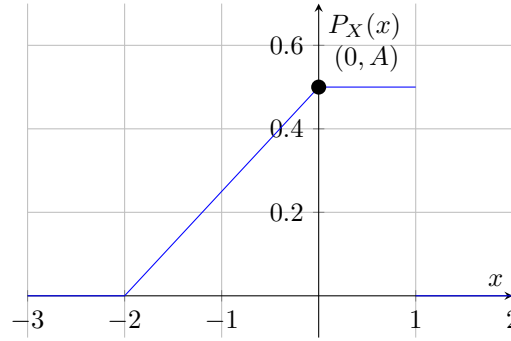


Gate 64.2022

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Consider a real valued source whose samples are independent and identically distributed random variables with the probability density function, $P_X(x)$, as shown in the figure.

Probability Density Function of X



Consider a 1 bit quantizer that maps positive samples to value α and others to value β . If α^* and β^* are the respective choices for α and β that minimize the mean square quantization error, then find $(\alpha^* - \beta^*)$

Solution: First we must find value of A. Area under the PDF graph should be equal to 1. Thus, we can say,

$$\left(\frac{1}{2} \cdot A \cdot 2\right) + (1 \cdot A) = 1 \quad (1)$$

$$A = 0.5 \quad (2)$$

Using the value of A, we can write $P_X(x)$ as a piecewise function given by,

$$p_X(x) = \begin{cases} 0.25x + 0.5 & -2 \leq x \leq 0 \\ 0.5 & 0 \leq x \leq 1 \end{cases} \quad (3)$$

Now, if X_q be the output of the quantizer, then,

$$X_q = \begin{cases} \alpha & 0 \leq x \leq 1 \\ \beta & -2 \leq x \leq 0 \end{cases} \quad (4)$$

Now, the quantization error, Q_e , is given by,

$$Q_e = X - X_q \quad (5)$$

Mean square quantization error,

$$MSQE = E(Q_e^2) \quad (6)$$

$$= E((X - X_q)^2) \quad (7)$$

$$= \int_{-\infty}^{\infty} (x - X_q)^2 \cdot P_X(x) \cdot dx \quad (8)$$

$$= \int_{-2}^0 (x - \beta)^2 \cdot P_X(x) \cdot dx + \int_0^1 (x - \alpha)^2 \cdot P_X(x) \cdot dx \quad (9)$$

$$= \int_{-2}^0 (x^2 + \beta^2 - 2x\beta) \cdot (0.25x + 0.5) \cdot dx \quad (10)$$

$$+ \int_0^1 (x - \alpha)^2 \cdot 0.5 \cdot dx \quad (11)$$

$$= \frac{\beta^2}{2} + \frac{2\beta}{3} - \frac{1}{3} + \frac{1}{6}[(1 - \alpha)^3 + \alpha^3] \quad (12)$$

$$= \left(\frac{\beta^2}{2} + \frac{2\beta}{3}\right) + \frac{1}{2}(\alpha^2 - \alpha) - \frac{1}{6} \quad (13)$$

$$= \frac{\alpha^2}{2} + \frac{\beta^2}{2} - \frac{\alpha}{2} + \frac{2\beta}{3} - \frac{1}{6} \quad (14)$$

Now, we can write MSQE as,

$$f(\mathbf{v}) = \mathbf{v}^\top \mathbf{A} \mathbf{v} + 2\mathbf{B} \mathbf{v} - \frac{1}{6} \quad (15)$$

Where,

$$\mathbf{v} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}; A = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}; B = \begin{bmatrix} -\frac{1}{4} & \frac{1}{3} \end{bmatrix} \quad (16)$$

Now, to minimize $MSQE$,

$$\frac{d}{d\mathbf{v}} f(\mathbf{v}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (17)$$

$$(A + A^\top) \mathbf{v} + 2\mathbf{B}^\top = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (18)$$

$$(19)$$

Since A is a symmetric matrix, $A^\top = A$. Therefore,

$$2A\mathbf{v} = -2B^\top \quad (20)$$

$$\mathbf{v} = -A^{-1}B^\top \quad (21)$$

$$= -\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ \frac{4}{3} \end{bmatrix} \quad (22)$$

$$= \begin{bmatrix} \frac{1}{2} \\ \frac{-2}{3} \end{bmatrix} \quad (23)$$

Now, to find $(\alpha^* - \beta^*)$,

$$(\alpha^* - \beta^*) = \begin{bmatrix} 1 & -1 \end{bmatrix} \mathbf{v} \quad (24)$$

$$= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{-2}{3} \end{bmatrix} \quad (25)$$

$$= \frac{7}{6} \quad (26)$$

Shown below are the PDF and CDF of a simulation of X , where X was simulated by taking inverse of CDF of a uniform distribution in the range $(0, 1)$

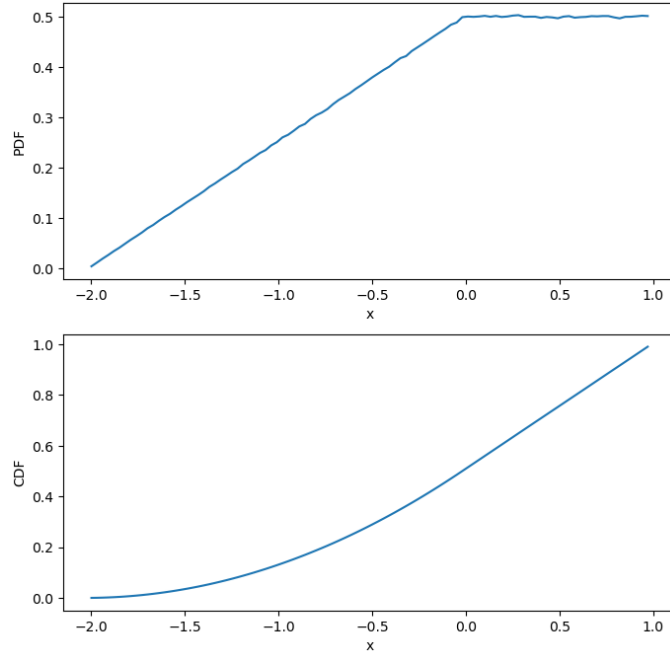


Figure 1:

Simulation steps:

1. Create an empty array, X , of a desired size n .
2. For every i in the range $[0, n)$, assign $X[i]$ to be $F_X^{-1}(u)$, where $F_X(x)$ is the CDF of X , and u is a random sample belonging to uniform distribution in the range $(0, 1)$.
3. Write all values stored in X onto a text file, say "vals.txt".
4. Using python, plot the PDF and CDF of the values stored in "vals.txt".