EE 416 – Image Processing and Computer Vision **(UH Manoa, Fall 2020)**

Homework Assignment #1

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Problem 1

a)

If the input were a Dirac impulse at $x_i = x_1 \Rightarrow x_o = Mx_1 \quad \therefore$

$$\delta(x - x_i) \xrightarrow{S} h(x; x_i) = \delta(x - x_o) = \delta(x - Mx_1) = \delta(x - Mx_i)$$

$$\therefore$$

$$h(x; x') = \delta(x - Mx')$$

b)

$$\begin{aligned} g(x) &= \int_{-\infty}^{\infty} f(x')h(x; x')dx' \\ &= \int_{-\infty}^{\infty} f(x')\delta(x - Mx')dx' \\ &= \int_{-\infty}^{\infty} f(x')\delta(Mx' - x)dx' \quad \text{, symmetry property.} \\ &= \int_{-\infty}^{\infty} f(x')\delta[M(x' - \frac{x}{M})]dx' \\ &= \frac{1}{|M|}f(\frac{x}{M}) \quad \text{, using scaling and sifting property of Dirac function.} \end{aligned}$$

c)

From the notes, section 1.4.2. A 1D pinhole models illustrate a magnification or mirroring system; this implies that is a shift-variant system. This is because the PSF does not depend solely on the difference between input and output coordinates.

Proof:

For g_1 :

$$\begin{aligned} g_1(x) =& f(x) \xrightarrow{S} g(x) \rightarrow shift(x_0) \\ g_{1_a}(x) =& f(x) \xrightarrow{S} g(x) = \frac{1}{|M|}f(\frac{x}{M}) \\ g_1(x) =& g_{1_a}(x) \rightarrow shift(x_0) = \frac{1}{|M|}f(\frac{x}{M} - x_0) \\ &\therefore \\ g_1(x) =& \frac{1}{|M|}f(\frac{x}{M} - x_0) \end{aligned}$$

For g_2 :

$$\begin{aligned}
g_2(x) =& f(x) \rightarrow shift(x_0) \xrightarrow{S} g(x) \\
g_{2_a}(x) =& f(x) \rightarrow shift(x_0) = f(x - x_0) \\
g_2(x) =& g_{2_a}(x) \xrightarrow{S} g(x) = \frac{1}{|M|} f(\frac{x - x_0}{M}) \\
&\therefore \\
g_2(x) =& \frac{1}{|M|} f(\frac{x - x_0}{M})
\end{aligned}$$

Therefore, we can conclude it is shift-variant because $g_1 \neq g_2$.

Problem 2

a)
If $g(t) \xleftrightarrow{CTFT} G(f) \Rightarrow g(t-a) \xleftrightarrow{CTFT} e^{-i2\pi af}G(f)$:

Proof:
We know that

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-i2\pi ft} dt$$

Let $z(t) = g(t-a)$ and $\tau = t - a \rightarrow d\tau = dt$; thus,

$$\begin{aligned}
z(t) &\xrightarrow{CTFT} Z(f) \\
Z(f) &= \int_{-\infty}^{\infty} g(t-a) e^{-i2\pi ft} dt = \\
&= \int_{-\infty}^{\infty} g(\tau) e^{-i2\pi f(\tau+a)} d\tau \\
&= \int_{-\infty}^{\infty} g(\tau) e^{-i2\pi f\tau} e^{-i2\pi fa} d\tau \\
&= e^{-i2\pi fa} \int_{-\infty}^{\infty} g(\tau) e^{-i2\pi f\tau} d\tau \\
&= e^{-i2\pi fa} G(f) \\
&= e^{-i2\pi fa} G(f) \\
&\therefore
\end{aligned}$$

$$z(t) \xleftrightarrow{CTFT} e^{-i2\pi fa}G(f) \Rightarrow g(t-a) \xleftrightarrow{CTFT} e^{-i2\pi fa}G(f)$$

b)
If $g(x,y) \xleftrightarrow{CSFT} G(u,v) \Rightarrow g(\frac{x}{a}, \frac{y}{b}) \xleftrightarrow{CSFT} |ab|G(au, bv)$:

Proof:
We know that

$$G(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-i2\pi(ux+vy)}dxdy$$

Let $z(x,y) = g(\frac{x}{a}, \frac{y}{b})$ and $\alpha = \frac{x}{a} \to d\alpha = \frac{1}{a}dx$; $\beta = \frac{y}{b} \to d\beta = \frac{1}{b}dy$ thus,

$$\begin{aligned}
z(x,y) &\xrightarrow{CSFT} Z(u,v) \\
Z(u,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\frac{x}{a}, \frac{y}{b})e^{-i2\pi(ux+vy)}dxdy = \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\alpha,\beta)e^{-i2\pi(au\alpha+bv\beta)}ad\alpha \cdot bd\beta \\
&= |ab|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\alpha,\beta)e^{-i2\pi(au\alpha+bv\beta)}d\alpha \cdot d\beta \\
&= |ab| \cdot G(au,bv)
\end{aligned}$$

Note that if $ab < 0, \quad da\alpha \cdot db\beta < 0$, will flip the integrals to $-\infty$ to ∞.

$$Z(u,v) = |ab| \cdot G(au,bv)$$

$$\therefore$$

$$z(x,y) \xleftrightarrow{CSFT} |ab| \cdot G(au,bv) \Rightarrow g(\frac{x}{a}, \frac{y}{b}) \xleftrightarrow{CSFT} |ab| \cdot G(au,bv)$$

c)

If $g(\begin{bmatrix} x \\ y \end{bmatrix}) \xleftrightarrow{CSFT} G(\begin{bmatrix} u \\ v \end{bmatrix}) \Rightarrow g(A\begin{bmatrix} x \\ y \end{bmatrix}) \xleftrightarrow{CSFT} |det(A)|^{-1}G((A^{-1})^T\begin{bmatrix} u \\ v \end{bmatrix})$

Proof:

We know that

$$G(\begin{bmatrix} u \\ v \end{bmatrix}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\begin{bmatrix} x \\ y \end{bmatrix})e^{-i2\pi\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}}dxdy$$

Let $z(x,y) = g(A\begin{bmatrix} x \\ y \end{bmatrix})$ and $\omega = A\begin{bmatrix} x \\ y \end{bmatrix} \to d\omega = |det(A)|dxdy$ thus,

$$\begin{aligned}
&\omega = A\begin{bmatrix} x \\ y \end{bmatrix} \Leftrightarrow \\
&\Leftrightarrow A^{-1}\omega = A^{-1}A\begin{bmatrix} x \\ y \end{bmatrix} \\
&\Leftrightarrow A^{-1}\omega = \begin{bmatrix} x \\ y \end{bmatrix} \\
&\Leftrightarrow (A^{-1}\omega)^T = (\begin{bmatrix} x \\ y \end{bmatrix})^T \\
&\Leftrightarrow \omega^T(A^{-1})^T = \begin{bmatrix} x & y \end{bmatrix}
\end{aligned}$$

$$z(x,y) \xrightarrow{CSFT} Z(u,v)$$

$$Z(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(A\begin{bmatrix}x\\y\end{bmatrix})e^{-i2\pi\begin{bmatrix}x & y\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}}dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\omega)e^{-i2\pi\omega^T(A^{-1})^T\begin{bmatrix}u\\v\end{bmatrix}}|det(A)|^{-1}d\omega$$

$$= |det(A)|^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\omega)e^{-i2\pi\omega^T(A^{-1})^T\begin{bmatrix}u\\v\end{bmatrix}}d\omega$$

$$= |det(A)|^{-1}G((A^{-1})^T\begin{bmatrix}u\\v\end{bmatrix})$$

$$\therefore$$

$$z(x,y) \xleftrightarrow{CSFT} |det(A)|^{-1}G((A^{-1})^T\begin{bmatrix}u\\v\end{bmatrix}) \Rightarrow g(A\begin{bmatrix}x\\y\end{bmatrix}) \xleftrightarrow{CSFT} |det(A)|^{-1}G((A^{-1})^T\begin{bmatrix}u\\v\end{bmatrix})$$

Problem 3

a)
$H(\rho) = |\rho| \quad \because$

$$g_\theta(r) = \int_{-\infty}^{\infty}|\rho|P_\theta(\rho)e^{i2\pi\rho r}d\rho =$$
$$= CTFT^{-1}|\rho|P_\theta(\rho)$$
$$= h(r) * P_\theta(r)$$
$$G_\theta(\rho) = |\rho|P_\theta(\rho)$$
$$= H(\rho)P_\theta(\rho)$$
$$|\rho|P_\theta(\rho) = H(\rho)P_\theta(\rho) \Rightarrow H(\rho) = |\rho|$$

b)
No, it is not practical. This is because $|\rho|$ is a ramp function; this means that it will take any frequencies as input which create noise. Thus, if we have more noise, it will create more inaccurate output.

c)

The new function is more practical than $H(\rho)$. This is because the rect function will filter the noises for high frequency.

A limitation of this new function is that it might filter for high frequencies.

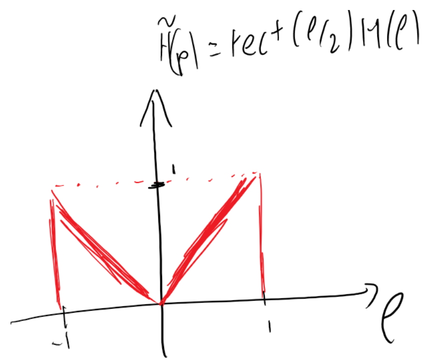

Figure 1: Sketch