From Cellular Automata to the Lattice Boltzmann Method

Theresa Pollinger, Marcial Gaißert
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Outline

Cellular Automata

Lattice Gas Automata

The Lattice Boltzmann Method

Cellular Automata

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- regular arrangement of cells
- each with discrete state
- updated simultaneously at each step
- update rules:
  - same for each cell
  - only dependent on close-by cells

a grid of cells

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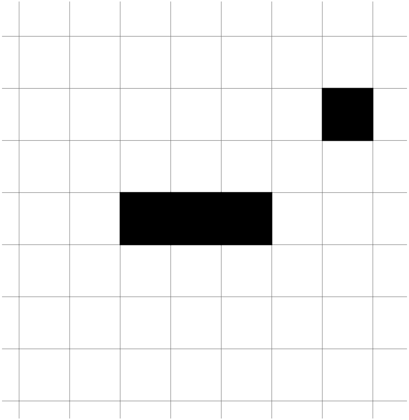

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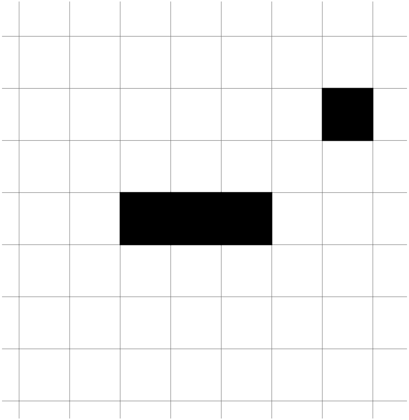

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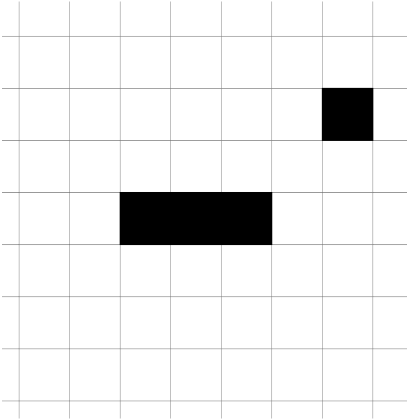

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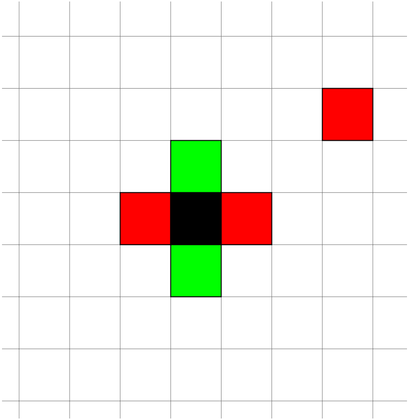

changes in first step

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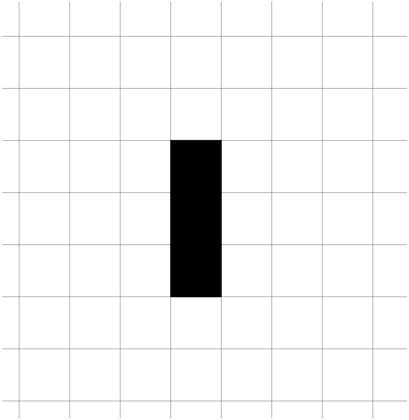

after first step

Cellular Automata – advantages and disadvantages

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Lattice Gas Automata

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- Regular arrangement of nodes
  - HPP: square lattice $\rightarrow$ does not lead to Navier-Stokes
  - FHP: triangular lattice
- 6 (7) cells per node
  - one for each direction
- for each step, do:
  - collision
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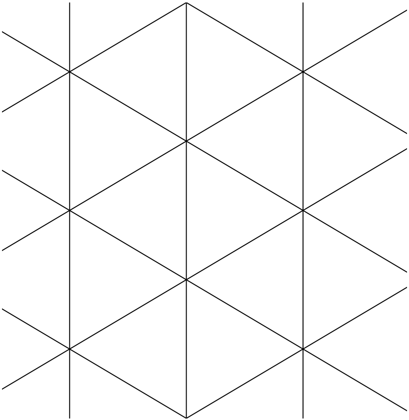

Triangular lattice

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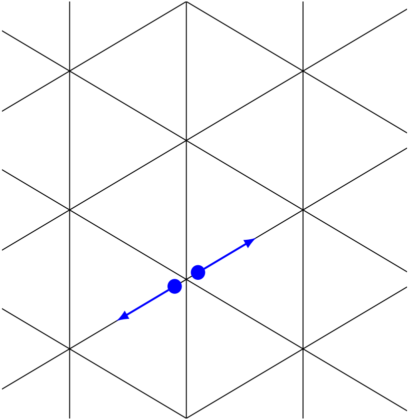

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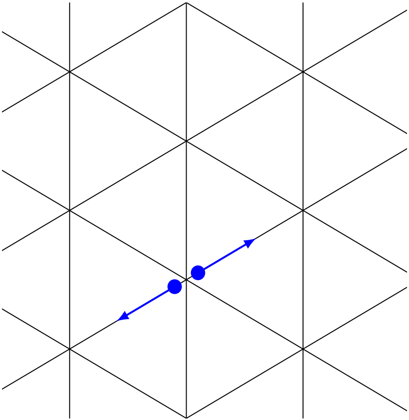

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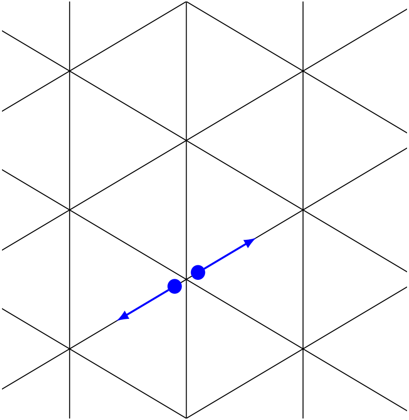

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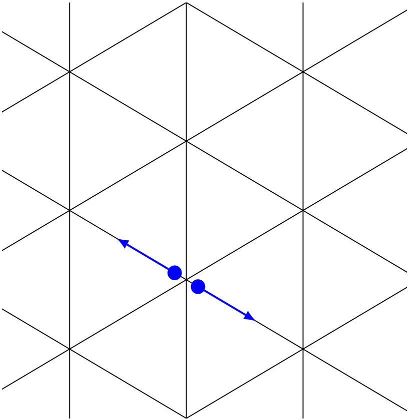

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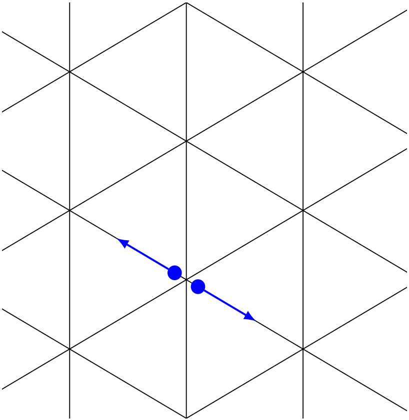

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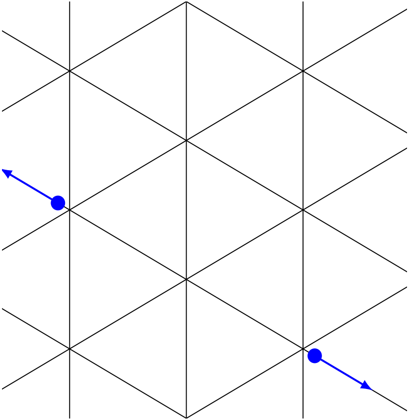

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Particle distribution function f_i:
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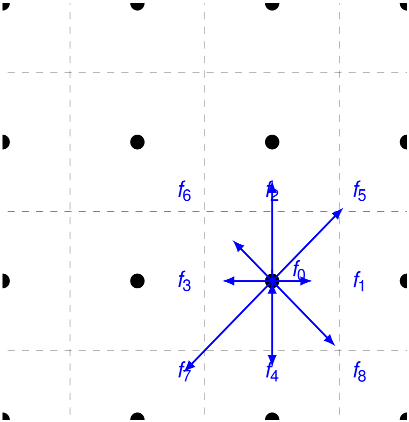

One collide step for a D2Q9-stencil

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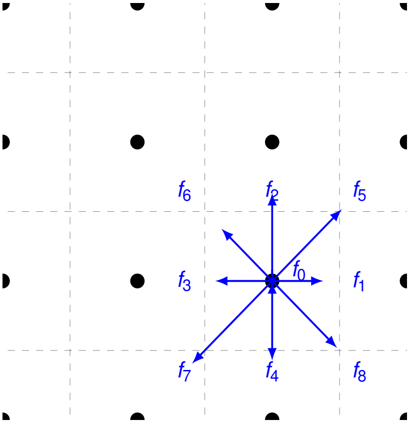

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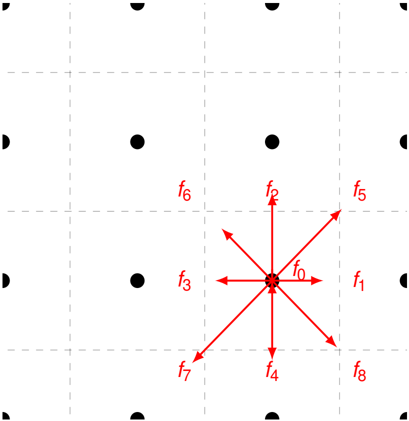

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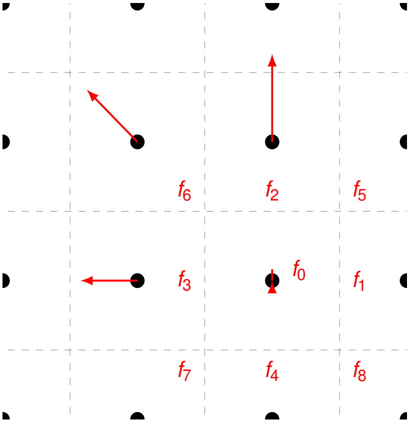

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The Lattice Boltzmann Equation

...is what we get when we apply both steps:

$$f_i(\boldsymbol{x} + \boldsymbol{c_i}\Delta t, t + \Delta t) = f_i(\boldsymbol{x}, t) + \Omega_i(\boldsymbol{x}, t)$$

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With collision operator $\Omega_i(\boldsymbol{x}, t)$, e.g.:

- BGK: $\Omega(f) = \frac{1}{\tau}(f_i^{\text{eq}} - f_i) = \omega(f_i^{\text{eq}} - f_i)$
- TRT: is often nicer but needs more formulas $\rightarrow$ please look up

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and the equilibrium particle distribution function $f_i^{\mathrm{eq}} = w_i \rho \left(1 + \frac{\vec{e}_i \vec{u}}{c_s^2} + \frac{(\vec{e}_i \vec{u})^2}{2c_s^4} - \frac{\vec{u}^2}{2c_s^2}\right)$

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$w_0 = \frac{4}{9}, w_{1,2,3,4} = \frac{1}{9}, w_{5,6,7,8} = \frac{1}{36}$ for D2Q9

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Cf. Boltzmann Equation

$$\frac{\partial f}{\partial t} + \frac{\boldsymbol{p}}{m} \cdot \nabla f = \left(\frac{\partial f}{\partial t}\right)_{coll}$$

Moments - the Interesting Outputs in LBM

Density and velocity are given locally

$$\rho(\boldsymbol{x}, t) = \sum_i f_i(\boldsymbol{x}, t)$$

$$\rho \boldsymbol{u}(\boldsymbol{x}, t) = \sum_i \boldsymbol{c_i} f_i(\boldsymbol{x}, t)$$

Nondimensionalization – setting the step size

= Normalization in the three physical dimensions time, space, mass

Usual choice:

$$\Delta x^* = 1 = C_x \cdot \Delta x$$
$$\Delta t^* = 1 = C_t \cdot \Delta t$$
$$\Delta \rho^* = 1 = C_\rho \cdot \Delta \rho$$

The kinematic velocity in lattice units is directly linked to ω or τ:

$$\nu^* = (\tau - 0.5)c_s^2 = (\frac{1}{\omega} - 0.5)c_s^2, \;\; c_s^2 = \frac{1}{3}$$

$\Rightarrow 0.5 < \tau$ ... and not much larger than 1!

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LBM – advantages and disadvantages

advantages	disadvantages
• (moderately) easy to implement • easy to parallelize • yields Navier-Stokes (by Chapman-Enskog analysis) • Galilean invariance	• needs parameter fine-tuning • explicit scheme $\rightarrow$ may need fine time resolution for stability, compared to other CFD methods

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Any questions?

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