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EE22BTECH11009

Consider a real valued source whose samples are independent and identically distributed random variables with the probability density function, $f(x)$, as shown in the figure.

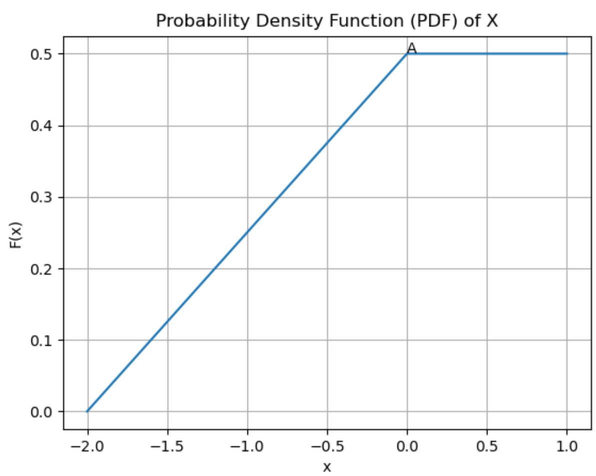

Figure 1:

Consider a 1 bit quantizer that maps positive samples to value α and others to value β. If α^* and β^* are the respective choices for α and β that minimize the mean square quantization error, then find $(\alpha^* - \beta^*)$

Solution: First we must find value of A. Area under the PDF graph should be equal to 1. Thus, we can say,

$$(\frac{1}{2} \cdot A \cdot 2) + (1 \cdot A) = 1 \tag{1}$$
$$A = 0.5 \tag{2}$$

Using the value of A, we can write $f_X(x)$ as a piecwise function given by,

$$f_X(x) = \begin{cases} 0.25x + 0.5 & -2 \le x \le 0 \\ 0.5 & 0 \le x \le 1 \end{cases} \tag{3}$$

Now, if X_q be the output of the quantizer, then,

$$X_q = \begin{cases} \alpha & 0 \le x \le 1 \\ \beta & -2 \le x \le 0 \end{cases} \tag{4}$$

Mean square quantization error,

$$\begin{aligned}
MSQ(Q_e) &= E(Q_e^2) && (5)\\
&= E((X - X_q)^2) && (6)\\
&= \int_{-\infty}^{\infty} (x - X_q)^2 \cdot f_X(x) \cdot dx && (7)\\
&= \int_{-2}^{0} (x - \beta)^2 \cdot f_X(x) \cdot dx + \int_{0}^{1} (x - \alpha)^2 \cdot f_X(x) \cdot dx && (8)\\
&= \int_{-2}^{0} (x^2 + \beta^2 - 2x\beta) \cdot (0.25x + 0.5) \cdot dx && (9)\\
&\quad + \int_{0}^{1} (x - \alpha)^2 \cdot 0.5 \cdot dx && (10)\\
&= \frac{\beta^2}{2} + \frac{2\beta}{3} - \frac{1}{3} + \frac{1}{6}[(1 - \alpha)^3 + \alpha^3] && (11)
\end{aligned}$$

Now, to minimize $MSQ(Q_e)$,

$$\frac{\partial}{\partial \beta} \cdot MSQ(Q_e) = 0 \tag{12}$$

$$\frac{\partial}{\partial \alpha} \cdot MSQ(Q_e) = 0 \tag{13}$$

1. First we will consider (12),

$$\begin{aligned}
\frac{\partial}{\partial \beta} \cdot MSQ(Q_e) &= 0 && (14)\\
\beta + \frac{2}{3} &= 0 && (15)\\
\beta &= -\frac{2}{3} && (16)
\end{aligned}$$

2. Now we will consider (13),

$$\frac{\partial}{\partial \alpha} \cdot MSQ(Q_e) = 0 \tag{17}$$

$$\frac{1}{6}[3(1-\alpha)^2(-1) + 3\alpha^2] = 0 \tag{18}$$

$$\alpha = \frac{1}{2} \tag{19}$$

Therefore the values of α and β for which the mean square error is minimum are,

$$\alpha^* = \frac{1}{2} \tag{20}$$

$$\beta^* = -\frac{2}{3} \tag{21}$$

Thus,

$$(\alpha^* - \beta^*) = \frac{7}{6} \tag{22}$$