From Cellular Automata to the Lattice Boltzmann Method

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Outline

Cellular Automata

Lattice Gas Automata

The Lattice Boltzmann Method

Cellular Automata

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- regular arrangement of cells
- each with discrete state
- updated simultaneously at each step
- update rules:
  - same for each cell
  - only dependent on close-by cells

a grid of cells

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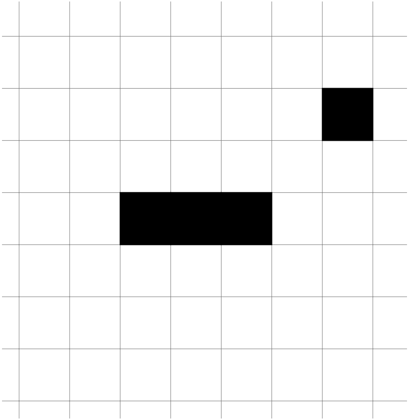

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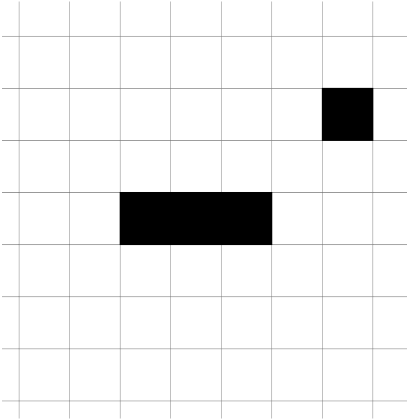

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	$2, 3$	■
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Rules of Conway's Game of Life

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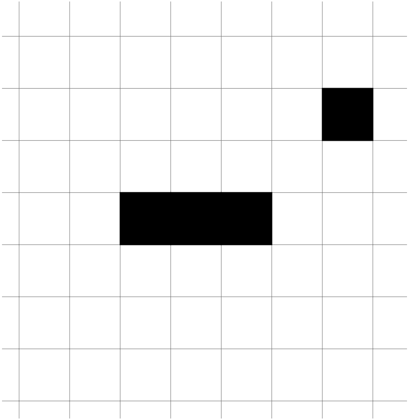

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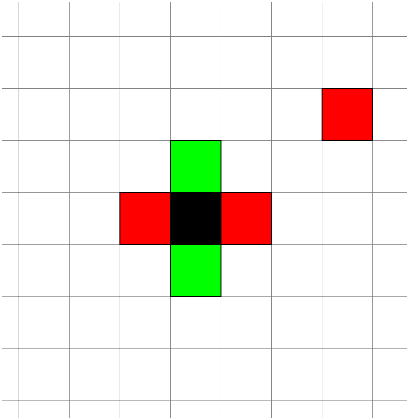

changes in first step

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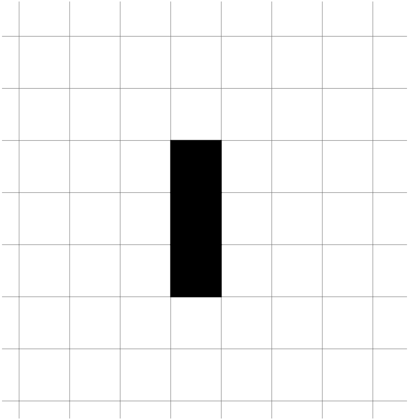

after first step

Cellular Automata – advantages and disadvantages

advantages	disadvantages
• simple structure • easy to implement • easy to parallelize	• difficult to design for given macroscopic behavior

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Lattice Gas Automata

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- Regular arrangement of nodes
  - HPP: square lattice $\rightarrow$ does not lead to Navier-Stokes
  - FHP: triangular lattice
- 6 (7) cells per node
  - one for each direction
- for each step, do:
  - collision
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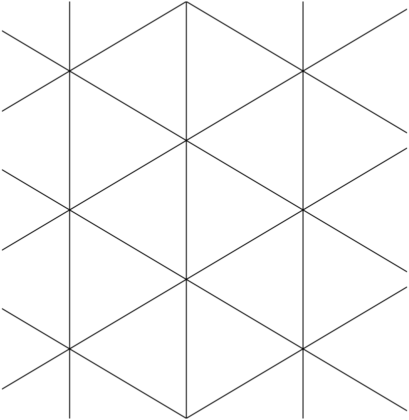

Triangular lattice

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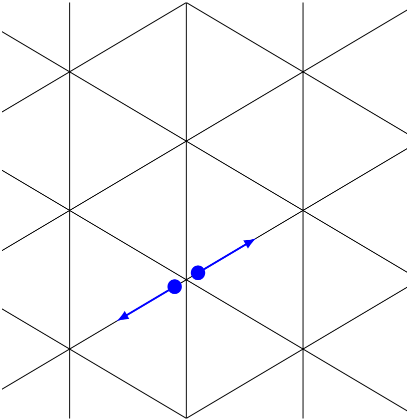

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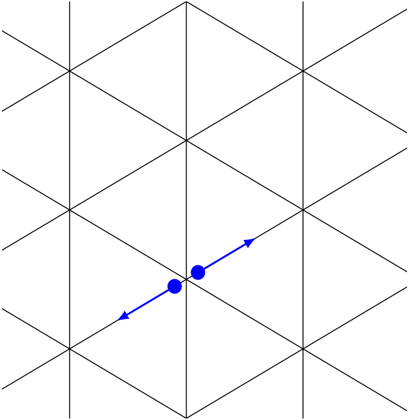

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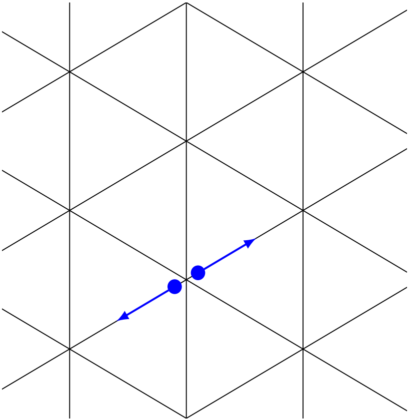

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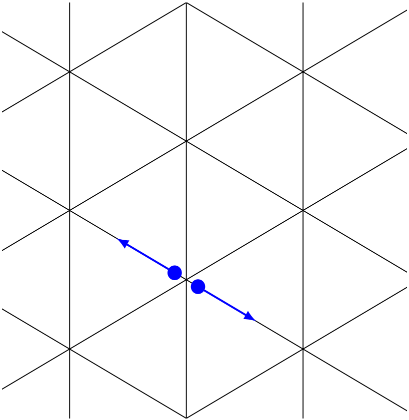

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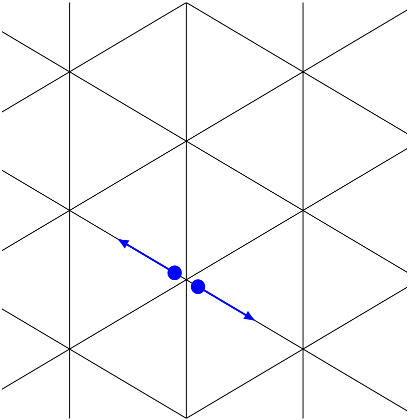

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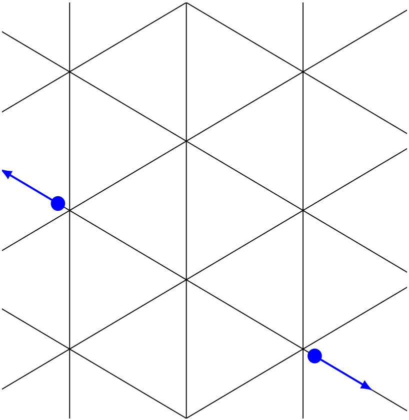

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Lattice Gas Automata – advantages and disadvantages

advantages	disadvantages
boolean state – no rounding errors	
easy to implement	statistical noise
easy to parallelize	lack of Galilean invariance
FHP yields Navier-Stokes	

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The Lattice Boltzmann Method

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Particle distribution function f_i:
discrete in space and time,
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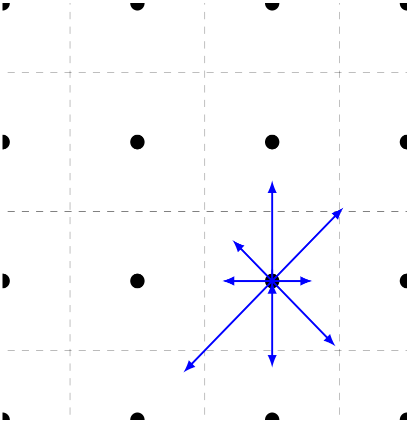
Particle distribution function f_i: discrete in space and time, continuous in probability values



One collide step for a D2Q9-stencil

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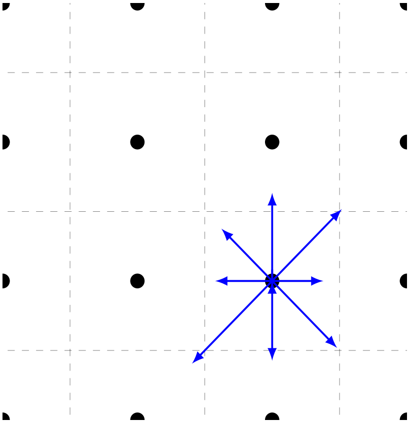

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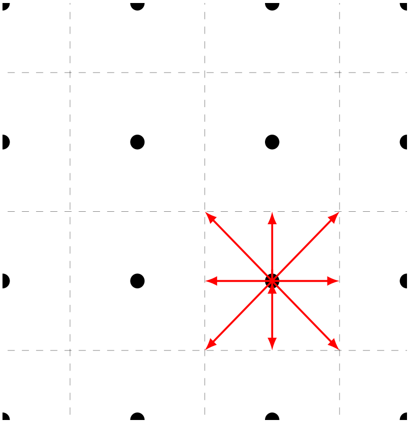

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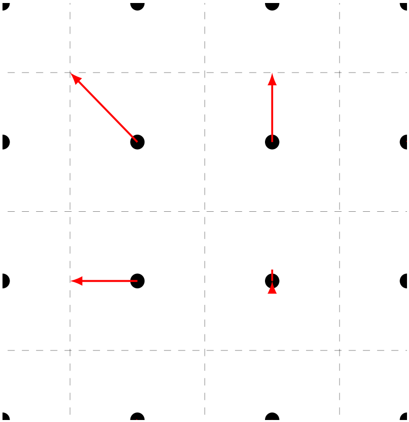

One stream step for a D2Q9-stencil

The Lattice Boltzmann Equation

...is what we get when we apply both steps:

$$f_i(\boldsymbol{x} + \boldsymbol{c_i}\Delta t, t + \Delta t) = f_i(\boldsymbol{x}, t) + \Omega_i(\boldsymbol{x}, t)$$

With collision operator $\Omega_i(\boldsymbol{x}, t)$, e.g.:

- BGK: $\Omega(f) = \frac{1}{\tau}(f_i^{eq} - f_i) = \omega(f_i^{eq} - f_i)$
- TRT:

and the equilibrium particle distribution function $f_i^{eq} =$

Cf. Boltzmann Equation

$$\frac{\partial f}{\partial t} + \frac{\boldsymbol{p}}{m} \cdot \nabla f = \left(\frac{\partial f}{\partial t}\right)_{coll}$$

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Moments - the Interesting Outputs in LBM

Density and velocity are given locally

$$\rho(\boldsymbol{x}, t) = \sum_i f_i(\boldsymbol{x}, t)$$

$$\rho \boldsymbol{u}(\boldsymbol{x}, t) = \sum_i \boldsymbol{c_i} f_i(\boldsymbol{x}, t)$$

Nondimensionalization – setting the step size

= Normalization in the three physical dimensions time, space, mass

Usual choice:

$$\Delta x^* = 1 = C_x \cdot \Delta x$$
$$\Delta t^* = 1 = C_t \cdot \Delta t$$
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The kinematic velocity in lattice units is directly linked to ω/τ:

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Thanks for listening.
Any questions?

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