

## Abstract

**Categories and Subject Descriptors** CR-number [subcategory]:  
third-level

**Keywords** gradual types, Clojure  
Tobin-Hochstadt and Felleisen [1, 2]

## References

- [1] S. Tobin-Hochstadt and M. Felleisen. The design and implementation of typed scheme. *SIGPLAN Not.*, 43(1):395–406, Jan. 2008. ISSN 0362-1340. URL <http://doi.acm.org/10.1145/1328897.1328486>.
- [2] S. Tobin-Hochstadt and M. Felleisen. Logical types for untyped languages. *SIGPLAN Not.*, 45(9):117–128, Sept. 2010. ISSN 0362-1340. URL <http://doi.acm.org/10.1145/1932681.1863561>.

$v ::= \#f \mid \#t$	Values
$e ::= x \mid v \mid (e \ e) \mid \lambda x^\tau. e \mid (\text{if } e \ e \ e) \mid (\text{let } (x \ e) \ e) \mid (\text{begin } e \ e) \mid (\text{cons } e \ e)$	Expressions
$\mathcal{R} ::= \{\omega \mid \psi\}$	Refinement Types
$\sigma, \tau ::= \#t \mid \#f \mid \mathbf{N} \mid (\bigcup \vec{\mathcal{R}}) \mid x : \mathcal{R} \rightarrow \mathcal{R} \mid \langle \mathcal{R}, \mathcal{R} \rangle$	Types
$\kappa ::= \omega \mid x$	Object Variables
$\psi ::= \tau_{\pi(\kappa)} \mid \bar{\tau}_{\pi(\kappa)} \mid \psi \supset \psi \mid \psi \wedge \psi \mid \psi \vee \psi \mid \mathbb{t} \mathbb{t} \mid \mathbb{f} \mathbb{f} \mid \pi(\kappa) \equiv \pi(\kappa)$	Propositions
$o ::= \pi(\kappa) \mid \emptyset$	Objects
$\pi ::= \vec{p}\vec{e}$	Paths
$pe ::= \mathbf{car} \mid \mathbf{cdr}$	Path Elements
$\Gamma ::= \vec{\psi}$	Proposition Environment

**Figure 1.** Syntax of Terms, Types, Propositions, and Objects

$$\begin{aligned}
\top &= \{\omega \mid \mathbb{t} \mathbb{t}\} \\
\perp &= \{\omega \mid \mathbb{f} \mathbb{f}\} \\
\mathbf{B} &= (\bigcup \{\omega \mid \#t_\omega\} \{\omega \mid \#f_\omega\})
\end{aligned}$$

**Figure 2.** Refinement Type Abbreviations

$\text{T-VAR} \quad \frac{\omega \notin \text{fv}(\Gamma) \quad \Gamma \vdash \tau_x}{\Gamma \vdash x : \{\omega \mid \tau_\omega \wedge \omega \equiv x\}}$	$\text{T-TRUE} \quad \Gamma \vdash \#t : \{\omega \mid \#t_\omega\}$	$\text{T-FALSE} \quad \Gamma \vdash \#f : \{\omega \mid \#f_\omega\}$	$\text{T-IF} \quad \frac{\omega_1 \notin \text{fv}(\Gamma) \quad \Gamma \vdash e_1 : \{\omega_1 \mid \psi_1\} \quad \Gamma, \#f_{\omega_1}, \psi_1 \vdash e_2 : \mathcal{R} \quad \Gamma, \#f_{\omega_1}, \psi_1 \vdash e_3 : \mathcal{R}}{\Gamma \vdash (\text{if } e_1 \ e_2 \ e_3) : \mathcal{R}}$
$\text{T-APP} \quad \frac{\omega \notin \text{fv}(\Gamma) \quad \mathcal{R} = \{\omega \mid \psi\} \quad \Gamma \vdash e : \{\omega_0 \mid (x : \mathcal{R} \rightarrow \mathcal{R}')_{\omega_0}\} \quad \Gamma \vdash e' : \{\omega \mid \psi\}}{\Gamma \vdash (e \ e') : \mathcal{R}[\omega/x]}$	$\text{T-SUBSUME} \quad \frac{\Gamma \vdash e : \mathcal{R} \quad \Gamma \vdash \mathcal{R} <: \mathcal{R}'}{\Gamma \vdash e : \mathcal{R}'}$	$\text{T-LET} \quad \frac{x, \omega_1 \notin \text{fv}(\Gamma) \quad \Gamma \vdash e_1 : \{\omega_1 \mid \psi_1\} \quad \Gamma, \psi_1[\omega_1/x] \vdash e : \mathcal{R}}{\Gamma \vdash (\text{let } (x \ e_1) \ e) : \mathcal{R}}$	$\text{T-CONS} \quad \frac{\Gamma \vdash e_1 : \mathcal{R}_1 \quad \Gamma \vdash e_2 : \mathcal{R}_2}{\Gamma \vdash (\text{cons } e_1 \ e_2) : \langle \mathcal{R}_1, \mathcal{R}_2 \rangle}$
$\text{T-CAR} \quad \frac{\omega, \omega_1 \notin \text{fv}(\Gamma) \quad \Gamma \vdash e : \{\omega_1 \mid \psi_1\} \quad \psi_1 \vdash \langle \top, \top \rangle_{\omega_1}}{\Gamma \vdash (\text{car } e) : \{\omega \mid \psi_1 \wedge \mathbf{car}(\omega_1) \equiv \omega\}}$	$\text{T-CDR} \quad \frac{\omega, \omega_1 \notin \text{fv}(\Gamma) \quad \Gamma \vdash e : \{\omega_1 \mid \psi_1\} \quad \psi_1 \vdash \langle \top, \top \rangle_{\omega_1}}{\Gamma \vdash (\text{cdr } e) : \{\omega \mid \psi_1 \wedge \mathbf{cdr}(\omega_1) \equiv \omega\}}$	$\text{T-ABS} \quad \frac{x \notin \text{fv}(\Gamma) \quad \mathcal{R} = \{\omega \mid \psi\} \quad \Gamma, \psi[x/\omega] \vdash e : \mathcal{R}'}{\Gamma \vdash \lambda x^\mathcal{R}. e : x : \mathcal{R} \rightarrow \mathcal{R}'}$	

**Figure 3.** Typing Rules

$\text{S-REFL} \quad \Gamma \vdash \tau <: \tau$	$\text{S-PAIR} \quad \frac{\Gamma \vdash \mathcal{R}_1 \sqsubseteq \mathcal{R}'_1 \quad \Gamma \vdash \mathcal{R}_2 \sqsubseteq \mathcal{R}'_2}{\Gamma \vdash \langle \mathcal{R}_1, \mathcal{R}_2 \rangle <: \langle \mathcal{R}'_1, \mathcal{R}'_2 \rangle}$	$\text{S-FUN} \quad \frac{x \notin \text{fv}(\Gamma) \quad \Gamma \vdash \mathcal{R}'_1 \sqsubseteq \mathcal{R}_1 \quad \Gamma \vdash \mathcal{R}_2 \sqsubseteq \mathcal{R}'_2}{\Gamma \vdash x : \mathcal{R}_1 \rightarrow \mathcal{R}_2 <: x : \mathcal{R}'_1 \rightarrow \mathcal{R}'_2}$
$\text{S-UNIONSUPER} \quad \frac{\exists i. \Gamma \vdash \tau <: \sigma_i}{\Gamma \vdash \tau <: (\bigcup \vec{\sigma}^i)}$	$\text{S-UNIONSUB} \quad \frac{\Gamma \vdash \tau_i <: \sigma^i}{\Gamma \vdash (\bigcup \vec{\tau}^i) <: \sigma}$	
$\text{SR-REFINE} \quad \frac{\omega \notin \text{fv}(\Gamma) \quad \Gamma \vdash \psi_1 \supset \psi_2}{\Gamma \vdash \{\omega \mid \psi_1\} \sqsubseteq \{\omega \mid \psi_2\}}$		

**Figure 4.** Subtyping rules

<b>L-ATOM</b> $\frac{\psi \in \Gamma}{\Gamma \vdash \psi}$	<b>L-TRUE</b> $\frac{}{\Gamma \vdash \text{tt}}$	<b>L-FALSE</b> $\frac{}{\Gamma \vdash \text{ff}}$	<b>L-ANDI</b> $\frac{\Gamma \vdash \psi_1 \quad \Gamma \vdash \psi_2}{\Gamma \vdash \psi_1 \wedge \psi_2}$	<b>L-ANDE</b> $\frac{\Gamma, \psi_1 \vdash \psi \text{ or } \Gamma, \psi_2 \vdash \psi}{\Gamma, \psi_1 \wedge \psi_2 \vdash \psi}$	<b>L-IMPLI</b> $\frac{\Gamma, \psi_1 \vdash \psi_2}{\Gamma \vdash \psi_1 \supset \psi_2}$	<b>L-IMPLE</b> $\frac{\Gamma \vdash \psi_1 \quad \Gamma \vdash \psi_1 \supset \psi_2}{\Gamma \vdash \psi_2}$
<b>L-ORI</b> $\frac{\Gamma \vdash \psi_1 \text{ or } \Gamma \vdash \psi_2}{\Gamma \vdash \psi_1 \vee \psi_2}$	<b>L-ORE</b> $\frac{\Gamma, \psi_1 \vdash \psi \quad \Gamma, \psi_2 \vdash \psi}{\Gamma, \psi_1 \wedge \psi_2 \vdash \psi}$	<b>L-SUB</b> $\frac{\Gamma \vdash \tau_{\pi(\kappa)} \quad \vdash \tau <: \sigma}{\Gamma \vdash \sigma_{\pi(\kappa)}}$	<b>L-SUBNOT</b> $\frac{\Gamma \vdash \bar{\sigma}_{\pi(\kappa)} \quad \vdash \tau <: \sigma}{\Gamma \vdash \bar{\tau}_{\pi(\kappa)}}$	<b>L-ALIAS</b> $\frac{\Gamma \vdash \psi \quad \Gamma \vdash \pi(\kappa_1) \equiv \kappa_2 \text{ or } \Gamma \vdash \kappa_2 \equiv \pi(\kappa_1)}{\Gamma \vdash \psi[\pi(\kappa_1)/\kappa_2]}$	<b>L-BOT</b> $\frac{\Gamma \vdash \perp_{\pi(\kappa)}}{\Gamma \vdash \psi}$	<b>L-UPDATE</b> $\frac{\Gamma \vdash \tau_{\pi'(\kappa)} \quad \Gamma \vdash \nu_{\pi(\pi'(\kappa))}}{\Gamma \vdash \text{update}(\Gamma, \tau, \nu, \pi)_{\pi'(\kappa)}}$

**Figure 5.** Proof System

$\Gamma[o/\kappa]$	$=$	$\overrightarrow{\psi[o/\kappa]}$	$\Gamma = \overrightarrow{\psi}$
$\psi_+   \psi_-[o/\kappa]$	$=$	$\psi_+[o/\kappa]   \psi_-[o/\kappa]$	
$\nu_{\pi(\kappa)}[\pi'(\kappa')/\kappa]$	$=$	$(\nu[\pi(\kappa')/\kappa])_{\pi(\pi'(\kappa'))}$	
$\nu_{\pi(\kappa)}[\emptyset/\kappa]_+$	$=$	$\text{tt}$	
$\nu_{\pi(\kappa)}[\emptyset/\kappa]_-$	$=$	$\text{ff}$	
$\nu_{\pi(\kappa)}[o/\kappa''']_+$	$=$	$\nu_{\pi(\kappa)}$	$\kappa \neq \kappa''' \text{ and } \kappa''' \notin \text{fv}(\nu)$
$\nu_{\pi(\kappa)}[o/\kappa''']_-$	$=$	$\text{tt}$	$\kappa \neq \kappa''' \text{ and } \kappa''' \in \text{fv}(\nu)$
$\text{tt}[o/\kappa]$	$=$	$\text{tt}$	$\kappa \neq \kappa''' \text{ and } \kappa''' \in \text{fv}(\nu)$
$\text{ff}[o/\kappa]$	$=$	$\text{ff}$	
$(\psi_1 \supset \psi_2)[o/\kappa]_+$	$=$	$\psi_1[o/\kappa]_- \supset \psi_2[o/\kappa]_+$	
$(\psi_1 \supset \psi_2)[o/\kappa]_-$	$=$	$\psi_1[o/\kappa]_+ \supset \psi_2[o/\kappa]_-$	
$(\psi_1 \vee \psi_2)[o/\kappa]$	$=$	$\psi_1[o/\kappa] \vee \psi_2[o/\kappa]$	
$(\psi_1 \wedge \psi_2)[o/\kappa]$	$=$	$\psi_1[o/\kappa] \wedge \psi_2[o/\kappa]$	
$(\pi'(\kappa') \equiv \pi''(y'))[\pi(\kappa''')/\kappa]$	$=$	$\pi'(\kappa')[\pi(\kappa''')/\kappa] \equiv \pi''(y')[\pi(\kappa''')/\kappa]$	
$(\pi(\kappa'') \equiv \pi'(\kappa'''))[\emptyset/\kappa]$	$=$	$\pi(\kappa'') \equiv \pi'(\kappa''')$	$\kappa \neq \kappa'' \text{ and } \kappa \neq \kappa'''$
$(\pi(\kappa) \equiv \pi'(\kappa''))[\emptyset/\kappa''']$	$=$	$\text{tt}$	
$\pi(\kappa)[\pi'(\kappa'')/\kappa]$	$=$	$\pi(\pi'(\kappa''))$	
$\pi(\kappa)[\emptyset/\kappa]$	$=$	$\emptyset$	
$\pi(\kappa)[o/\kappa''']$	$=$	$\pi(\kappa)$	$\kappa \neq \kappa'''$
$\emptyset[o/\kappa]$	$=$	$\emptyset$	
$\{\omega \mid \psi\}[o/\kappa]$	$=$	$\{\omega \mid \psi[o/\kappa]\}$	$\kappa \neq \omega$
$\{\omega \mid \psi\}[o/\kappa]$	$=$	$\{\omega \mid \psi\}$	
$(x:\sigma \rightarrow \tau)[o/\kappa]$	$=$	$x:\sigma[o/\kappa] \rightarrow \tau[o/\kappa]$	$x \neq \kappa$
$(x:\sigma \rightarrow \tau)[o/\kappa]$	$=$	$x:\sigma[o/\kappa] \rightarrow \tau$	

**Figure 6.** Substitution

$\delta_\tau(\text{number?})$	$=$	$x:\mathbb{T} \rightarrow \{\omega \mid \mathbf{B}_\omega \wedge \overline{\#\mathbf{f}}_\omega \supset \mathbf{N}_x \wedge \#\mathbf{f}_\omega \supset \overline{\mathbf{N}}_x\}$
$\delta_\tau(\text{boolean?})$	$=$	$x:\mathbb{T} \rightarrow \{\omega \mid \mathbf{B}_\omega \wedge \overline{\#\mathbf{f}}_\omega \supset \mathbf{B}_x \wedge \#\mathbf{f}_\omega \supset \overline{\mathbf{B}}_x\}$
$\delta_\tau(\text{cons?})$	$=$	$x:\mathbb{T} \rightarrow \{\omega \mid \mathbf{B}_\omega \wedge \overline{\#\mathbf{f}}_\omega \supset \langle \mathbb{T}, \mathbb{T} \rangle_x \wedge \#\mathbf{f}_\omega \supset \overline{\langle \mathbb{T}, \mathbb{T} \rangle}_x\}$
$\delta_\tau(\text{add1})$	$=$	$x:\mathbf{N} \rightarrow \mathbf{N}$
$\delta_\tau(\text{zero?})$	$=$	$x:\mathbf{N} \rightarrow \mathbf{B}$

**Figure 7.** Constant Typing

$$\begin{array}{lll}
\text{update}(\Gamma, \langle \{\omega \mid \psi\}, \mathcal{R} \rangle, \nu, \pi :: \mathbf{car}) & = & \langle \{\omega \mid \psi \wedge \nu_{\pi(\omega)}\}, \mathcal{R} \rangle \\
\text{update}(\Gamma, \langle \mathcal{R}, \{\omega \mid \psi\} \rangle, \nu, \pi :: \mathbf{cdr}) & = & \langle \mathcal{R}, \{\omega \mid \psi \wedge \nu_{\pi(\omega)}\} \rangle \\
\text{update}(\Gamma, \tau, \sigma, \epsilon) & = & \text{restrict}(\Gamma, \tau, \sigma) \\
\text{update}(\Gamma, \tau, \bar{\sigma}, \epsilon) & = & \text{remove}(\Gamma, \tau, \bar{\sigma}) \\
\\ 
\text{restrict}(\Gamma, \tau, \sigma) & = & \perp \quad \text{if } \nexists v. \Gamma \vdash v : \{\omega \mid \sigma_\omega\} \text{ and } \Gamma \vdash v : \{\omega \mid \tau_\omega\} \\
\text{restrict}(\Gamma, (\bigcup \vec{\tau}), \sigma) & = & (\bigcup \text{restrict}(\Gamma, \tau, \sigma)) \\
\text{restrict}(\Gamma, \tau, \sigma) & = & \tau \quad \text{if } \Gamma \vdash \tau <: \sigma \\
\text{restrict}(\Gamma, \tau, \sigma) & = & \sigma \quad \text{otherwise} \\
\\ 
\text{remove}(\Gamma, \tau, \sigma) & = & \perp \quad \text{if } \Gamma \vdash \tau <: \sigma \\
\text{remove}(\Gamma, (\bigcup \vec{\tau}), \sigma) & = & (\bigcup \text{remove}(\Gamma, \tau, \sigma)) \\
\text{remove}(\Gamma, \tau, \sigma) & = & \tau \quad \text{otherwise}
\end{array}$$

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**Figure 8.** Type Update