

From Cellular Automata to the Lattice Boltzmann Method

Theresa Pollinger, Marcial Gaißert
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Outline

Cellular Automata

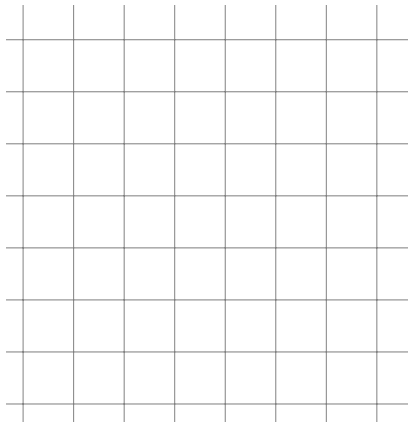
Lattice Gas Automata

The Lattice Boltzmann Method

Cellular Automata

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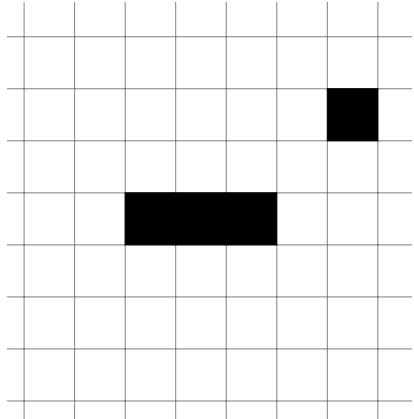
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- each with discrete state
- updated simultaneously at each step
- update rules:
 - same for each cell
 - only dependent on close-by cells



a grid of cells

Cellular Automata

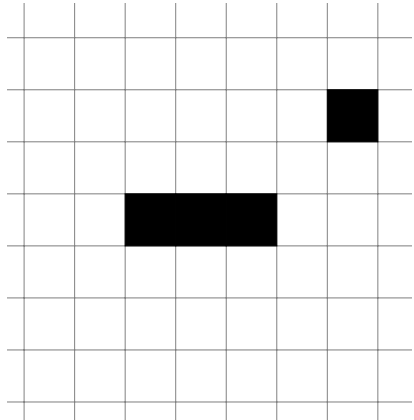
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a grid of cells with state

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	> 3	
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Rules of Conway's Game of Life

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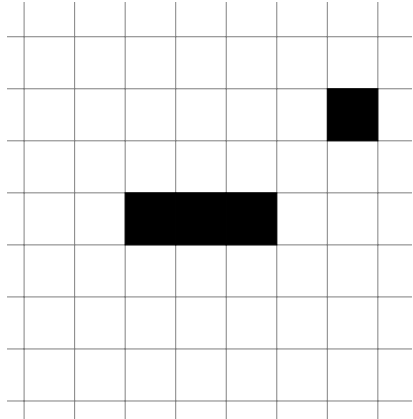
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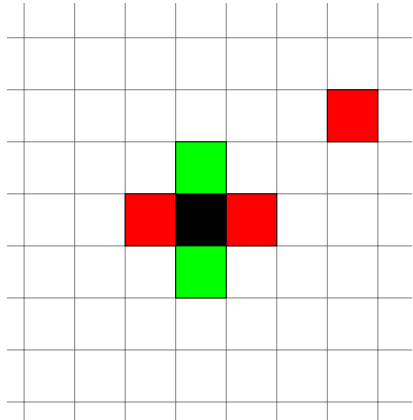
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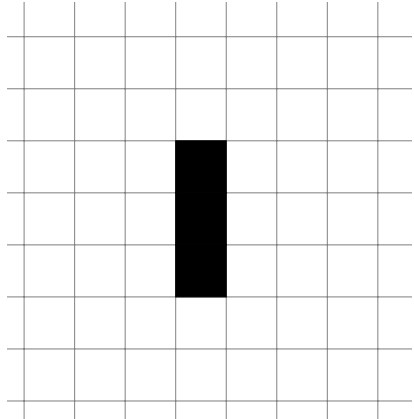
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changes in first step

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after first step

Cellular Automata – advantages and disadvantages

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- simple structure
- easy to implement
- easy to parallelize

disadvantages

- difficult to design for given macroscopic behavior

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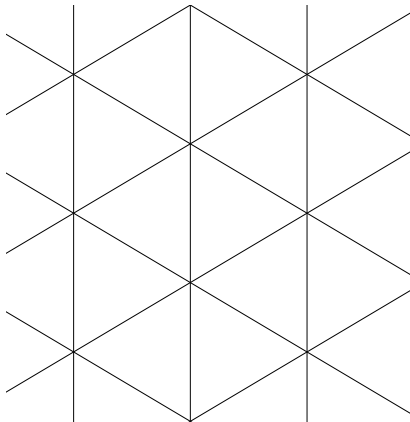
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 - HPP: square lattice
→ does not lead to Navier-Stokes
 - FHP: triangular lattice
- 6 (7) cells per node
 - one for each direction
- for each step, do:
 - collision
 - propagation

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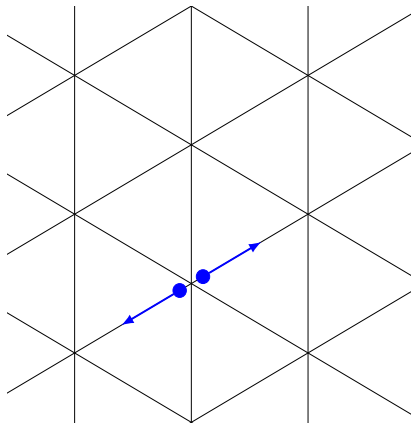
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Triangular lattice

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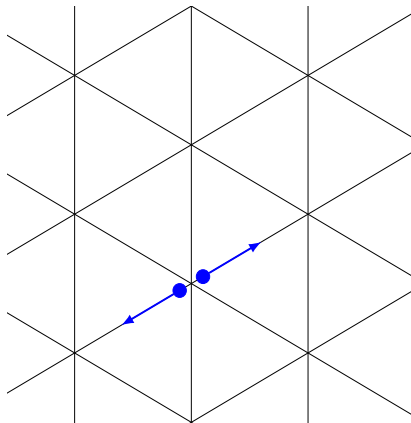
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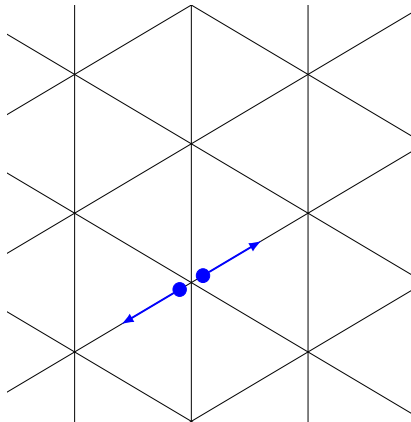
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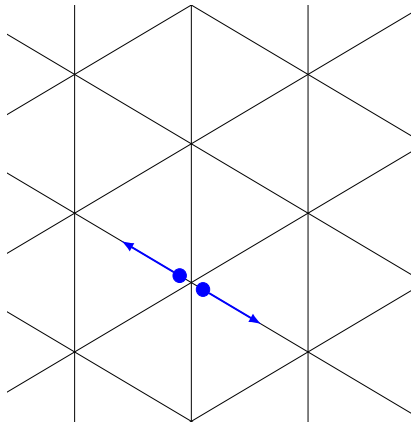
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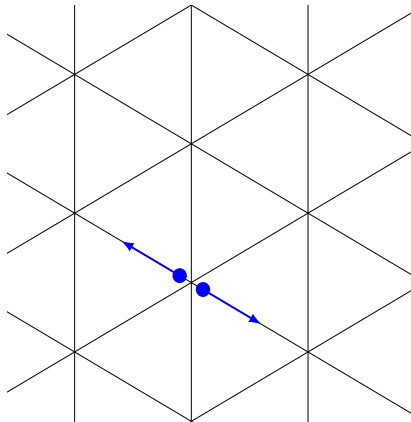
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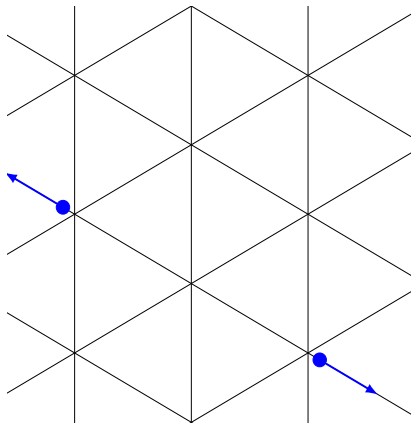
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- boolean state – no rounding errors
- easy to implement
- easy to parallelize
- FHP yields Navier-Stokes

disadvantages

- statistical noise
- lack of Galilean invariance

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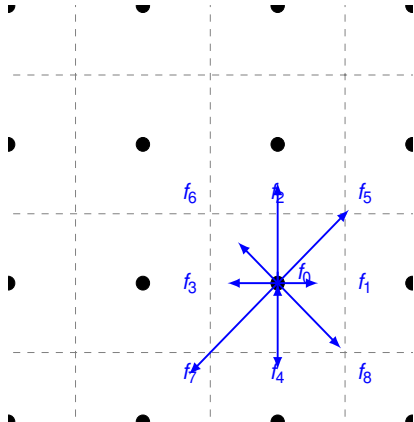
The Lattice Boltzmann Method

Collide Step

Particle distribution function f_i :
discrete in space and time,
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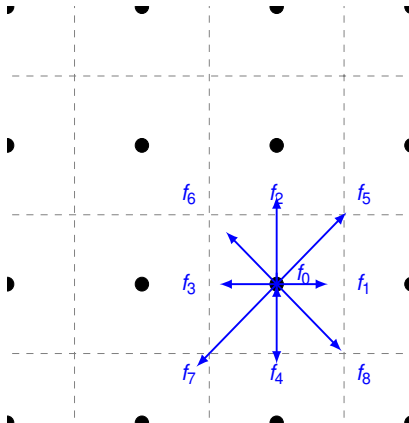
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One collide step for a D2Q9-stencil

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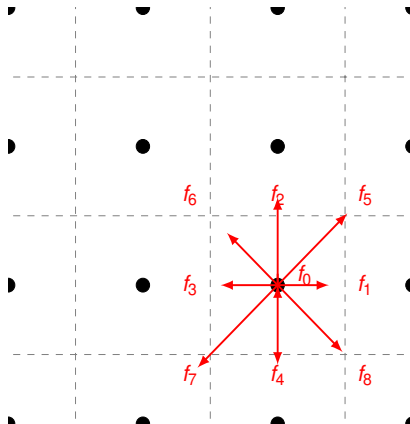
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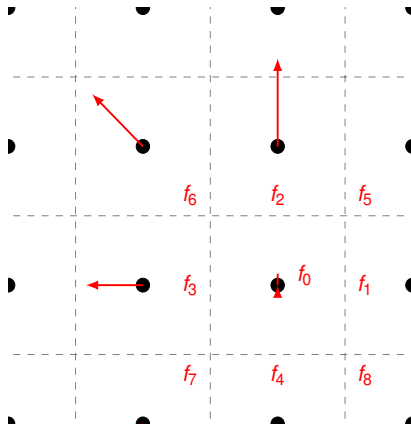
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The Lattice Boltzmann Equation

...is what we get when we apply both steps:

$$f_i(\mathbf{x} + \mathbf{c}_i \Delta t, t + \Delta t) = f_i(\mathbf{x}, t) + \Omega_i(\mathbf{x}, t)$$

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With collision operator $\Omega_i(\mathbf{x}, t)$, e.g.:

- BGK: $\Omega(f) = \frac{1}{\tau}(f_i^{\text{eq}} - f_i) = \omega(f_i^{\text{eq}} - f_i)$
- TRT: is often nicer but needs more formulas → please look up

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and the equilibrium particle distribution function $f_i^{\text{eq}} = w_i \rho \left(1 + \frac{\vec{e}_i \vec{u}}{c_s^2} + \frac{(\vec{e}_i \vec{u})^2}{2c_s^4} - \frac{\vec{u}^2}{2c_s^2} \right)$

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 $w_0 = \frac{4}{9}, w_{1,2,3,4} = \frac{1}{9}, w_{5,6,7,8} = \frac{1}{36}$ for D2Q9

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Cf. Boltzmann Equation

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla f = \left(\frac{\partial f}{\partial t} \right)_{coll}$$

Moments - the Interesting Outputs in LBM

Density and velocity are given locally

$$\rho(\mathbf{x}, t) = \sum_i f_i(\mathbf{x}, t)$$

$$\rho \mathbf{u}(\mathbf{x}, t) = \sum_i \mathbf{c}_i f_i(\mathbf{x}, t)$$

Nondimensionalization – setting the step size

= Normalization in the three physical dimensions time, space, mass

Usual choice:

$$\Delta x^* = 1 = C_x \cdot \Delta x$$

$$\Delta t^* = 1 = C_t \cdot \Delta t$$

$$\Delta \rho^* = 1 = C_\rho \cdot \Delta \rho$$

The kinematic velocity in lattice units is directly linked to ω or τ :

$$v^* = (\tau - 0.5)c_s^2 = \left(\frac{1}{\omega} - 0.5\right)c_s^2, \quad c_s^2 = \frac{1}{3}$$

$\Rightarrow 0.5 < \tau \dots$ and not much larger than 1!

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LBM – advantages and disadvantages

advantages

- (moderately) easy to implement
- easy to parallelize
- yields Navier-Stokes (by Chapman-Enskog analysis)
- Galilean invariance

disadvantages

- needs parameter fine-tuning
- explicit scheme → may need fine time resolution for stability, compared to other CFD methods

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- [1] T. Krüger, H. Kusumaatmaja, A. Kuzmin, O. Shardt, G. Silva, and E. M. Vigen, *The lattice boltzmann method: Principles and practice*. Springer, Dec. 11, 2016, 705 pp., ISBN: 978-3-319-44649-3.