Cryptography
Week #7:
Computational Complexity & Hard problems

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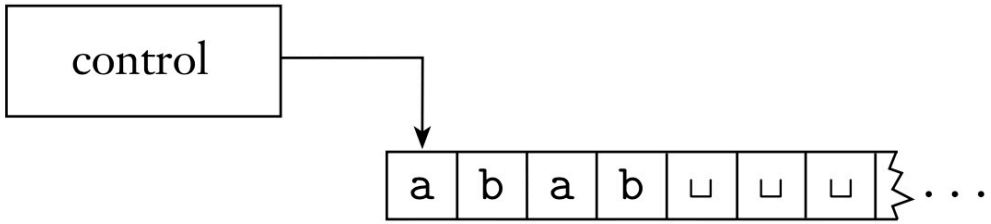
Complexity & Cryptography

Given that the model of secrecy used by modern cryptography en not compatible with “secrecy by obfuscation” its security must rely on the Computational Complexity results.

Turing Machine



1. A Turing machine can both write on the tape and read from it.
2. The read–write head can move both to the left and to the right.
3. The tape is infinite.
4. The special states for rejecting and accepting take effect immediately.

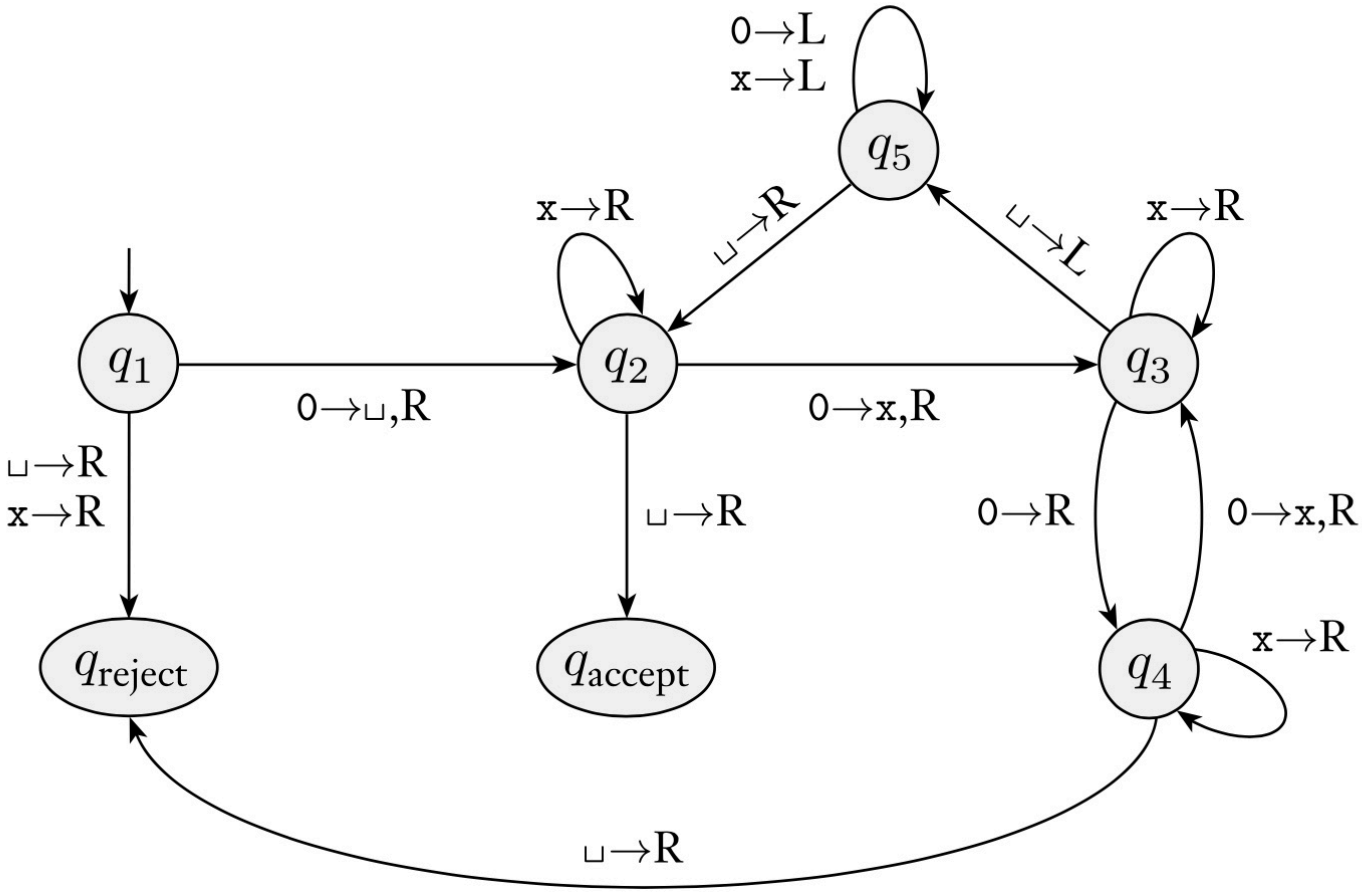
Turing-recognisable language

Call a language L **Turing-recognisable** if some Turing machine A recognises it. That is, if given a word $w \in L$ as input the TM always come to a stop giving a positive answer.

Turing-decidable language

Call a language **Turing-decidable** or simply decidable if some Turing machine decides it.

Some (actually the vast majority) languages are **not decidable** neither **not recognisable**.
The obvious example is the language of Turing Machines that halt with an empty input. (The "halting problem").



Turing machine that decides $A = \{0^{2n} | n \geq 0\}$.

Church–Turing thesis

All models of enough expressiveness are equivalent.

Computational Complexity

The computational complexity of a task is the mesure of the time necessary to accomplish the task as a function of the size of the input.

This is normally done using a Turing Machine using a single tape as reference, but (as long as they are deterministic) other models do not give results that correspond to different classes of complexity.

We will work on the assumption that the choice of the model of computation used is irrelevant in what complexity classification is concern[1].

[1] Kind of a Church-Turing thesis v2.0.

Landau notation

The "big O" notation

$$f(n) = \mathcal{O}(g(n))$$

$$\exists k > 0 \, \exists n_0 \, \forall n > n_0 : |f(n)| \leq k \, g(n)$$

When $f(n) = \mathcal{O}(g(n))$, we say that $g(n)$ is an upper bound for $f(n)$, or more precisely, that $g(n)$ is an asymptotic upper bound for $f(n)$, to emphasise that we are suppressing constant factors.

Landau notation

- $f_1(n) = 5n^3 + 2n^2 + 55 = \mathcal{O}(n^3)$
- $f_1(n) = \mathcal{O}(n^4)$
- $f_1(n) \neq \mathcal{O}(n^2)$, because, $\forall k \, \forall n_0 \, \exists n > n_0 : |f_1(n)| > k \, n^2$
- $f_2(n) = 3n \log_2 n + 5n \log_2 \log_2 n + 2 = \mathcal{O}(n \log n)$

Landau Notation

The "small o" notation

$$f(n) = o(g(n))$$

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

Complexity Theory

Traditionally Complexity Theory considers only a special kind of programs that takes an input and and returns an answer of acceptance or rejection of that input. The set of accepted words by one of these programs is called *the language defined by the program* and we say that the program *decides the language* in the sense that it decides, for each word fed as input, if it belongs to the language or not.
Although this model does not covers all the problems that are necessary to study, most of the problems may be divided in components that follow in this definition of "decision program". Thus, we may see the study of the complexity of these "decision programs" as the study of lower bounds for complexity of a more general set of programs.

Complexity Classes

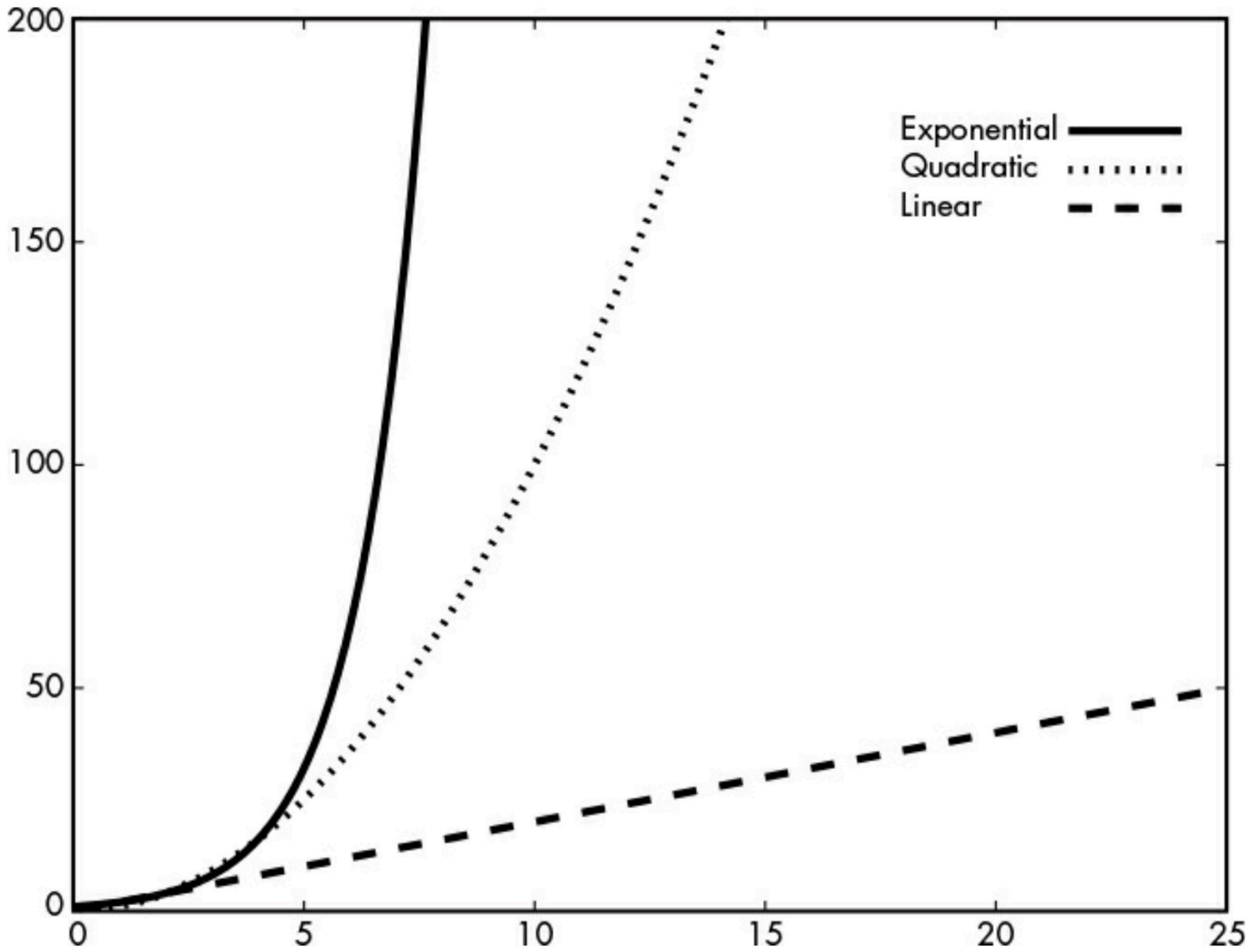
TIME

Let $f : \mathbb{N} \to \mathbb{R}^+$ be a function. Define the time complexity class, TIME($t(n)$), to be the collection of all languages that are decidable by an $\mathcal{O}(t(n))$ time Turing machine.

SPACE

Let $t : \mathbb{N} \to \mathbb{R}^+$ be a function. Define the space complexity class, SPACE($t(n)$), to be the collection of all languages that are decidable by an Turing machine using a $\mathcal{O}(t(n))$ space in their tape.

How important is the complexity in practice?



Complexity Classes

It is easy to see that if a function f is in $\mathsf{TIME}(g(n))$ then it cannot but be in $\mathsf{SPACE}(g(n))$.

$$\mathsf{TIME}(n^2) \subseteq \mathsf{SPACE}(n^2)$$

The most "important" of the complexity classes is P:

The class P

$$\mathsf{P} = \bigcup_{k \in \mathbb{N}} \mathsf{TIME}(n^k)$$

Thus P is the class of complexity of all the functions that can be decided in a polynomial time by a Turing Machine.

Examples of languages in P

RelPrime Given two integers, decide if their greatest common divider is 1, i.e. if they are coprime. The Euclides' algorithm solves the problem in time $\mathcal{O}(n)$.

PrimeP Given an integer decide if it is a prime number. Proven in 2002 (*PRIMES is in P*, Agrawal, Kayal and Saxena, 2002). The AKS algorithm runs in time $\mathcal{O}(n^{12}\log(n^{12})) = \tilde{\mathcal{O}}(n^{12})$[2]. Although this a polynomial asymptotic complexity, in practice, and for the size of numbers one wants to use it, the AKS is outperformed by many algorithms of non polynomial complexity.

Any CFL Any context-free language have a CFG that recognises it and CYK parser can thus operate in time $\mathcal{O}(n^3)$.

[2] This is the "soft-O" notation: $\tilde{\mathcal{O}}(f(n)) = \mathcal{O}(f(n)\log f(n))$.

Complexity Classes

In the same manner:

The class PSPACE

$$\text{PSPACE} = \bigcup_{k \in \mathbb{N}} \text{SPACE}(n^k)$$

PSPACE is the class of complexity of the functions that can be decided by a Turing Machine using polynomial space on its tape.

Complexity Classes

The second most "important" of the complexity classes is NP. NP does **not** stand for "non-polynomial" but for **"nondeterministic polynomial"**, that is languages that (these conditions are equivalent):

- can be decided by a nondeterministic Turing Machine in polynomial time;
- can be verified in polynomial time by a (deterministic) Turing Machine.

By "verified" one means that a Turing Machine with input $\langle w, c \rangle$ can decide if w is part of the language using c as additional information (normally called a witness). The time dependence of the machine only takes in account the size of w and not c.

Examples of languages in NP

Of course, P $\in$ NP, and thus all the previous examples are in NP.

COMPOSITES Given an integer decide if it belongs to $\{n \mid \exists p, q, (p, q < n) \wedge (n = pq)\}$. Trivially is in NP because a verifier is straightforward to write using the list of factors as witness. Although many believe that the problem is in P, we still do not know any algorithm that ensure that.

CLIQUE Given a graph G and an integer k, decide if G has a k-clique as a subgraph. As a witness its enough to give the list of the vertices of the k-clique.

SUBSET-SUM $\left\{ \langle S, t \rangle \mid S = \{x_1, \ldots, x_k\} \wedge S' \subseteq S \wedge \sum_{x \in S'} x = t \right\}$

DISCRETELOG Given p, a and n decide if there is k s.t.

$$a^k \equiv n \pmod{p}.$$

Complexity Classes

NTIME

Let $f : \mathbb{N} \to \mathbb{R}^+$ be a function. Define the time complexity class, NTIME($t(n)$), to be the collection of all languages that are decidable by an $\mathcal{O}(t(n))$ time nondeterministic Turing machine.

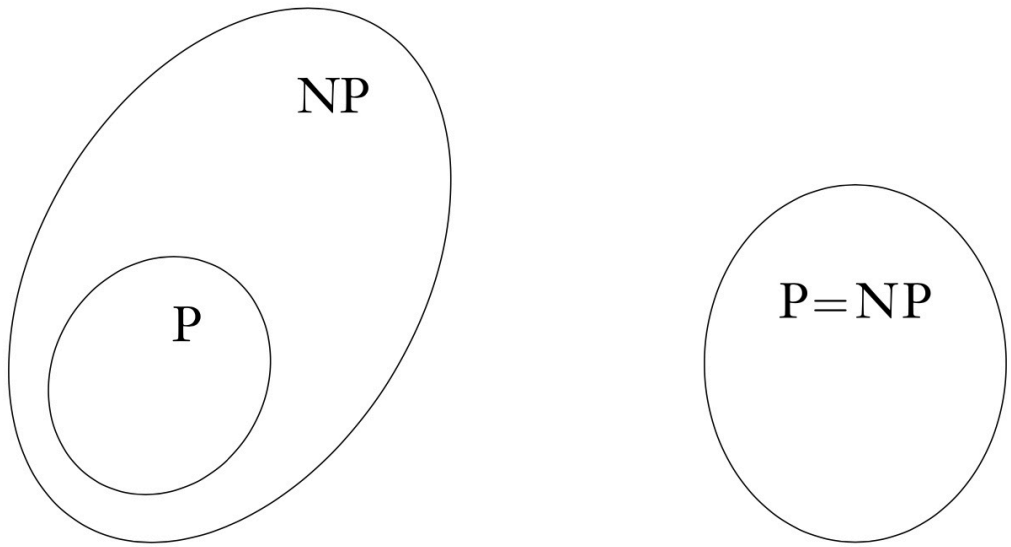
NP

$$\mathsf{NP} = \bigcup_{k \in \mathbb{N}} \mathsf{NTIME}(n^k)$$

P vs NP

P = the class of languages with quick membership decision.

NP = the class of languages with quick membership verification.



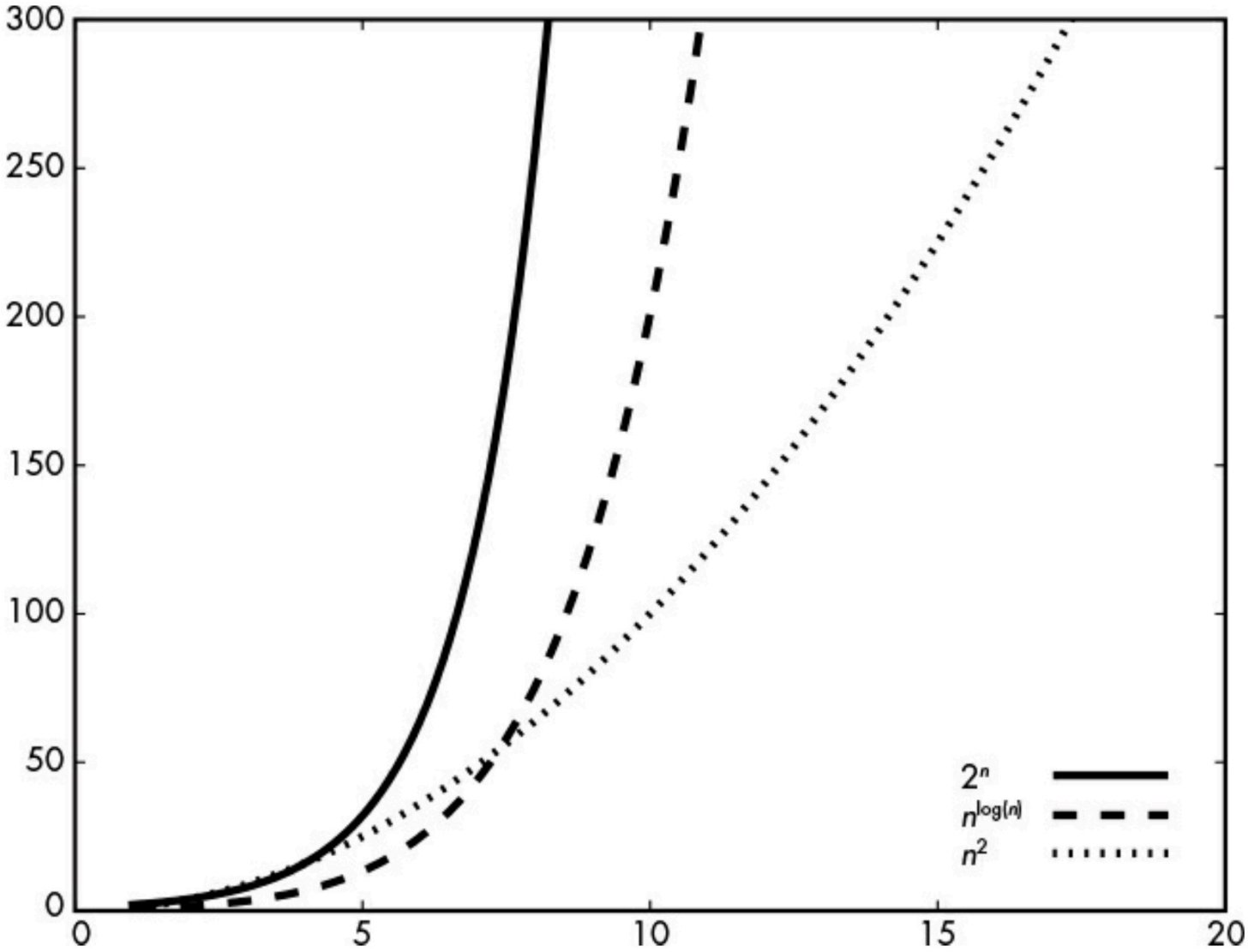
One of these is correct! Which one?

The best deterministic method currently known for deciding languages in NP uses exponential time.

EXPTIME

$$\mathsf{NP} \subseteq \mathsf{EXPTIME} = \bigcup_{k \in \mathbb{N}} \mathsf{TIME}\left(2^{n^k}\right)$$

SuperPolyTime is not ExpTime



NP-Completness

Polynomial time computable function

A function $f : \Sigma^\star \rightarrow \Sigma^\star$ is a **polynomial time computable function** if some polynomial time Turing machine M exists that halts with just $f(w)$ on its tape, when started on any input w.

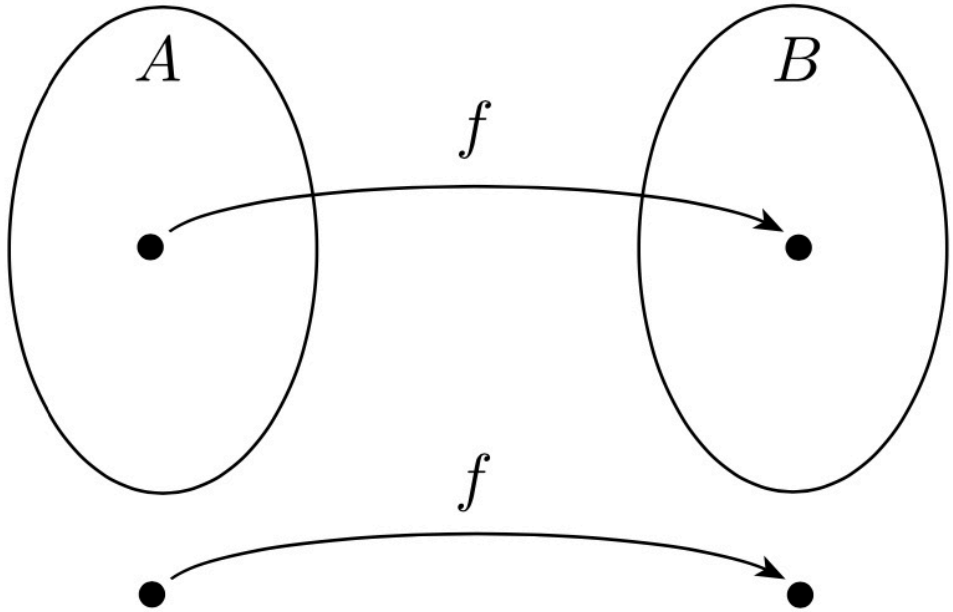
NP-Completness

Polynomial time reduction

Language A is **polynomial time reducible** to language B, written $A \leq_P B$, if a polynomial time computable function $f : \Sigma^\star \to \Sigma^\star$, where

$$\forall w \; w \in A \iff f(w) \in B.$$

The function f is called the **polynomial time reduction of A to B**.



NP-Completness

Theorem

If $A \leq_P B$ and $B \in \mathrm{P}$, then $A \in \mathrm{P}$.

NP-complete

A language B is NP-complete if it satisfies:

1. B is in NP, and
2. $\forall A \in \mathrm{NP}(A \leq_P B)$. (we say that B is NP-hard)

Theorem

If B is NP-complete and $B \in \mathrm{P}$, then $\mathrm{P} = \mathrm{NP}$.

Theorem

If B is NP-complete and $B \leq_P C$ for $C \in \mathrm{NP}$, then C is NP-complete.

NP-Completness

The set of NP-complete languages (problems) is the set of hardest languages (problems) in NP.

Examples of NP-complete languages

Traveling salesman problem Given a graph with weights labelling each edge decide if there is a path through all the vertices with a total sum not exceeding a given x.

Clique

Knapsack Given set of integers S and two additional integers x and y decide if there is a $Q \subseteq S$ such that $x \leq \sum_{q \in Q} q \leq y$.

$$\mathsf{P} \subseteq \mathsf{NP} \subseteq \mathsf{PSPACE} = \mathsf{NPSPACE} \subseteq \mathsf{EXPTIME}$$

The Factoring Problem

prime number

A number p is prime if its only positive factors are 1 and p.

Fundamental Theorem of Arithmetic

For any positive integer n there is a unique factorisation of n as a product of increasing primes.

The factoring problem consists of finding the prime numbers p and q given a large number, $N = pq$. The widely used RSA algorithms are based on the fact that factoring a number is difficult. In fact, the hardness of the factoring problem is what makes RSA encryption and signature schemes secure. This problem is indeed **hard**, yet probably **not NP-complete**.

How to factor an integer?

To factor the number n we can try to divide n by every $i \in \{2, \ldots, n-1\}$.
This will take a time $\mathcal{O}(n)$.
But we can do better...
We can try to divide n by every $i \in \{2, \ldots, \lfloor\sqrt{n}\rfloor\}$. We have reduced the time to $\mathcal{O}\left(n^{\frac{1}{2}}\right)$
Still, we can improve.
We can only try to divide n by the primes that are smaller than $\sqrt{n}$. By the prime number theorem we know that the number of primes below $\sqrt{n}$ is approximately $\frac{\sqrt{n}}{\log\sqrt{n}}$. This makes the task faster. But still we need 2^{120} operations for a 256-bit integer. Not good enough!

The fastest factoring algorithm is the **general number field sieve (GNFS)**, with an average time to operate for a number n of

$$e^{1.91 n^{\frac{1}{3}} (\log n)^{\frac{1}{3}}}.$$

Factoring a 1024-bits integer	2^{70} operations
Factoring a 2048-bits integer	2^{90} operations
Factoring a 4096-bits integer	2^{128} operations

1. In 2005, after about 18 months of computation — and thanks to the power of a cluster of 80 processors, with a total effort equivalent to 75 years of computation on a single processor—a group of researchers factored a 663-bit (200-decimal digit) number.
2. In 2009, after about two years and using several hundred processors, with a total effort equivalent to about 2000 years of computation on a single processor, another group of researchers factored a 768-bit (232-decimal digit) number.

Thus the estimates are **very optimistic** regarding the possible performance of computers and algorithms.

So we have a problem that is in NP and that looks hard, but is it as hard as the hardest NP problems? In other words, is factoring NP- complete? **Probably not.**

Factoring may then be slightly easier than NP-complete in theory, but as far as cryptography is concerned, it's hard enough, and even more reliable than NP-complete problems. Indeed, it's **easier** to build cryptosystems on top of the factoring problem than NP-complete problems, because it's hard to know exactly how hard it is to break a cryptosystem based on some NP-complete problems—in other words, how many bits of security you'd get.

On the other hand... we may have to deal with **quantum computers**...

... and if we are not careful, factoring can became easy!

To factor 17976931348623159077293051907890247336179769789423065
72734300811577393438199338429869825571741982572789172586381937
09265819186026626180659730665062710995556578639447715608415186
89565284169198292110720231716536912489048151238855803905342712
50992903154492623247093152632560831325404614070528728327909
15388014592 takes just a few seconds because
its factors are: 2^{800}, 641, 6700417, 167773885276849215533569 and
37414057161322375957408148834323969.

The Discrete Logarithm Problem

Consider the multiplicative group $\mathbb{Z}_p^*$ (with p prime). The **DLP** consists, given g and x, in finding y such that $g^y = x$ in $\mathbb{Z}_p^*$, i.e.

$$g^y \equiv x \pmod{p}.$$

In this conditions **DLP** seems as hard as the integer factoring problem.


TO BE CONTINUED...