Question 1.5.9

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Given triangle ABC with vertices,

$$\mathbf{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} \tag{1}$$

Find the points of contact, $\mathbf{E}_3$ and $\mathbf{F}_3$, of the incircle with sides AC and AB respectively.

Solution

Required to find points of contact, $\mathbf{E}_3$ and $\mathbf{F}_3$, of incircle with sides AC and AB respectively. From previous questions we know the coordinates of the incircle are :

$$\mathbf{I} = \begin{pmatrix} \frac{-53-11\sqrt{37}+7\sqrt{61}+\sqrt{2257}}{12} \\ \frac{5-\sqrt{37}+5\sqrt{61}-\sqrt{2257}}{12} \end{pmatrix} \tag{2}$$

Radius of incircle is :

$$r = \frac{185 + 41\sqrt{37} - 37\sqrt{61} - \sqrt{2257}}{6\sqrt{74}} \tag{3}$$

Equation of incircle is :

$$\|\mathbf{x} - \mathbf{I}\|^2 = r^2 \tag{4}$$

points $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are :

$$\mathbf{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} \tag{5}$$

Parametric equation of AC is :

$$\mathbf{x} = \mathbf{A} + k\mathbf{m} \tag{6}$$

where,

$$\mathbf{m} = \mathbf{A} - \mathbf{C} \tag{7}$$

Substituting (6) in (4) :

$$\|\mathbf{A} + k\mathbf{m} - \mathbf{I}\|^2 = r^2 \tag{8}$$

$$\implies (k\mathbf{m} + (\mathbf{A} - \mathbf{I}))(k\mathbf{m} + (\mathbf{A} - \mathbf{I})) = r^2 \tag{9}$$

$$\implies k^2\|\mathbf{m}\|^2 + 2k\mathbf{m}^\top(\mathbf{A} - \mathbf{I}) + \|\mathbf{A} - \mathbf{I}\|^2 = r^2 \tag{10}$$

The discriminant of the quadratic is,

$$\Delta = 4(\mathbf{m}^\top(\mathbf{A} - \mathbf{I}))^2 - 4\|\mathbf{m}\|^2(\|\mathbf{A} - \mathbf{I}\|^2 - r^2) \tag{11}$$

Upon substituting values into (11) we find that,

$$\Delta = 0 \tag{12}$$

Therefore AC is tangent to the incircle and the value of k is given as :

$$k = -\frac{\mathbf{m}^\top(\mathbf{A} - \mathbf{I})}{\|\mathbf{m}\|^2} \tag{13}$$

Similarly for AB, we can replace $\mathbf{C}$ with $\mathbf{B}$ in (11) and upon substituting values we get,

$$\Delta = 0 \tag{14}$$

Therefore we can use (13) and find the points of contact with AC and AB by substituting k back into their parametric equations.

Parametric equation of AB is :

$$x = \mathbf{A} + k\mathbf{m} \tag{15}$$

where,

$$\mathbf{m} = \mathbf{A} - \mathbf{B} \tag{16}$$

1) For finding $\mathbf{E}_3$, substitute A, C and I into (13),

$$k = -\frac{\mathbf{m}^\top(\mathbf{A} - \mathbf{I})}{\|\mathbf{m}\|^2} \tag{17}$$

$$= -\frac{(\mathbf{A} - \mathbf{C})^\top(\mathbf{A} - \mathbf{I})}{\|\mathbf{A} - \mathbf{C}\|^2} \tag{18}$$

Substituting the values, we get,

$$k = \frac{-4 - \sqrt{37} + \sqrt{61}}{2} \tag{19}$$

Substituting (19) into (6), we get the value of $\mathbf{E}_3$,

$$\mathbf{E}_3 = \begin{pmatrix} \frac{-2-\sqrt{37}+\sqrt{61}}{2} \\ \frac{-6-\sqrt{37}+\sqrt{61}}{2} \end{pmatrix} \tag{20}$$

2) For finding $\mathbf{F}_3$, substitute A, B and I into (13),

$$k = -\frac{\mathbf{m}^\top(\mathbf{A} - \mathbf{I})}{\|\mathbf{m}\|^2} \tag{21}$$

$$= -\frac{(\mathbf{A} - \mathbf{B})^\top(\mathbf{A} - \mathbf{I})}{\|\mathbf{A} - \mathbf{B}\|^2} \tag{22}$$

Substituting the values, we get,

$$k = \frac{-37 - 4\sqrt{37} + \sqrt{2257}}{74} \tag{23}$$

Substituting (23) into (15), we get the value of $\mathbf{F}_3$,

$$\mathbf{F}_3 = \begin{pmatrix} \frac{-111-20\sqrt{37}+5\sqrt{2257}}{74} \\ \frac{185+28\sqrt{37}-7\sqrt{2257}}{74} \end{pmatrix} \tag{24}$$

Diagram is shown below,

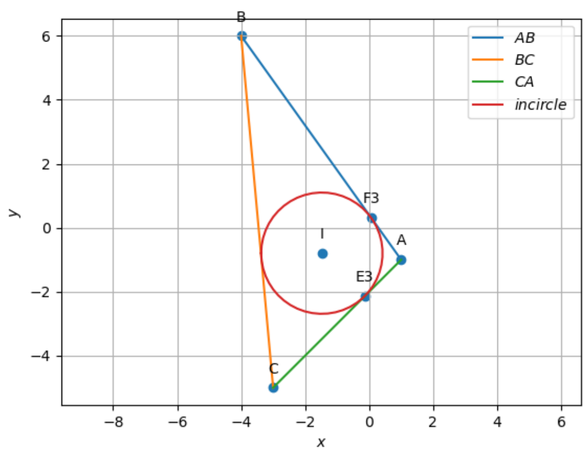

Fig. 2. Points of contact of incircle