

From Cellular Automata to the Lattice Boltzmann Method

BGCE Honours Project

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Outline

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waLBerla – Implementation of the Lattice Boltzmann Method

LBM – Short Overview

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Thermal LBM

Introduction

Earth Mantle Convection II

- Creeping flow, according to **weismueller.2015**

$$\left. \begin{array}{l} \mu \approx 10^{21} \text{ Pa} \cdot \text{s} \\ u \approx 3 \text{ cm/a} \\ h = 2867 \text{ km} \\ \rho \approx 4000 \text{ kg/m}^3 \end{array} \right\} Re = \frac{\rho \cdot u \cdot h}{\mu} \approx 10^{-20} \ll 1$$

- Dominant viscous forces, negligible inertial forces
- *Navier-Stokes* equations simplify to *Stokes* equations

$$\begin{aligned} \mu \Delta \mathbf{u} - \nabla p + \mathbf{f} &= 0 \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

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Derivation of a Benchmark II

- Construct a divergence-free velocity field

$$\begin{bmatrix} u \\ v \end{bmatrix} = \nabla^\perp \Phi = \begin{bmatrix} \sin(2\pi x) \cos(\pi y) \\ -2 \cos(2\pi x) \sin(\pi y) \end{bmatrix}$$

- Construct the pressure field according to the *Stokes* equations with $\mu = 1$

$$\mathbf{F} = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} -\Delta u \\ -\Delta v \end{bmatrix} + \nabla p = \begin{bmatrix} 0 \\ ? \end{bmatrix}$$

$$\Rightarrow p = \frac{5}{2} \pi \cos(2\pi x) \cos(\pi y)$$

- Calculate the force field

$$\mathbf{F} = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} 0 \\ -12.5\pi^2 \cos(2\pi x) \sin(\pi y) \end{bmatrix}$$

- Extend the domain to $\Omega = (0, 1)^3$ to obtain a pseudo-3D benchmark

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TerraNeo – Implementation of the Finite Element Method

Preparations

- Use CLOCK_MONOTONIC from the system environment for runtime measurement
- Implement “random-noise” initialization
- Implement an approximated L2 norm to ensure comparability with LBM
⇒ Verification by taking the norm of the analytical velocity field
- Calculate the accuracy based on the approximated L2 norm

$$\frac{\| \mathbf{u} - \mathbf{u}_{\text{exact}} \|}{\| \mathbf{u}_{\text{exact}} \|}$$

- Use the tolerance 10^{-12} between two iterations to identify the steady state

waLBerla – Implementation of the Lattice Boltzmann Method

LBM - very short overview

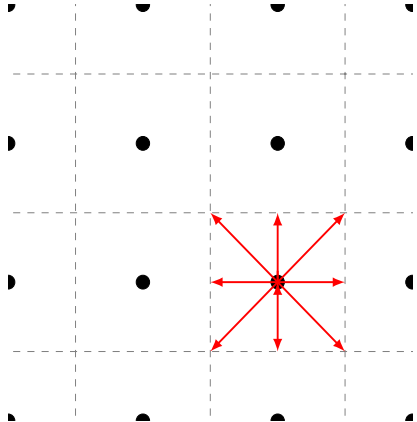
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particle distribution function

$$f_i = c_j \cdot w_j$$

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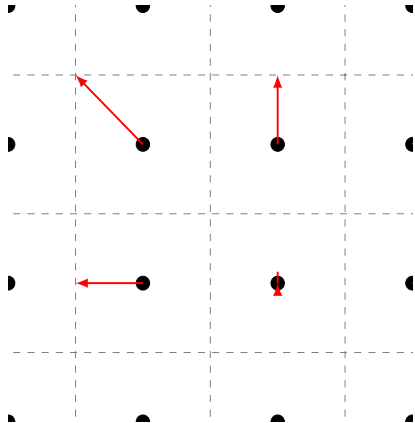


One streaming step for a D2Q9-stencil

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Collide step: application of the lattice Boltzmann equation

$$f_i(\mathbf{x} + \mathbf{c}_i \Delta t, t + \Delta t) = f_i(\mathbf{x}, t) + \Omega_i(\mathbf{x}, t)$$

Density and velocity are given locally

$$\rho(\mathbf{x}, t) = \sum_i f_i(\mathbf{x}, t)$$

$$\rho \mathbf{u}(\mathbf{x}, t) = \sum_i \mathbf{c}_i f_i(\mathbf{x}, t)$$

Source: **kruger.2016**

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Boundary Conditions : Expectations

- Free-slip boundary conditions (Neumann Boundary Conditions)
- No-slip / UBB boundary conditions (Dirichlet boundary conditions)

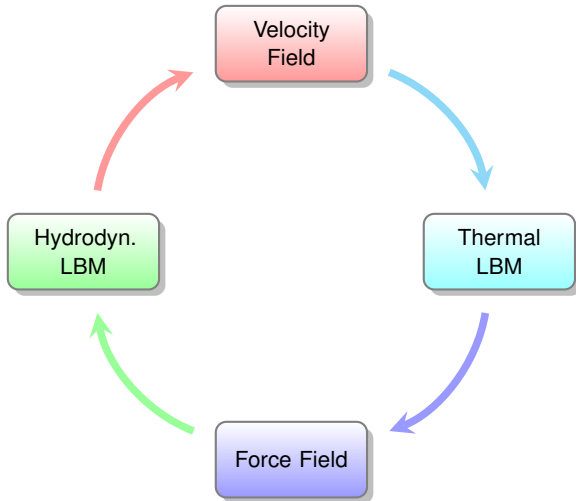
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Thermal LBM : Principle



Thanks for listening.
Any questions?

References

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