Elements Finis
TP 4

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Exercice 1

$$\begin{cases} -\Delta u + u = f \text{ dans } \Omega \\ \dfrac{\partial u}{\partial n} = 0 \text{ sur } \partial\Omega \end{cases}$$

$$-\int v\Delta u + \int vu = \int vf$$

On suppose que $u \in C^2(\overline{\Omega})$ et $v \in C^1(\overline{\Omega})$. On peut donc se servir d'une formule de Green, ce qui nous amène à

$$\int_\Omega \nabla v \cdot \nabla u + \int_\Omega vu - \int_{\partial\Omega} v\nabla u = \int_\Omega vf$$

Et donc, grace à la condition de Neumann,

$$\int_\Omega \nabla v \cdot \nabla u + \int_\Omega vu = \int_\Omega vf$$

Exercice 2

$$\begin{cases} \text{Trouver } u \in H^1 \text{ telle que} \\ a(u,v) = \ell(v)\ \forall v \in H^1 \end{cases}$$

Où

$$a(u,v) = \int_\Omega \nabla v \cdot \nabla u + \int_\Omega vu$$
$$\ell(v) = \int_\Omega vf$$

Soit ϕ_n une base de V_h.

$$\begin{cases} \text{Trouver } u_h \in V_h \text{ telle que} \\ a(u_h,\varphi_j) = \ell(\varphi_j)\ \forall j \in 1...N \end{cases}$$

Exercice 3

Soit $u_h = \sum u_i\varphi_i$. Alors,

$$\begin{aligned} a(u_h, \varphi_j) &= a\left(\sum u_i\varphi_i, \varphi_j\right) \\ &= \sum a\left(u_i\varphi_i, \varphi_j\right) \\ &= \sum a\left(\varphi_i, \varphi_j\right) u_i \end{aligned}$$

Donc, $a(u_h, \varphi_j) = \ell(\varphi_j)$ implique que

$$\sum a\left(\varphi_i, \varphi_j\right) u_i = \ell(\varphi_j)$$

C'est à dire,

$$(M + K)u = f$$

Où

$$\begin{aligned} M_{ij} &= \int_\Omega \varphi_i\varphi_j \\ K_{ij} &= \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \\ f_i &= \int_\Omega \varphi_i f \end{aligned}$$

On note que M et K sont symétriques.

Exercice 4

$$\begin{aligned} \lambda_1(x, y) &= \frac{1}{D}(y_{23}(x - x_3) - x_{23}(y - y_3)) \\ \lambda_2(x, y) &= \frac{1}{D}(y_{31}(x - x_1) - x_{31}(y - y_1)) \\ \lambda_3(x, y) &= \frac{1}{D}(y_{12}(x - x_2) - x_{12}(y - y_2)) \\ D &= x_{23}y_{31} - x_{31}y_{23} \end{aligned}$$

1.

$$\int_T \varphi_i(x, y)\varphi_j(x, y)dxdy$$

Soit S_1, S_2, S_3 les sommets 3 de T. On pose $\varphi_{S_i} = \lambda_i$. Grace au symétrie et le fait que seulement les φ appartenant au même triangle ont du support non vide, il faudra calculer juste les 6 intégrales suivantes :

$$\int_T \varphi_{S_i}(x,y)\varphi_{S_j}(x,y)dxdy$$

On commence avec $i = 1, j = 2$:

$$\int_T \varphi_{S_1}(x,y)\varphi_{S_2}(x,y)dxdy = \int_T \lambda_1(x,y)\lambda_2(x,y)dxdy$$

On effectue le changement de variables suivant :

$$\begin{aligned}
x &= x_{13}u + x_{23}v + x_3 \\
y &= y_{13}u + y_{23}v + y_3
\end{aligned}$$

Le jabocien est donc $|D| = D$, car on traverse les sommets des triangles comme il faut.

On a donc,

$$\begin{aligned}
\lambda_1 &= \frac{1}{D}(y_{23}(x - x_3) - x_{23}(y - y_3)) \\
&= \frac{1}{D}(y_{23}(x_{13}u + x_{23}v + x_3 - x_3) - x_{23}(y_{13}u + y_{23}v + y_3 - y_3)) \\
&= \frac{1}{D}(y_{23}(x_{13}u + x_{23}v) - x_{23}(y_{13}u + y_{23}v)) \\
&= \frac{1}{D}(y_{23}x_{13}u + y_{23}x_{23}v - x_{23}y_{13}u - x_{23}y_{23}v) \\
&= \frac{1}{D}(u(y_{23}x_{13} - x_{23}y_{13}) + v(y_{23}x_{23} - x_{23}y_{23})) \\
&= \frac{1}{D}(u(y_{23}x_{13} - x_{23}y_{13})) \\
&= u
\end{aligned}$$

$$
\begin{aligned}
\lambda_2 &= \frac{1}{D}(y_{31}(x-x_1) - x_{31}(y-y_1)) \\
&= \frac{1}{D}(y_{31}(x_{13}u + x_{23}v + x_3 - x_1) - x_{31}(y_{13}u + y_{23}v + y_3 - y_1)) \\
&= \frac{1}{D}(y_{31}(x_{13}u + x_{23}v + x_{31}) - x_{31}(y_{13}u + y_{23}v + y_{31})) \\
&= \frac{1}{D}(y_{31}x_{13}u + y_{31}x_{23}v + y_{31}x_{31} - x_{31}y_{13}u - x_{31}y_{23}v - x_{31}y_{31}) \\
&= \frac{1}{D}(u(y_{31}x_{13} - x_{31}y_{13}) + v(y_{31}x_{23} - x_{31}y_{23}) + y_{31}x_{31} - x_{31}y_{31}) \\
&= \frac{1}{D}(u(y_{31}x_{13} - x_{31}y_{13}) + v(y_{31}x_{23} - x_{31}y_{23}) + y_{31}x_{31} - x_{31}y_{31}) \\
&= \frac{1}{D}(u(-y_{13}x_{13} + x_{13}y_{13}) + v(y_{31}x_{23} - x_{31}y_{23})) \\
&= \frac{1}{D}(v(y_{31}x_{23} - x_{31}y_{23})) \\
&= v
\end{aligned}
$$

$$
\begin{aligned}
\lambda_3 &= \frac{1}{D}(y_{12}(x-x_2) - x_{12}(y-y_2)) \\
&= \frac{1}{D}(y_{12}(x_{13}u + x_{23}v + x_3 - x_2) - x_{12}(y_{13}u + y_{23}v + y_3 - y_2)) \\
&= \frac{1}{D}(y_{12}(x_{13}u + x_{23}v + x_{32}) - x_{12}(y_{13}u + y_{23}v + y_{32})) \\
&= \frac{1}{D}(y_{12}x_{13}u + y_{12}x_{23}v + y_{12}x_{32} - x_{12}y_{13}u - x_{12}y_{23}v - x_{12}y_{32}) \\
&= \frac{1}{D}(u(y_{12}x_{13} - x_{12}y_{13}) + v(y_{12}x_{23} - x_{12}y_{23}) + y_{12}x_{32} - x_{12}y_{32}) \\
&= \frac{1}{D}(u(y_{21}x_{31} - x_{21}y_{31}) + v(y_{12}x_{32} - x_{12}y_{32}) + y_{12}x_{32} - x_{12}y_{32}) \\
&= -u - v + 1 = 1 - u - v
\end{aligned}
$$

Enfin on a

$$
\begin{aligned}
\int_T \lambda_1(x,y)\lambda_2(x,y)dxdy &= \int_T \lambda_1(u,v)\lambda_2(u,v)Ddudv \\
&= D\int_0^1 \int_0^{1-u} uvdvdu \\
&= D\int_0^1 u[v^2/2]_0^{1-u}du \\
&= D/2\int_0^1 u(1-u)^2du \\
&= D/2\int_0^1 u(u^2-2u+1)du \\
&= D/2\int_0^1 (u^3-2u^2+u)du \\
&= D/2[u^4/4-2u^3/3+u^2/2]_0^1 \\
&= D/2[1/4-2/3+1/2] \\
&= D/2[3/12-8/12+6/12] = D/2[1/12] = D/24
\end{aligned}
$$

$$
\begin{aligned}
\int_T \lambda_1(x,y)\lambda_3(x,y)dxdy &= \int_T \lambda_1(u,v)\lambda_3(u,v)Ddudv \\
&= D\int_0^1\int_0^{1-u} u(1-u-v)dvdu \\
&= D\int_0^1\int_0^{1-u} (u-u^2-vu)dvdu \\
&= D\int_0^1 [uv-u^2v-uv^2/2]_0^{1-u}du \\
&= D\int_0^1 [u(1-u)-u^2(1-u)-u(1-u)^2/2]_0^{1-u}du \\
&= D\int_0^1 [u-u^2-u^2+u^3-u(1-2u+u^2)/2]du \\
&= D\int_0^1 [u-2u^2+u^3-(u-2u^2+u^3)/2]du \\
&= D\int_0^1 [u-2u^2+u^3-u/2+u^2-u^3/2]du \\
&= D\int_0^1 [u/2-u^2+u^3/2]du \\
&= D/2\int_0^1 [u-2u^2+u^3]du \\
&= D/2[u^2/2-2u^3/3+u^4/4]_0^1 \\
&= D/2[1/2-2/3+1/4] = D/2[6/12-8/12+3/12] \\
&= D/2[1/12] = D/24
\end{aligned}
$$

$$
\begin{aligned}
\int_T \lambda_1^2(x,y)dxdy &= \int_T \lambda_1^2(u,v)Ddudv \\
&= D\int_0^1\int_0^{1-u} u^2 dvdu \\
&= D\int_0^1 [u^2v]_0^{1-u}du \\
&= D\int_0^1 [u^2(1-u)]du \\
&= D\int_0^1 [u^2-u^3]du \\
&= D[u^3/3-u^4/4]_0^1 \\
&= D[1/3-1/4] \\
&= D[4/12-3/12] \\
&= D/12
\end{aligned}
$$

$$
\begin{aligned}
\int_T \lambda_3^2(x,y)dxdy &= \int_T \lambda_3^2(u,v)Ddudv \\
&= D\int_0^1\int_0^{1-u} (1-u-v)^2 dvdu \\
&= D\int_0^1 [-(1-u-v)^3/3]_0^{1-u}du \\
&= D\int_0^1 [(1-u)^3/3]du \\
&= D[-(1-u)^4/12]_0^1 = D/12
\end{aligned}
$$

2. Donc on a, pour un seul triangle,

$$
M_T = \begin{bmatrix} D/12 & D/24 & D/24 \\ D/24 & D/12 & D/24 \\ D/24 & D/24 & D/12 \end{bmatrix} = \frac{D}{24}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \frac{|\tau|}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}
$$

Exercice 6

1.

$$\nabla\lambda_1(x,y) = \begin{pmatrix} \frac{1}{D}\partial_x(y_{23}(x-x_3) - x_{23}(y-y_3)) \\ \frac{1}{D}\partial_y(y_{23}(x-x_3) - x_{23}(y-y_3)) \end{pmatrix} = \begin{pmatrix} \frac{1}{D}y_{23} \\ \frac{1}{D}x_{23} \end{pmatrix}$$

$$\nabla\lambda_2(x,y) = \begin{pmatrix} \frac{1}{D}y_{31} \\ \frac{1}{D}x_{31} \end{pmatrix}$$

$$\nabla\lambda_3(x,y) = \begin{pmatrix} \frac{1}{D}y_{12} \\ \frac{1}{D}x_{12} \end{pmatrix}$$

$$\begin{aligned}
\int_T \nabla\lambda_1 \cdot \nabla\lambda_1 &= \int_T \frac{1}{D^2}(y_{23}^2 + x_{23}^2) \\
&= \frac{1}{D^2}(y_{23}^2 + x_{23}^2)\int_T dxdy \\
&= \frac{1}{D^2}(y_{23}^2 + x_{23}^2)\frac{D}{2} \\
&= \frac{1}{2D}(y_{23}^2 + x_{23}^2) \\
\int_T \nabla\lambda_2 \cdot \nabla\lambda_2 &= \frac{1}{2D}(y_{31}^2 + x_{31}^2) \\
\int_T \nabla\lambda_3 \cdot \nabla\lambda_3 &= \frac{1}{2D}(y_{12}^2 + x_{12}^2) \\
\int_T \nabla\lambda_1 \cdot \nabla\lambda_2 &= \frac{1}{2D}(y_{23}y_{31} + x_{23}x_{31}) \\
\int_T \nabla\lambda_1 \cdot \nabla\lambda_3 &= \frac{1}{2D}(y_{23}y_{12} + x_{23}x_{12}) \\
\int_T \nabla\lambda_2 \cdot \nabla\lambda_3 &= \frac{1}{2D}(y_{31}y_{12} + x_{31}x_{12})
\end{aligned}$$

$$K_T = \frac{1}{2D}\begin{bmatrix} (y_{23}^2 + x_{23}^2) & (y_{23}y_{31} + x_{23}x_{31}) & (y_{23}y_{12} + x_{23}x_{12}) \\ (y_{23}y_{31} + x_{23}x_{31}) & (y_{31}^2 + x_{31}^2) & (y_{31}y_{12} + x_{31}x_{12}) \\ (y_{23}y_{12} + x_{23}x_{12}) & (y_{31}y_{12} + x_{31}x_{12}) & (y_{12}^2 + x_{12}^2) \end{bmatrix}$$

$$
\begin{aligned}
x_1 &= 1 & y_1 &= 0 \\
x_2 &= 0 & y_2 &= 1 \\
x_3 &= 0 & y_3 &= 0 \\
x_{23} &= 0 & y_{23} &= 1 \\
x_{31} &= -1 & y_{31} &= 0 \\
x_{12} &= 1 & y_{12} &= -1
\end{aligned}
$$

$$
K_T = \frac{1}{2D} \begin{bmatrix} (1^2 + 0^2) & (10 + 0(-1)) & (1(-1) + 01) \\ (10 + 0(-1)) & (0^2 + (-1)^2) & (0(-1) + (-1)1) \\ (1(-1) + 01) & (0(-1) + (-1)1) & ((-1)^2 + 1^2) \end{bmatrix}
$$

$$
K_T = \frac{1}{2D} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}
$$