Question 1.5.9

EE22BTECH11054 - Umair Parwez

Required to find points of contact, E_3 and F_3, of incircle with sides AC and AB respectively.

From previous questions we know the coordinates of the incircle are :

$$I = \begin{pmatrix} \frac{-53-11\sqrt{37}+7\sqrt{61}+\sqrt{2257}}{12} \\ \\ \frac{5-\sqrt{37}+5\sqrt{61}-\sqrt{2257}}{12} \end{pmatrix} \tag{1}$$

Radius of incircle is :

$$r = \frac{185 + 41\sqrt{37} - 37\sqrt{61} - \sqrt{2257}}{6\sqrt{74}} \tag{2}$$

Equation of incircle is :

$$\|x - I\|^2 = r^2 \tag{3}$$

points A, B and C are :

$$A = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, B = \begin{pmatrix} -4 \\ 6 \end{pmatrix}, C = \begin{pmatrix} -3 \\ -5 \end{pmatrix} \tag{4}$$

Parametric equation of AC is :

$$x = A + k(A - C) \tag{5}$$

Substituting (5) in (3) :

$$\|A + k(A - C) - I\|^2 = r^2 \tag{6}$$

$$(A + k(A - C) - I) \cdot (A + k(A - C) - I) = r^2 \tag{7}$$

$$k^2\|A - C\|^2 + 2k(\|A\|^2 - A^TC - I^TA + I^TC) + \|A - I\|^2 \tag{8}$$

Substituting values of A, C and I into (8) we get the following quadratic equation :

$$2k^2 + 4.545k + 2.58216 = 0 \tag{9}$$

On solving the quadratic, we find that k has only one value,

$$k = \frac{-4 - \sqrt{37} + \sqrt{61}}{2} \tag{10}$$

Substituting (10) back into (5), we get point of contact with AC,

$$E_3 = \begin{pmatrix} \frac{-2-\sqrt{37}+\sqrt{61}}{2} \\ \\ \frac{-6-\sqrt{37}+\sqrt{61}}{2} \end{pmatrix} \tag{11}$$

Now let us find the other point of contact, with AB.

Parametric equation of AB is :

$$x = A + k(A - B) \tag{12}$$

Substituting (12) in (3) :

$$\|A + k(A - B) - I\|^2 = r^2 \tag{13}$$

$$(A + k(A - B) - I) \cdot (A + k(A - B) - I) = r^2 \tag{14}$$

$$k^2\|A - B\|^2 + 2k(\|A\|^2 - A^TB - I^TA + I^TB) + \|A - I\|^2 \tag{15}$$

Substituting values of A, B and I into (15) we get the following quadratic equation :

$$74k^2 + 27.6463k + 2.58216 = 0 \tag{16}$$

On solving the quadratic, we find that k has only one value,

$$k = \frac{-37 - 4\sqrt{37} + \sqrt{2257}}{74} \tag{17}$$

Substituting (17) back into (12), we get point of contact with AB,

$$F_3 = \begin{pmatrix} \frac{-111-20\sqrt{37}+5\sqrt{2257}}{74} \\ \\ \frac{185+28\sqrt{37}-\sqrt{2257}}{74} \end{pmatrix} \tag{18}$$

Diagram is shown on next page.

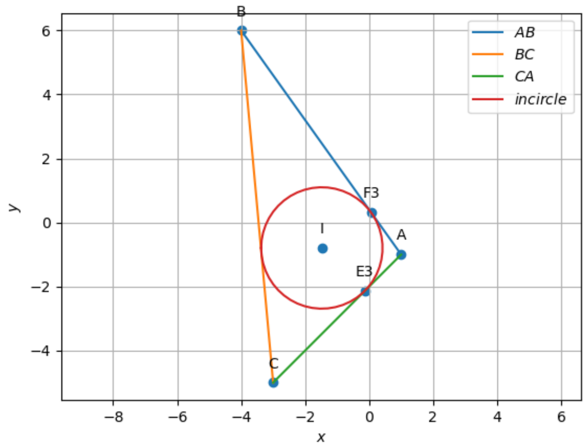

Fig. 0. Points of contact of incircle