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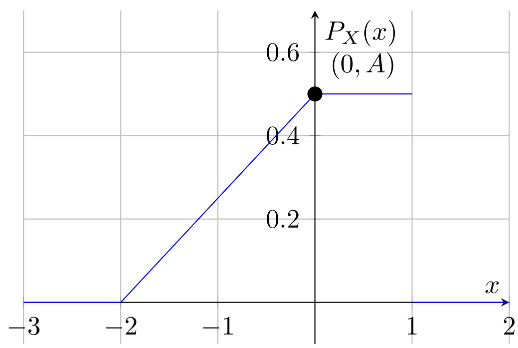
Umair Parwez
EE22BTECH11009

Consider a real valued source whose samples are independent and identically distributed random variables with the probability density function, $P_X(x)$, as shown in the figure.

Probability Density Function of X



Consider a 1 bit quantizer that maps positive samples to value α and others to value β. If α^* and β^* are the respective choices for α and β that minimize the mean square quantization error, then find $(\alpha^* - \beta^*)$

Solution: First we must find value of A. Area under the PDF graph should be equal to 1. Thus, we can say,

$$\left(\frac{1}{2} \cdot A \cdot 2\right) + (1 \cdot A) = 1 \tag{1}$$

$$A = 0.5 \tag{2}$$

Using the value of A, we can write $P_X(x)$ as a piecwise function given by,

$$p_X(x) = \begin{cases} 0.25x + 0.5 & -2 \leq x \leq 0 \\ 0.5 & 0 \leq x \leq 1 \end{cases} \tag{3}$$

Now, if X_q be the output of the quantizer, then,

$$X_q = \begin{cases} \alpha & 0 \leq x \leq 1 \\ \beta & -2 \leq x \leq 0 \end{cases} \tag{4}$$

Now, the quantization error, Q_e, is given by,

$$Q_e = X - X_q \tag{5}$$

Mean square quantization error,

$$\begin{align}
MSQE &= E(Q_e^2) \tag{6}\\
&= E((X - X_q)^2) \tag{7}\\
&= \int_{-\infty}^{\infty} (x - X_q)^2 \cdot P_X(x) \cdot dx \tag{8}\\
&= \int_{-2}^{0} (x - \beta)^2 \cdot P_X(x) \cdot dx + \int_{0}^{1} (x - \alpha)^2 \cdot P_X(x) \cdot dx \tag{9}\\
&= \int_{-2}^{0} (x^2 + \beta^2 - 2x\beta) \cdot (0.25x + 0.5) \cdot dx \tag{10}\\
&\quad + \int_{0}^{1} (x - \alpha)^2 \cdot 0.5 \cdot dx \tag{11}\\
&= \frac{\beta^2}{2} + \frac{2\beta}{3} - \frac{1}{3} + \frac{1}{6}[(1-\alpha)^3 + \alpha^3] \tag{12}\\
&= (\frac{\beta^2}{2} + \frac{2\beta}{3}) + \frac{1}{2}(\alpha^2 - \alpha) - \frac{1}{6} \tag{13}\\
&= \frac{\alpha^2}{2} + \frac{\beta^2}{2} - \frac{\alpha}{2} + \frac{2\beta}{3} - \frac{1}{6} \tag{14}
\end{align}$$

Now, we can write MSQE as,

$$f(\mathbf{v}) = \mathbf{v}^\intercal \mathbf{A} \mathbf{v} + 2\mathbf{B}\mathbf{v} - \frac{1}{6} \tag{15}$$

Where,

$$\mathbf{v} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}; A = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}; B = \begin{bmatrix} \frac{-1}{4} & \frac{1}{3} \end{bmatrix} \tag{16}$$

Now, to minimize $MSQE$,

$$\begin{align}
\frac{d}{d\mathbf{v}} f(\mathbf{v}) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{17}\\
(A + A^\intercal)\mathbf{v} + 2B^\intercal &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{18}\\
& \tag{19}
\end{align}$$

Since A is a symmetric matrix, $A^\intercal = A$. Therefore,

$$\begin{aligned} 2A\mathbf{v} &= -2B^\intercal && (20) \\ \mathbf{v} &= -A^{-1}B^\intercal && (21) \\ &= -\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{-1}{4} \\ \frac{1}{3} \end{bmatrix} && (22) \\ &= \begin{bmatrix} \frac{1}{2} \\ \frac{-2}{3} \end{bmatrix} && (23) \end{aligned}$$

Now, to find $(\alpha^* - \beta^*)$,

$$\begin{aligned} (\alpha^* - \beta^*) &= \begin{bmatrix} 1 & -1 \end{bmatrix} \mathbf{v} && (24) \\ &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{-2}{3} \end{bmatrix} && (25) \\ &= \frac{7}{6} && (26) \end{aligned}$$

Shown below are the PDF and CDF of a simulation of X, where X was simulated by taking inverse of CDF of a uniform distribution in the range $(0, 1)$

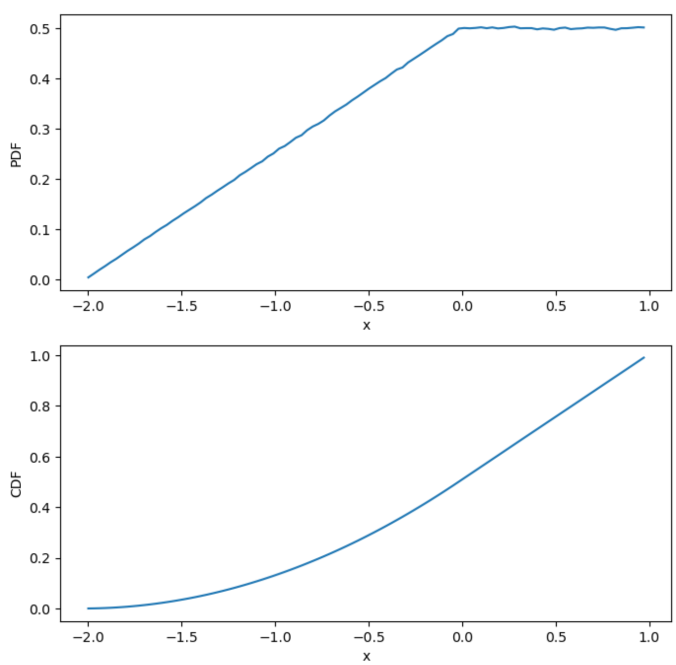


Figure 1:

Simulation steps:

1. Define a function that generates a sample u belonging to uniform distribution between $[0, 1]$, and returns $F_X^{-1}(u)$, where,

$$F_X^{-1}(x) = \begin{cases} 2\sqrt{2x} - 2 0 \leq x \leq 0.5 \\ 2x - 1 0.5 \leq x \leq 1 \end{cases} \tag{27}$$

   is the inverse of the CDF of X

2. Create an empty array, $Xvals$, of a desired size n.

3. For every i in the range $[0, n)$, assign $Xvals[i]$ to be $F_X^{-1}(u)$ using the function defined earlier.

4. Now, the array $Xvals$ is a collection of n samples having same PDF as the desired random variable X.

5. Write all the values stored in $Xvals$ onto a file say "vals.txt"

6. In python, using the numpy histograms function, plot the PDF and CDF of the values stored in "vals.txt".