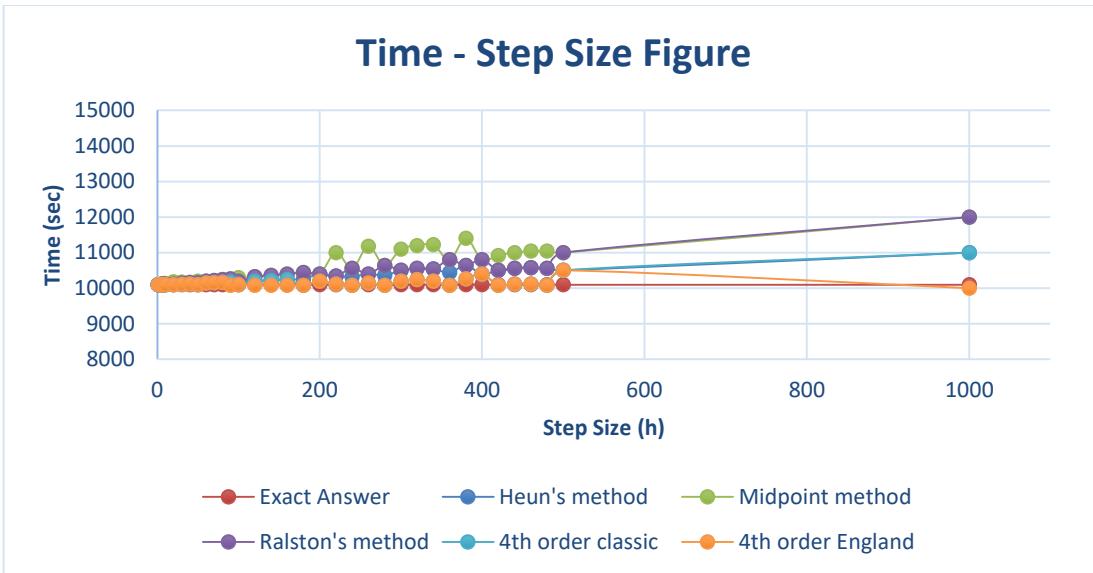


Figure 2: Number of patents on nanotechnology with time

### 3.1 Solution of the ODE with Runge-Kutta 2<sup>nd</sup> order

As shown in Figure 2, the initial height is 5m and time starts from 0 till when the tank gets empty. We will check the level of water by several step sizes (h) of time; as we want to realize when the tank will get empty, we should check when the height of water becomes 0. In the Table 1., we will check when tank gets empty by using three methods of Runge-Kutta 2<sup>nd</sup> order; ratio of tap's diameter to tank's diameter is 0.01 and  $g=9.81 \text{ m} / \text{s}^2$ .

Exact answer of the problem is calculated and is:  $t=1.0096 \text{ e4 sec}$ .

Table 1 Title of the Table

$h$	Heun's	Midpoint	Ralston's	$\delta_H(t)$	$\delta_M(t)$	$\delta_R(t)$
200	<b>10400</b>	<b>10400</b>	<b>10400</b>	<b>0.0301%</b>	<b>0.0301%</b>	<b>0.0301%</b>
100	<b>10200</b>	<b>10300</b>	<b>10200</b>	<b>0.0103%</b>	<b>0.0202%</b>	<b>0.0103%</b>
50	<b>10150</b>	<b>10200</b>	<b>10150</b>	<b>0.0053%</b>	<b>0.0103%</b>	<b>0.0053%</b>
10	<b>10110</b>	<b>10130</b>	<b>10110</b>	<b>0.0014%</b>	<b>0.0034%</b>	<b>0.0014%</b>
5	<b>10100</b>	<b>10105</b>	<b>10105</b>	<b>0.0004%</b>	<b>0.0009%</b>	<b>0.0009%</b>

### 3.2 Solution of the ODE with Runge-Kutta 4th order

In Runge-Kutta 4<sup>th</sup> order we will face the problem with the same assumptions as 2<sup>nd</sup> order with the classic and England's method.

Table 2 Title of the Table

<b>h</b>	Classic method	England's method	<b><math>\delta_C(t)</math></b>	<b><math>\delta_E(t)</math></b>
200	<b>10200</b>	<b>10200</b>	<b>0.0103%</b>	<b>0.0103%</b>
100	<b>10100</b>	<b>10100</b>	<b>0.0004%</b>	<b>0.0004%</b>
50	<b>10100</b>	<b>10100</b>	<b>0.0004%</b>	<b>0.0004%</b>
5	<b>10100</b>	<b>10100</b>	<b>0.0004%</b>	<b>0.0004%</b>
1	<b>10097</b>	<b>10097</b>	<b>0.0001%</b>	<b>0.0001%</b>

### Second Problem

In the second problem we want to measure distance passed by a dropped ball from height= $h_0$  with arbitrary height  $h_0$ , location, mass and geographical latitude.

To obtain equation of motion, we use 2 coordinate systems. First coordinate system is station and located at the centre of Earth and the second one is located on Earth's surface and moving with Earth's angular velocity.

The only external force applied to the ball is gravity. With respect to Newton's second law, we have:

$$F = ma \rightarrow -mg\hat{k} = m(\hat{a}_{o'} + \hat{a}_{P/o'} + \hat{\alpha} \times \hat{r} + \hat{\omega} \times (\hat{\omega} \times \hat{r}) + 2\hat{\omega} \times \hat{v})$$

Where  $\hat{\alpha}$  angular acceleration is assumed to be zero,  $\hat{\omega}$  is angular velocity,  $\hat{r}$  is vector defining the ball location in moving system,  $\hat{a}_{o'}$  is moving system's acceleration and  $\hat{a}_{P/o'}$  is ball's acceleration in moving system.

The term  $2\hat{\omega} \times \hat{v}$  is known as Coriolis acceleration; term  $\hat{\omega} \times (\hat{\omega} \times \hat{r})$  is known as centrifugal acceleration and term  $\hat{\alpha} \times \hat{r}$  is known as tangential acceleration.

After calculating above terms we reach 3 differential equations:

$$\ddot{x} - R\omega^2 \sin^2 \lambda - 2\dot{y}\omega \cos \lambda - x\omega^2 \cos^2 \lambda - z\omega^2 \sin \lambda \cos \lambda = 0$$

$$\ddot{y} - y\omega^2 \sin^2 \lambda + 2\dot{x}\omega \cos \lambda - y\omega^2 \cos^2 \lambda + 2\dot{z}\omega \sin \lambda = 0$$

$$\ddot{z} + R\omega^2 \sin \lambda \cos \lambda - 2\dot{y}\omega \sin \lambda - x\omega^2 \sin \lambda \cos \lambda - z\omega^2 \sin^2 \lambda = -g$$

With initial conditions:

$$x_0 = y_0 = 0$$

$$z_0 = h_0$$

$$\dot{x}_0 = \dot{y}_0 = \dot{z}_0 = 0$$

In order to solve 2<sup>nd</sup> order ODE we should make some changes to make them similar to 1<sup>st</sup> order ODE; therefore, we can use ODE45 command in Matlab.

$$x = u_1, \dot{x} = u_2, \ddot{x} = \dot{u}_2 \rightarrow u_2 = \dot{u}_1$$

$$y = u_3, \dot{y} = u_4, \ddot{y} = \dot{u}_4 \rightarrow u_4 = \dot{u}_3$$

$$z = u_5, \dot{z} = u_5, \ddot{z} = \dot{u}_6 \rightarrow u_6 = \dot{u}_5$$

$$A^* \begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \dot{u}_4 \\ \dot{u}_5 \\ \dot{u}_6 \end{pmatrix} = B^* \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix} + C \quad \Rightarrow \quad \begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \dot{u}_4 \\ \dot{u}_5 \\ \dot{u}_6 \end{pmatrix} = A^{-1} * [B^* \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix} + C]$$

Where A, B and C are matrix of coefficients that are made from the 3 equations.

By defining these matrixes in Matlab we can solve the 2<sup>nd</sup> order ODEs we saw above.

## **4 CONCLUSION**

In this note, we use a general formula of Runge-Kutta methods in order 2 and 4 to solve two problems in mechanical Engineering. Results will be gathered in order to be compared. In the comparison we will discuss about accuracy of each method.

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