Question 1.5.9

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Given triangle ABC with vertices,

$$\mathbf{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} \tag{1}$$

find the points of contact, $\mathbf{E}_3$ and $\mathbf{F}_3$, of the incircle with sides AC and AB respectively.

Solution

Required to find points of contact, $\mathbf{E}_3$ and $\mathbf{F}_3$, of incircle with sides AC and AB respectively.

From previous questions we know the coordinates of the incircle are :

$$\mathbf{I} = \begin{pmatrix} \frac{-53-11\sqrt{37}+7\sqrt{61}+\sqrt{2257}}{12} \\ \\ \frac{5-\sqrt{37}+5\sqrt{61}-\sqrt{2257}}{12} \end{pmatrix} \tag{2}$$

Radius of incircle is :

$$r = \frac{185 + 41\sqrt{37} - 37\sqrt{61} - \sqrt{2257}}{6\sqrt{74}} \tag{3}$$

Equation of incircle is :

$$\|\mathbf{x} - \mathbf{I}\|^2 = r^2 \tag{4}$$

points $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are :

$$\mathbf{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} \tag{5}$$

Parametric equation of AC is :

$$\mathbf{x} = \mathbf{A} + k(\mathbf{A} - \mathbf{C}) \tag{6}$$

Substituting (5) in (3) :

$$\|\mathbf{A} + k(\mathbf{A} - \mathbf{C}) - \mathbf{I}\|^2 = r^2 \tag{7}$$

$$(\mathbf{A} + k(\mathbf{A} - \mathbf{C}) - \mathbf{I}) \cdot (\mathbf{A} + k(\mathbf{A} - \mathbf{C}) - \mathbf{I}) = r^2 \tag{8}$$

$$k^2 \|\mathbf{A} - \mathbf{C}\|^2 + 2k(\|\mathbf{A}\|^2 - \mathbf{A}^T\mathbf{C} - \mathbf{I}^T\mathbf{A} + \mathbf{I}^T\mathbf{C}) + \|\mathbf{A} - \mathbf{I}\|^2 = r^2 \tag{9}$$

Since AC is tangent to the incircle, the discriminant of the obtained quadratic equation is zero and the value of k is given as :

$$k = -\frac{(\|\mathbf{A}\|^2 - \mathbf{A}^T\mathbf{C} - \mathbf{I}^T\mathbf{A} + \mathbf{I}^T\mathbf{C})}{\|\mathbf{A} - \mathbf{C}\|^2} \tag{10}$$

Upon substituting the values of $\mathbf{A}$, $\mathbf{C}$ and $\mathbf{I}$ into (10) we get,

$$k = \frac{-4 - \sqrt{37} + \sqrt{61}}{2} \tag{11}$$

Substituting (11) back into (5), we get point of contact with AC,

$$\mathbf{E}_3 = \begin{pmatrix} \frac{-2-\sqrt{37}+\sqrt{61}}{2} \\ \\ \frac{-6-\sqrt{37}+\sqrt{61}}{2} \end{pmatrix} \tag{12}$$

Now let us find the other point of contact, with AB.

Parametric equation of AB is :

$$x = \mathbf{A} + k(\mathbf{A} - \mathbf{B}) \tag{13}$$

We can get the value of k by replacing $\mathbf{C}$ with $\mathbf{B}$ in Eq(10). Upon substituting the values, we get,

$$k = \frac{-37 - 4\sqrt{37} + \sqrt{2257}}{74} \tag{14}$$

Substituting (14) back into (13), we get point of contact with AB,

$$\mathbf{F}_3 = \begin{pmatrix} \frac{-111-20\sqrt{37}+5\sqrt{2257}}{74} \\ \\ \frac{185+28\sqrt{37}-\sqrt{2257}}{74} \end{pmatrix} \tag{15}$$

Diagram is shown on next page.

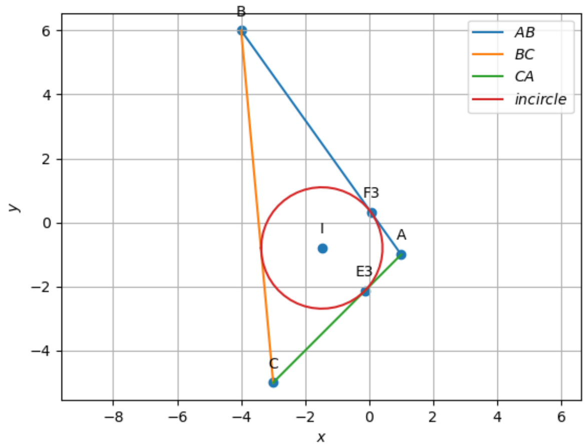

Fig. 0. Points of contact of incircle