Question 1.5.9

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Question 1.5.9:
Given triangle ABC with vertices,

$$\mathbf{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} \tag{1}$$

Find the points of contact, $\mathbf{E}_3$ and $\mathbf{F}_3$, of the incircle with sides AC and AB respectively.

Solution:
Required to find points of contact, $\mathbf{E}_3$ and $\mathbf{F}_3$, of incircle with sides AC and AB respectively. From previous questions we know the coordinates of the incircle are :

$$\mathbf{I} = \begin{pmatrix} \frac{-53-11\sqrt{37}+7\sqrt{61}+\sqrt{2257}}{12} \\ \frac{5-\sqrt{37}+5\sqrt{61}-\sqrt{2257}}{12} \end{pmatrix} \tag{2}$$

$$= \begin{pmatrix} -1.47756 \\ -0.79495 \end{pmatrix} \tag{3}$$

Radius of incircle is :

$$r = \frac{185 + 41\sqrt{37} - 37\sqrt{61} - \sqrt{2257}}{6\sqrt{74}} \tag{4}$$

$$= 1.89689 \tag{5}$$

Equation of incircle is :

$$\|\mathbf{x} - \mathbf{I}\|^2 = r^2 \tag{6}$$

points $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are :

$$\mathbf{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} \tag{7}$$

Parametric equation of any line is of the form :

$$\mathbf{x} = \mathbf{A} + k\mathbf{m} \tag{8}$$

We can substitute (8) in (6), and we get :

$$\|\mathbf{A} + k\mathbf{m} - \mathbf{I}\|^2 = r^2 \tag{9}$$

$$\implies (k\mathbf{m} + (\mathbf{A} - \mathbf{I}))(k\mathbf{m} + (\mathbf{A} - \mathbf{I})) = r^2 \tag{10}$$

$$\implies k^2\|\mathbf{m}\|^2 + 2k\mathbf{m}^\top(\mathbf{A} - \mathbf{I}) + \|\mathbf{A} - \mathbf{I}\|^2 = r^2 \tag{11}$$

The discriminant of the above quadratic is,

$$\Delta = 4(\mathbf{m}^\top(\mathbf{A} - \mathbf{I}))^2 - 4\|\mathbf{m}\|^2(\|\mathbf{A} - \mathbf{I}\|^2 - r^2) \tag{12}$$

If,

$$\Delta = 0 \tag{13}$$

Then the line given by (8) is tangent to the incircle and the value of k will be given by :

$$k = -\frac{\mathbf{m}^\top(\mathbf{A} - \mathbf{I})}{\|\mathbf{m}\|^2} \tag{14}$$

Upon substituting k back into (8), we will obtain the point of contact of the line with the incircle.

1) Finding $\mathbf{E}_3$:
   Parametric equation of AC is,

$$\mathbf{x} = \mathbf{A} + k\mathbf{m} \tag{15}$$

where,

$$\mathbf{m} = \mathbf{A} - \mathbf{C} \tag{16}$$

First we confirm value of Δ for AC,

$$\Delta = 4(\mathbf{m}^\top(\mathbf{A} - \mathbf{I}))^2 - 4\|\mathbf{m}\|^2(\|\mathbf{A} - \mathbf{I}\|^2 - r^2) \tag{17}$$

$$= 4(9.09004)^2 - 4(32)(6.18035 - 3.59819) \tag{18}$$

$$= 0 \tag{19}$$

Therefore, AC is tangent to the incircle, and value of k is given by,

$$k = -\frac{\mathbf{m}^\top(\mathbf{A} - \mathbf{I})}{\|\mathbf{m}\|^2} \tag{20}$$

$$= -\frac{(\mathbf{A} - \mathbf{C})^\top(\mathbf{A} - \mathbf{I})}{\|\mathbf{A} - \mathbf{C}\|^2} \tag{21}$$

Substituting the values, we get,

$$k = \frac{-4 - \sqrt{37} + \sqrt{61}}{2} \tag{22}$$

Substituting (22) into (15), we get the value of $\mathbf{E}_3$,

$$\mathbf{E}_3 = \begin{pmatrix} \frac{-2-\sqrt{37}+\sqrt{61}}{2} \\ \frac{-6-\sqrt{37}+\sqrt{61}}{2} \end{pmatrix} \tag{23}$$

2) Finding $\mathbf{F}_3$:
Parametric equation of AB is,

$$\mathbf{x} = \mathbf{A} + k\mathbf{m} \tag{24}$$

where,

$$\mathbf{m} = \mathbf{A} - \mathbf{B} \tag{25}$$

First we confirm value of Δ for AB,

$$\Delta = 4(\mathbf{m}^\top(\mathbf{A} - \mathbf{I}))^2 - 4\left\|\mathbf{m}\right\|^2\left(\left\|\mathbf{A} - \mathbf{I}\right\|^2 - r^2\right) \tag{26}$$

$$= 4(13.82315)^2 - 4(74)(6.18035 - 3.59819) \tag{27}$$

$$= 0 \tag{28}$$

Therefore, AB is tangent to the incircle, and value of k is given by,

$$k = -\frac{\mathbf{m}^\top(\mathbf{A} - \mathbf{I})}{\left\|\mathbf{m}\right\|^2} \tag{29}$$

$$= -\frac{(\mathbf{A} - \mathbf{B})^\top(\mathbf{A} - \mathbf{I})}{\left\|\mathbf{A} - \mathbf{B}\right\|^2} \tag{30}$$

Substituting the values, we get,

$$k = \frac{-37 - 4\sqrt{37} + \sqrt{2257}}{74} \tag{31}$$

Substituting (31) into (24), we get the value of $\mathbf{F}_3$,

$$\mathbf{F}_3 = \begin{pmatrix} \frac{-111-20\sqrt{37}+5\sqrt{2257}}{74} \\ \frac{185+28\sqrt{37}-7\sqrt{2257}}{74} \end{pmatrix} \tag{32}$$

Diagram is shown below,

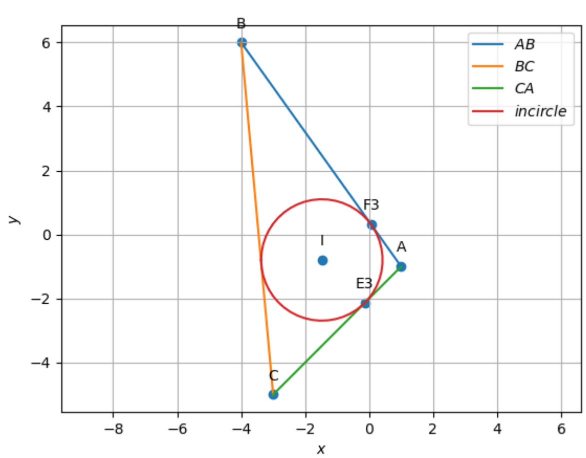

Fig. 2. Points of contact of incircle