From Cellular Automata to the Lattice Boltzmann Method

BGCE Honours Project

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Outline

The Lattice Boltzmann Method

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Stream Step

Discrete
particle distribution function

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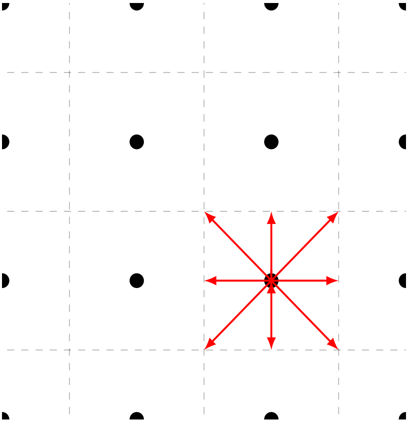

One streaming step for a D2Q9-stencil

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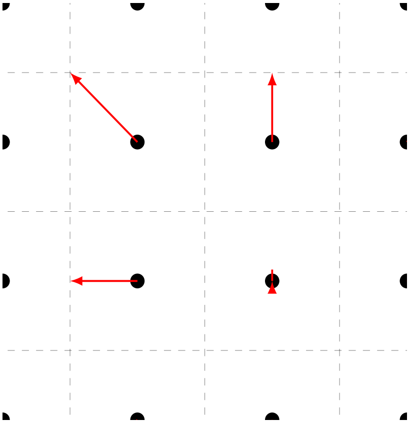

One streaming step for a D2Q9-stencil

The Lattice Boltzmann Equation

Applied as the collide step:

$$f_i(\boldsymbol{x} + \boldsymbol{c_i}\Delta t, t + \Delta t) = f_i(\boldsymbol{x}, t) + \Omega_i(\boldsymbol{x}, t)$$

Moments - the Interesting Outputs in LBM

Density and velocity are given locally

$$\rho(\boldsymbol{x}, t) = \sum_i f_i(\boldsymbol{x}, t)$$

$$\rho\,\boldsymbol{u}(\boldsymbol{x}, t) = \sum_i \boldsymbol{c_i} f_i(\boldsymbol{x}, t)$$

Source: **kruger.2016**

Earth Mantle Convection II

- Creeping flow, according to **weismueller.2015**

$$\left.\begin{aligned} \mu &\approx 10^{21}\,\mathrm{Pa}\cdot\mathrm{s} \\ u &\approx 3\,\mathrm{cm/a} \\ h &= 2867\,\mathrm{km} \\ \rho &\approx 4000\,\mathrm{kg/m^3} \end{aligned}\right\} Re = \frac{\rho \cdot u \cdot h}{\mu} \approx 10^{-20} \ll 1$$

- Dominant viscous forces, negligible inertial forces
- *Navier-Stokes* equations simplify to *Stokes* equations

$$\begin{aligned} \mu\Delta\boldsymbol{u} - \nabla p + \boldsymbol{f} &= 0 \\ \nabla\cdot\boldsymbol{u} &= 0 \end{aligned}$$

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Derivation of a Benchmark II

- Construct a divergence-free velocity field

$$\begin{bmatrix} u \\ v \end{bmatrix} = \nabla^{\perp}\Phi = \begin{bmatrix} \sin(2\pi x)\cos(\pi y) \\ -2\cos(2\pi x)\sin(\pi y) \end{bmatrix}$$

- Construct the pressure field according to the *Stokes* equations with $\mu = 1$

$$\boldsymbol{F} = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} -\Delta u \\ -\Delta v \end{bmatrix} + \nabla p = \begin{bmatrix} 0 \\ ? \end{bmatrix}$$

$$\Rightarrow p = \frac{5}{2}\pi\cos(2\pi x)\cos(\pi y)$$

- Calculate the force field

$$\boldsymbol{F} = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} 0 \\ -12.5\pi^2\cos(2\pi x)\sin(\pi y) \end{bmatrix}$$

- Extend the domain to $\Omega = (0,1)^3$ to obtain a pseudo-3D benchmark

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Preparations

- Use `CLOCK_MONOTONIC` from the system environment for runtime measurement
- Implement “random-noise” initialization
- Implement an approximated L2 norm to ensure comparability with LBM
  $\Rightarrow$ Verification by taking the norm of the analytical velocity field
- Calculate the accuracy based on the approximated L2 norm

$$\frac{\left\| \boldsymbol{u} - \boldsymbol{u}_{\text{exact}} \right\|}{\left\| \boldsymbol{u}_{\text{exact}} \right\|}$$

- Use the tolerance 10^{-12} between two iterations to identify the steady state

LBM - very short overview

Collide step: application of the lattice Boltzmann equation

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- No-slip / UBB boundary conditions (Dirichlet boundary conditions)

Both are expected to converge with order 2 in Δx

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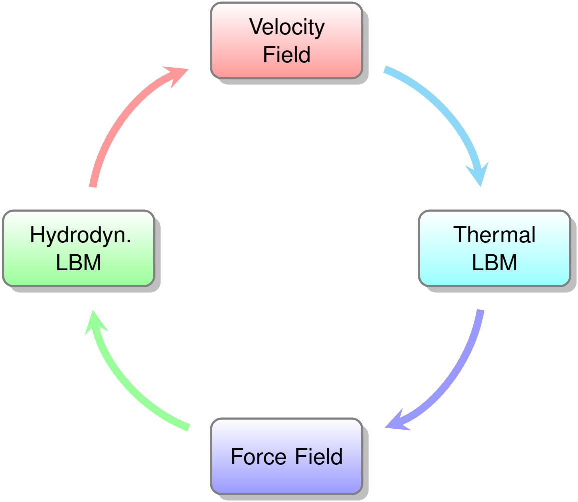
Thermal LBM : Principle



Thanks for listening.
Any questions?

References

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