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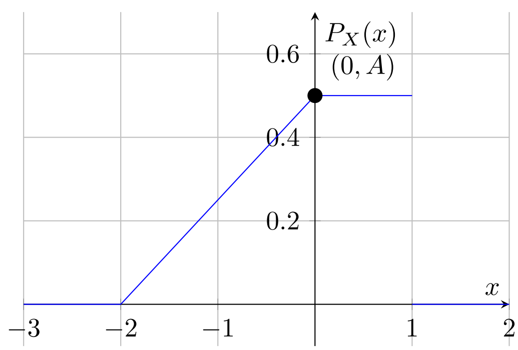
EE22BTECH11009

Consider a real valued source whose samples are independent and identically distributed random variables with the probability density function, $P_X(x)$, as shown in the figure.

Probability Density Function of X



Consider a 1 bit quantizer that maps positive samples to value α and others to value β. If α^* and β^* are the respective choices for α and β that minimize the mean square quantization error, then find $(\alpha^* - \beta^*)$

Solution: First we must find value of A. Area under the PDF graph should be equal to 1. Thus, we can say,

$$(\frac{1}{2} \cdot A \cdot 2) + (1 \cdot A) = 1 \tag{1}$$

$$A = 0.5 \tag{2}$$

Using the value of A, we can write $P_X(x)$ as a piecwise function given by,

$$P_X(x) = \begin{cases} 0.25x + 0.5 & -2 \leq x \leq 0 \\ 0.5 & 0 \leq x \leq 1 \end{cases} \tag{3}$$

Now, if X_q be the output of the quantizer, then,

$$X_q = \begin{cases} \alpha & 0 \leq x \leq 1 \\ \beta & -2 \leq x \leq 0 \end{cases} \tag{4}$$

Now, the quantization error, Q_e, is given by,

$$Q_e = X - X_q \tag{5}$$

Mean square quantization error,

$$\begin{aligned}
MSQE &= E(Q_e^2) && (6)\\
&= E((X - X_q)^2) && (7)\\
&= \int_{-\infty}^{\infty} (x - X_q)^2 \cdot P_X(x) \cdot dx && (8)\\
&= \int_{-2}^{0} (x - \beta)^2 \cdot P_X(x) \cdot dx + \int_{0}^{1} (x - \alpha)^2 \cdot P_X(x) \cdot dx && (9)\\
&= \int_{-2}^{0} (x^2 + \beta^2 - 2x\beta) \cdot (0.25x + 0.5) \cdot dx && (10)\\
&\quad + \int_{0}^{1} (x - \alpha)^2 \cdot 0.5 \cdot dx && (11)\\
&= \frac{\beta^2}{2} + \frac{2\beta}{3} - \frac{1}{3} + \frac{1}{6}[(1 - \alpha)^3 + \alpha^3] && (12)\\
&= (\frac{\beta^2}{2} + \frac{2\beta}{3}) + \frac{1}{2}(\alpha^2 - \alpha) - \frac{1}{6} && (13)\\
&= f(\alpha, \beta) && (14)
\end{aligned}$$

Now, to minimize $MSQE$,

$$\nabla f(\alpha, \beta) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{15}$$

$$\begin{bmatrix} \frac{\partial f(\alpha,\beta)}{\partial \alpha} \\ \frac{\partial f(\alpha,\beta)}{\partial \beta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{16}$$

$$\begin{bmatrix} \alpha - \frac{1}{2} \\ \beta + \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{17}$$

Therefore the values of α and β for which the mean square error is minimum are,

$$\alpha^* = \frac{1}{2} \tag{18}$$

$$\beta^* = -\frac{2}{3} \tag{19}$$

Thus,

$$(\alpha^* - \beta^*) = \frac{7}{6} \tag{20}$$