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# Math for Machine Learning

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## Linear algebra - Week 4

# W4 Lesson 1



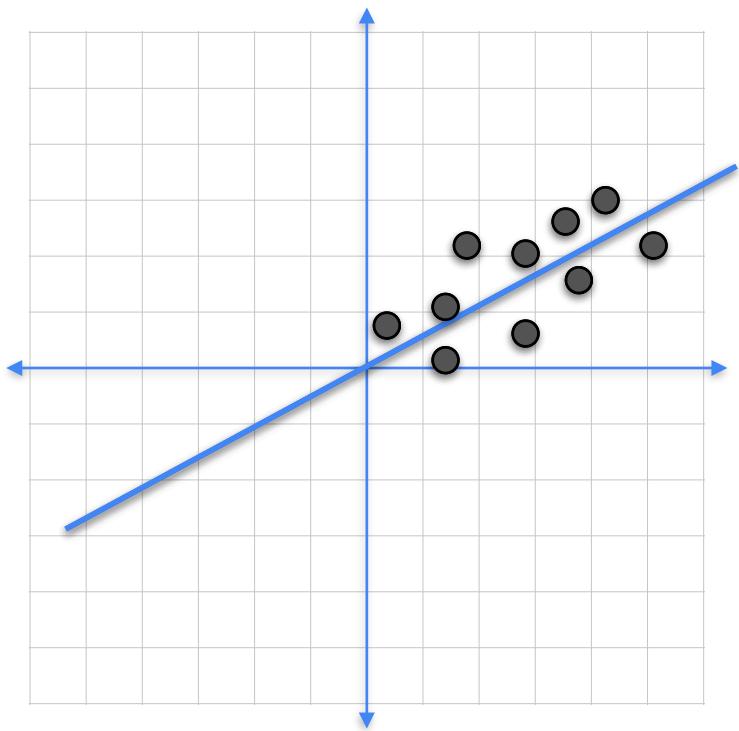
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## Determinants and Eigenvectors

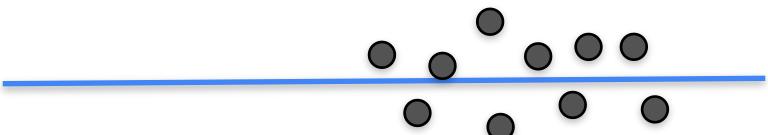
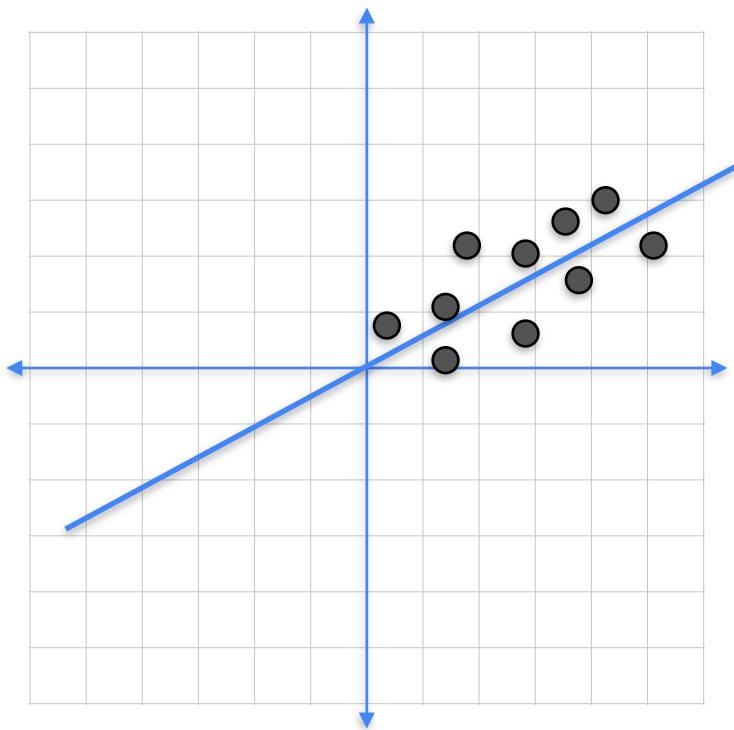
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# Machine learning motivation

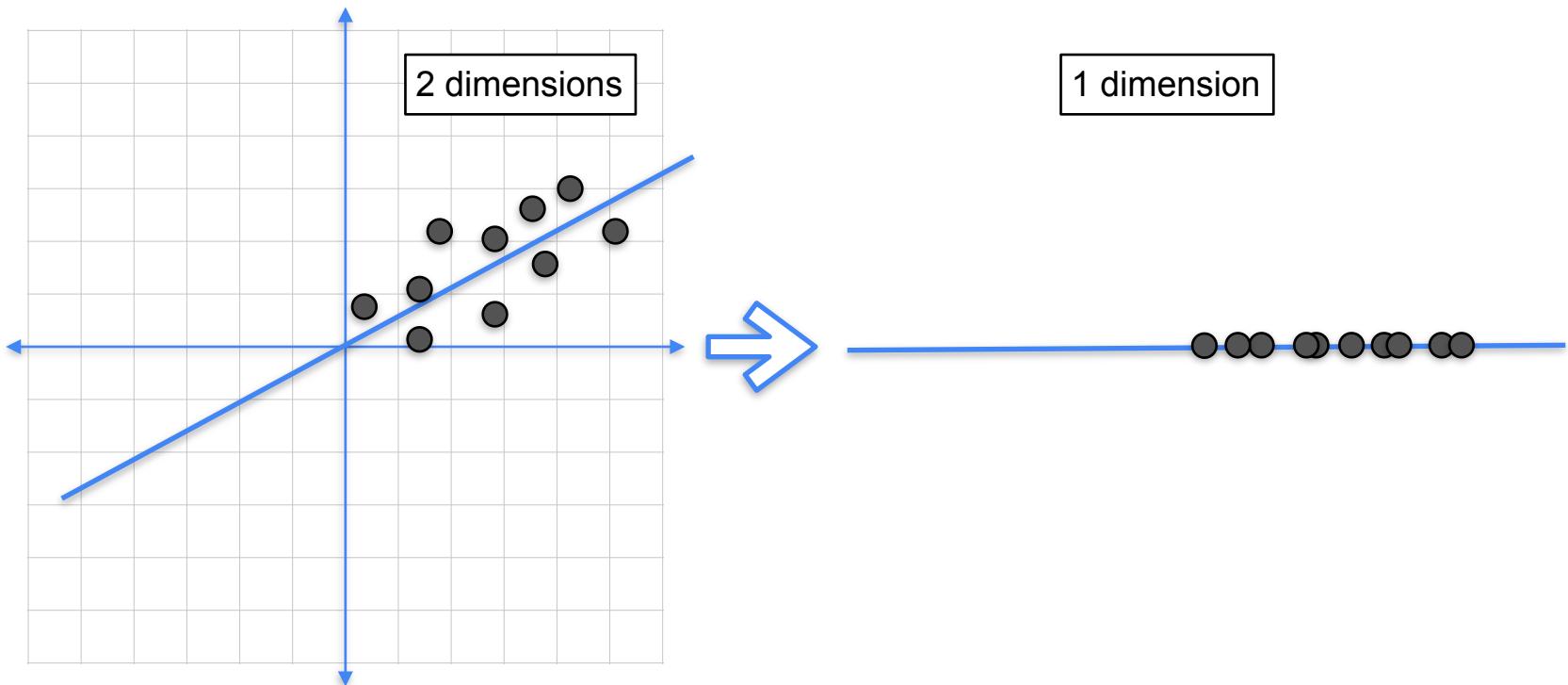
# Principal Component Analysis



# Principal Component Analysis

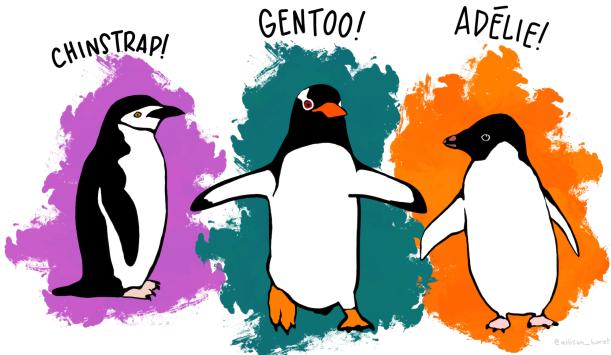


# Principal Component Analysis



# Principal Component Analysis

- Reduce dimensions (columns) of dataset
- Preserve as much information as possible



species	culmen_length_mm	culmen_depth_mm	flipper_length_mm	body_mass_g	PC1	PC2	species
Adelie	40.6	17.2	187.0	3475.0	1.353843	-0.422253	Adelie
Adelie	38.9	17.8	181.0	3625.0	1.760446	-0.350965	Adelie
Adelie	35.7	16.9	185.0	3150.0	2.005766	-1.113797	Adelie
Gentoo	50.0	15.3	220.0	5550.0	-2.585758	0.061768	Gentoo
Adelie	34.5	18.1	187.0	2900.0	2.438111	-0.786227	Adelie

# What to expect?



Linear transformation



Characterize your transformation

# What to expect?



Linear transformation



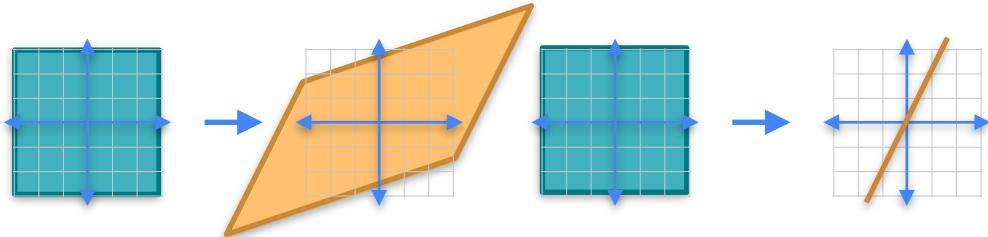
Singular / Non-singular



Characterize your transformation

Non-singular

3	1
1	2



Singular

1	1
2	2

# What to expect?



Linear transformation

1

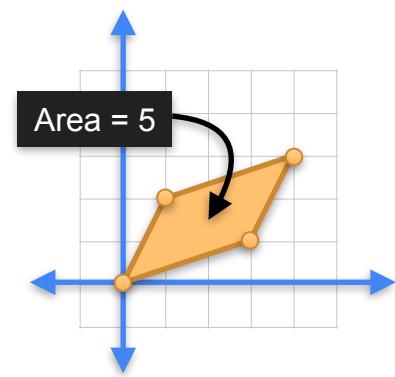
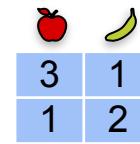
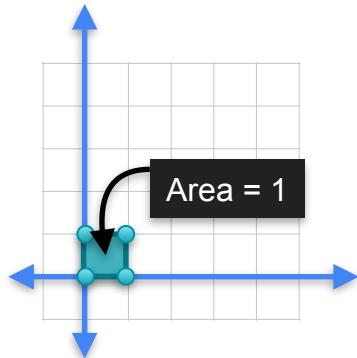
2

3

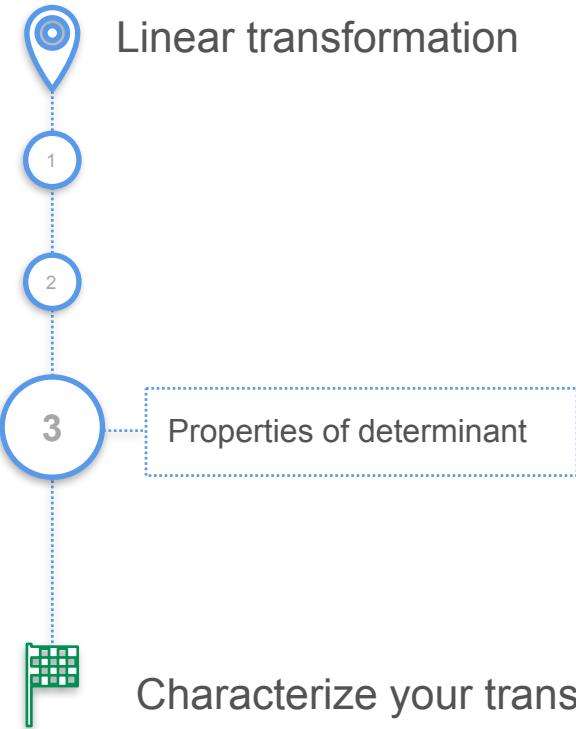
Determinant as area



Characterize your transformation



# What to expect?



$$\begin{matrix} 3 & 1 \\ 1 & 2 \end{matrix}^{-1} = \begin{matrix} 0.4 & -0.2 \\ -0.2 & 0.6 \end{matrix}$$

Det = 5      Det = 0.2 =  $\frac{1}{5}$

# What to expect?



Linear transformation



Singular / Non-singular



Determinant as area

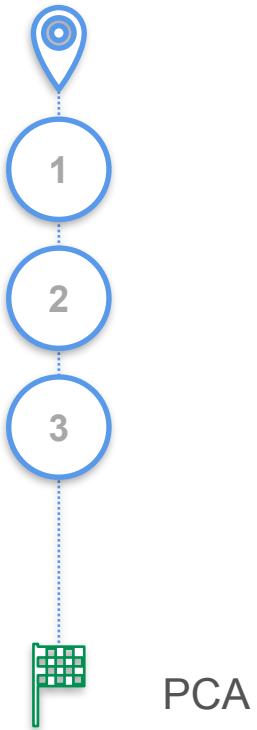
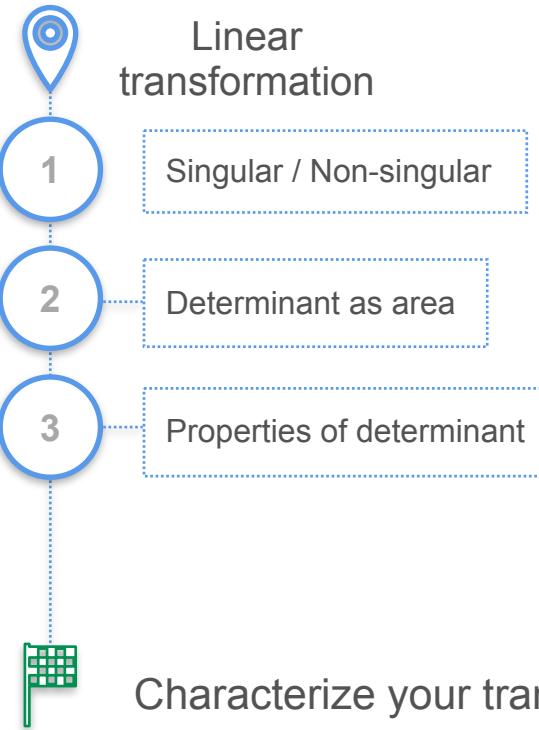


Properties of determinant

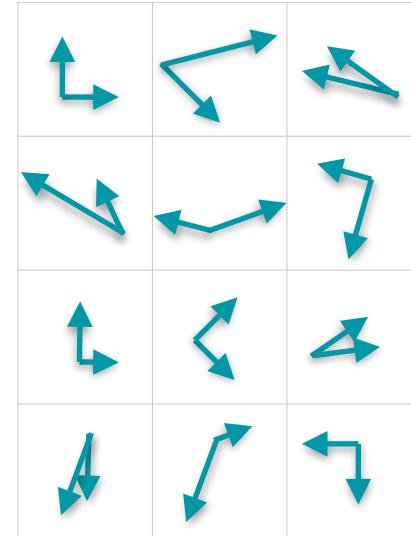
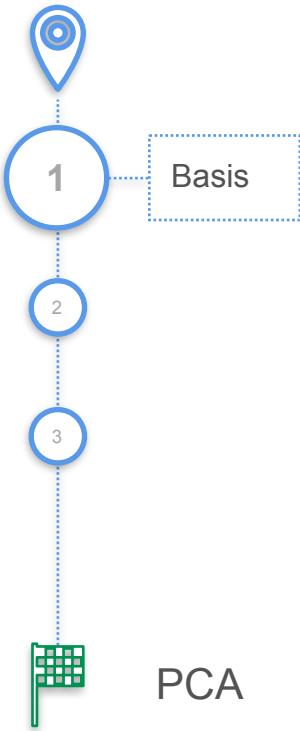
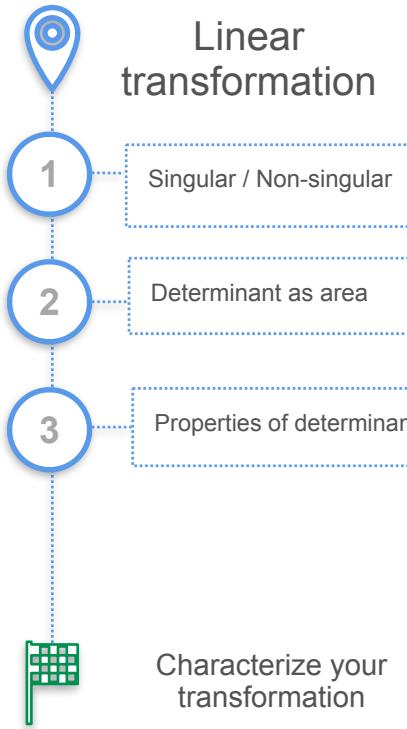


Characterize your transformation

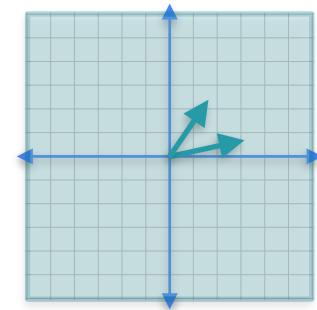
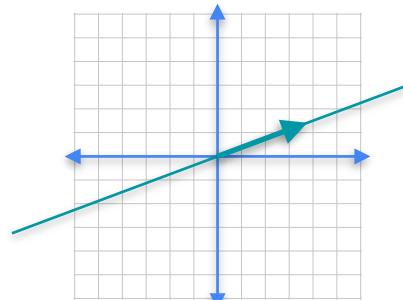
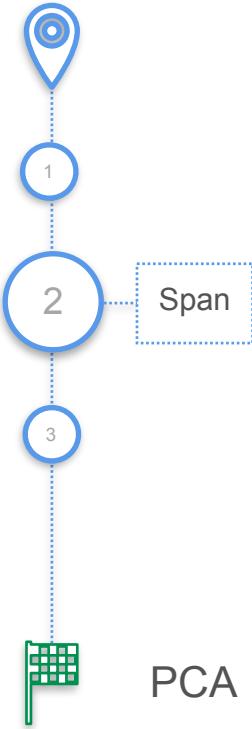
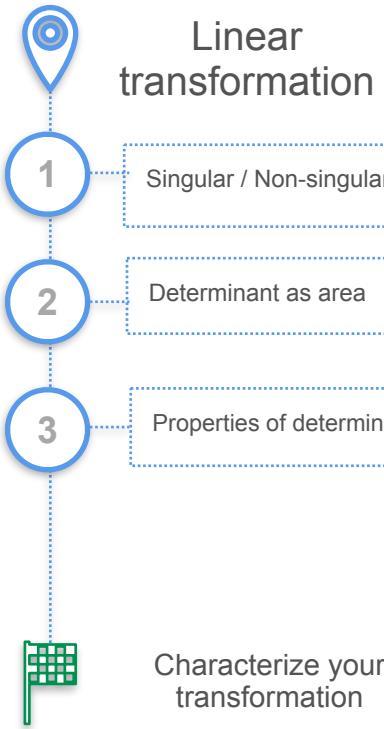
# What to expect?



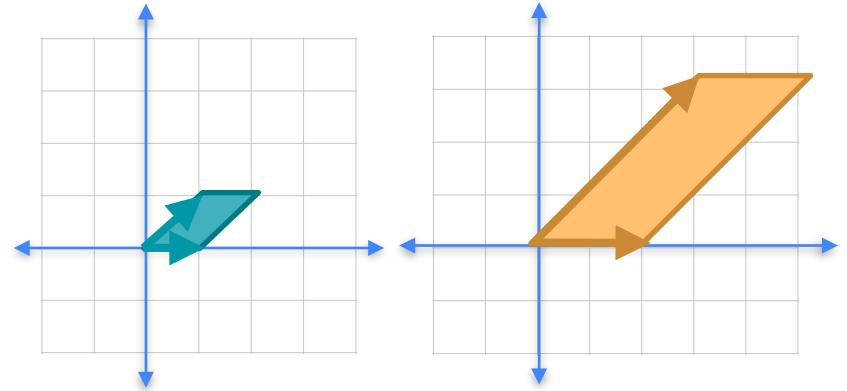
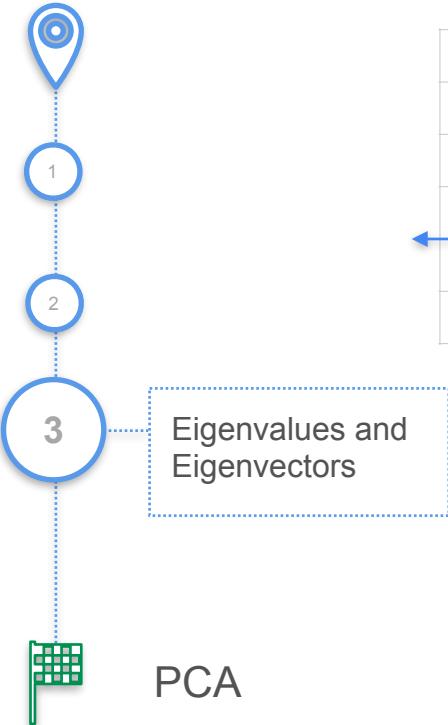
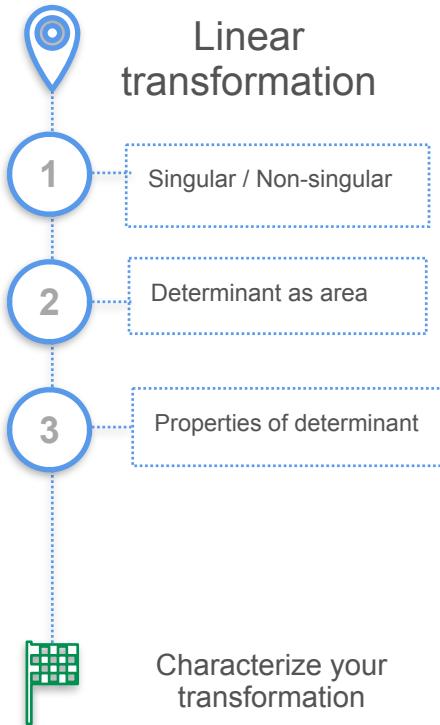
# What to expect?



# What to expect?



# What to expect?



$$\begin{matrix} 2 & 1 & 1 \\ 0 & 3 & 0 \end{matrix} = \begin{matrix} 2 \\ 0 \end{matrix}$$

$(1,0) \rightarrow (2,0)$

$$A v_1 = \lambda_1 v_1$$

# What to expect?



Linear transformation

1

Singular / Non-singular

2

Determinant as area

3

Properties of determinant



Characterize your transformation



1

Basis

2

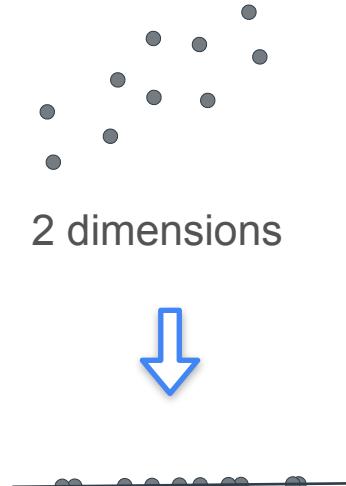
Span

3

Eigenvalues and  
Eigenvectors



PCA



2 dimensions



1 dimension



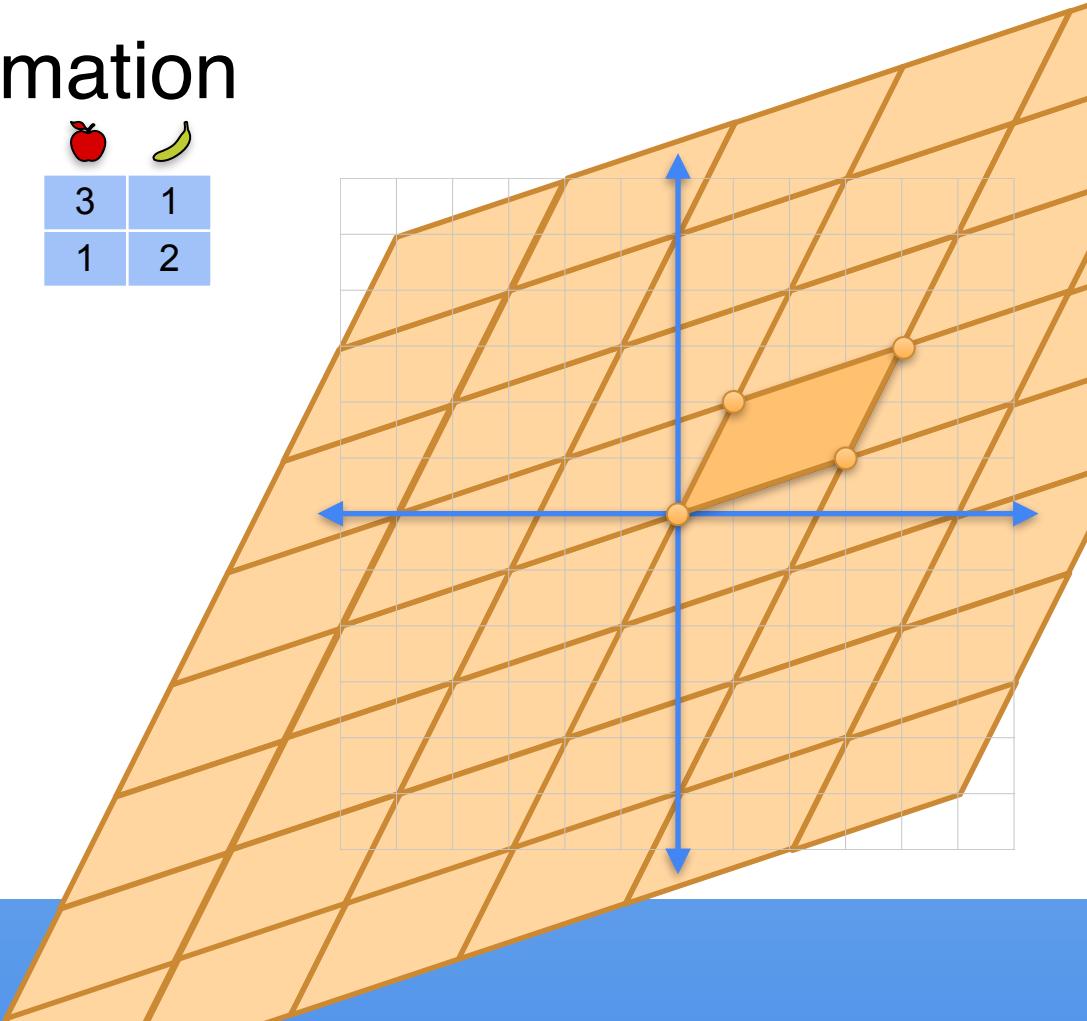
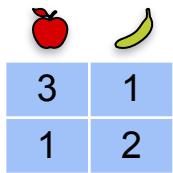
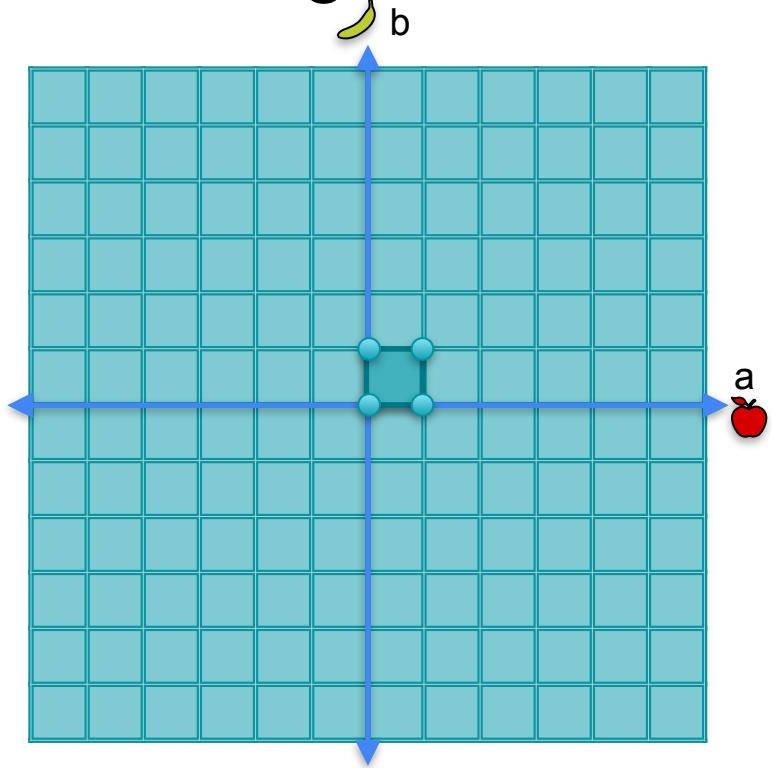
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## Determinants and Eigenvectors

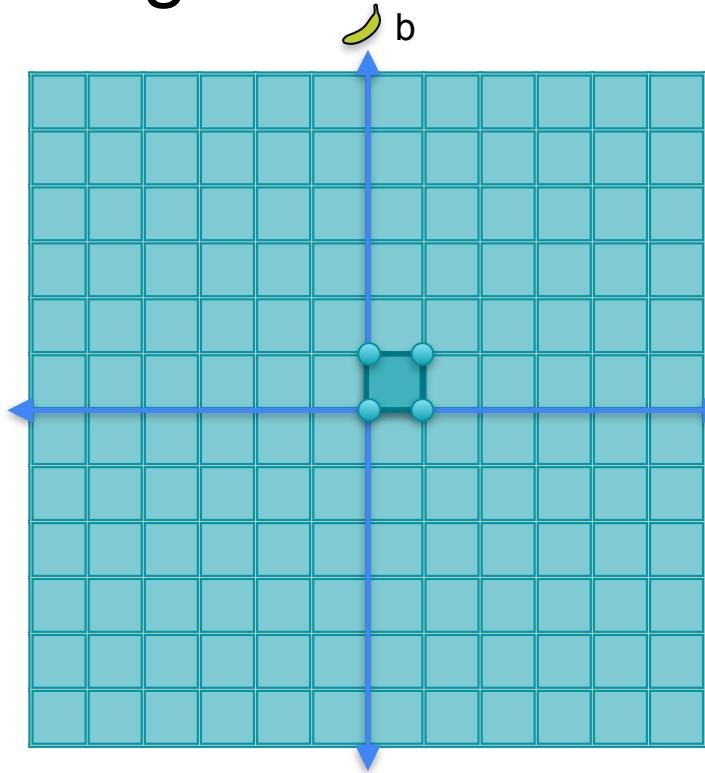
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**Singularity and rank of linear transformations**

# Non-singular transformation



# Singular transformation

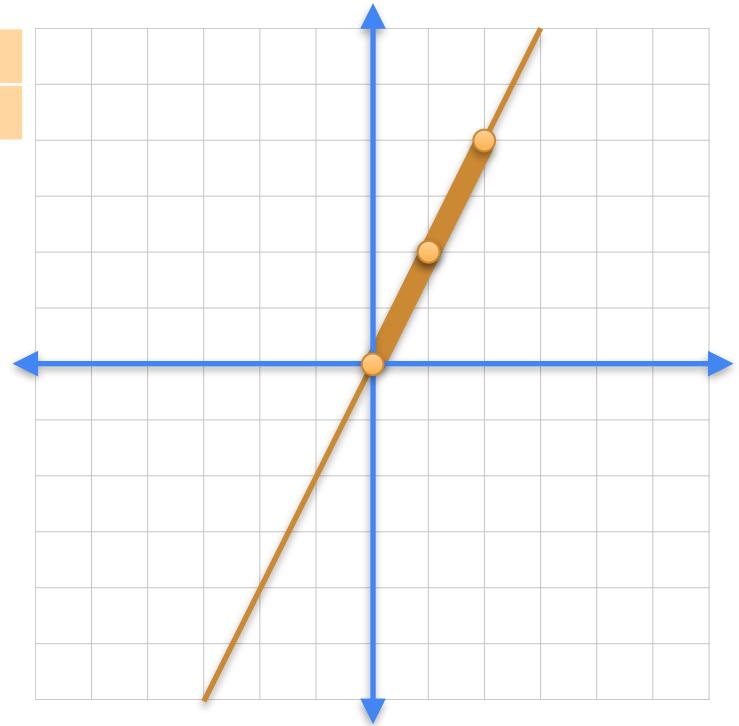


A red apple and a yellow banana are shown above a matrix equation. The equation consists of three matrices: a 2x2 matrix with columns [1, 2] and rows [1, 2], a 1x2 matrix with columns [1, 1], and a 2x1 matrix with rows [1, 1]. An equals sign follows the first two matrices, and the result is a 2x1 matrix with rows [2, 4].

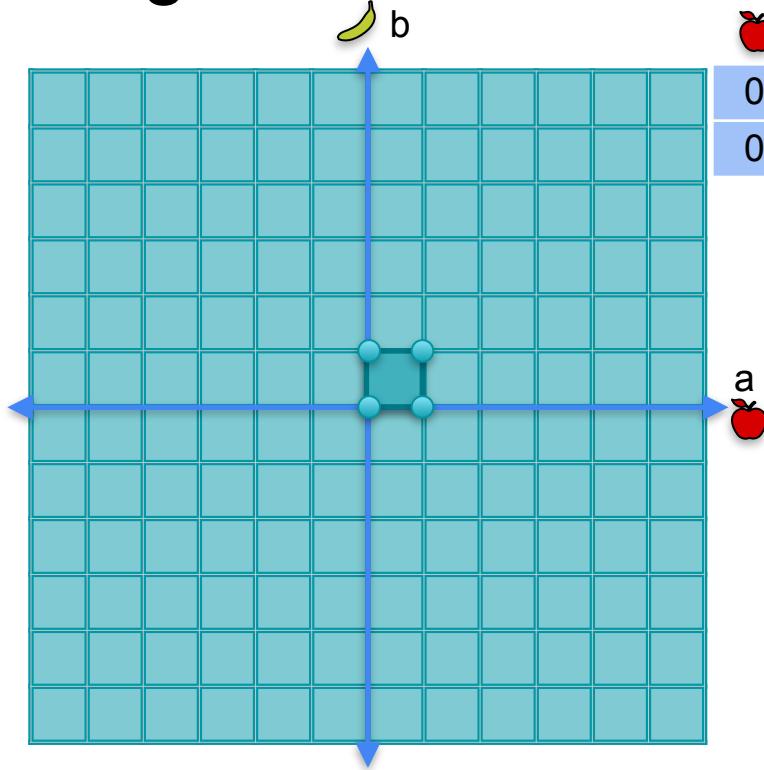
$$\begin{matrix} 1 & 1 \\ 2 & 2 \end{matrix} \begin{matrix} 1 \\ 1 \end{matrix} = \begin{matrix} 2 \\ 4 \end{matrix}$$

Mapping of points:

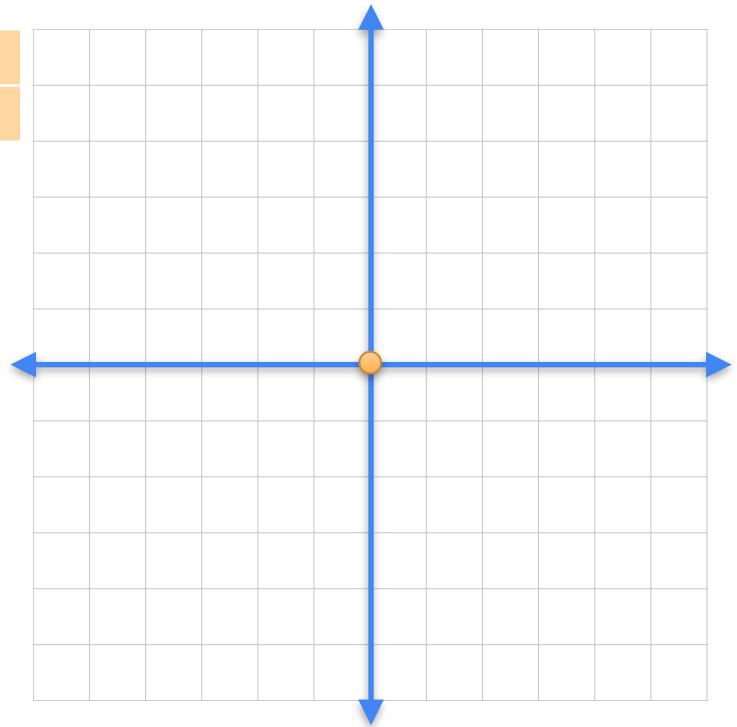
- (0,0)  $\rightarrow$  (0,0)
- (1,0)  $\rightarrow$  (1,2)
- (0,1)  $\rightarrow$  (1,2)
- (1,1)  $\rightarrow$  (2,4)



# Singular transformation



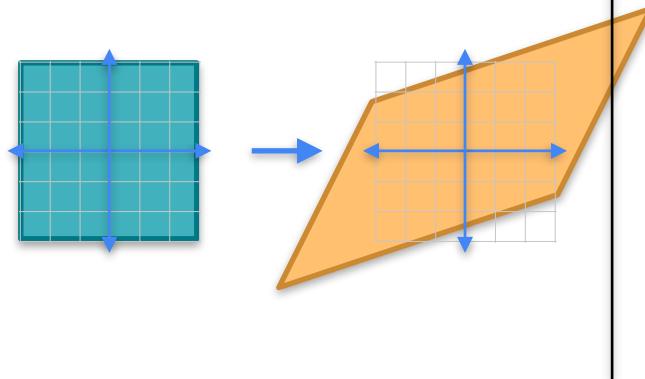
$$\begin{matrix} \text{apple} & \text{banana} \\ 0 & 0 \\ 0 & 0 \end{matrix} \begin{matrix} a \\ b \end{matrix} = \begin{matrix} 0 \\ 0 \end{matrix}$$



# Singular and non-singular transformations

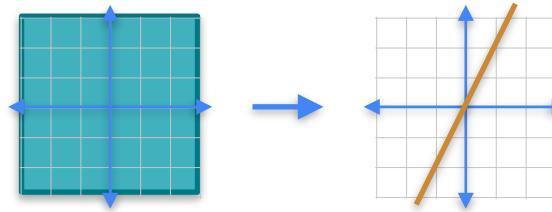
Non-singular

3	1
1	2



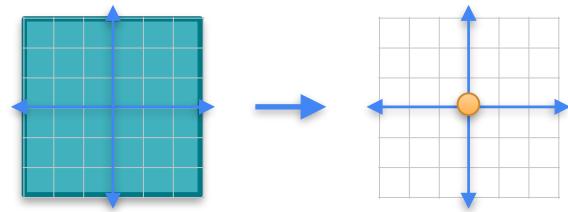
Singular

1	1
2	2



Singular

0	0
0	0



# Rank of linear transformations

Rank 2

3	1
1	2



3	1
1	2

1	1
2	2

Rank 1

1	1
2	2



0	0
0	0

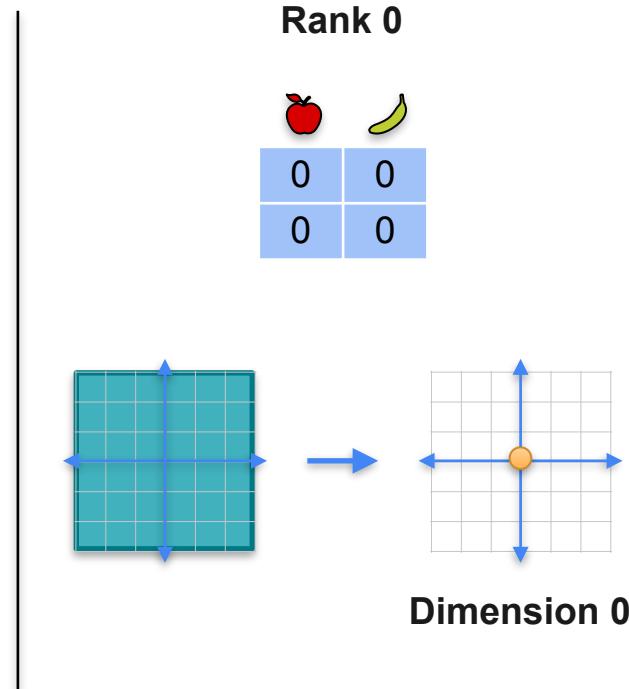
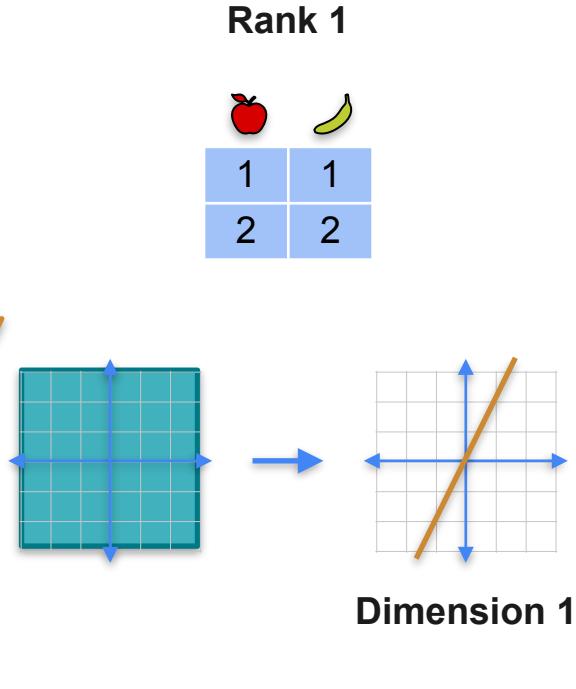
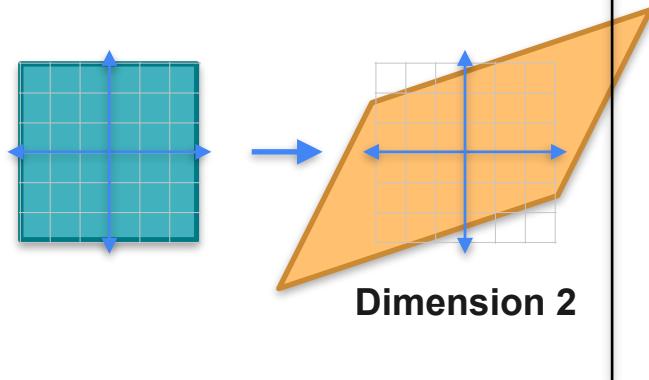
0	0
0	0

Rank 0

0	0
0	0



0	0
0	0





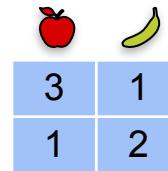
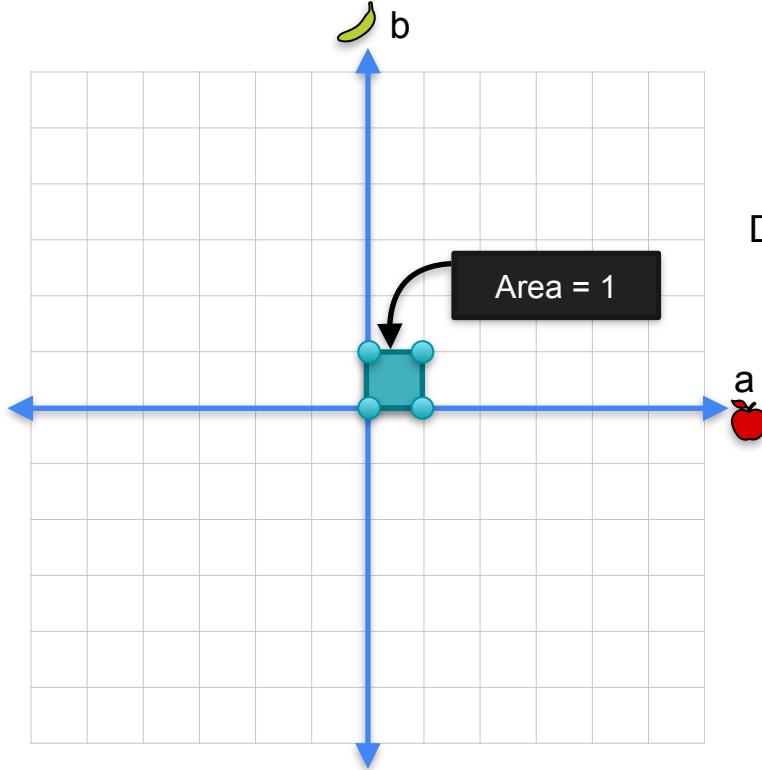
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# Determinants and Eigenvectors

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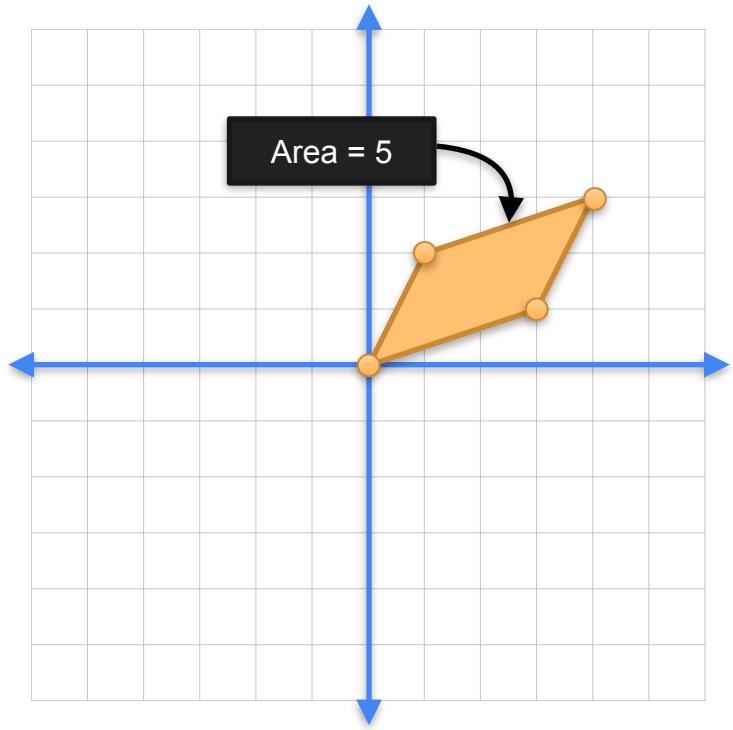
**Determinant as an area**

# Determinant as an area

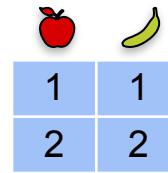
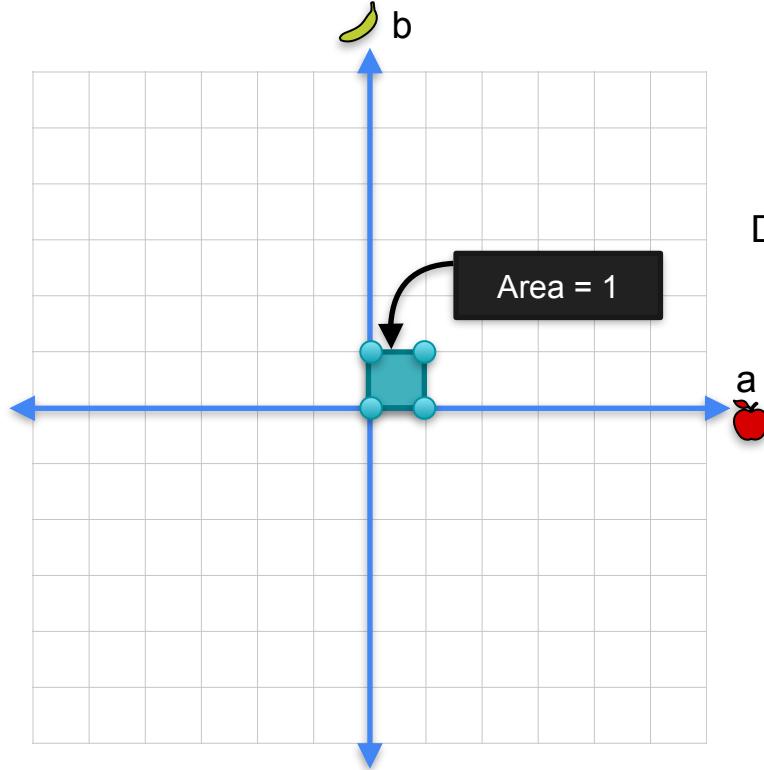


$$\text{Det} = 3 \cdot 2 - 1 \cdot 1$$

$$\text{Det} = 5$$

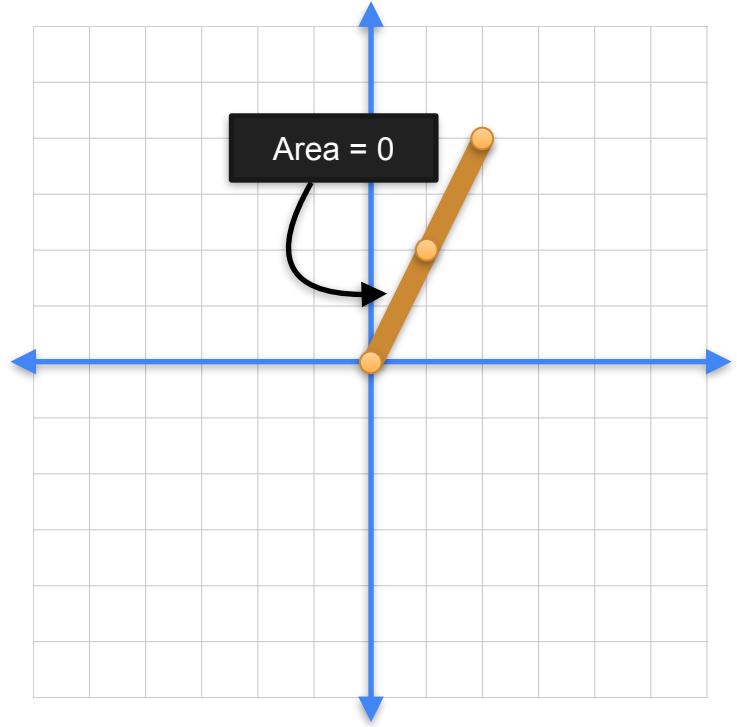


# Determinant as an area

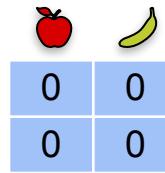
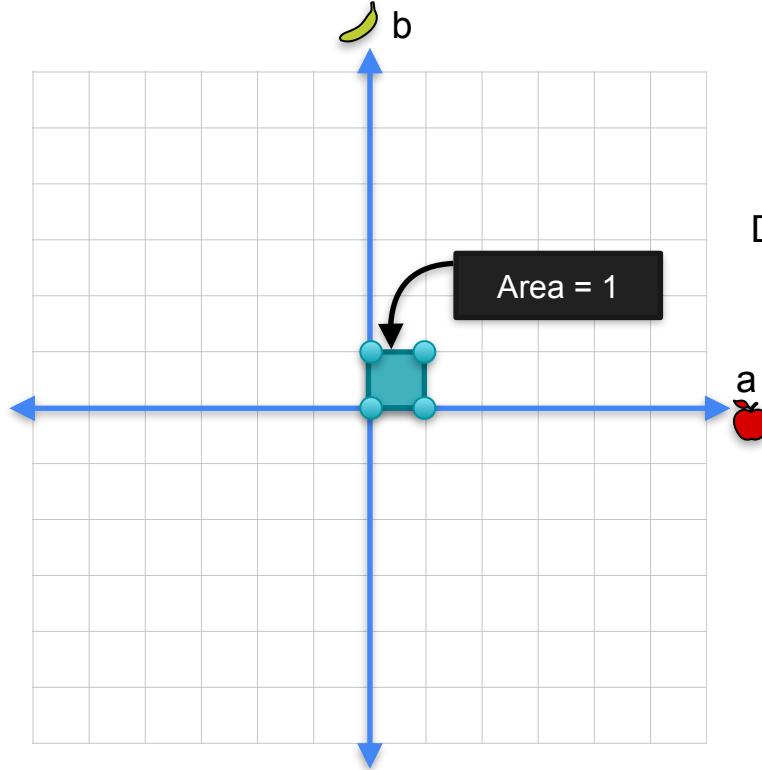


$$\text{Det} = 1 \cdot 2 - 1 \cdot 2$$

$$\text{Det} = 0$$

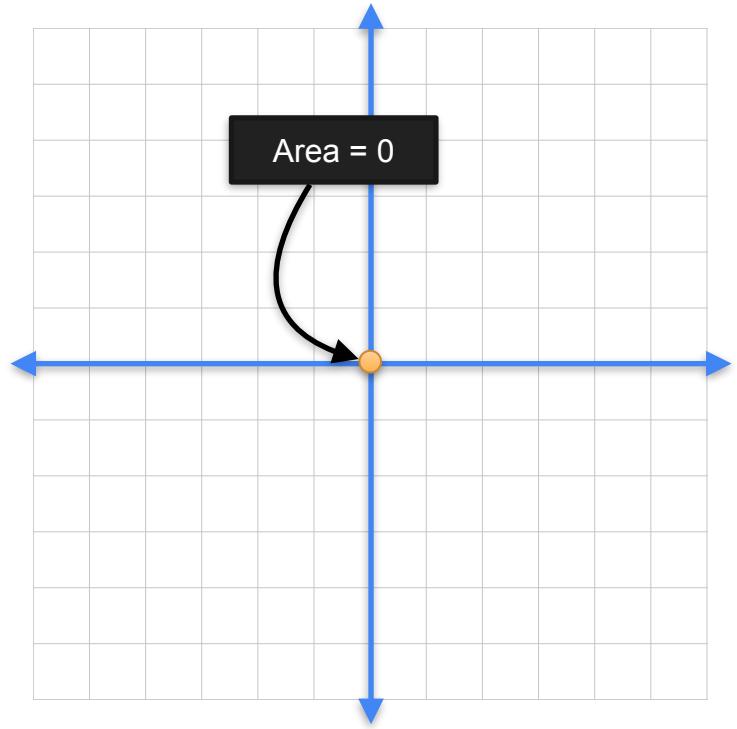


# Determinant as an area



$$\text{Det} = 0 \cdot 0 - 0 \cdot 0$$

$$\text{Det} = 0$$

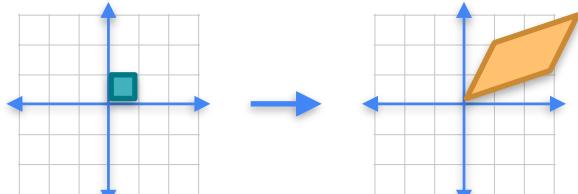


# Determinant as an area

Non-singular

3	1
1	2

$$\text{Determinant} = 5$$

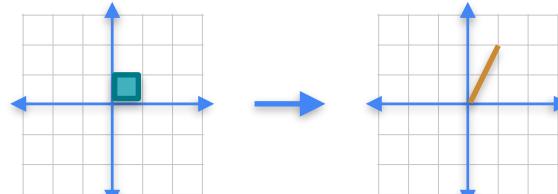


$$\text{Area} = 5$$

Singular

1	1
2	2

$$\text{Determinant} = 0$$

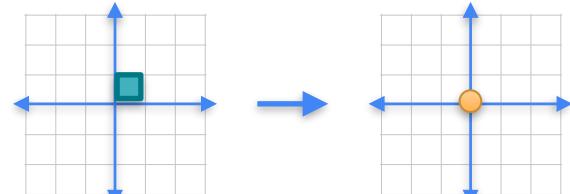


$$\text{Area} = 0$$

Singular

0	0
0	0

$$\text{Determinant} = 0$$



$$\text{Area} = 0$$

# Negative determinants?

	
3	1
1	2

$$\text{Det} = 3 \cdot 2 - 1 \cdot 1$$

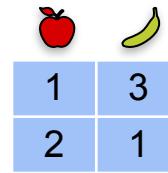
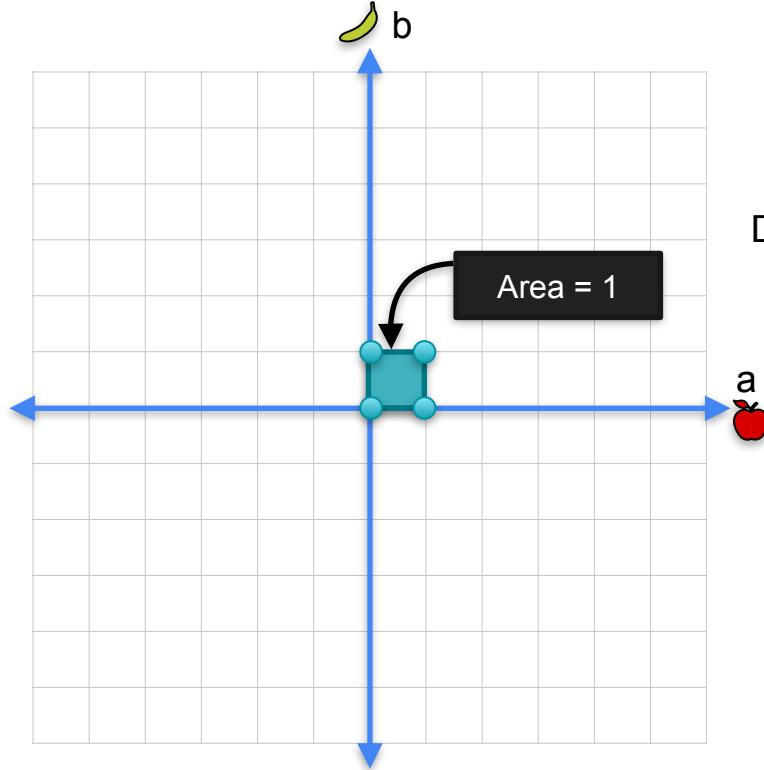
$$\text{Det} = 5$$

	
1	3
2	1

$$\text{Det} = 1 \cdot 1 - 3 \cdot 2$$

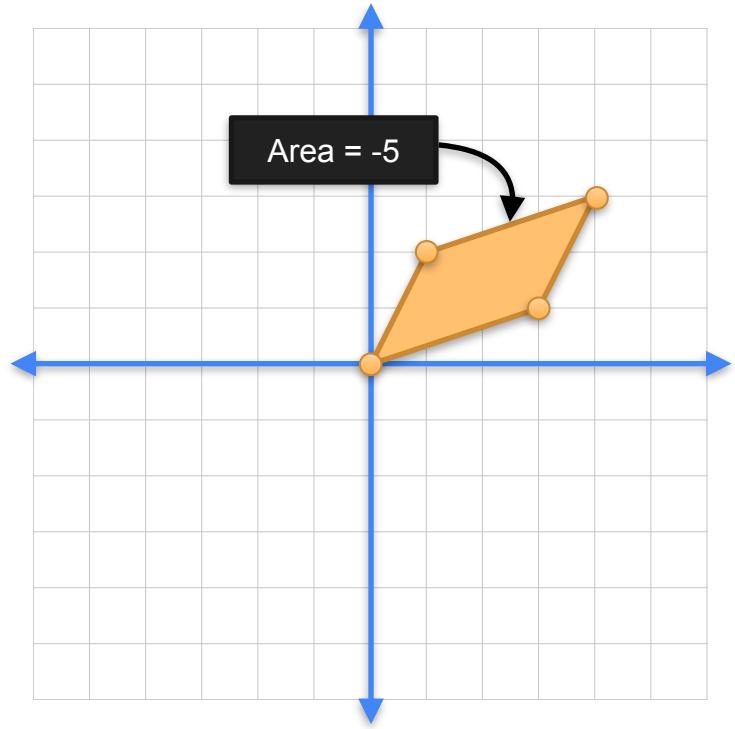
$$\text{Det} = -5$$

# Determinant as an area

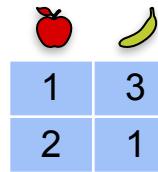
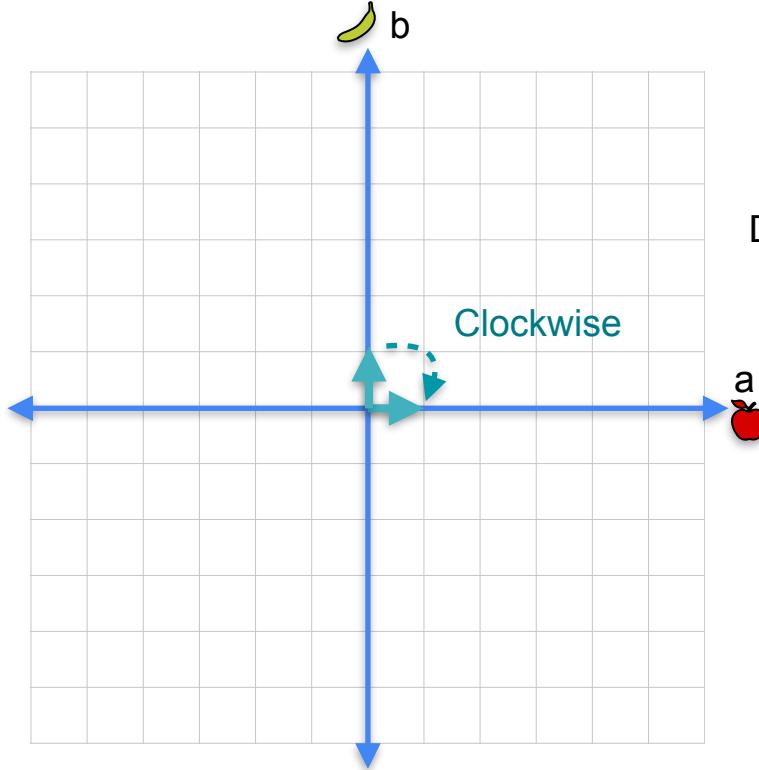


$$\text{Det} = 1 \cdot 1 - 3 \cdot 2$$

$$\text{Det} = -5$$



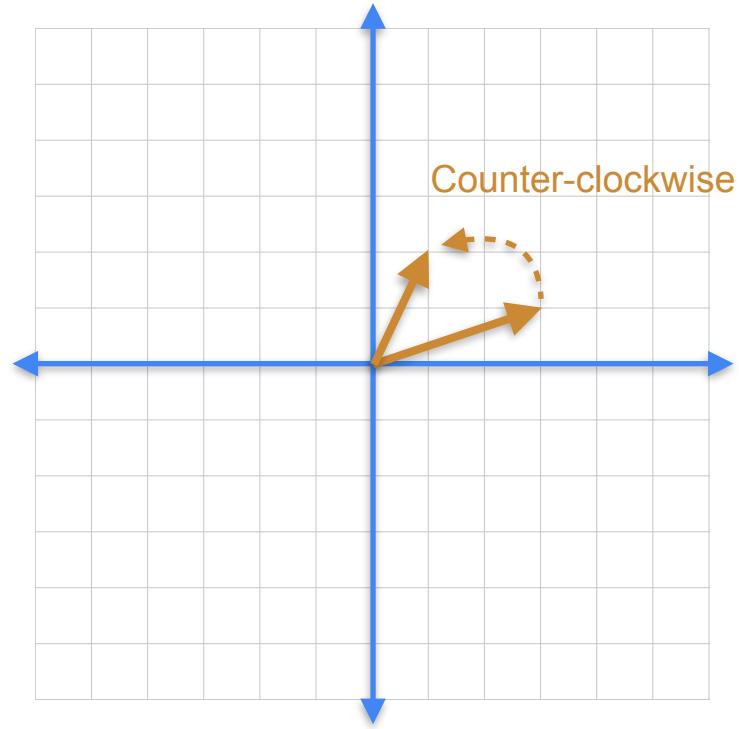
# Determinant as an area



$$\text{Det} = 1 \cdot 1 - 3 \cdot 2$$

$$\text{Det} = -5$$

Negative





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## Determinants and Eigenvectors

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### Determinant of a product

# Determinant of a product

3	1
1	2

5	2
1	2

=

16	8
7	6

$$\det = 5$$

$$3 \cdot 2 - 1 \cdot 1$$

$$\det = 8$$

$$5 \cdot 2 - 2 \cdot 1$$

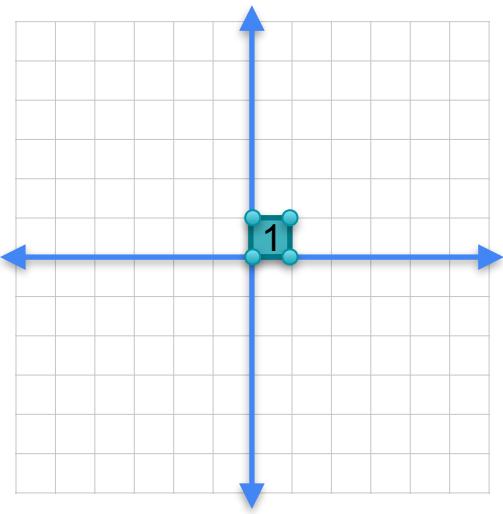
$$\det = 40$$

$$16 \cdot 6 - 8 \cdot 7$$

# Determinant of a product

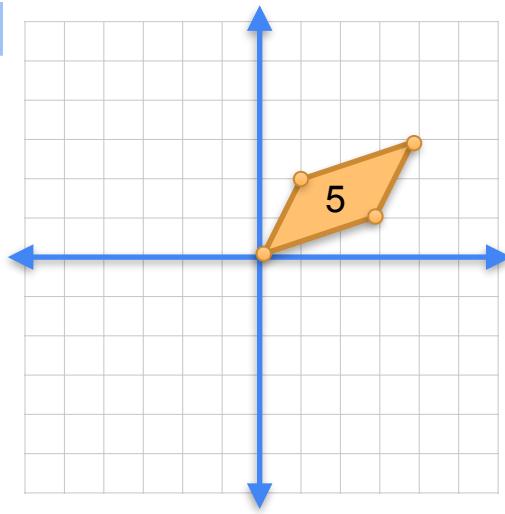
$$\det(AB) = \det(A) \det(B)$$

# Determinant of a product



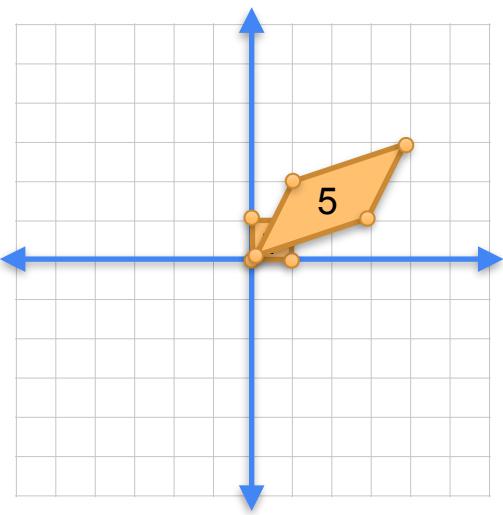
$$\begin{array}{|c|c|} \hline 3 & 1 \\ \hline 1 & 2 \\ \hline \end{array}$$

Det = 5



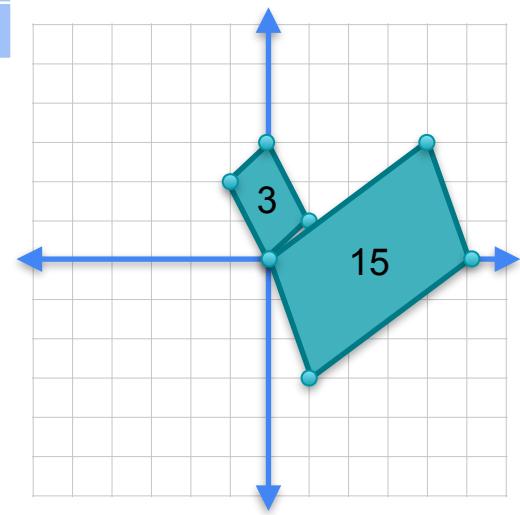
Area blows up by 5

# Determinant of a product



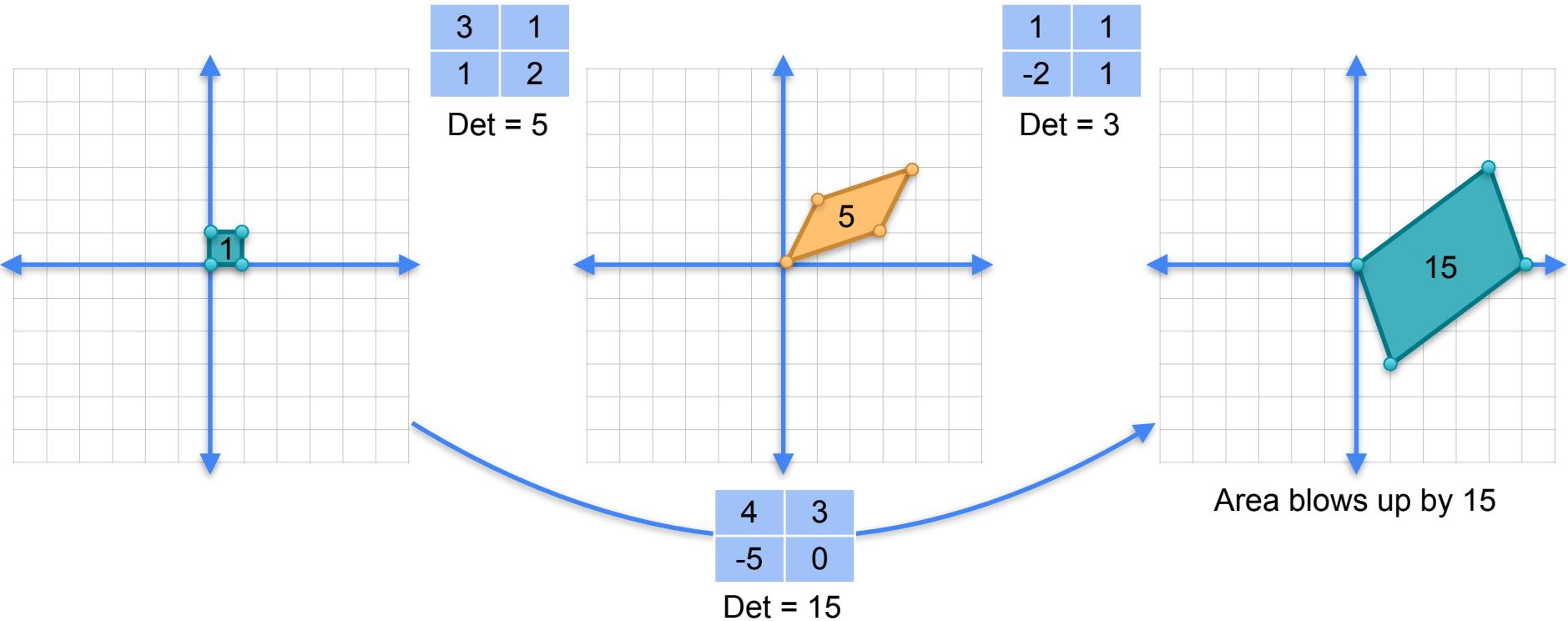
$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline -2 & 1 \\ \hline \end{array}$$

Det = 3



Area blows up by 3

# Determinant of a product



# Quiz

- The product of a singular and a non-singular matrix (in any order) is:
  - Singular
  - Non-singular
  - Could be either one

# Solution

- If A is non-singular and B is singular, then  $\det(AB) = \det(A) \times \det(B) = 0$ , since  $\det(B) = 0$ . Therefore  $\det(AB) = 0$ , so AB is **singular**.

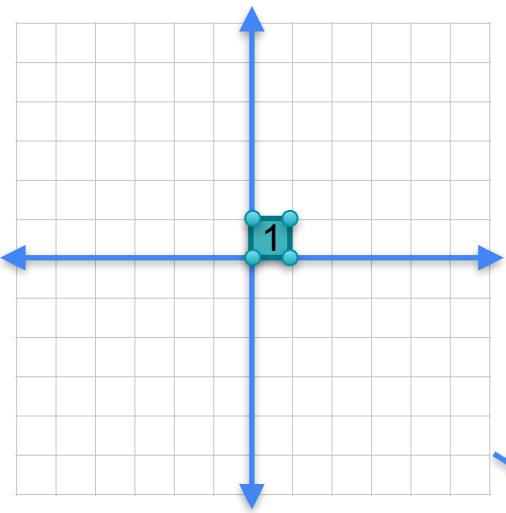
# When one factor is zero

$$5 \cdot 0 = 0$$

# When one factor is singular...

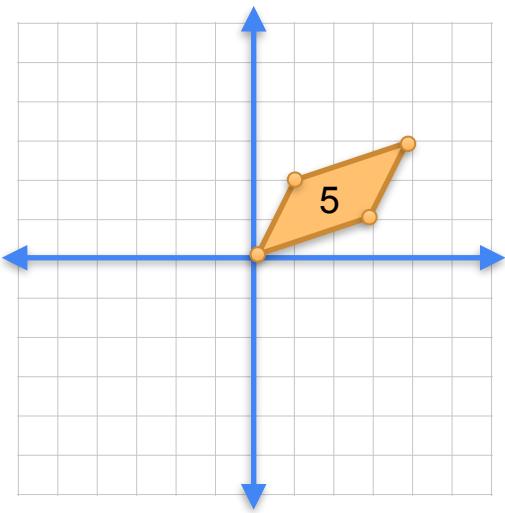
Non-singular	Singular	Singular	
$\begin{array}{ c c } \hline 3 & 1 \\ \hline 1 & 2 \\ \hline \end{array}$ Det = 5	$\begin{array}{ c c } \hline 1 & 2 \\ \hline 1 & 2 \\ \hline \end{array}$ Det = 0	$=$	$\begin{array}{ c c } \hline 4 & 8 \\ \hline 3 & 6 \\ \hline \end{array}$ Det = 0

# If one factor is singular...



3	1
1	2

Det = 5

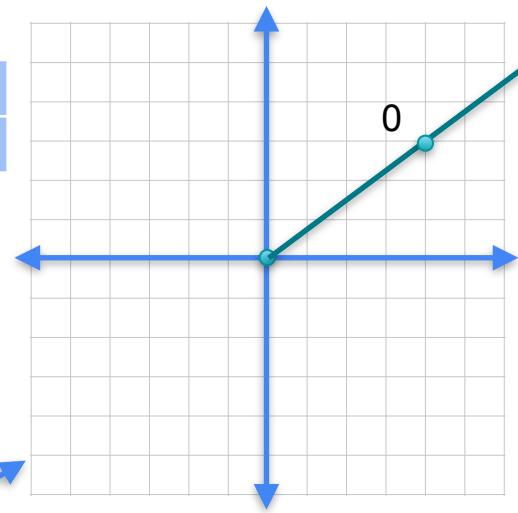


1	2
1	2

Det = 0

4	8
3	6

Det = 0



Area blows up by 0



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## Determinants and Eigenvectors

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### Determinant of inverse

# Quiz

- Find the determinants of the following matrices

0.4	-0.2
-0.2	0.6

0.25	-0.25
-0.125	0.625

# Solution

$$\text{Det} \begin{array}{|c|c|} \hline 0.4 & -0.2 \\ \hline -0.2 & 0.6 \\ \hline \end{array} = (0.4)(0.6) - (-0.2)(-0.2) = 0.2$$

$$\text{Det} \begin{array}{|c|c|} \hline 0.25 & -0.25 \\ \hline -0.125 & 0.625 \\ \hline \end{array} = (0.25)(0.625) - (-0.125)(-0.25) = 0.125$$

# Determinant of an inverse

$$\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}^{-1} = \begin{vmatrix} 0.4 & -0.2 \\ -0.2 & 0.6 \end{vmatrix}$$

$$\det = 5$$

$$5^{-1} = 0.2$$

$$\begin{vmatrix} 5 & 2 \\ 1 & 2 \end{vmatrix}^{-1} = \begin{vmatrix} 0.25 & -0.25 \\ -0.125 & 0.625 \end{vmatrix}$$

$$\det = 8$$

$$8^{-1} = 0.125$$

$$\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = \begin{vmatrix} ? & ? \\ ? & ? \end{vmatrix}$$

$$\det = 0$$

$$0^{-1} = ???$$

# Determinant of an inverse

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

# Why?

$$\det(AB) = \det(A) \det(B)$$

Why is this?

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

$$\begin{aligned} \det(AA^{-1}) &= \det(A) \det(A^{-1}) \\ \det(I) &= \det(A) \det(A^{-1}) \\ 1 &= \det(A) \det(A^{-1}) \\ 1 &= \frac{1}{\det(A)} \end{aligned}$$

# Determinant of the identity matrix

$$\det \begin{array}{|cc|} \hline 1 & 0 \\ 0 & 1 \\ \hline \end{array} = 1 \cdot 1 - 0 \cdot 0 = 1$$

$$\det(I) = 1$$

# W4 Lesson 2



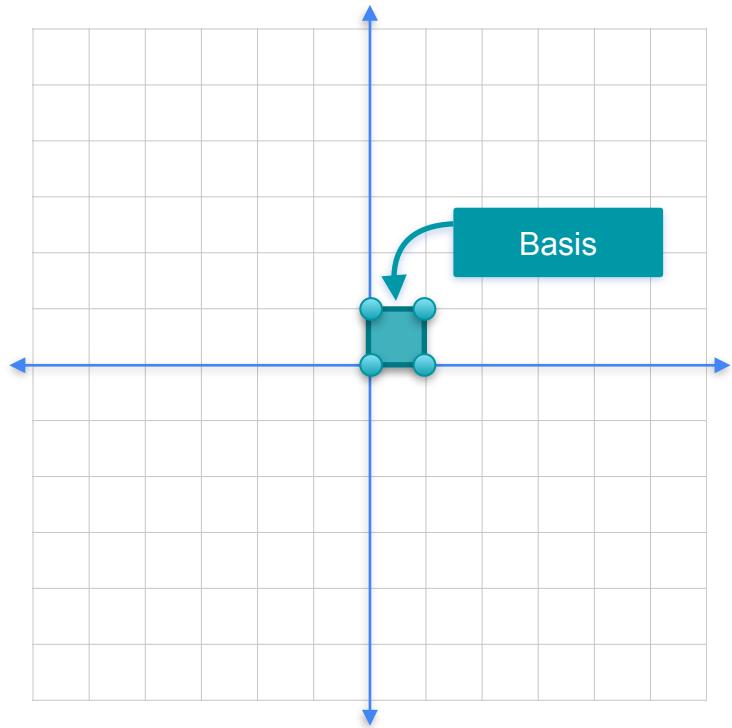
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# Determinants and Eigenvectors

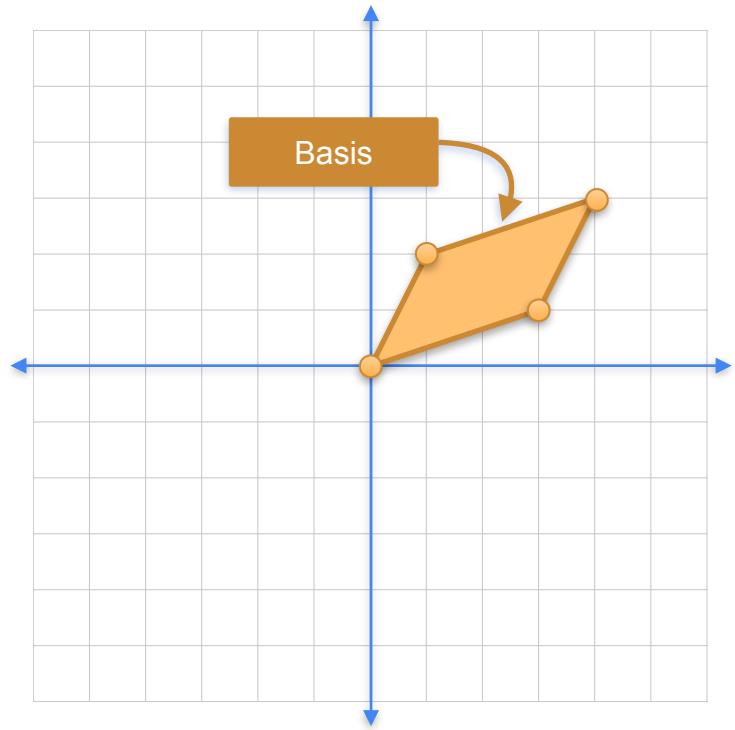
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## Bases

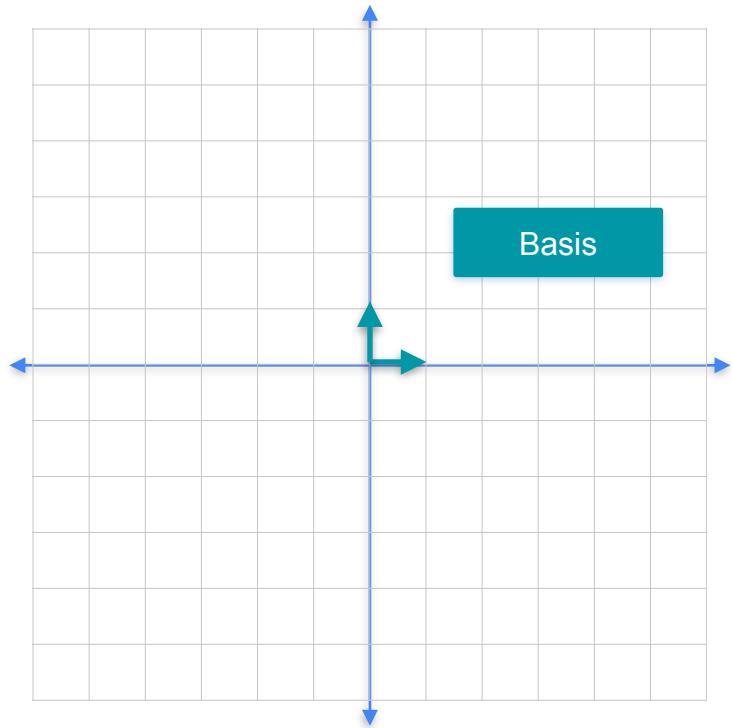
# Bases



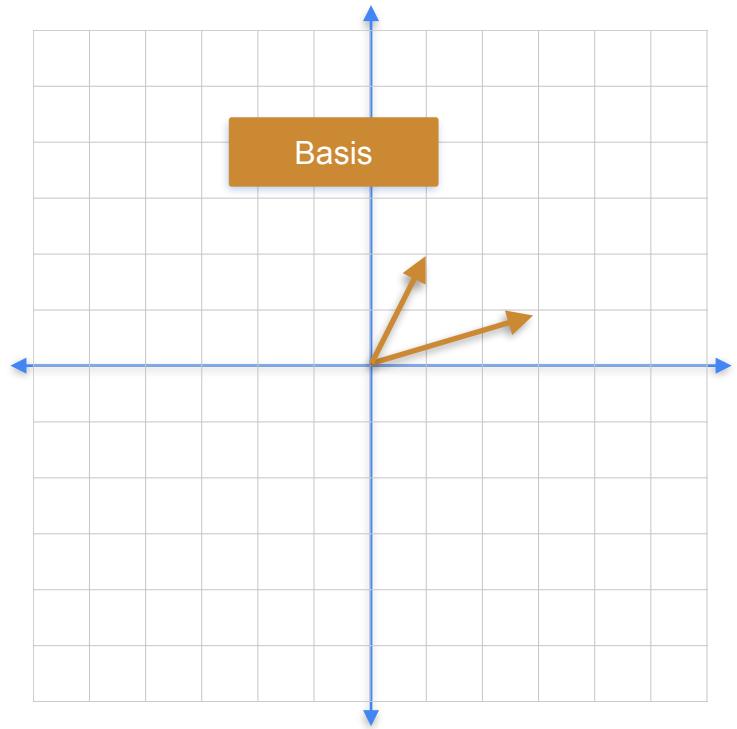
3	1
1	2



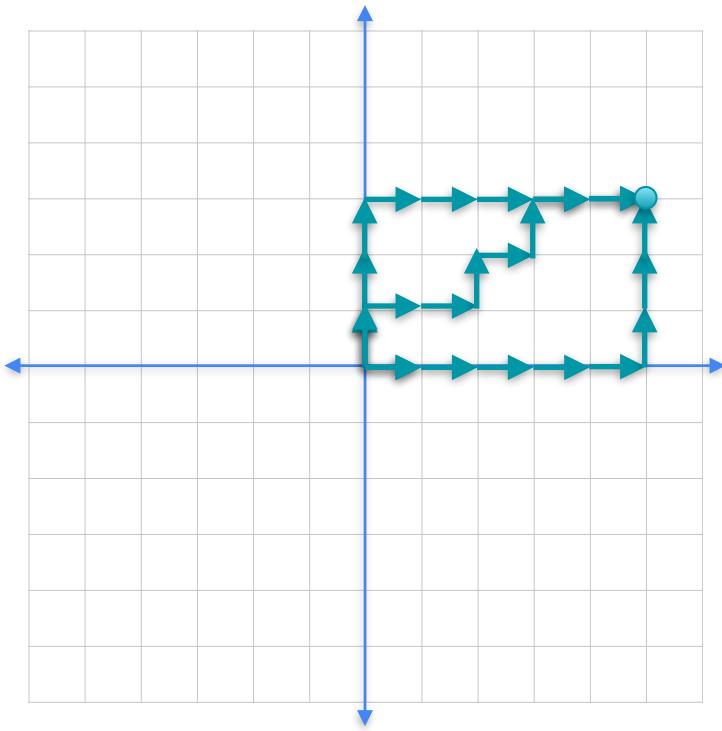
# Bases



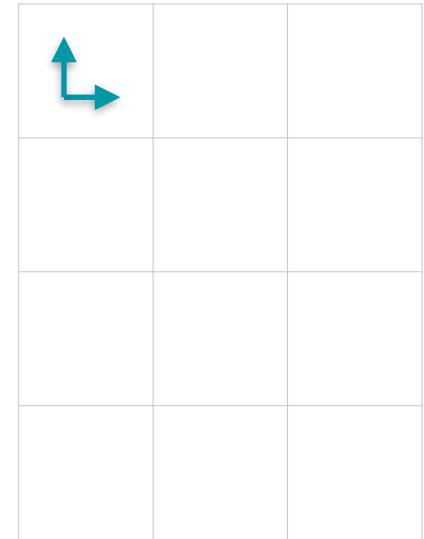
3	1
1	2



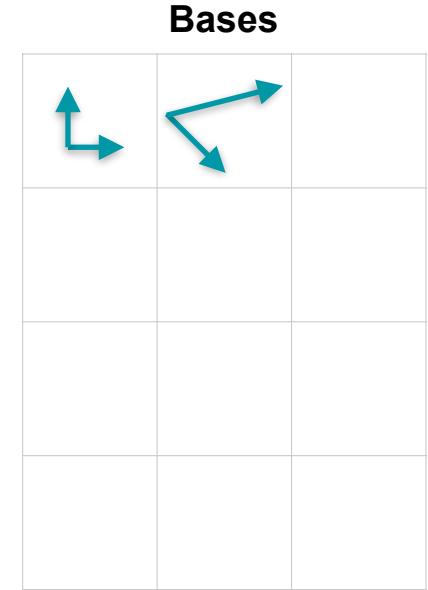
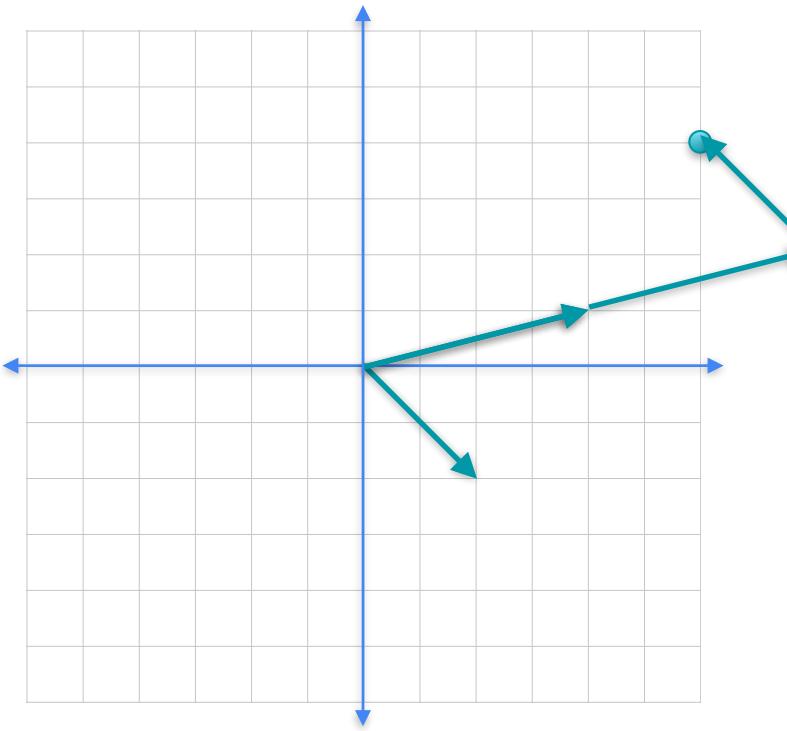
# Bases



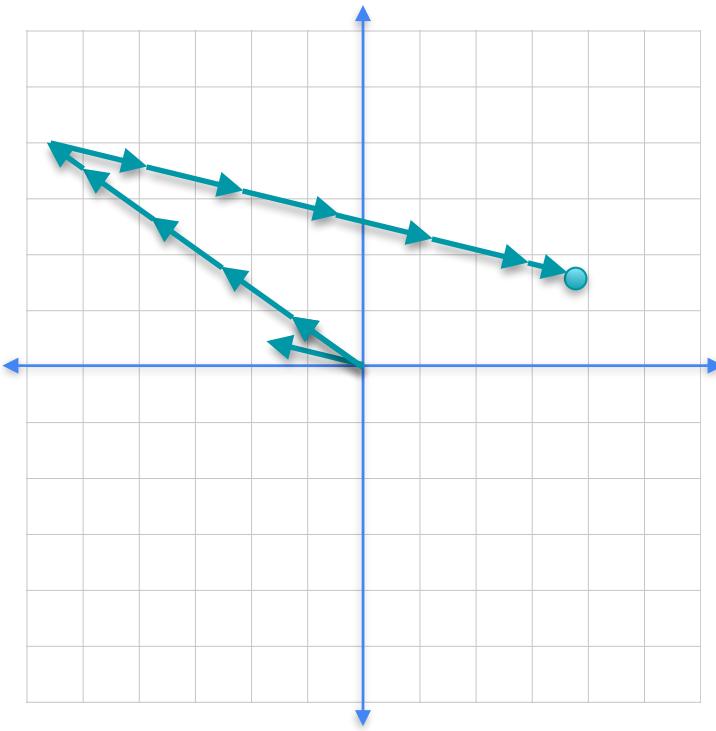
Bases



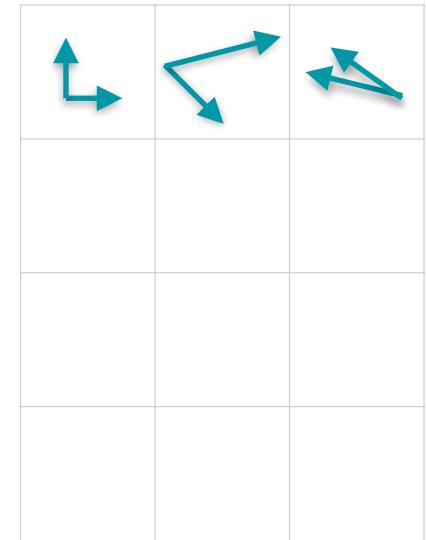
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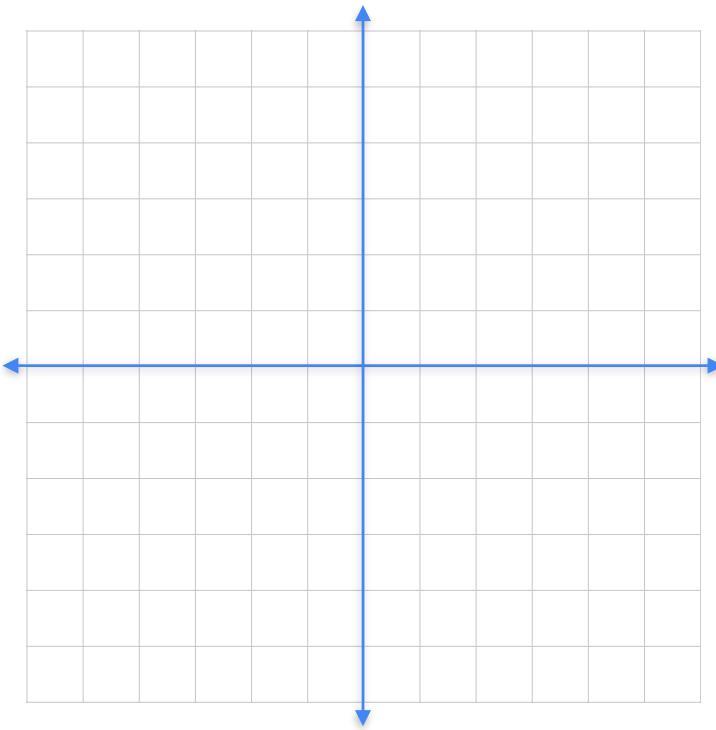
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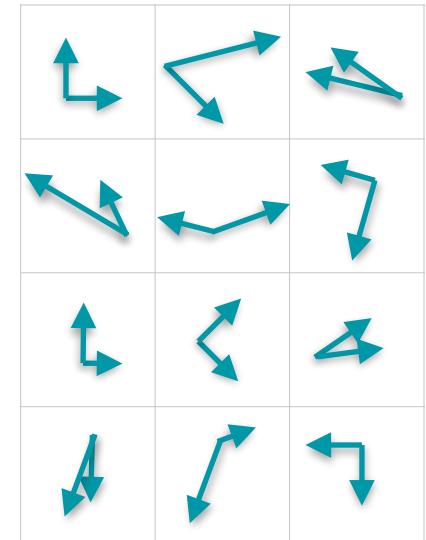
**Bases**



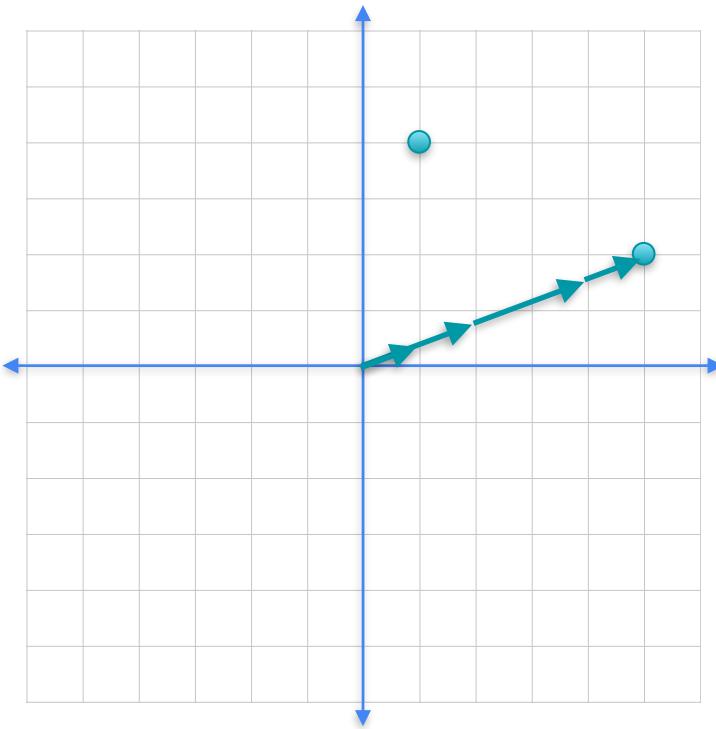
# Bases



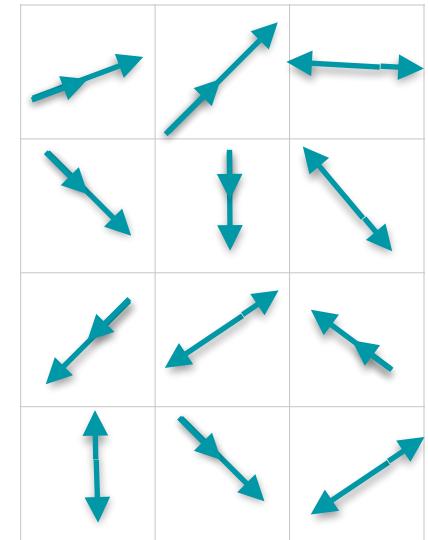
**Bases**



# What is not a basis?



Not bases





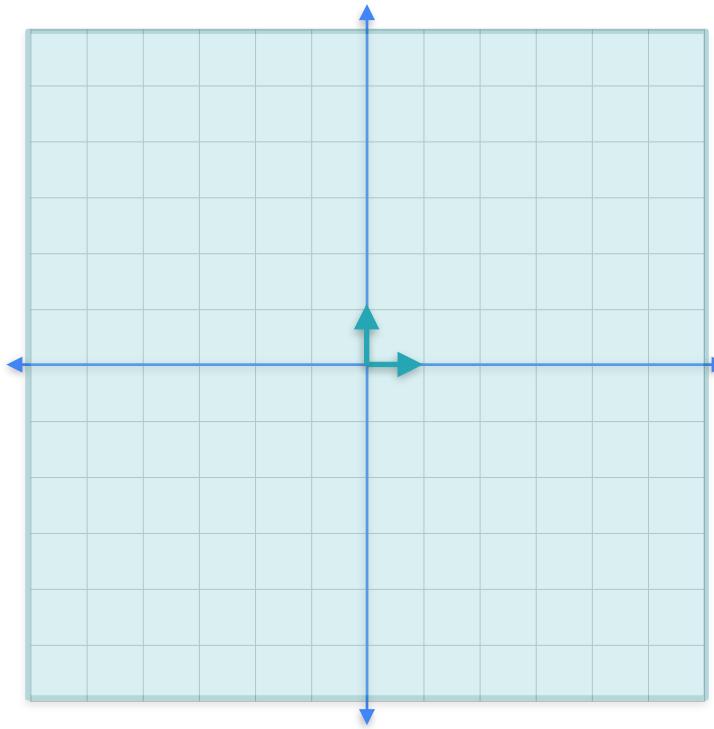
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# Determinants and Eigenvectors

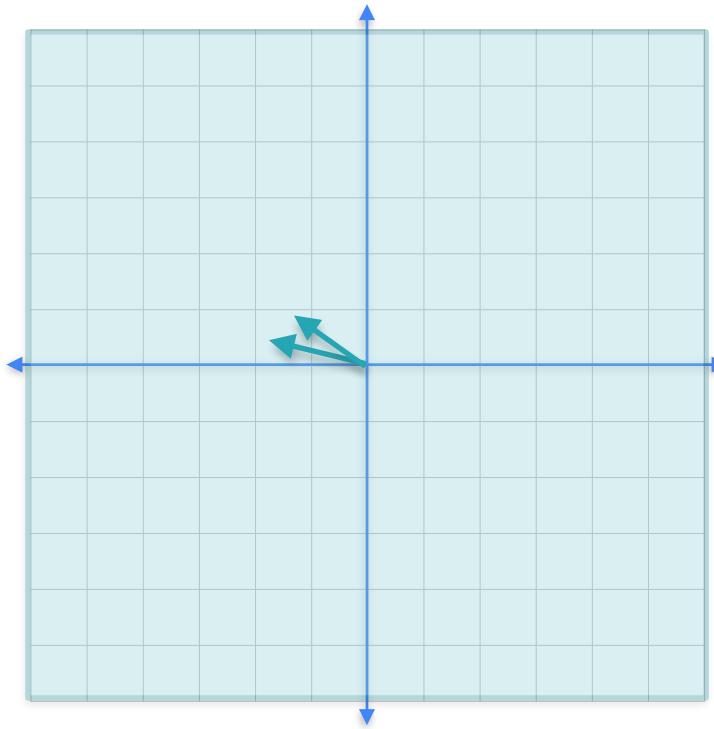
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## Span

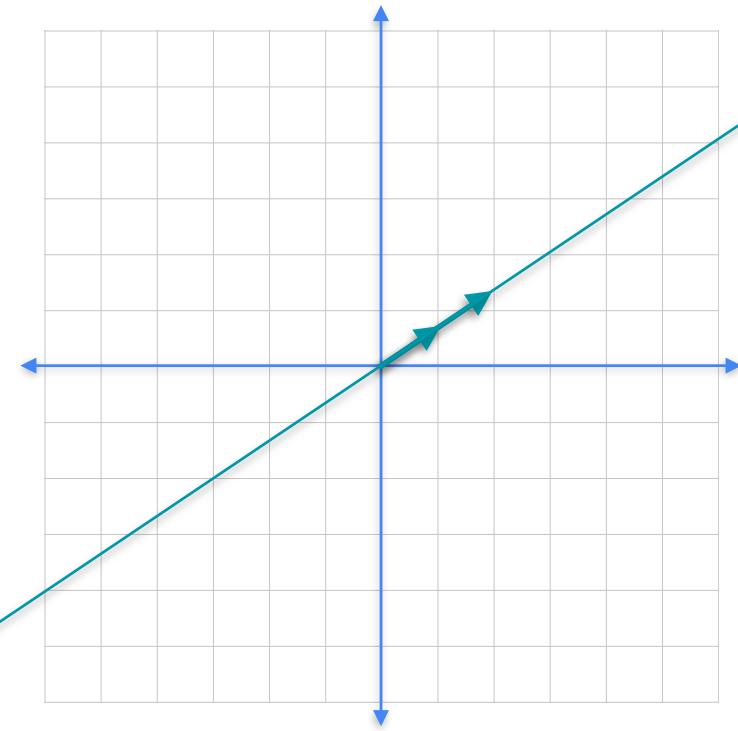
# Span



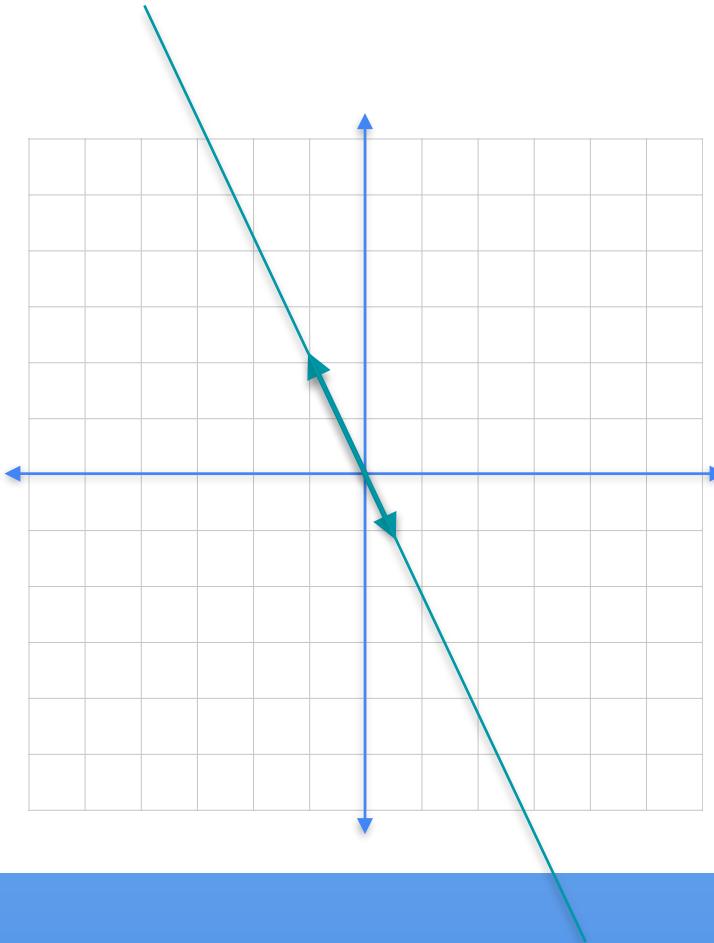
# Span



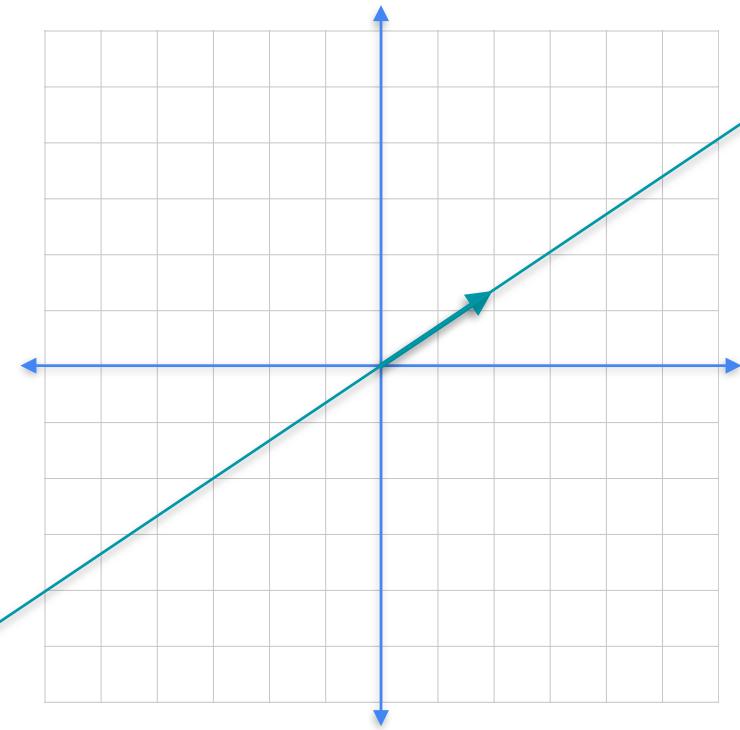
# Span



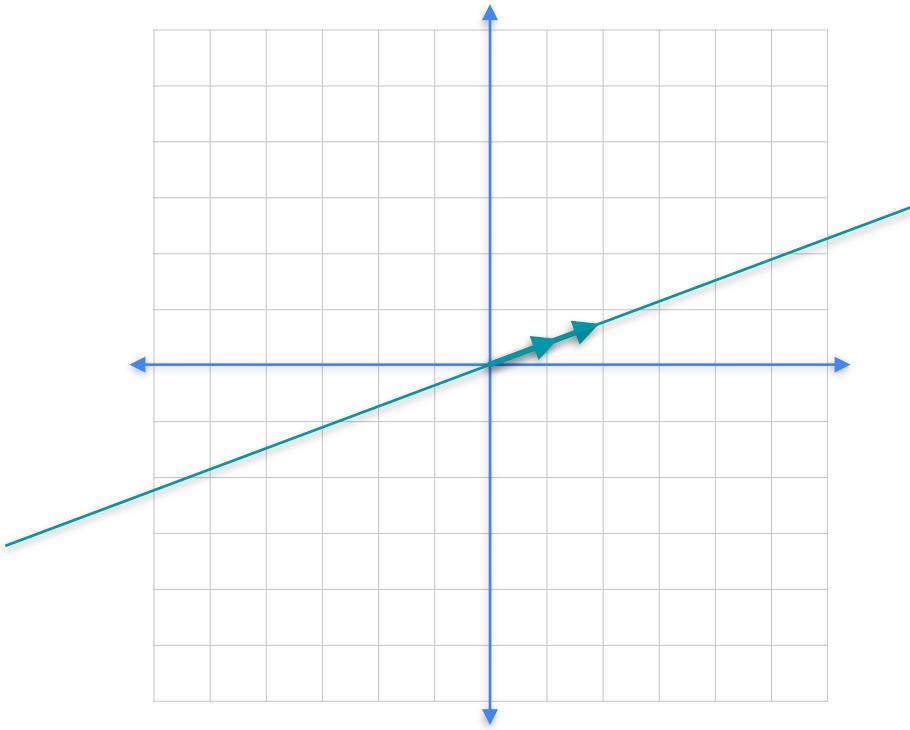
# Span



# Span

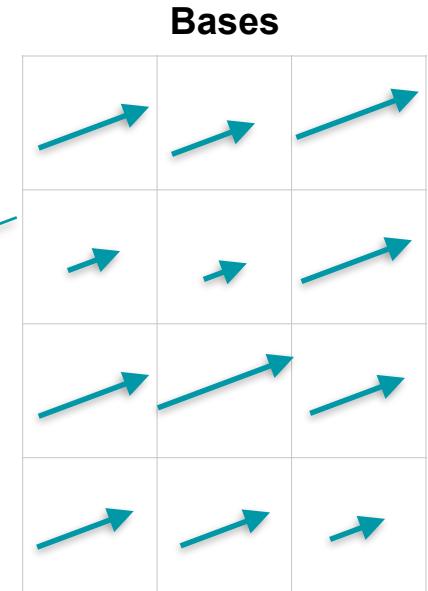
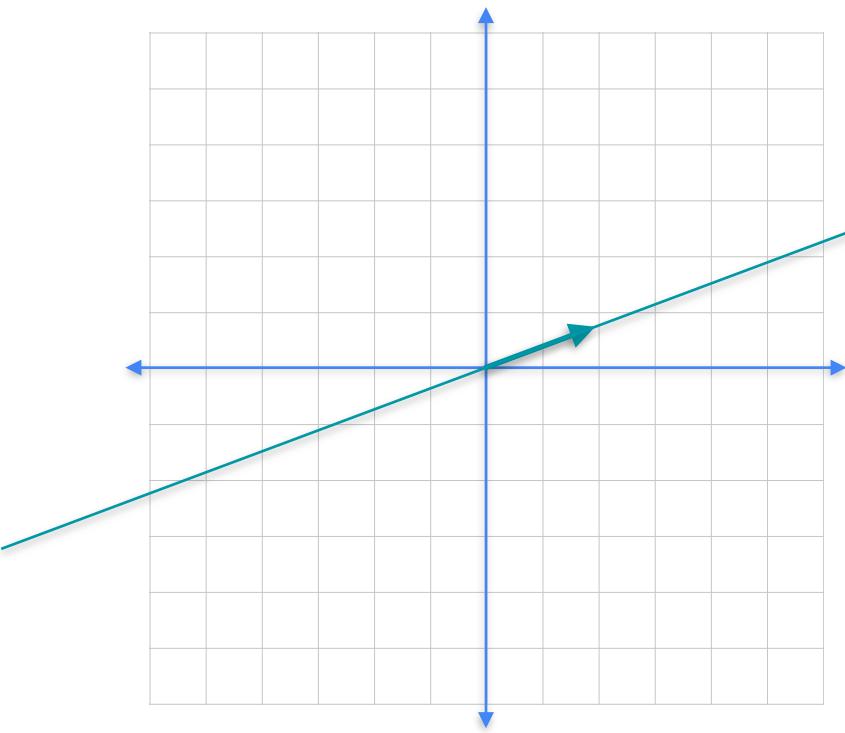


# Is this a basis?

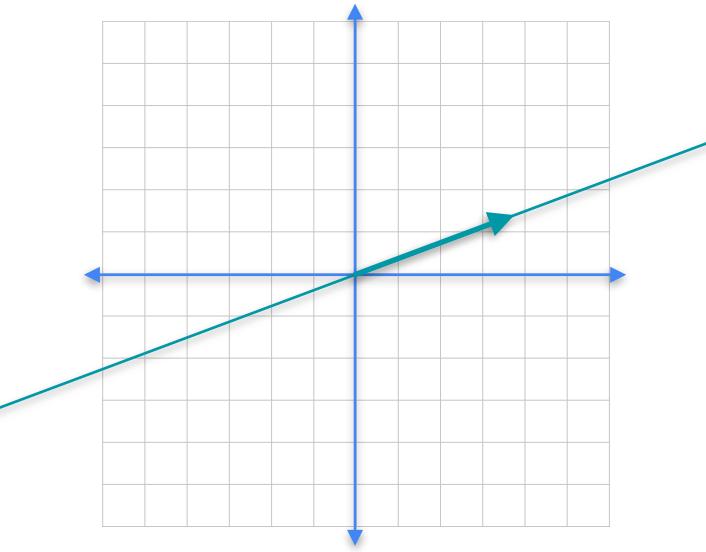


No

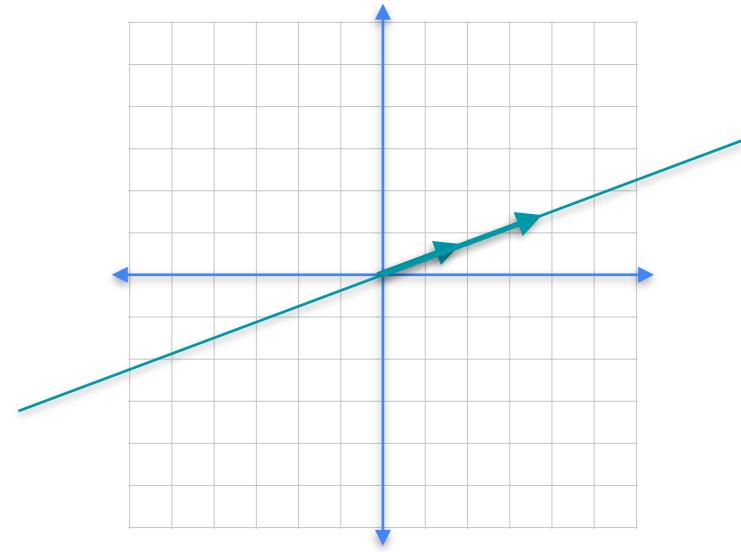
# Is this a basis for something?



# A basis is a minimal spanning set

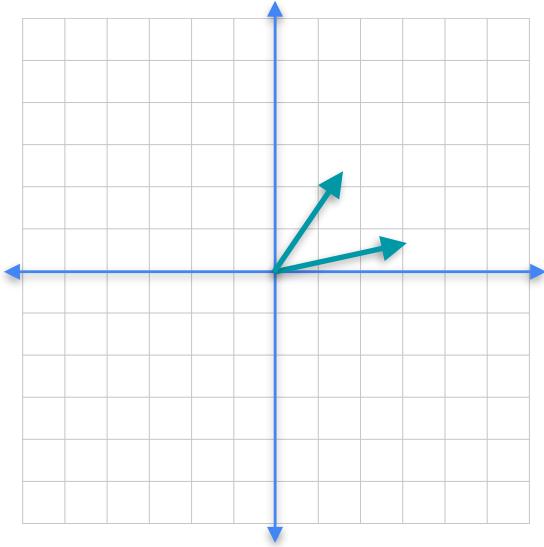


Basis

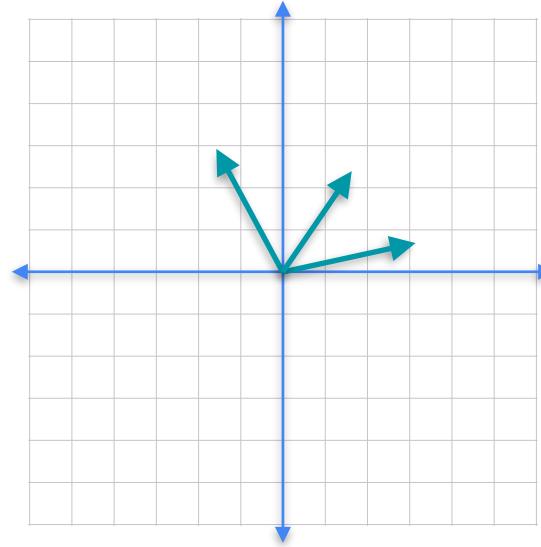


Not a basis

# A basis is a minimal spanning set

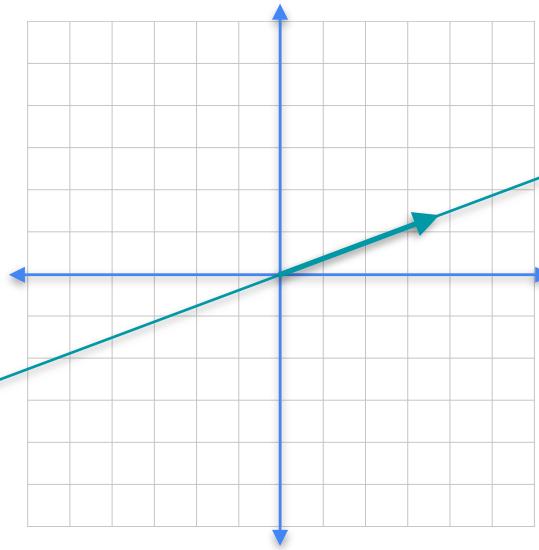


Basis

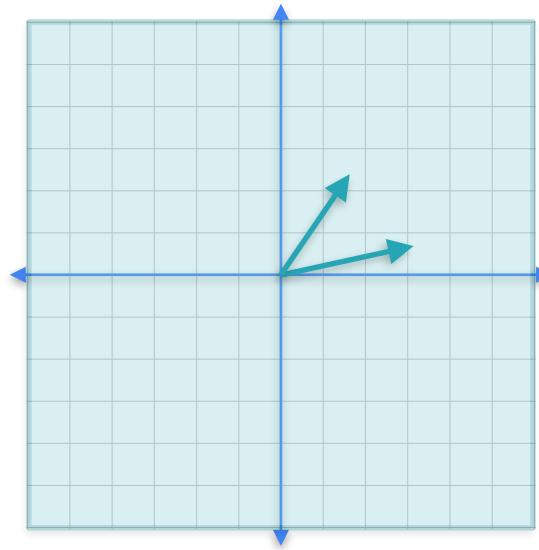


Not a basis

# Number of elements in the basis is the dimension

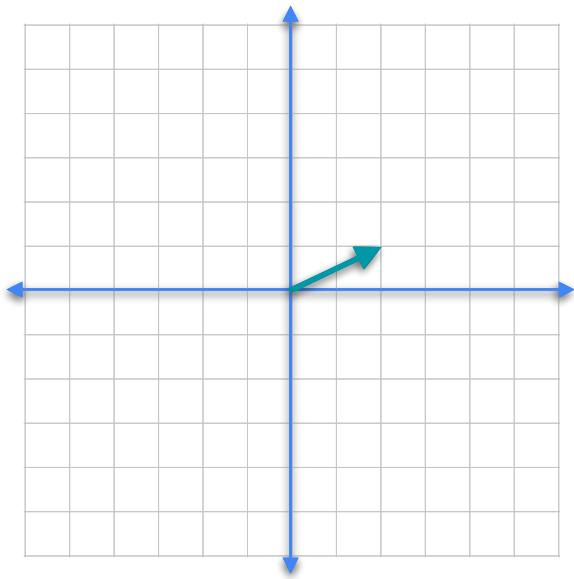


Dimensions: 1  
1 element in the basis



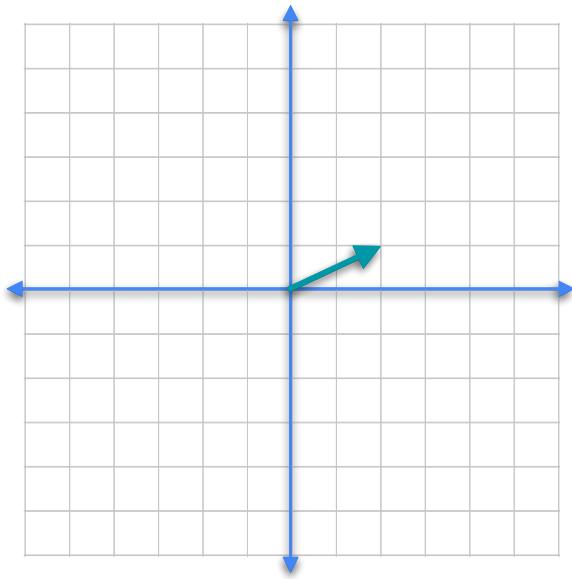
Dimensions: 2  
2 elements in the basis

# Linearly independent and linearly dependent vectors

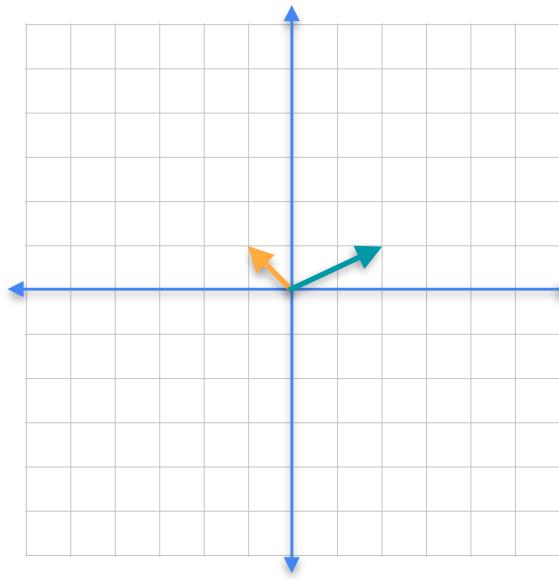


Linearly independent

# Linearly independent and linearly dependent vectors

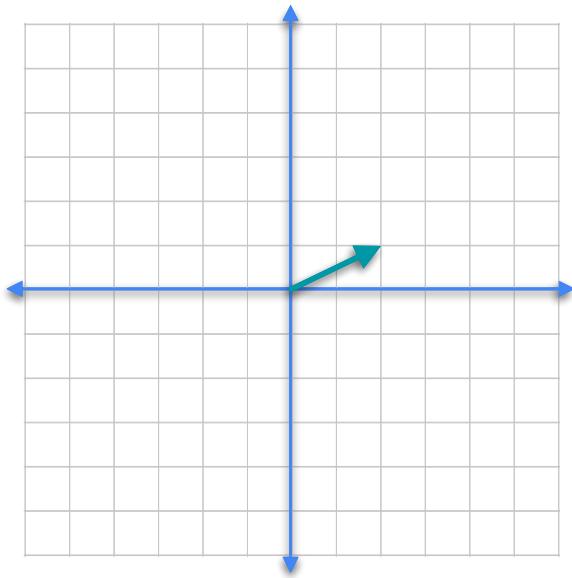


Linearly independent

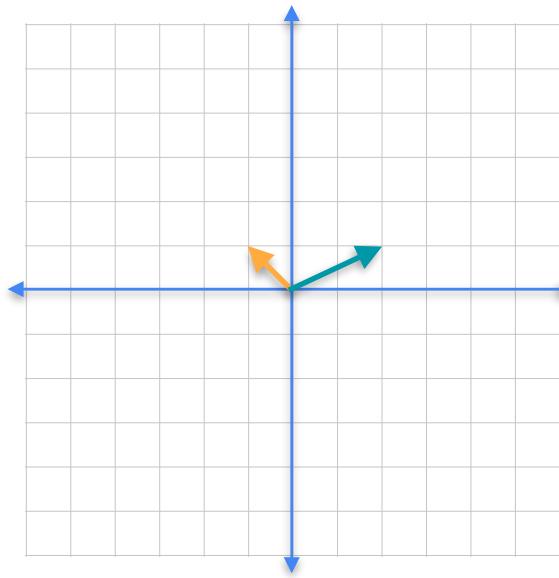


Linearly independent

# Linearly independent and linearly dependent vectors

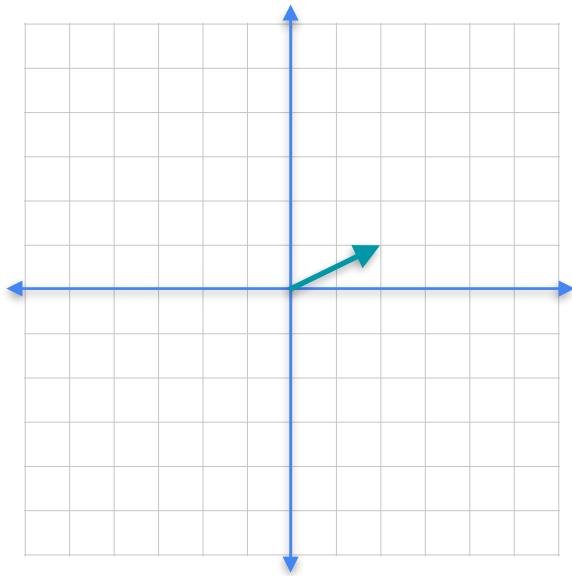


Linearly independent

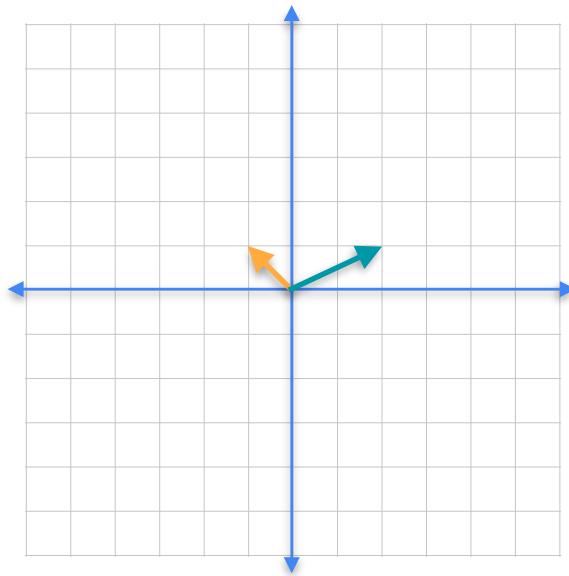


Linearly independent

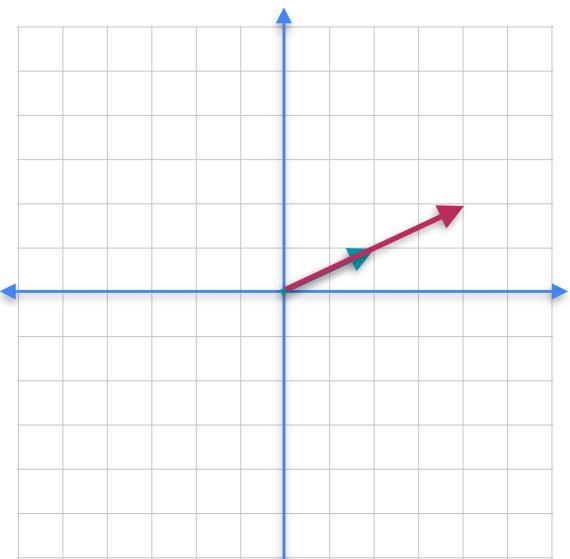
# Linearly independent and linearly dependent vectors



Linearly independent

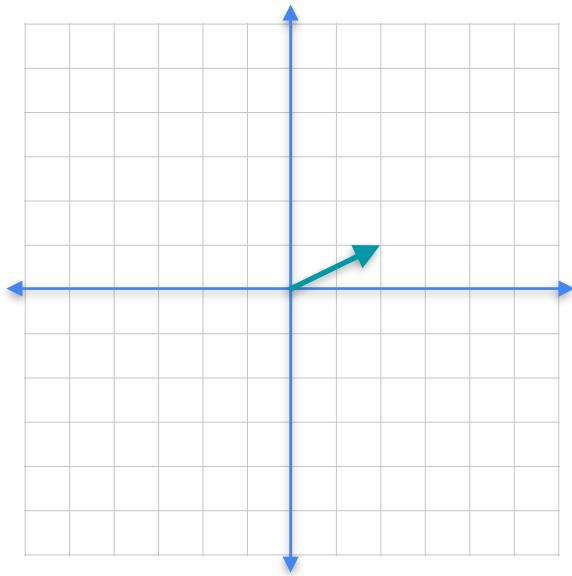


Linearly independent

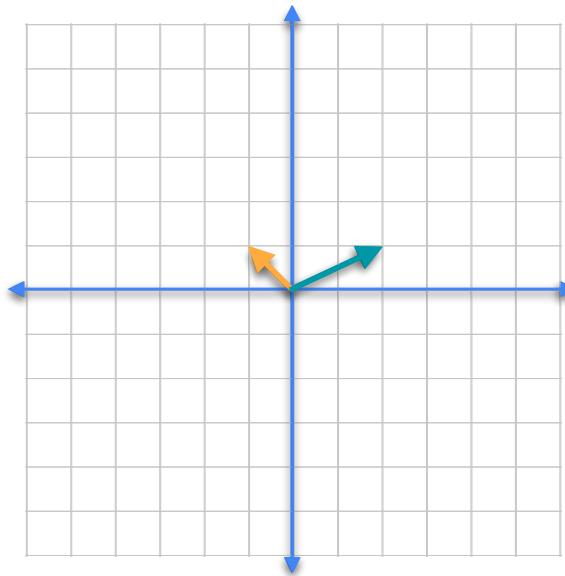


Linearly dependent

# Linearly independent and linearly dependent vectors

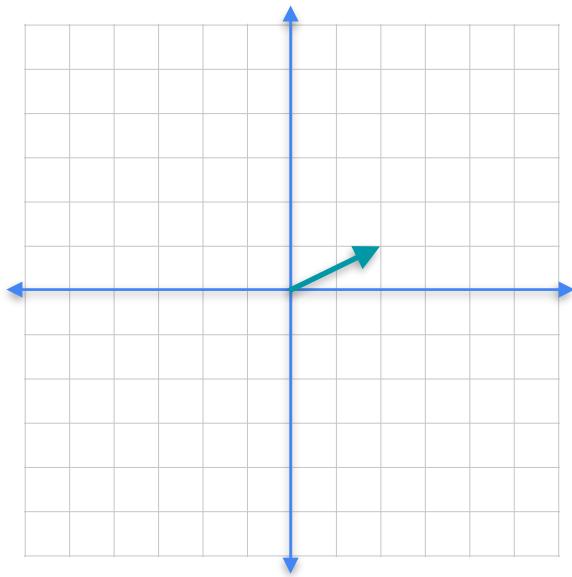


Linearly independent

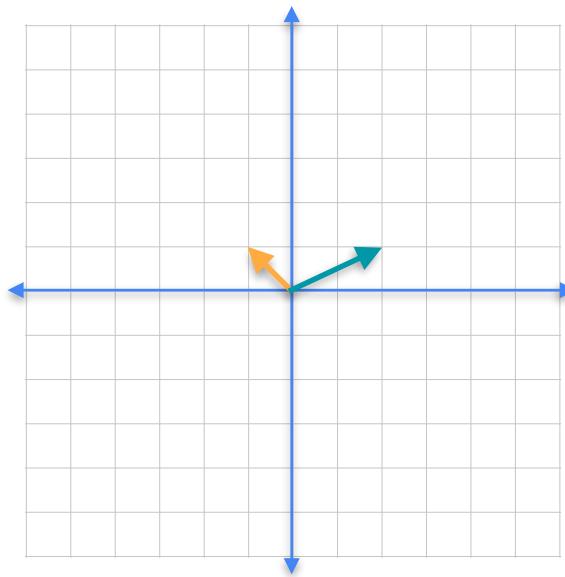


Linearly independent

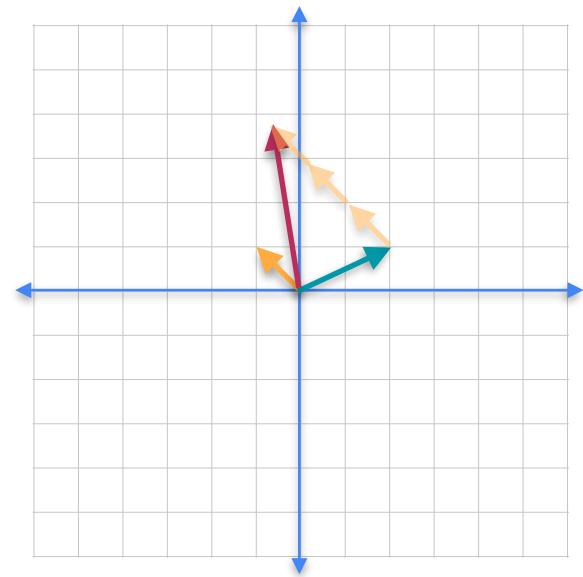
# Linearly independent and linearly dependent vectors



Linearly independent

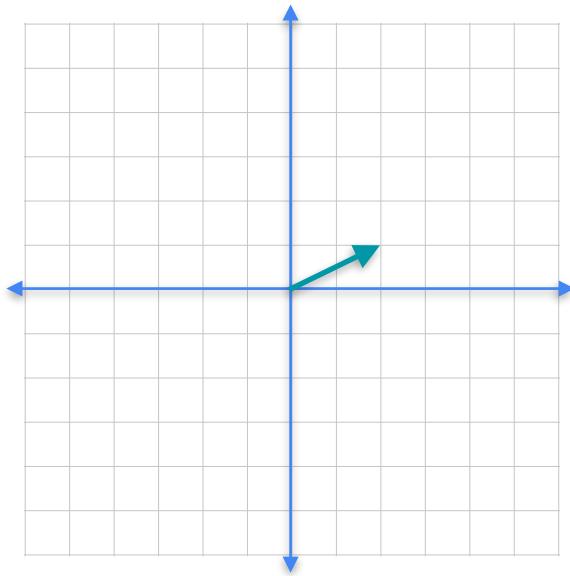


Linearly independent

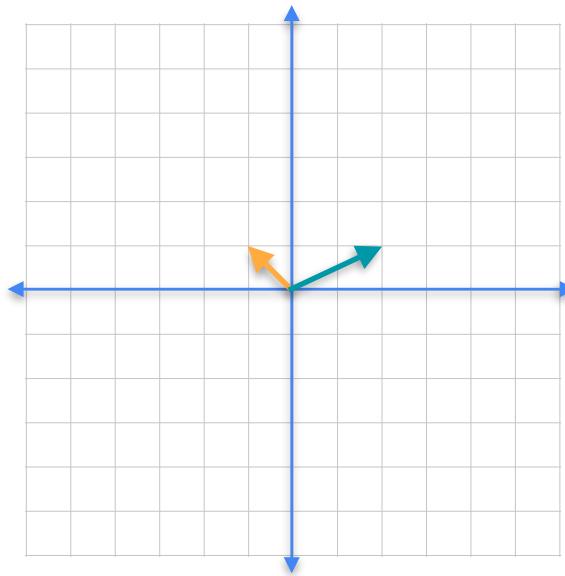


Linearly dependent

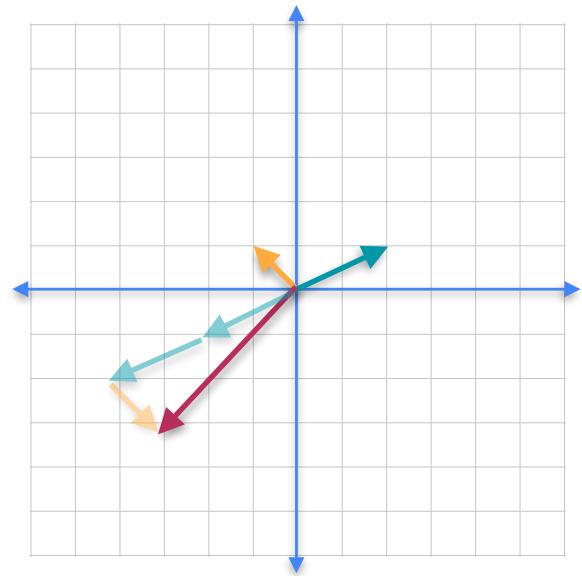
# Linearly independent and linearly dependent vectors



Linearly independent

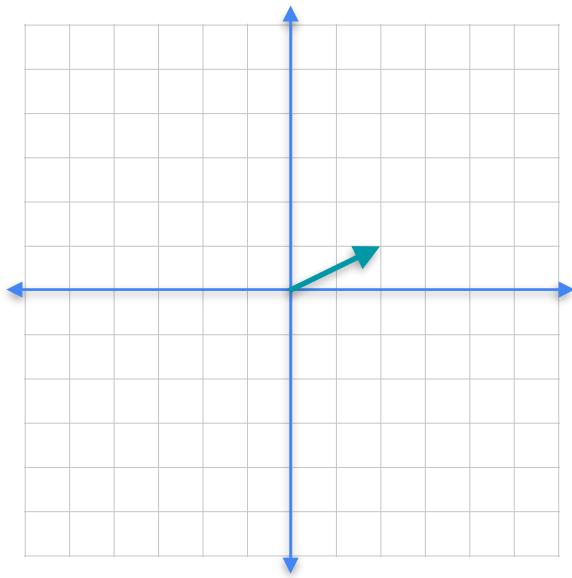


Linearly independent

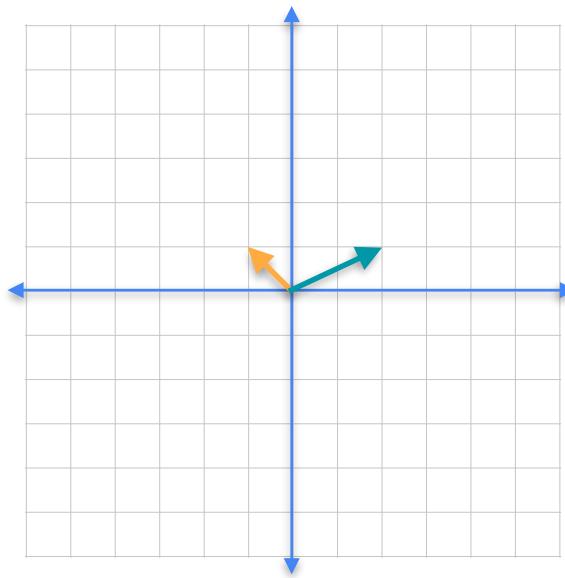


Linearly dependent

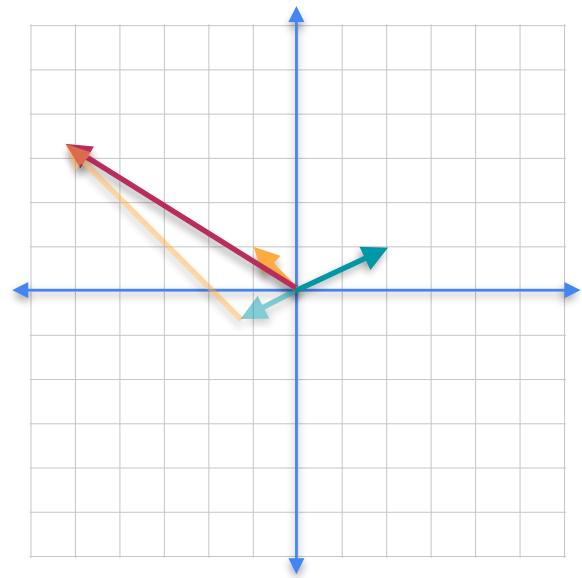
# Linearly independent and linearly dependent vectors



Linearly independent

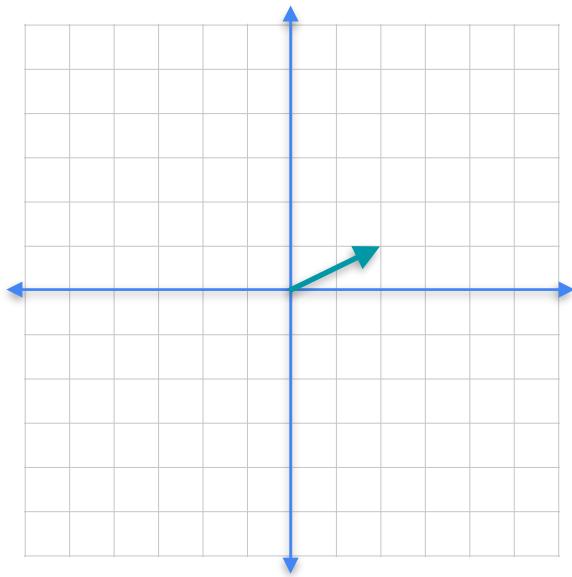


Linearly independent

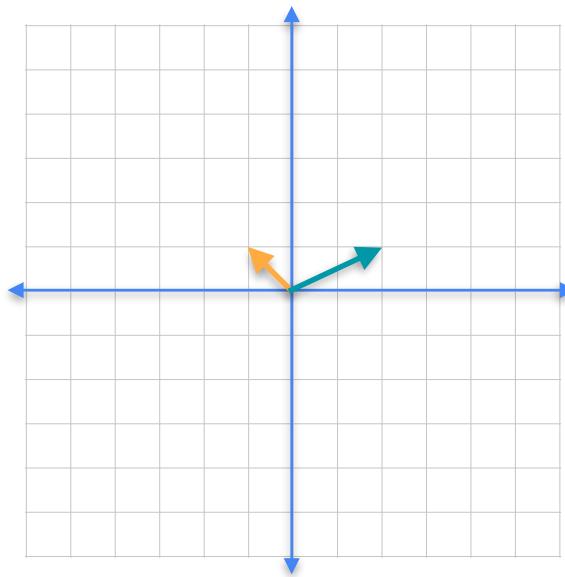


Linearly dependent

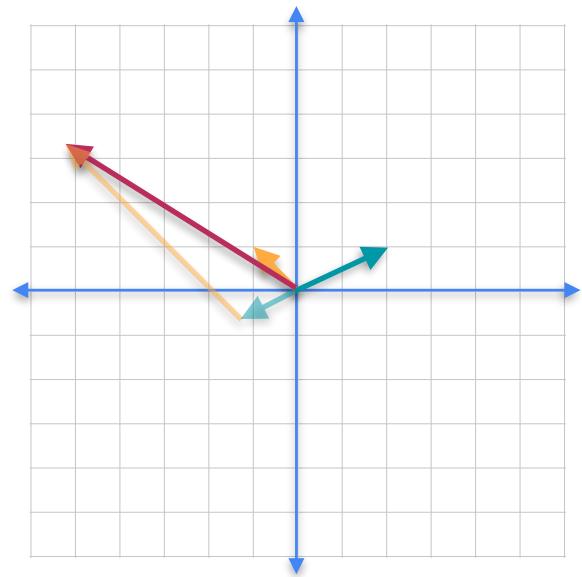
# Linearly independent and linearly dependent vectors



Linearly independent

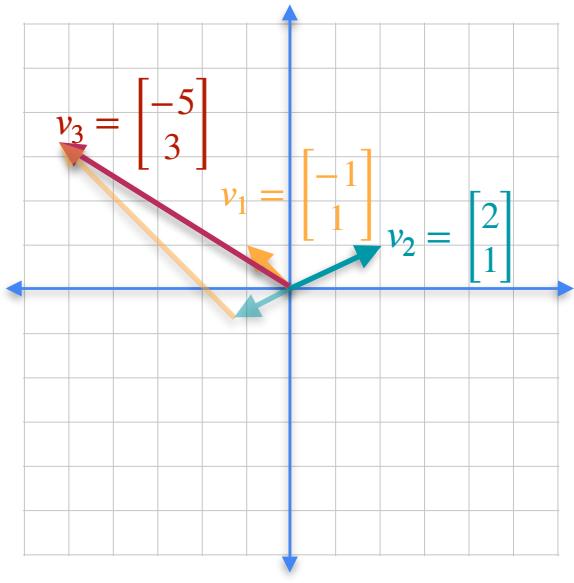


Linearly independent



Linearly dependent

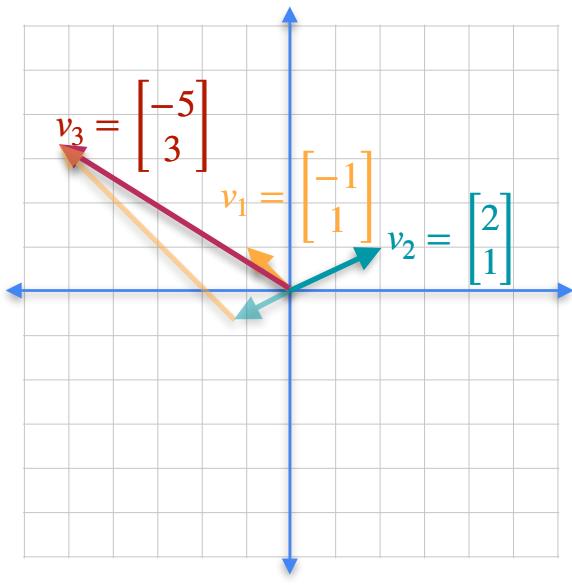
# Let's see how to check for linear dependence



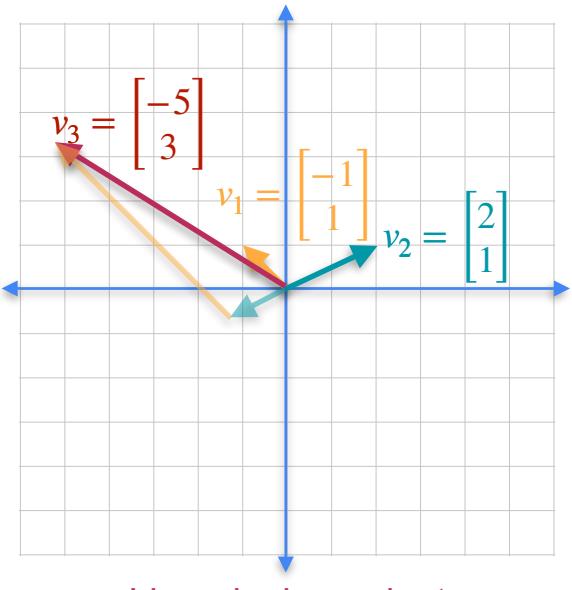
$$\alpha + \beta =$$

Linearly dependent

# Let's see how to check for linear dependence



# Let's see how to check for linear dependence

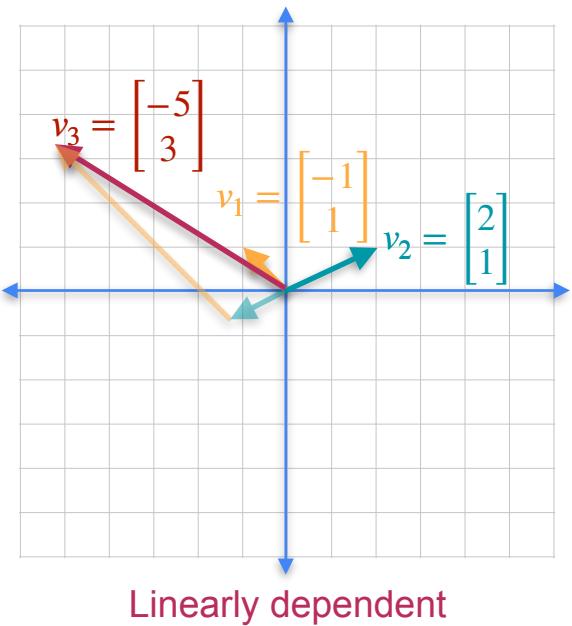


$$\alpha v_1 + \beta v_2 = v_3$$
$$\alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

- 1
- 2

$$-\alpha + 2\beta = -5$$
$$\alpha + \beta = 3$$

# Let's see how to check for linear dependence



$v_3$  is a linear combination  
of  $v_1$  and  $v_2$

$$\alpha v_1 + \beta v_2 = v_3$$

$$\alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

1  
2

$$-\alpha + 2\beta = -5$$
$$\alpha + \beta = 3$$

1 + 2

$$3\beta = -2 \rightarrow \beta = -\frac{2}{3}$$

2

$$\alpha - \frac{2}{3} = 3 \rightarrow \alpha = \frac{11}{3}$$

# Quiz

Are these vectors linearly independent?

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

# Solution

Are these vectors linearly independent?

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

1                  -1                  =                  Linearly dependent

# Solution

Are these vectors linearly independent?

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Linearly dependent

# Solution

Are these vectors linearly independent?

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Not a basis!

Linearly independent

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Linearly independent

$$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Linearly independent

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

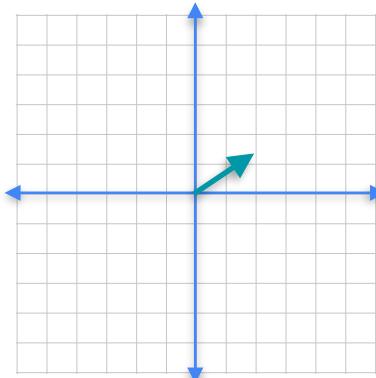
# Basis: a formal definition

A basis is a set of vectors that:

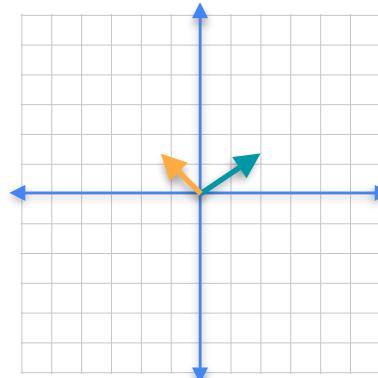
- Spans a vector space
- Is linearly independent



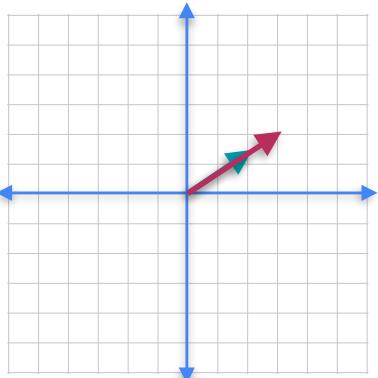
Not all sets of N vectors are a basis  
for N-dimensional space



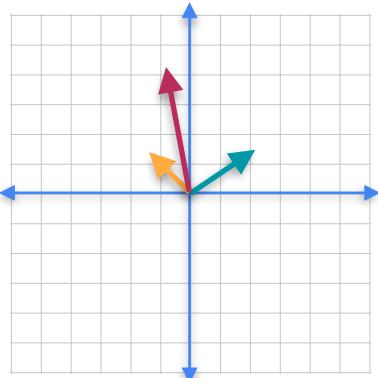
Spans a line  
Linearly independent  
Is a basis



Spans the plane  
Linearly independent  
Is a basis



Spans a line  
Linearly dependent  
Not a basis



Spans the plane  
Linearly dependent  
Not a basis



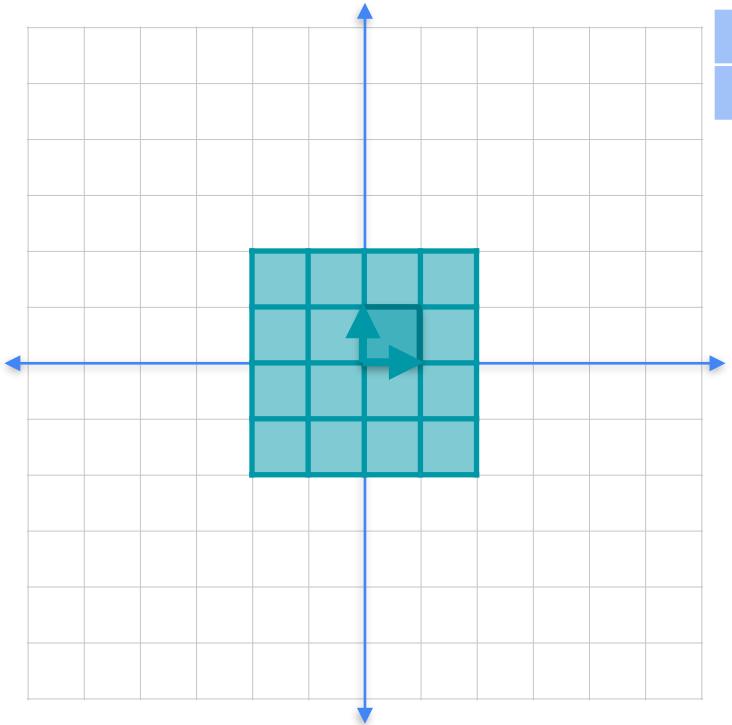
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# Determinants and Eigenvectors

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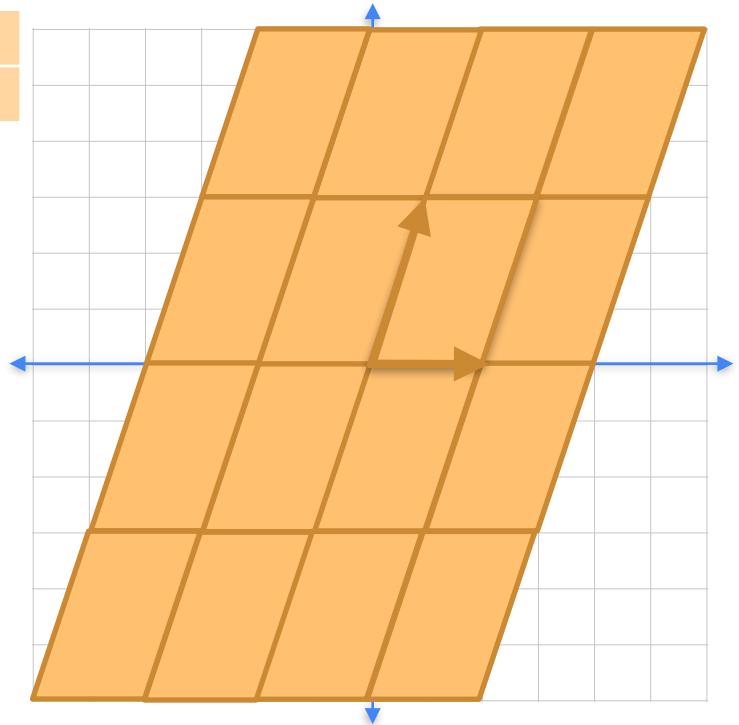
## Eigenbasis

# Basis

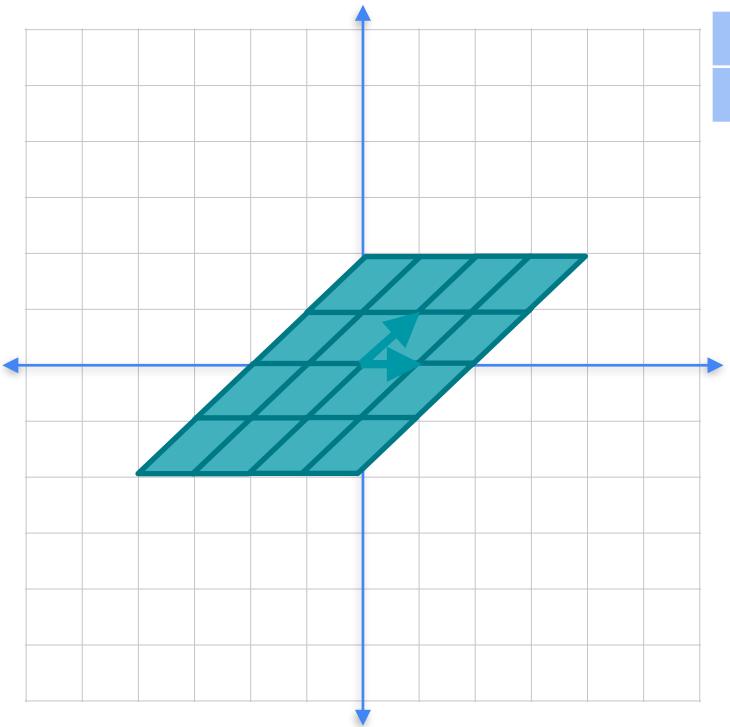


$$\begin{matrix} 2 & 1 & 0 \\ 0 & 3 & 1 \end{matrix} = \begin{matrix} 1 \\ 3 \end{matrix}$$

$$\begin{aligned} (1,0) &\rightarrow (2,0) \\ (0,1) &\rightarrow (1,3) \end{aligned}$$

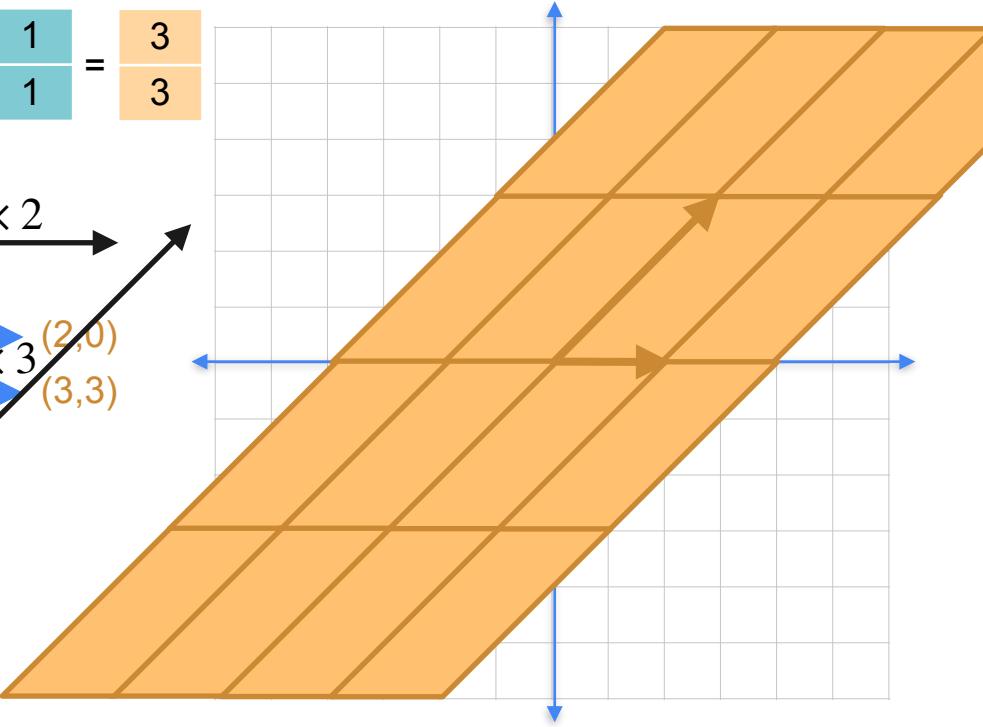


# Eigenbasis

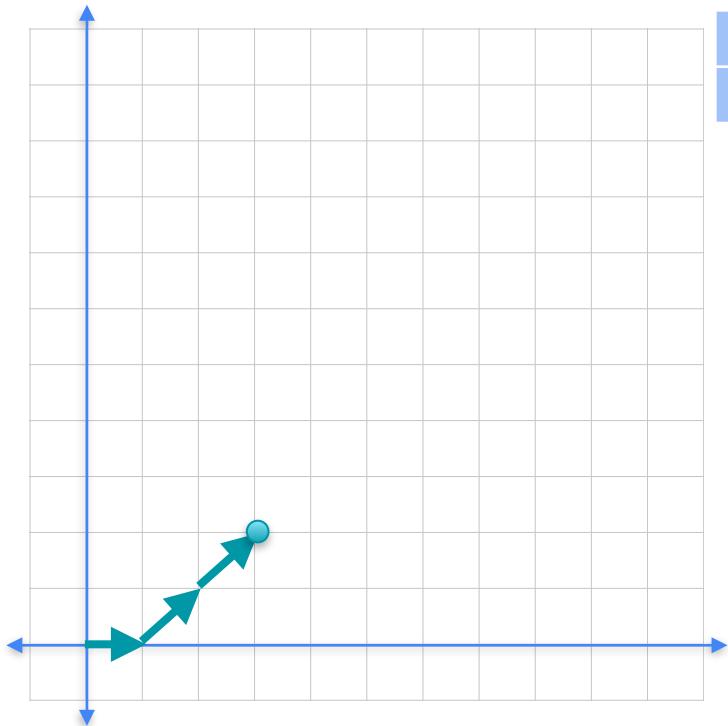


$$\begin{matrix} 2 & 1 & 1 \\ 0 & 3 & 1 \end{matrix} = \begin{matrix} 3 & \\ 3 & \end{matrix}$$

$$\begin{array}{c} \times 2 \\ \xrightarrow{(1,0)} (2,0) \\ \xrightarrow{(1,1)} (3,3) \end{array}$$

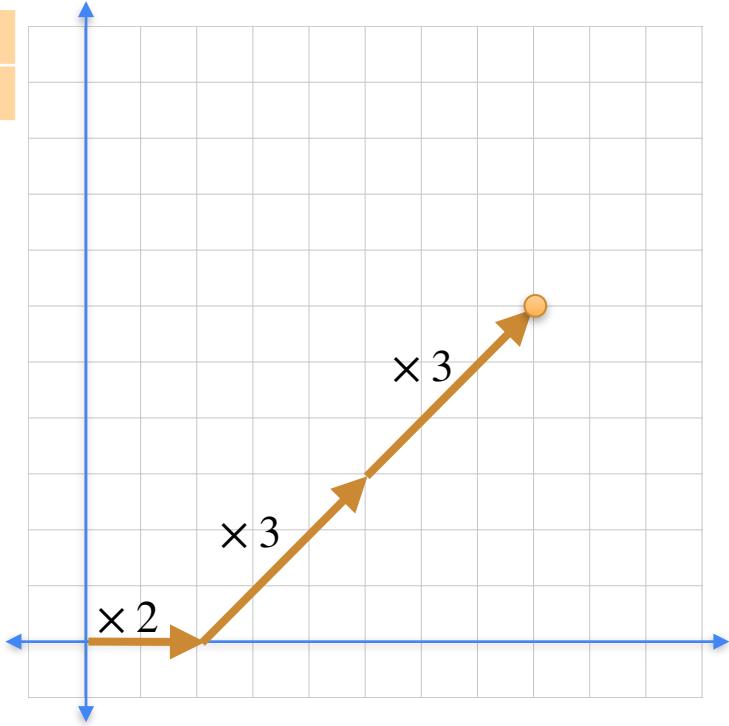


# Eigenbasis



$$\begin{matrix} 2 & 1 & 3 \\ 0 & 3 & 2 \end{matrix} = \begin{matrix} 8 \\ 6 \end{matrix}$$

$(3,2) \rightarrow (8,6)$





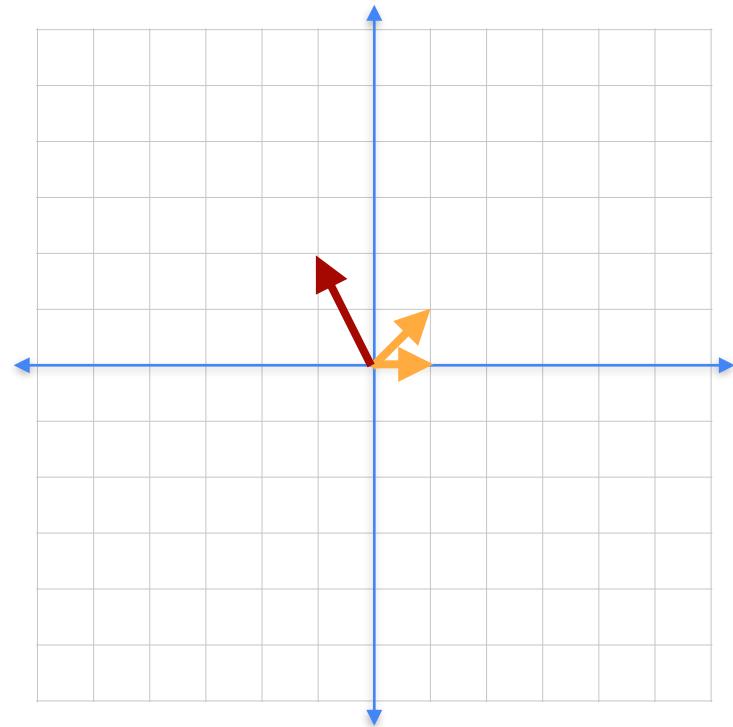
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## Determinants and Eigenvectors

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# Eigenvalues and Eigenvectors

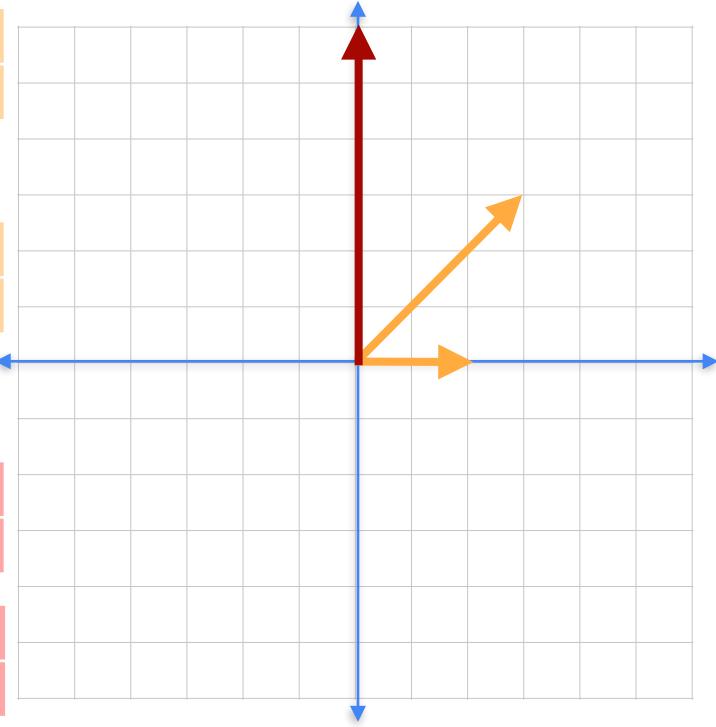
# Eigenvalues and eigenvectors



$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 6 \end{bmatrix} = \lambda \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$



# Eigenvalues and eigenvectors

$$\begin{matrix} 2 & 1 & 1 \\ 0 & 3 & 0 \end{matrix} = \begin{matrix} 2 & 1 \\ 0 & 0 \end{matrix}$$

8 multiplications

2 multiplications

Matrix Multiplication  
More work

$$A v_1$$

Scalar Multiplication  
Less work

$$\lambda_1 v_1$$

First pair

$$\begin{matrix} 2 & 1 & 1 \\ 0 & 3 & 1 \end{matrix} = \begin{matrix} 3 & 1 \\ 1 & 1 \end{matrix}$$

$$A v_2$$

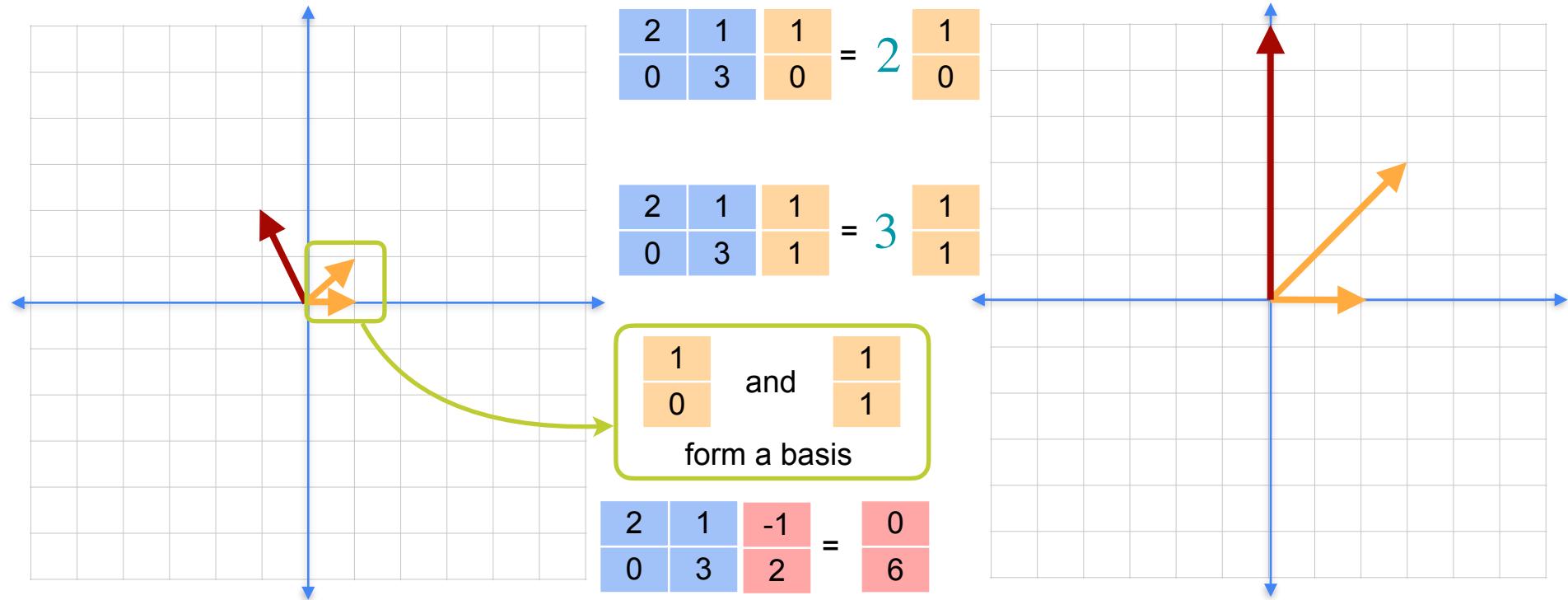
$$\lambda_2 v_2$$

Second pair

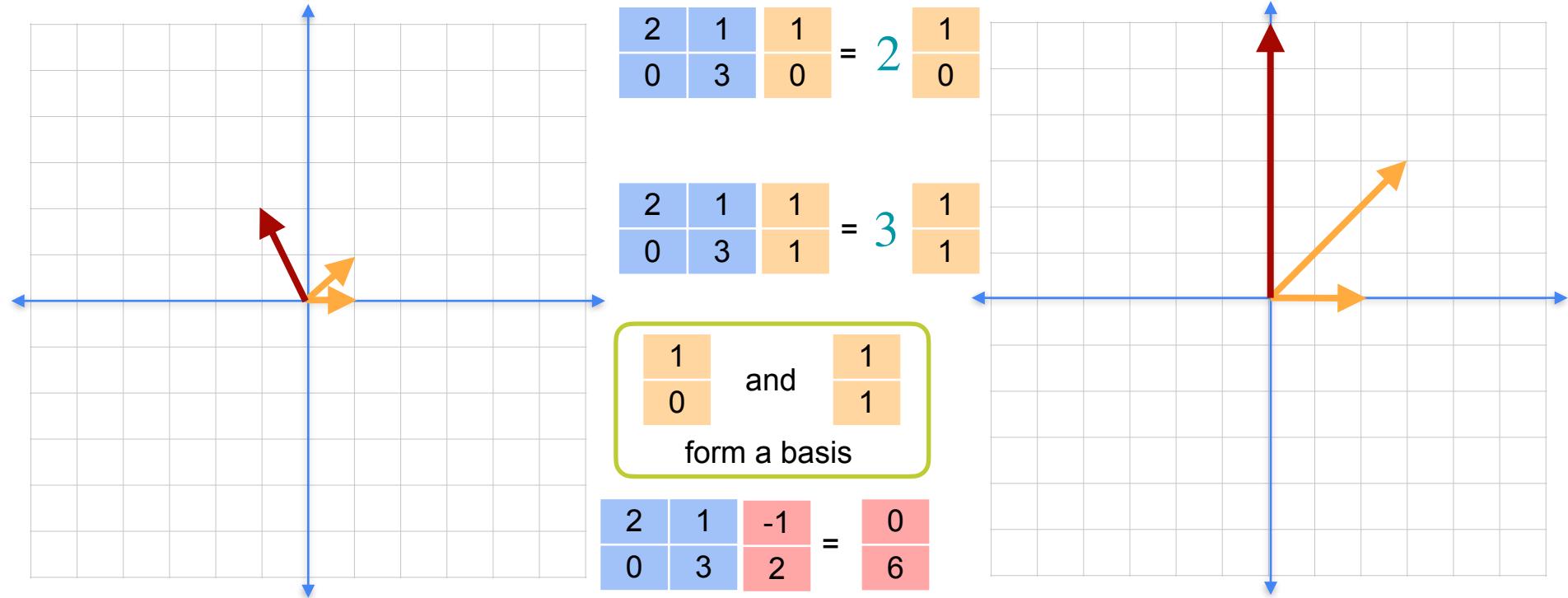
Eigenvalues

Eigenvectors

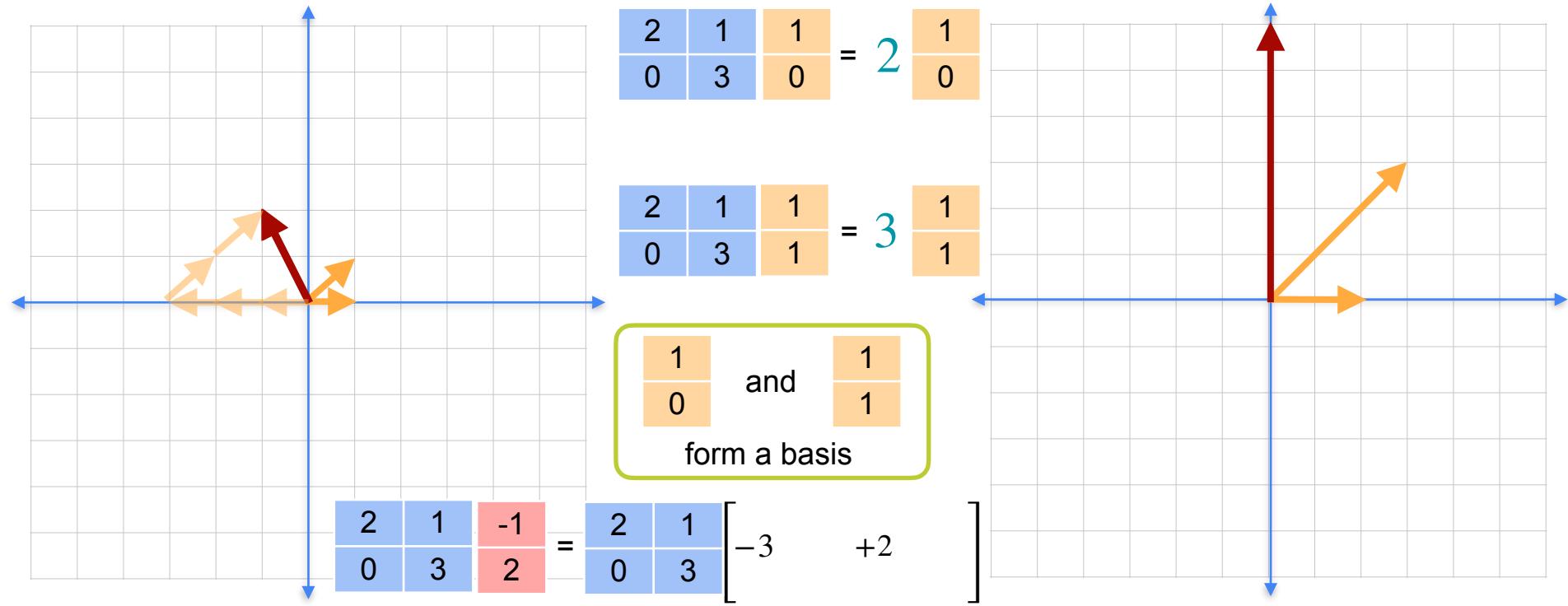
# Eigenvalues and eigenvectors



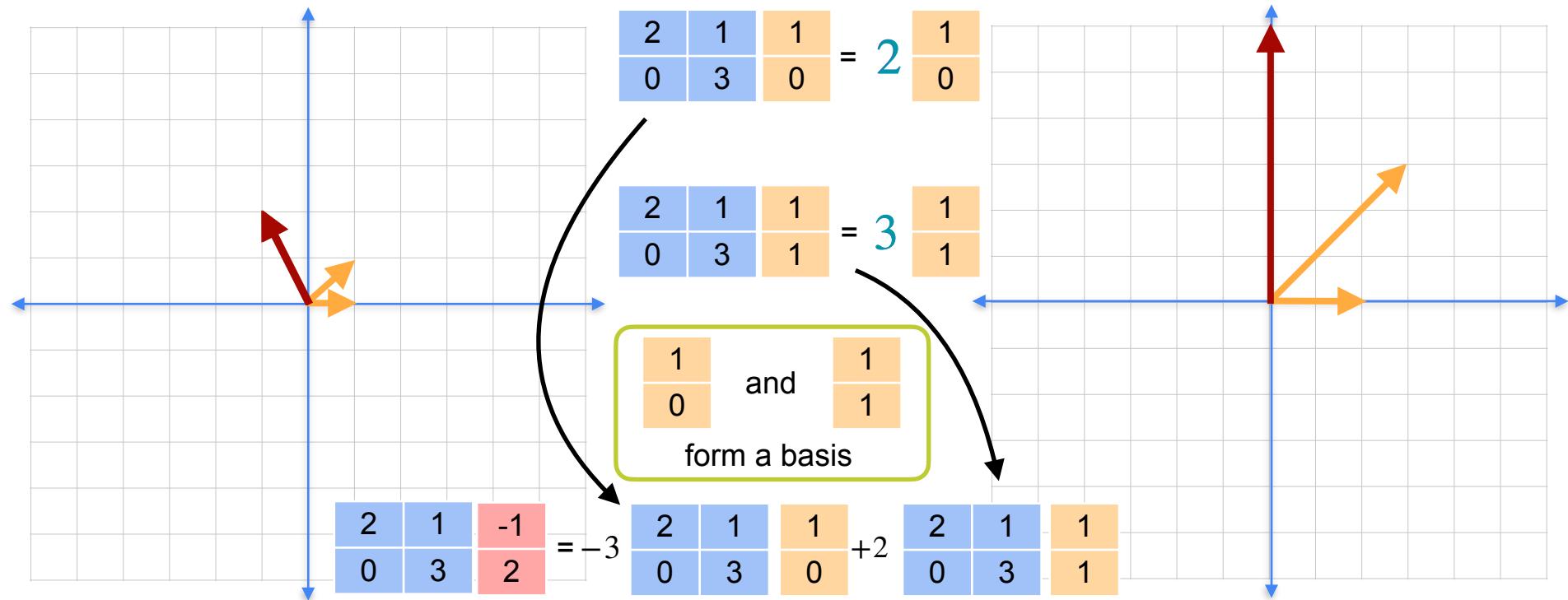
# Eigenvalues and eigenvectors



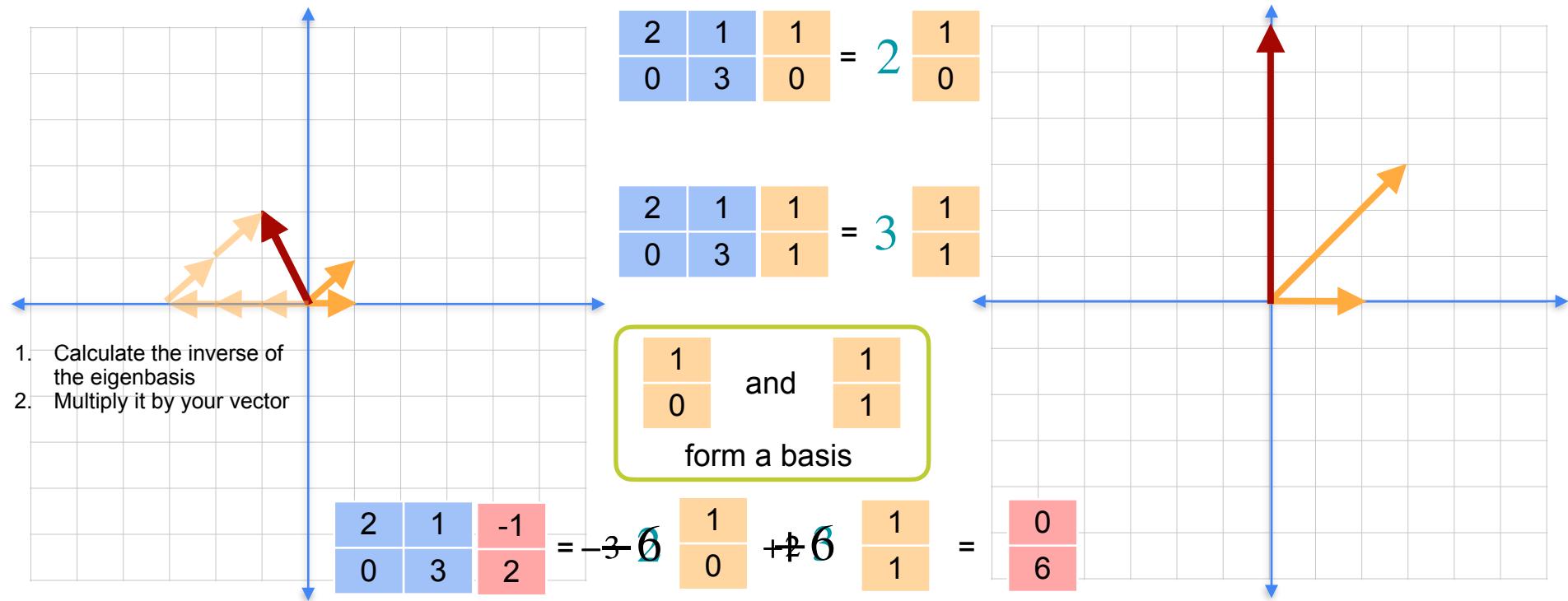
# Eigenvalues and eigenvectors



# Eigenvalues and eigenvectors



# Eigenvalues and eigenvectors



# Eigenvalues and eigenvectors

- $Av = \lambda v$  for each eigenvector / eigenvalue
- Eigenvectors: direction of stretch
- Eigenvalues: how much stretch
- Eigenbasis: the set of a matrix's eigenvectors, can be arranged as a matrix with one eigenvector in each column
- Save work and characterize a transformation



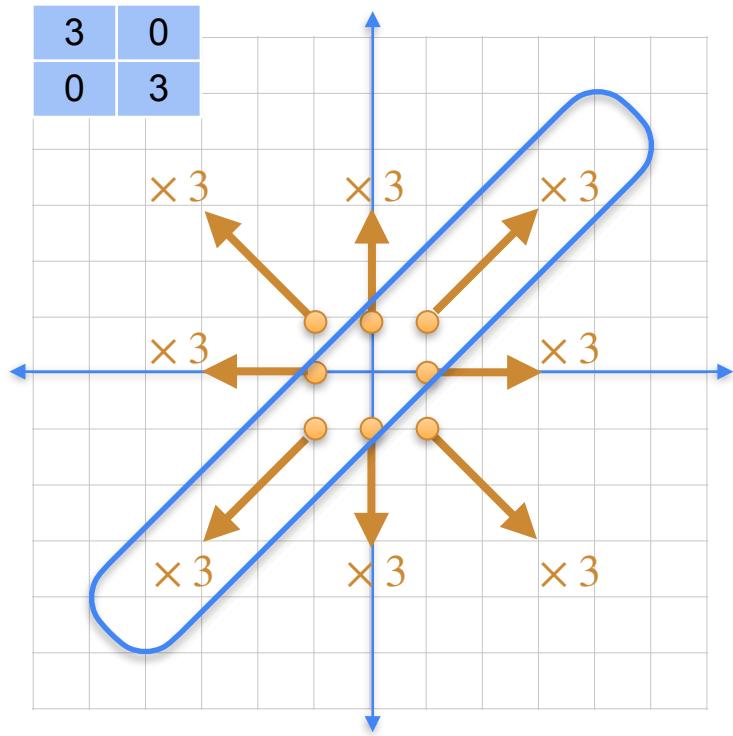
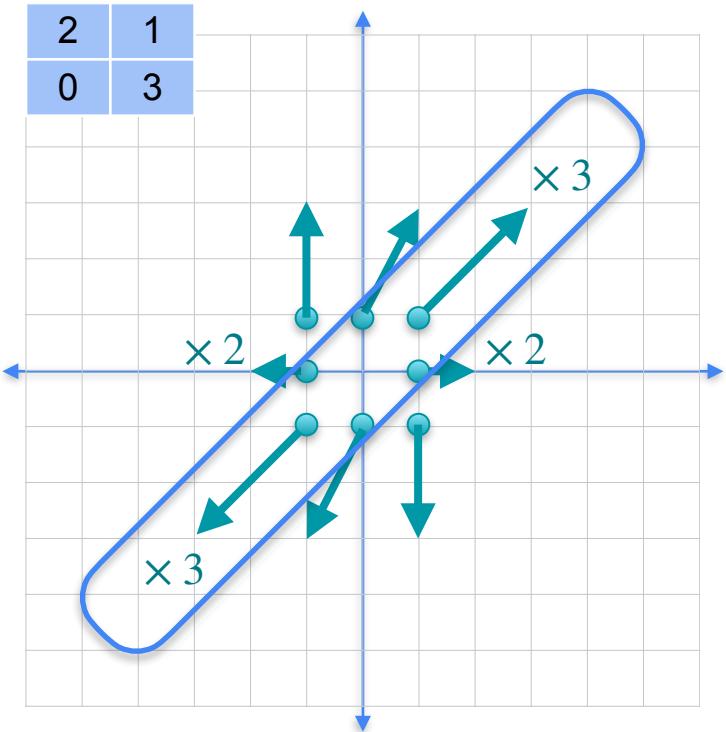
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## Determinants and Eigenvectors

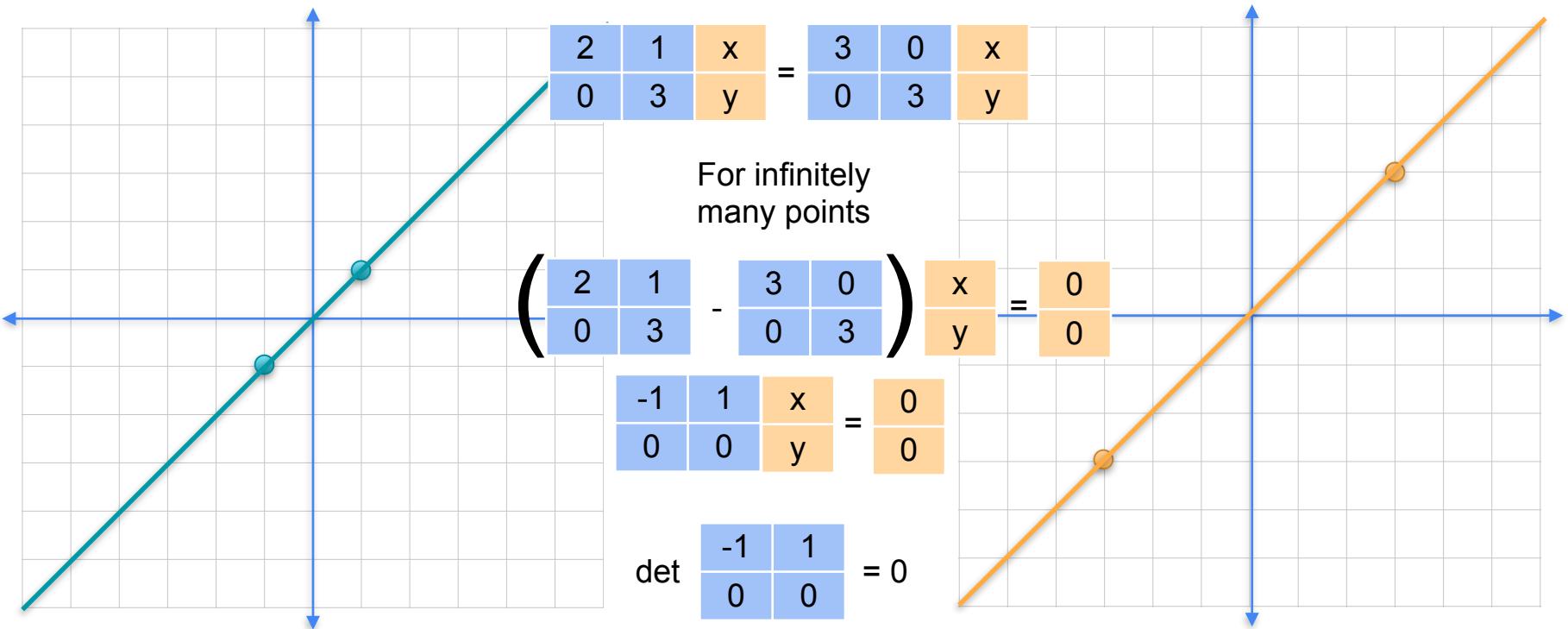
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**Calculating eigenvalues and eigenvectors**

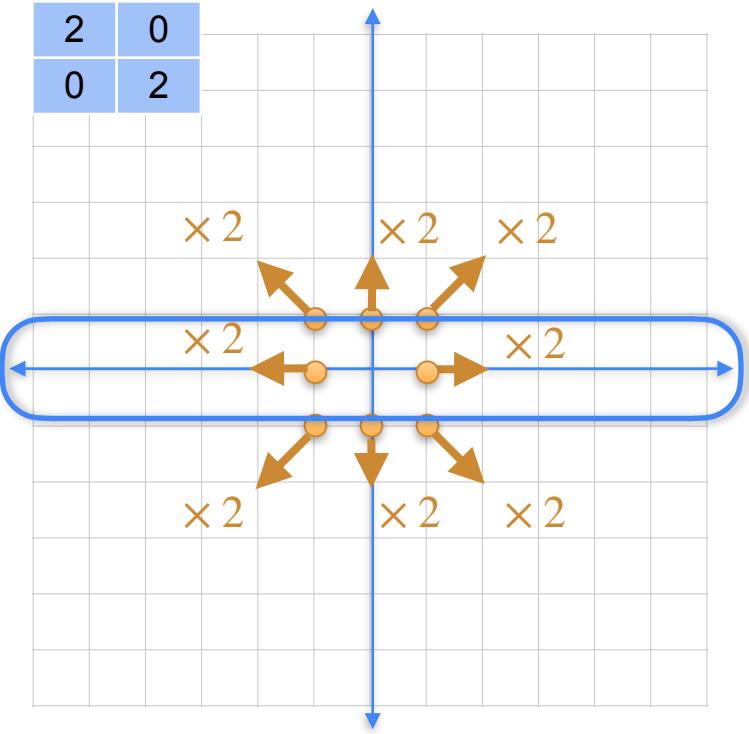
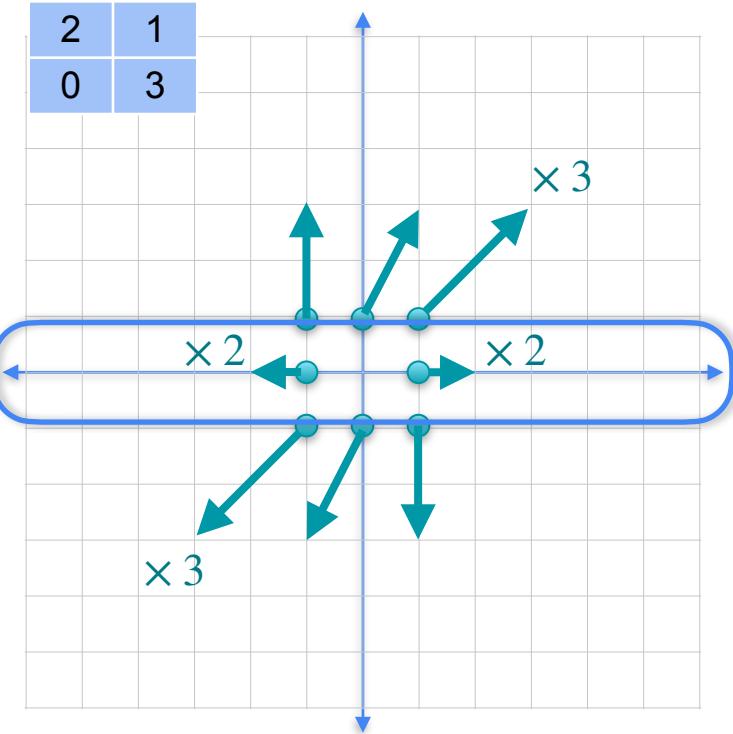
# Finding eigenvalues



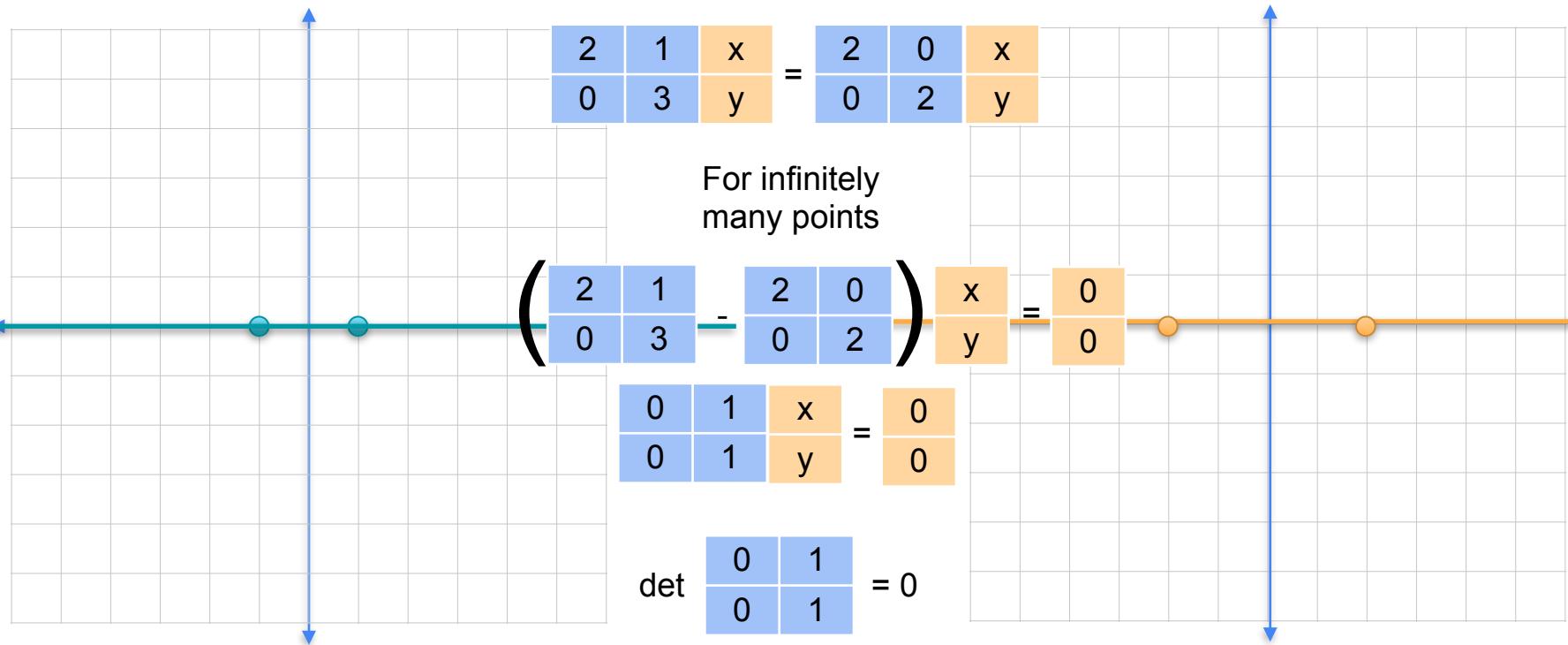
# Finding eigenvalues



# Finding eigenvalues



# Finding eigenvalues



# Finding eigenvalues

If  $\lambda$  is an eigenvalue:

$$\begin{array}{|c|c|c|} \hline 2 & 1 & x \\ \hline 0 & 3 & y \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline \lambda & 0 & x \\ \hline 0 & \lambda & y \\ \hline \end{array}$$

For infinitely many (x,y)

$$\begin{array}{|c|c|c|} \hline 2-\lambda & 1 & x \\ \hline 0 & 3-\lambda & y \\ \hline \end{array} = \begin{array}{|c|c|} \hline 0 \\ \hline 0 \\ \hline \end{array}$$

Has infinitely many solutions

$$\det \begin{array}{|c|c|} \hline 2-\lambda & 1 \\ \hline 0 & 3-\lambda \\ \hline \end{array} = 0$$

Characteristic polynomial

$$(2 - \lambda)(3 - \lambda) - 1 \cdot 0 = 0$$

$$\lambda = 2$$

$$\lambda = 3$$

# Finding eigenvectors

Eigenvalues:  $\lambda = 2$   
 $\lambda = 3$

Solve the equations

$$\begin{array}{|c|c|c|} \hline 2 & 1 & x \\ \hline 0 & 3 & y \\ \hline \end{array} = 2 \begin{array}{|c|c|} \hline x \\ \hline y \\ \hline \end{array}$$
$$2x + y = 2x$$
$$0x + 3y = 2y$$
$$x = 1$$
$$y = 0$$
$$\begin{array}{|c|} \hline 1 \\ \hline 0 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|} \hline 2 & 1 & x \\ \hline 0 & 3 & y \\ \hline \end{array} = 3 \begin{array}{|c|c|} \hline x \\ \hline y \\ \hline \end{array}$$
$$2x + y = 3x$$
$$0x + 3y = 3y$$
$$x = 1$$
$$y = 1$$
$$\begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline \end{array}$$

# Quiz

- Find the eigenvalues and eigenvectors of this matrix:

9	4
4	3

# Solution

- Eigenvalues: 11, 1
- Eigenvectors: (2,1), (-1,2)

9	4
4	3

- The characteristic polynomial is

$$\det \begin{array}{|cc|} \hline 9-\lambda & 4 \\ 4 & 3-\lambda \\ \hline \end{array} = (9 - \lambda)(3 - \lambda) - 4 \cdot 4 = 0$$

- Which factors as  $\lambda^2 - 12\lambda + 11 = (\lambda - 11)(\lambda - 1)$

- The solutions are  $\lambda = 11$   
 $\lambda = 1$

# Finding eigenvalues

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & 0 \end{bmatrix}$$

$$\lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

**Characteristic polynomial:**  $\det(A - \lambda I) = 0$

$$\det \begin{bmatrix} 2 - \lambda & 1 & -1 \\ 1 & -\lambda & -3 \\ -1 & -3 & -\lambda \end{bmatrix} = 0$$

$$(2 - \lambda)\lambda^2 + 3 + 3 - 9(2 - \lambda) + \lambda + \lambda = -\lambda^3 + 2\lambda^2 + 11\lambda - 12 = 0$$

$$-(\lambda + 3)(\lambda - 1)(\lambda - 4) = 0$$

**Eigenvalues:**  $-3, 1, 4$

# Finding eigenvalues

$$A = \begin{matrix} & \begin{matrix} 2 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & 0 \end{matrix} \end{matrix} \quad \text{Eigenvalues: } -3, 1, 4$$

$$Av = \lambda v$$

$$\underbrace{\begin{matrix} 2 & 1 & -1 & x_1 \\ 1 & 0 & -3 & x_2 \\ -1 & -3 & 0 & x_3 \end{matrix}}_{= 4} = 4 \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix}$$
$$\begin{matrix} 2x_1 + x_2 - x_3 \\ x_1 - 3x_3 \\ -x_1 - 3x_2 \end{matrix} = \begin{matrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{matrix}$$

# Finding eigenvalues

$$A = \begin{array}{|ccc|} \hline & 2 & 1 & -1 \\ & 1 & 0 & -3 \\ & -1 & -3 & 0 \\ \hline \end{array}$$

$$Av = \lambda v$$

$$\underbrace{\begin{array}{|ccc|c|} \hline & 2 & 1 & -1 & x_1 \\ & 1 & 0 & -3 & x_2 \\ & -1 & -3 & 0 & x_3 \\ \hline \end{array}}_{\begin{array}{l} 2x_1 + x_2 - x_3 \\ x_1 - 3x_3 \\ -x_1 - 3x_2 \end{array}} = 4 \begin{array}{|c|} \hline x_1 \\ x_2 \\ x_3 \\ \hline \end{array}$$

Eigenvalues:  $-3, 1, 4$

$$\begin{array}{rcl} 2x_1 + x_2 - x_3 & = & 4x_1 \\ x_1 - 3x_3 & = & 4x_2 \\ -x_1 - 3x_2 & = & 4x_3 \end{array}$$

$$\begin{array}{rcl} R_1 & -2x_1 + x_2 - x_3 & = 0 \\ R_2 & x_1 - 4x_2 - 3x_3 & = 0 \\ R_3 & -x_1 - 3x_2 - 4x_3 & = 0 \end{array}$$

$$\begin{array}{rcl} R_2 + R_3 & & 3R_1 + R_3 \\ -7x_2 - 7x_3 = 0 & & -7x_1 - 7x_3 = 0 \\ x_2 = -x_3 & & x_1 = -x_3 \end{array}$$

$$\begin{array}{l} x_1 = k \\ x_2 = k \\ x_3 = -k \end{array}$$

infinite solutions  
of this form

$$\begin{array}{ll} x_1 = 1 & x_1 = 2 \\ x_2 = 1 & x_2 = 2 \\ x_3 = -1 & x_3 = -2 \end{array}$$

this works! so does this!

**Eigenvector:**  $\begin{array}{|c|} \hline 1 \\ 1 \\ -1 \\ \hline \end{array}$

# Finding eigenvalues

$$A = \begin{matrix} \begin{array}{|c|c|c|} \hline 2 & 1 & -1 \\ \hline 1 & 0 & -3 \\ \hline -1 & -3 & 0 \\ \hline \end{array} \end{matrix}$$

**Eigenvalues**       $\lambda_1 = 4$        $\lambda_2 = 1$        $\lambda_3 = -3$

**Eigenvectors**

$\begin{matrix} 1 \\ 1 \\ -1 \end{matrix}$	$\begin{matrix} 0 \\ 1 \\ 1 \end{matrix}$	$\begin{matrix} 2 \\ -1 \\ 1 \end{matrix}$
--	---	--

# Note on dimensions

Eigenvalues → Determinant → Square Matrix

9	4
4	3



9	4	5
4	3	-2





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# Determinants and Eigenvectors

---

**On the number of  
eigenvectors**

# Number of eigenvectors

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & 0 \end{bmatrix}$$

3 by 3 matrix

?

3 distinct eigenvalues

?

3 distinct eigenvectors

Eigenvalues

$$\lambda_1 = 4 \quad \lambda_2 = 1 \quad \lambda_3 = -3$$



Eigenvectors

$$\begin{array}{ccc} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \end{array}$$

# Repeated eigenvalues - Example 1

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{pmatrix}$$

Characteristic polynomial =  $\det(A - \lambda I) = \det$

$$\begin{pmatrix} 2 - \lambda & 0 & 0 \\ 1 & 4 - \lambda & 0.5 \\ 0 & 0 & 2 - \lambda \end{pmatrix}$$

$$(2 - \lambda)^2(4 - \lambda) + 0 + 0 - 0 - 0 - 0 = 0$$

Eigenvalues: 4, 2, 2 Repeated eigenvalue

# Repeated eigenvalues - Example 1

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{Eigenvalue: } 4$$

$$Av = 4v$$

$$\underbrace{\begin{pmatrix} 2 & 0 & 0 & x_1 \\ -1 & 4 & -0.5 & x_2 \\ 0 & 0 & 2 & x_3 \end{pmatrix}}_{\begin{pmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 2x_3 \end{pmatrix}} = 4 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{pmatrix}$$

# Repeated eigenvalues - Example 1

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{Eigenvalue: } 4$$

$$Av = 4v$$

$$\underbrace{\begin{pmatrix} 2 & 0 & 0 & x_1 \\ -1 & 4 & -0.5 & x_2 \\ 0 & 0 & 2 & x_3 \end{pmatrix}}_{\begin{matrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 2x_3 \end{matrix}} = 4 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{pmatrix}$$

$$\begin{aligned} 2x_1 &= 4x_1 \\ -x_1 + 4x_2 - 0.5x_3 &= 4x_2 \\ 2x_3 &= 4x_3 \end{aligned}$$

$$\begin{aligned} -2x_1 &= 0 \\ -x_1 - 0.5x_3 &= 0 \\ -2x_3 &= 0 \\ \rightarrow x_1 &= 0 \\ x_2 &= \text{any number} \\ \rightarrow x_3 &= 0 \end{aligned}$$

Eigenvector

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

# Repeated eigenvalues - Example 1

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{Eigenvalue: } 2$$

$$Av = 2v$$

$$\underbrace{\begin{pmatrix} 2 & 0 & 0 & x_1 \\ -1 & 4 & -0.5 & x_2 \\ 0 & 0 & 2 & x_3 \end{pmatrix}}_{\begin{pmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 2x_3 \end{pmatrix}} = 2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{pmatrix}$$

# Repeated eigenvalues - Example 1

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{Eigenvalue: } 2$$

$$Av = 2v$$

$$\underbrace{\begin{pmatrix} 2 & 0 & 0 & x_1 \\ -1 & 4 & -0.5 & x_2 \\ 0 & 0 & 2 & x_3 \end{pmatrix}}_{\begin{matrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 2x_3 \end{matrix}} = 2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{pmatrix}$$

$$\begin{aligned} 2x_1 &= 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 &= 2x_2 \\ 2x_3 &= 2x_3 \end{aligned}$$

$$\begin{aligned} 0 &= 0 \\ -x_1 + 2x_2 - 0.5x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

$$x_1 = 2x_2 - 0.5x_3$$

$$\begin{array}{ll} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} & \begin{array}{l} x_1 = 2 \\ x_2 = 1 \\ x_3 = 0 \end{array} \\ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} & \begin{array}{l} x_1 = 1 \\ x_2 = 1 \\ x_3 = 2 \end{array} \end{array}$$

Point in different directions  
Different eigenvectors

# Repeated eigenvalues - Example 1

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{pmatrix}$$

Eigenvalues       $\lambda_1 = 4$        $\lambda_2 = 2$        $\lambda_3 = 2$

Eigenvectors

$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$
---	---	---

# Repeated eigenvalues - Example 2

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{pmatrix}$$

# Repeated eigenvalues - Example 2

$$A = \begin{matrix} \begin{array}{|ccc|} \hline & 2 & 0 \\ & -1 & 4 \\ & 4 & 0 \\ \hline & 0 & -0.5 \\ & 2 & \end{array} \end{matrix}$$

Characteristic polynomial =  $\det(A - \lambda I)$  =  $\det$

$$\begin{matrix} \begin{array}{|ccc|} \hline & 2-\lambda & 0 \\ & 1 & 4-\lambda \\ & -4 & 0 \\ \hline & 0 & 0.5 \\ & & 2-\lambda \\ \hline \end{array} \end{matrix}$$

$$(2 - \lambda)^2(4 - \lambda) + 0 + 0 - 0 - 0 - 0$$

Eigenvalues: 4, 2, 2    Repeated eigenvalue

# Repeated eigenvalues - Example 2

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{pmatrix} \quad \text{Eigenvalue: } 4$$

$$Av = 4v$$

$$\underbrace{\begin{pmatrix} 2 & 0 & 0 & x_1 \\ -1 & 4 & -0.5 & x_2 \\ 4 & 0 & 2 & x_3 \end{pmatrix}}_{\begin{pmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 4x_1 + 2x_3 \end{pmatrix}} = 4 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{pmatrix}$$

# Repeated eigenvalues - Example 2

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{pmatrix} \quad \text{Eigenvalue: } 4$$

$$Av = 4v$$

$$\underbrace{\begin{pmatrix} 2 & 0 & 0 & x_1 \\ -1 & 4 & -0.5 & x_2 \\ 4 & 0 & 2 & x_3 \end{pmatrix}}_{2x_1} = 4 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{pmatrix}$$

$$\begin{pmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 4x_1 + 2x_3 \end{pmatrix}$$

$$\begin{aligned} 2x_1 &= 4x_1 \\ -x_1 + 4x_2 - 0.5x_3 &= 4x_2 \\ 4x_1 + 2x_3 &= 4x_3 \end{aligned}$$

$$\begin{aligned} -2x_1 &= 0 \\ -x_1 - 0.5x_3 &= 0 \\ 4x_1 - 2x_3 &= 0 \end{aligned}$$

$$x_1 = 0 \quad x_3 = 0 \quad x_2 = \text{any number}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{aligned} x_1 &= 0 \\ x_2 &= 1 \\ x_3 &= 0 \end{aligned}$$

Same as before!

# Repeated eigenvalues - Example 2

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{pmatrix} \quad \text{Eigenvalue: } 2$$

$$Av = 2v$$

$$\underbrace{\begin{pmatrix} 2 & 0 & 0 & x_1 \\ -1 & 4 & -0.5 & x_2 \\ 4 & 0 & 2 & x_3 \end{pmatrix}}_{\begin{pmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 4x_1 + 2x_3 \end{pmatrix}} = 2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{pmatrix}$$

# Repeated eigenvalues - Example 2

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{pmatrix} \quad \text{Eigenvalue: } 2$$

$$Av = 2v$$

$$\underbrace{\begin{pmatrix} 2 & 0 & 0 & x_1 \\ -1 & 4 & -0.5 & x_2 \\ 4 & 0 & 2 & x_3 \end{pmatrix}}_{\begin{pmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 4x_1 + 2x_3 \end{pmatrix}} = 2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{pmatrix}$$

$$\begin{aligned} 2x_1 &= 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 &= 2x_2 \\ 4x_1 + 2x_3 &= 2x_3 \end{aligned}$$

$$\begin{aligned} 0 &= 0 & 0 \\ -x_1 + 2x_2 - 0.5x_3 &= 0 & k \\ 4x_1 &= 0 & 4k \end{aligned}$$

$$x_1 = 0 \qquad x_3 = 4x_2$$

$$\begin{array}{c|c} \begin{matrix} 0 \\ 1 \\ 4 \end{matrix} & \begin{matrix} x_1 = 0 \\ x_2 = 1 \\ x_3 = 4 \end{matrix} \\ \hline \begin{matrix} 0 \\ 0.5 \\ 2 \end{matrix} & \begin{matrix} x_1 = 0 \\ x_2 = 0.5 \\ x_3 = 2 \end{matrix} \end{array}$$

$$\begin{pmatrix} 0 \\ k \\ 4k \end{pmatrix}$$

On the same line  
Same eigenvector

# Repeated eigenvalues - Example 2

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{pmatrix} \quad \text{Eigenvalue: } 2$$

$$Av = 2v$$

$$\underbrace{\begin{pmatrix} 2 & 0 & 0 & x_1 \\ -1 & 4 & -0.5 & x_2 \\ 4 & 0 & 2 & x_3 \end{pmatrix}}_{\begin{pmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 4x_1 + 2x_3 \end{pmatrix}} = 2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{pmatrix}$$

$$\begin{aligned} 2x_1 &= 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 &= 2x_2 \\ 4x_1 + 2x_3 &= 2x_3 \end{aligned}$$

$$\begin{aligned} 0 &= 0 \\ -x_1 + 2x_2 - 0.5x_3 &= 0 \\ 4x_1 &= 0 \end{aligned}$$

$$x_1 = 0 \qquad x_3 = 4x_2$$

$$\begin{pmatrix} 0 \\ k \\ 4k \end{pmatrix}$$

# Repeated eigenvalues - Example 2

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{pmatrix}$$

Eigenvalues

$$\lambda_1 = 4$$

$$\lambda_2 = 2$$

$$\lambda_3 = 2$$

Eigenvectors

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$$

Can't create an eigenbasis  
from this matrix

# Summary

a	b
c	d

Eigenvalues

$\lambda_1, \lambda_2$

a	b	c
d	e	f
g	h	i

$\lambda_1, \lambda_2, \lambda_3$

If  $\lambda_1 \neq \lambda_2$  → 2 eigenvectors  
(2 different directions)

If  $\lambda_1 = \lambda_2$  → 1 eigenvector  
(1 direction)  
→ 2 eigenvectors  
(2 different directions)

If  $\lambda_1 \neq \lambda_2 \neq \lambda_3$  → 3 eigenvectors  
(3 different directions)

If  $\lambda_1 = \lambda_2 \neq \lambda_3$  → 2 eigenvectors  
(2 different directions)  
→ 3 eigenvectors  
(3 different directions)

If  $\lambda_1 = \lambda_2 = \lambda_3$  → 1 eigenvector  
(1 direction)  
→ 2 eigenvectors  
(2 different directions)  
→ 3 eigenvectors  
(3 different directions)



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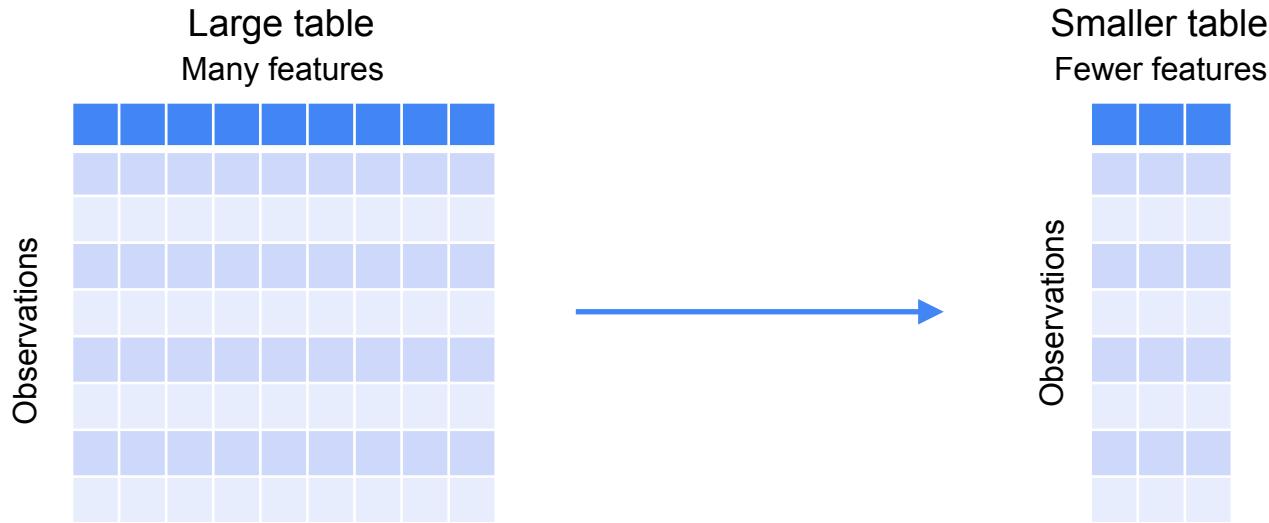
## Determinants and Eigenvectors

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**Dimensionality reduction  
and projection**

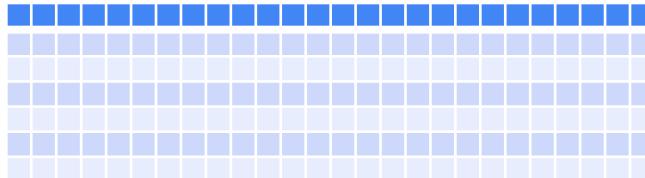
# Dimensionality Reduction

- Reduce dimensions (# of columns) of dataset
- Preserve as much information as possible



# Dimensionality Reduction

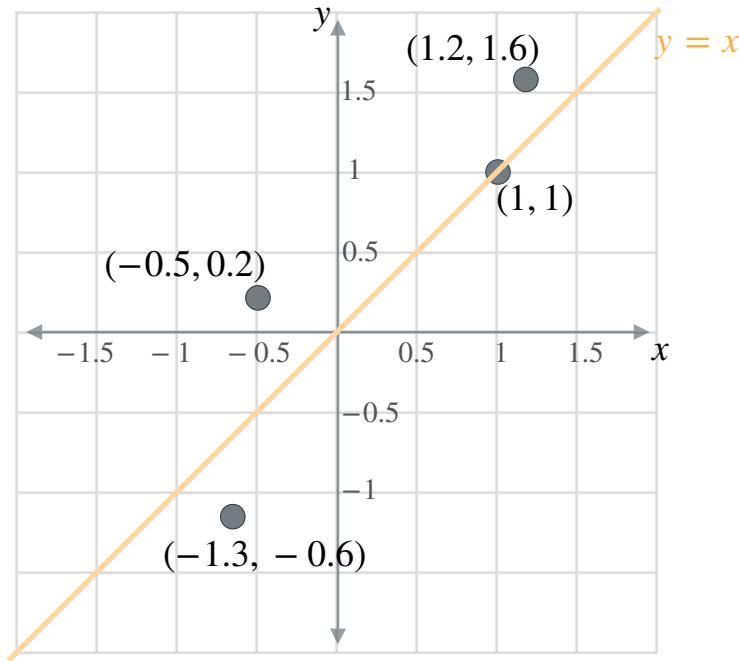
- Leads to smaller datasets
- Easier to visualize



Customer Age	Account Age	Days Since Login	Total Purchases	Total \$ Spent
23	1 month	10 days	1	\$100
71	45 months	2 days	Easy approach - just delete columns Loses valuable information	\$100
54	30 months	15 days	2	\$70
36	22 months	12 days	4	\$210

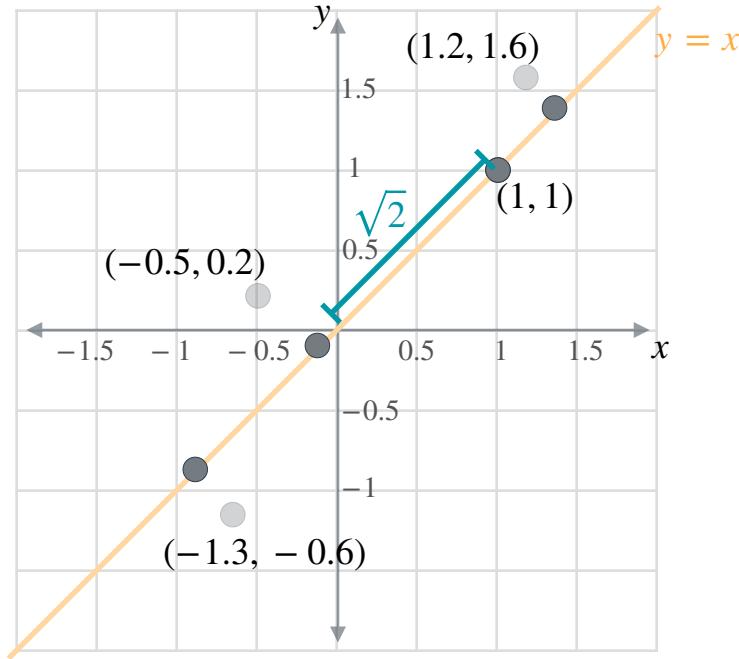
# Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6



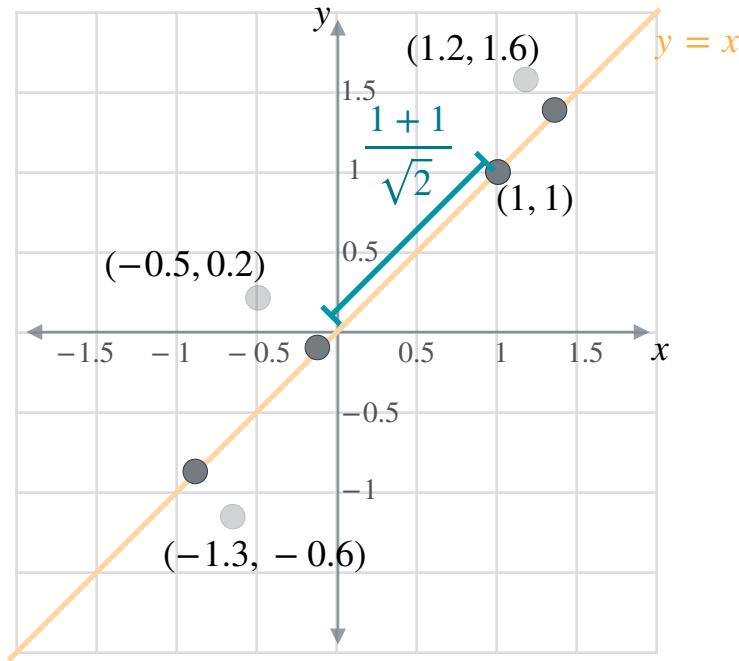
# Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6



# Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6



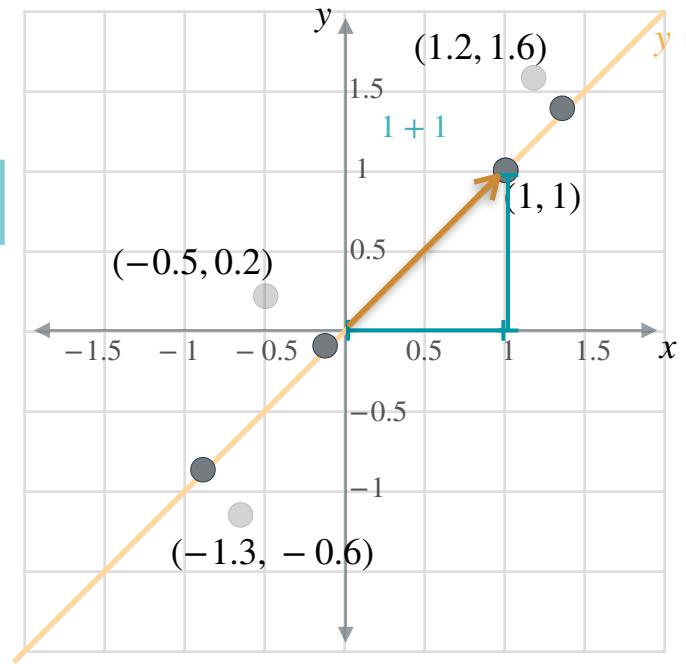
# Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

1
1

$$= (1 + 1)$$



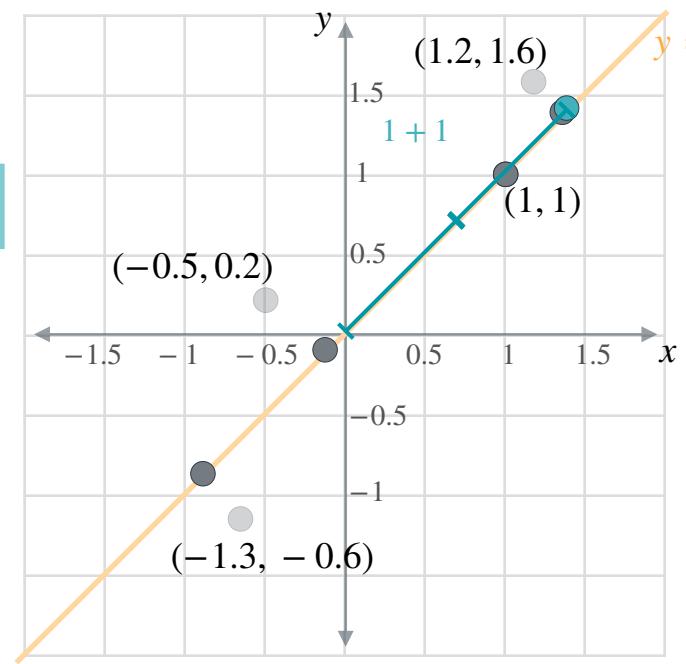
# Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

1
1

$$= (1 + 1)$$

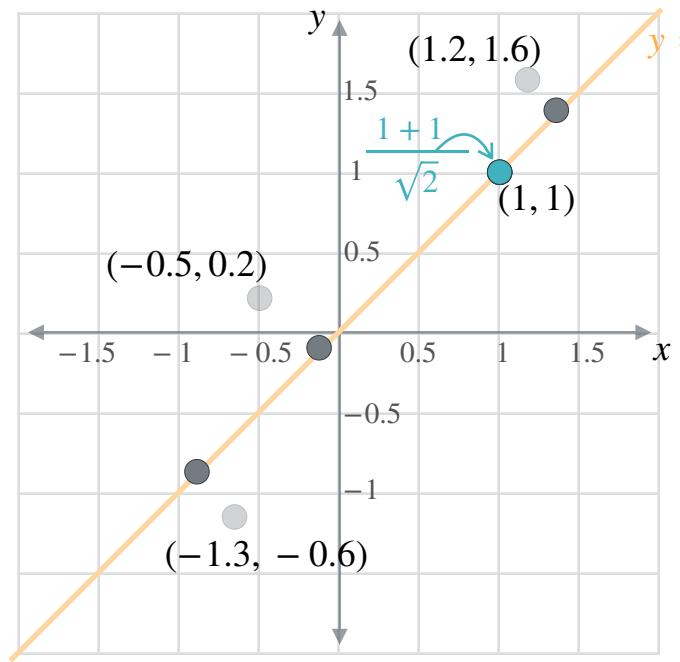


# Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

$$\begin{matrix} 1 \\ 1 \end{matrix} \frac{1}{\sqrt{2}} =$$

$$(1+1)/\sqrt{2}$$



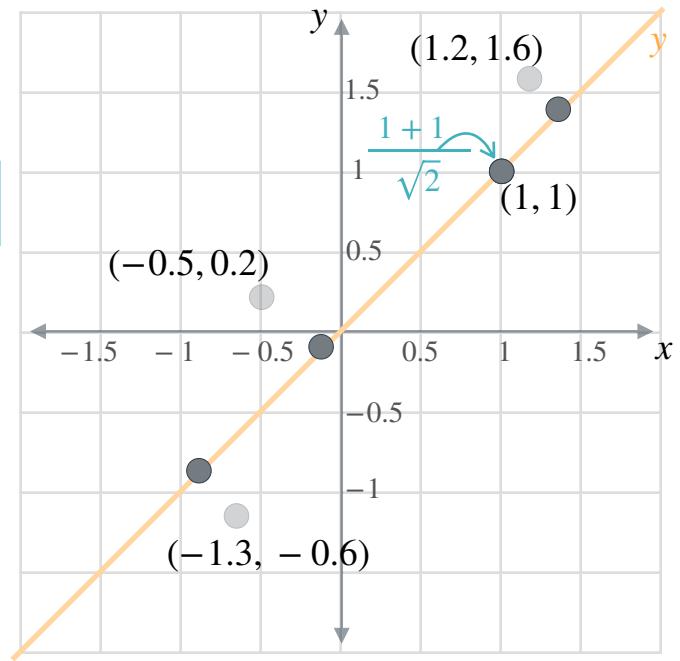
# Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

Norm of 1

$$\frac{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\sqrt{2}} = \frac{1}{\left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\|_2}$$

$$(1+1)/\sqrt{2}$$

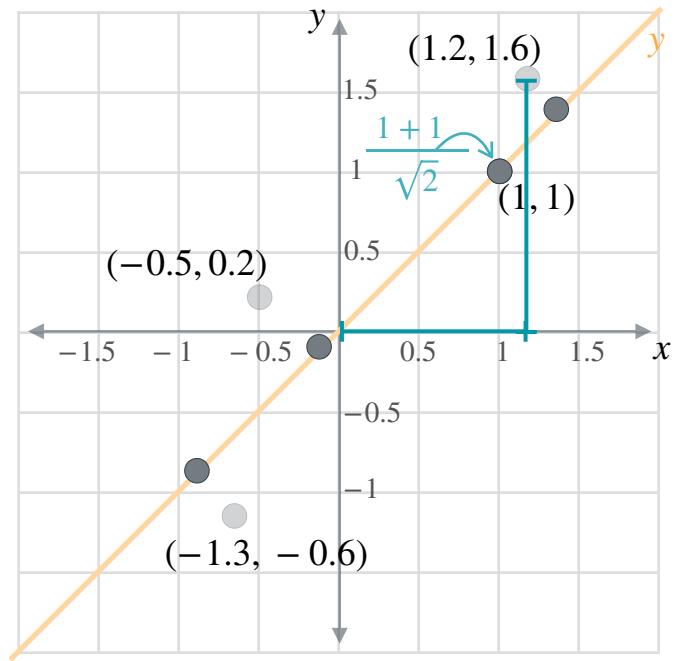


# Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

$$\begin{matrix} 1 \\ 1 \end{matrix} \frac{1}{\sqrt{2}} =$$

$$\begin{matrix} (1+1)/\sqrt{2} \\ 1 \end{matrix}$$

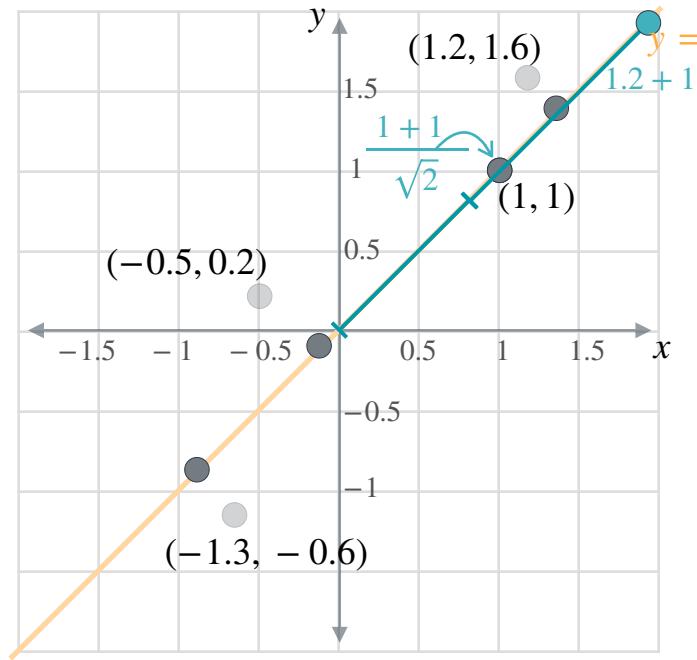


# Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

$$\begin{matrix} 1 \\ 1 \end{matrix} \frac{1}{\sqrt{2}} =$$

$$\begin{matrix} (1+1)/\sqrt{2} \\ (1.2+1.6) \end{matrix}$$

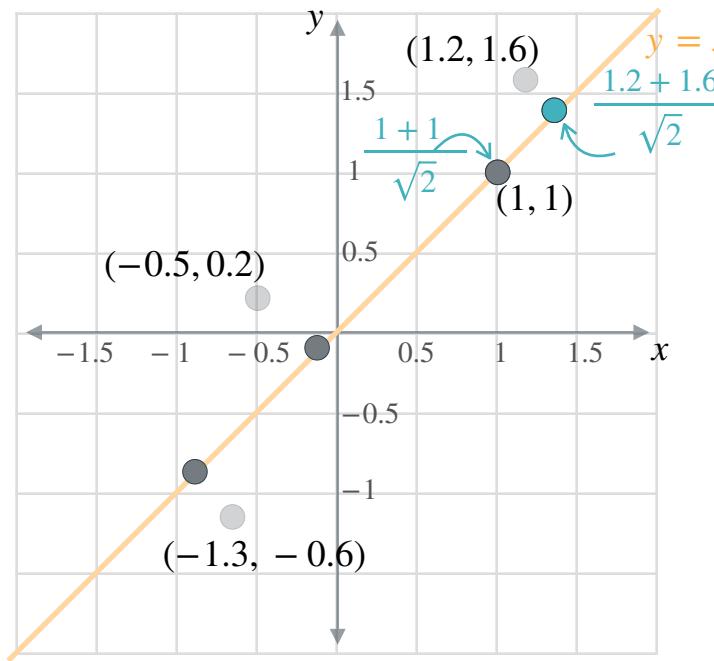


# Projections

x	y
1.0	1.0
1.2	1.6
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$$\begin{matrix} 1 \\ 1 \end{matrix} \frac{1}{\sqrt{2}} =$$

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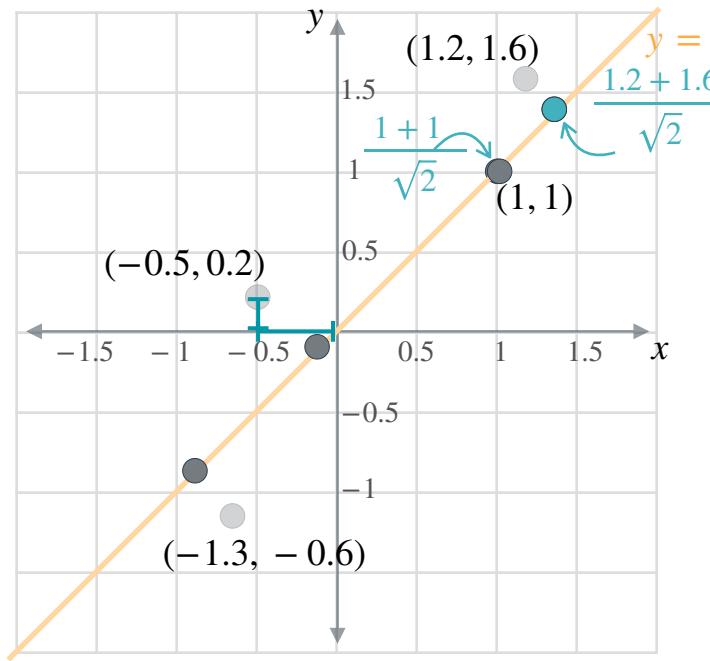


# Projections

x	y
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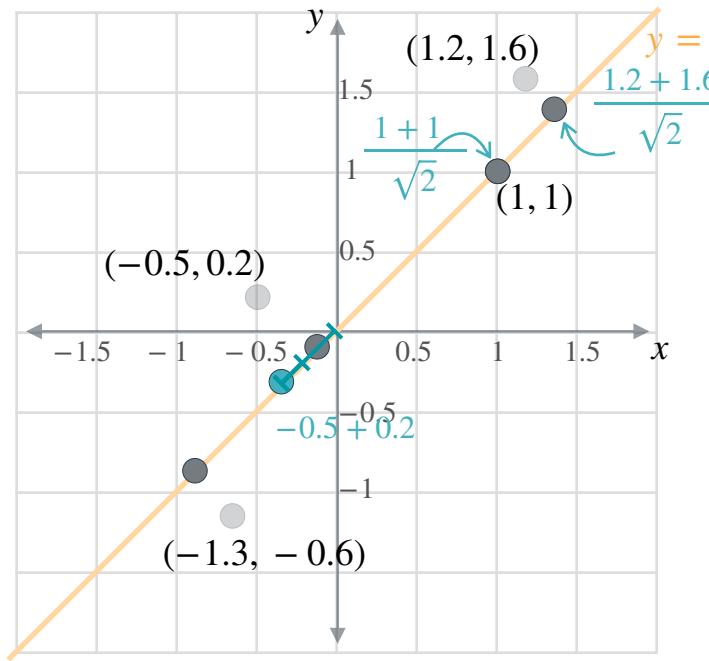


# Projections

x	y
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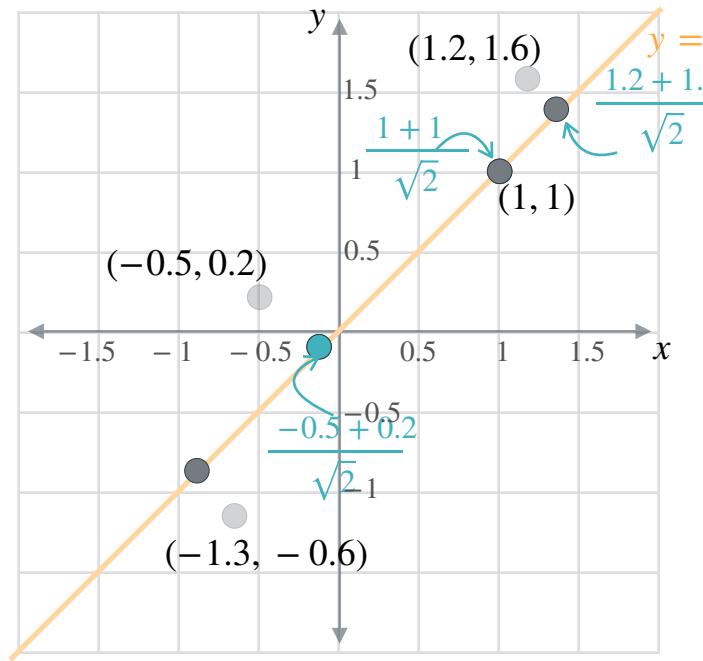


# Projections

x	y
1.0	1.0
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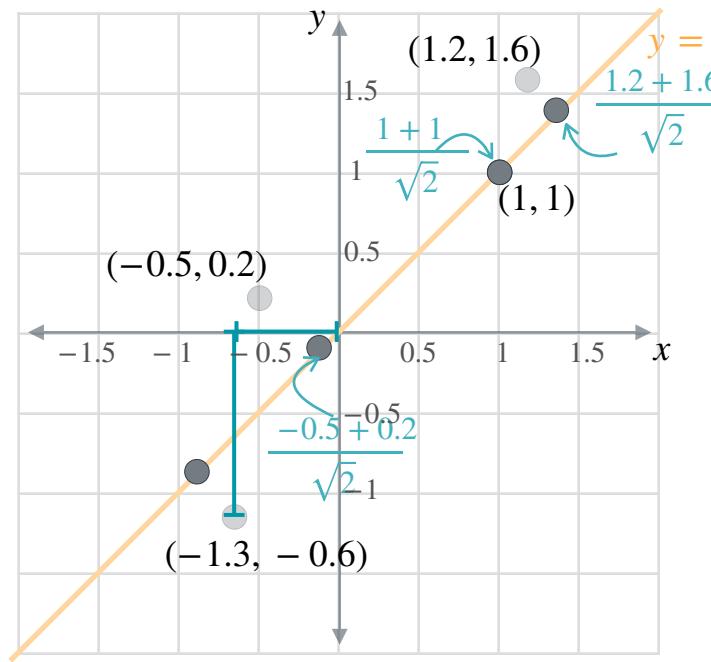


# Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
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$$\begin{matrix} 1 \\ 1 \end{matrix} \frac{1}{\sqrt{2}} =$$

$$\begin{matrix} (1+1)/\sqrt{2} \\ (1.2+1.6)/\sqrt{2} \\ (-0.5+0.2)/\sqrt{2} \end{matrix}$$

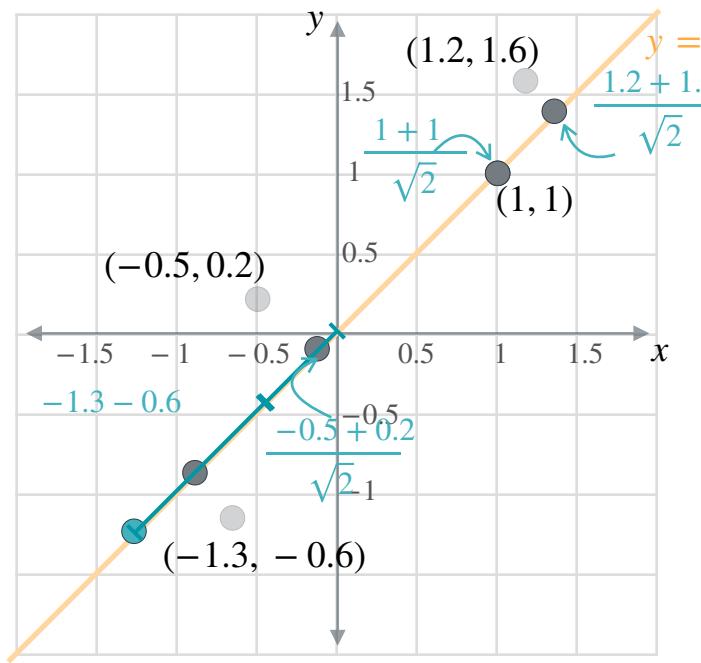


# Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

$$\begin{matrix} 1 \\ 1 \end{matrix} \frac{1}{\sqrt{2}} =$$

$$\begin{matrix} (1+1)/\sqrt{2} \\ (1.2+1.6)/\sqrt{2} \\ (-0.5+0.2)/\sqrt{2} \\ (-1.3-0.6) \end{matrix}$$



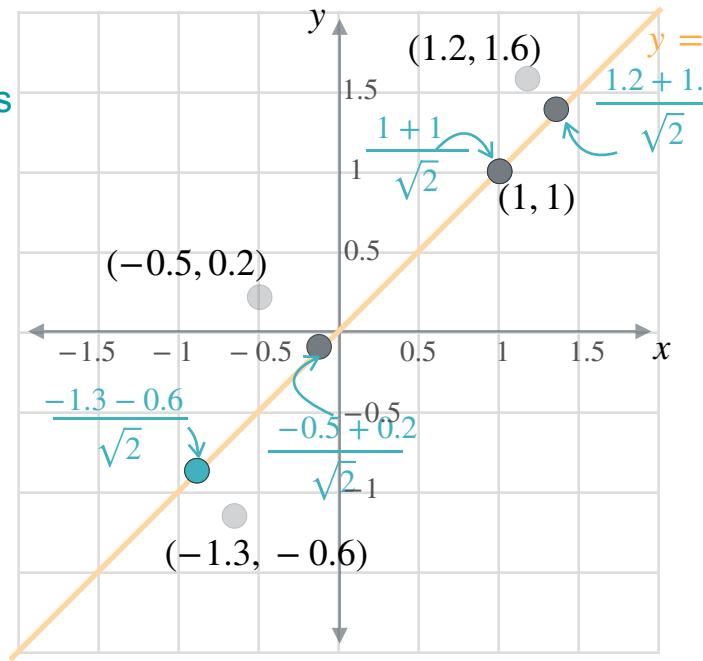
# Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

$$\begin{matrix} 1 \\ 1 \end{matrix} \frac{1}{\sqrt{2}} =$$

Final coordinates

$$\begin{aligned} & (1+1)/\sqrt{2} \\ & (1.2+1.6)/\sqrt{2} \\ & (-0.5+0.2)/\sqrt{2} \\ & (-1.3-0.6)/\sqrt{2} \end{aligned}$$



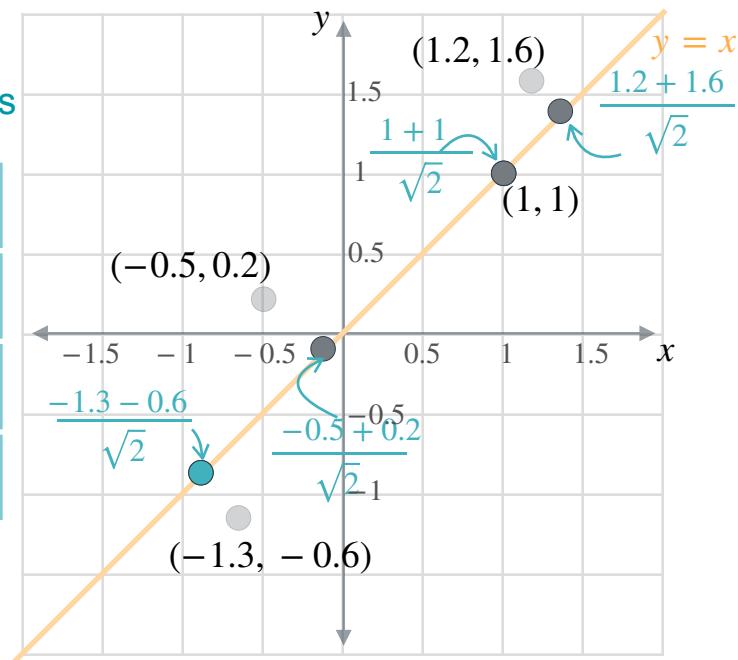
# Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

$$\begin{matrix} 1 \\ 1 \end{matrix} \frac{1}{\sqrt{2}} =$$

Final coordinates

1.4142
1.9799
-0.2121
-1.344



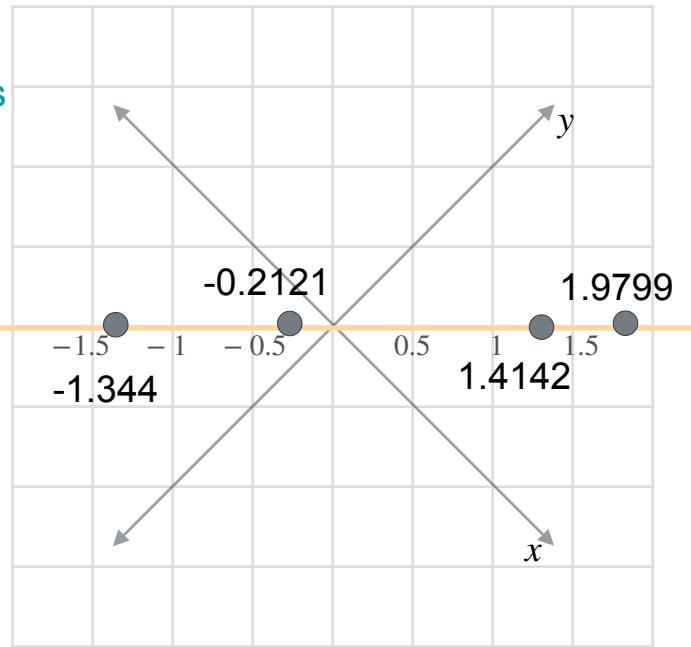
# Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} =$$

Final coordinates

1.4142
1.9799
-0.2121
-1.344



# Projections

To project a matrix  $A$  onto a vector  $v$

$$A_P = A \frac{v}{\|v\|_2}$$

$r \times 1 \quad r \times c \quad c \times 1$

# Projections

To project a matrix  $A$  onto vectors  $v_1$  and  $v_2$

$$A_P = A \underbrace{\begin{bmatrix} \frac{v_1}{\|v_1\|_2} & \frac{v_2}{\|v_2\|_2} \end{bmatrix}}_{V}$$

$r \times 2$        $r \times c$        $c \times 2$

# Projections

To project a matrix  $A$  onto vectors  $v_1$  and  $v_2$

$$A_P = A \underbrace{\begin{bmatrix} \frac{v_1}{\|v_1\|_2} & \frac{v_2}{\|v_2\|_2} \end{bmatrix}}_{V}$$

$r \times 2$        $\boxed{r} \times c$        $c \times \boxed{2}$

# Projections

To project a matrix  $A$  onto vectors  $v_1$  and  $v_2$

$$A_P = \textcolor{brown}{A} \textcolor{green}{V}$$

$$r \times 2 \quad r \times c \quad c \times 2$$



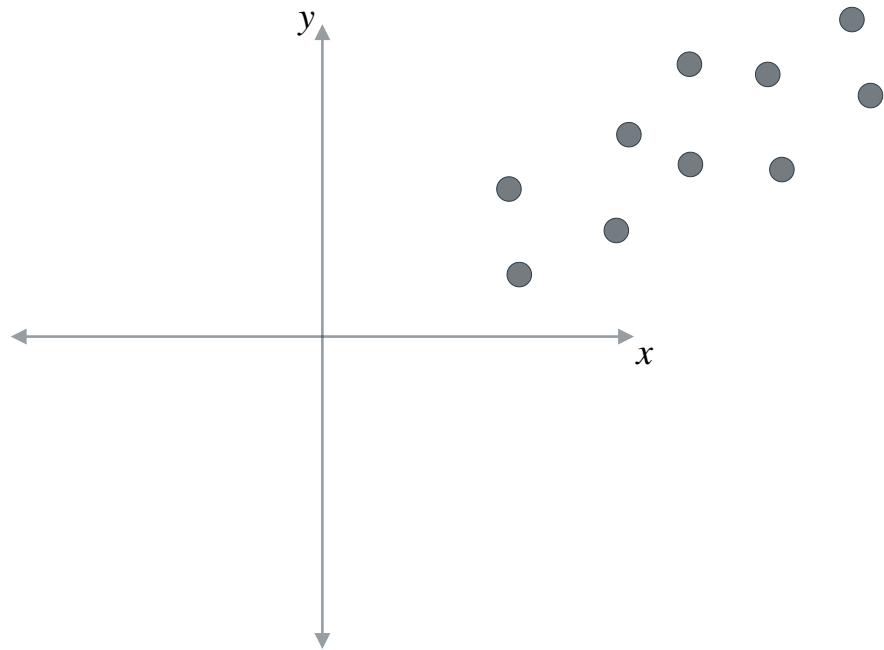
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## Determinants and Eigenvectors

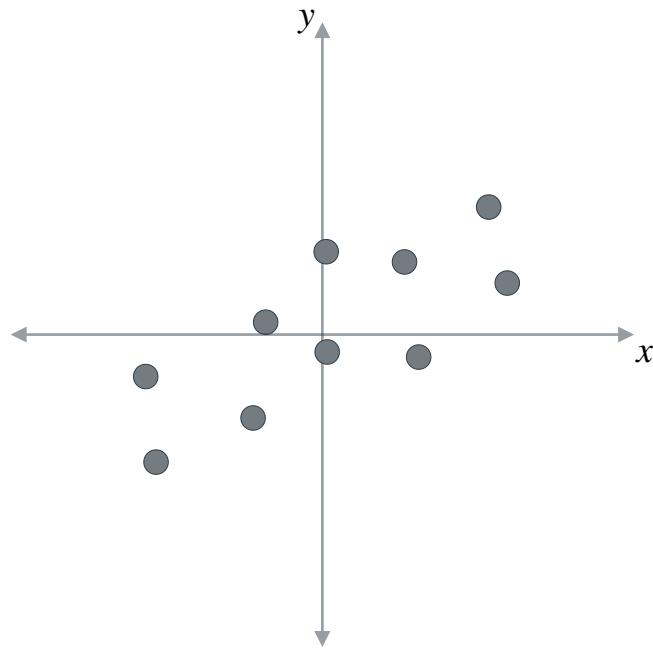
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### Motivating PCA

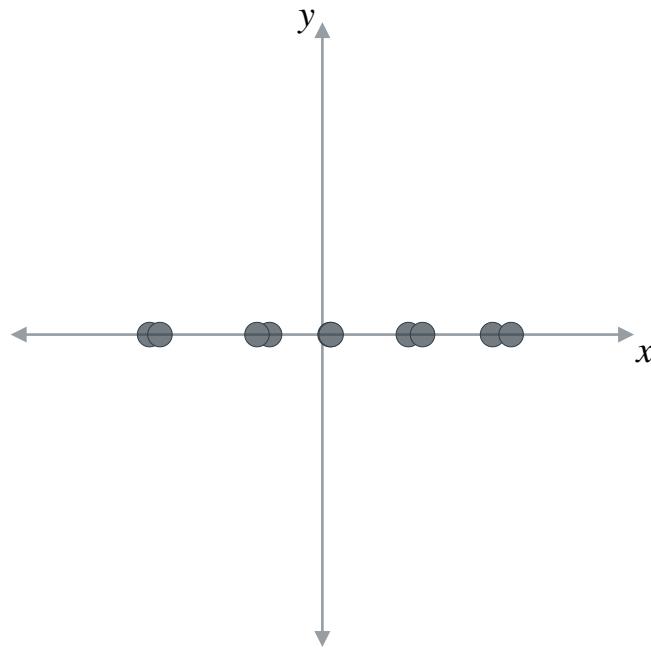
# Dimensionality Reduction



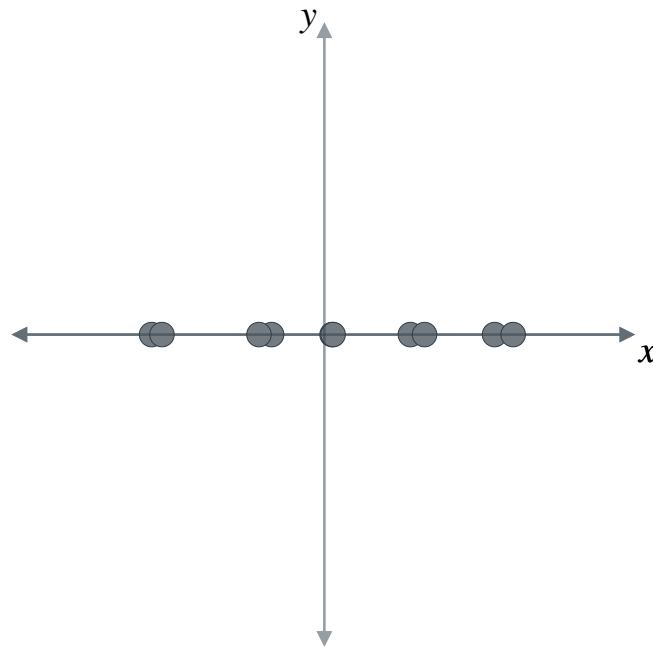
# Principal Component Analysis (PCA)



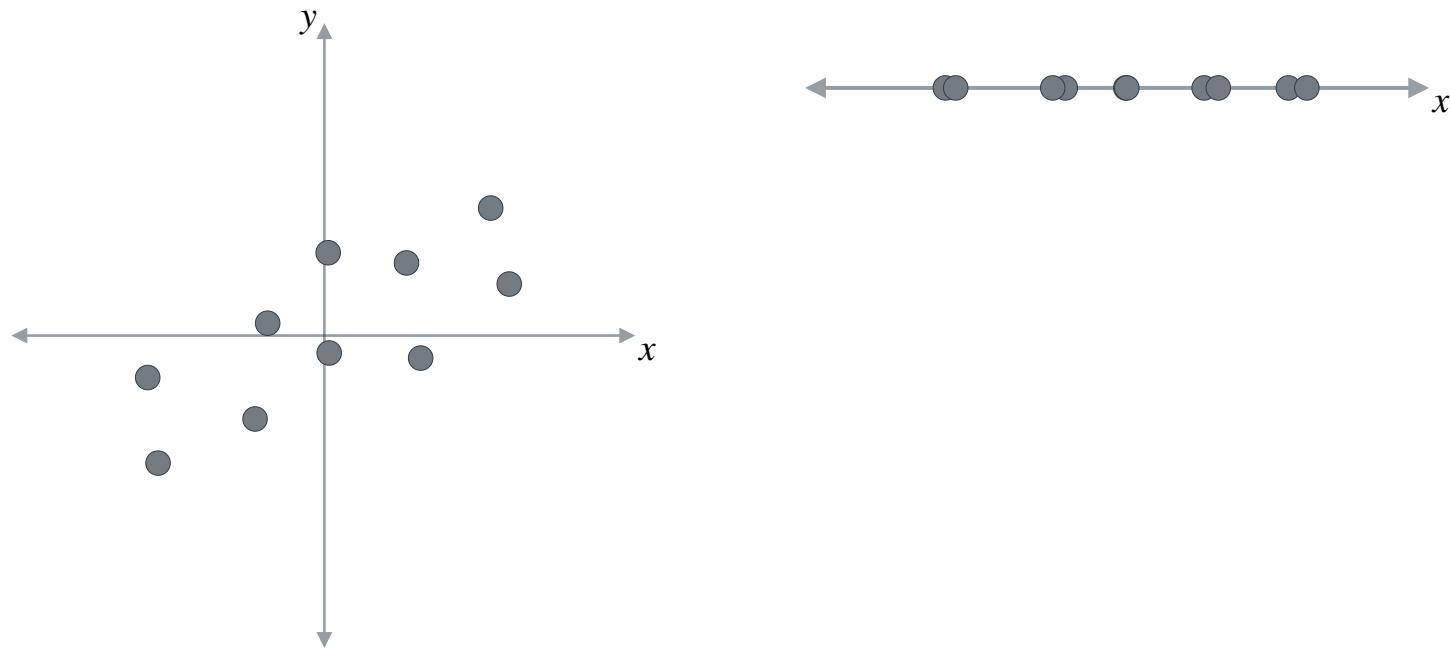
# Principal Component Analysis (PCA)



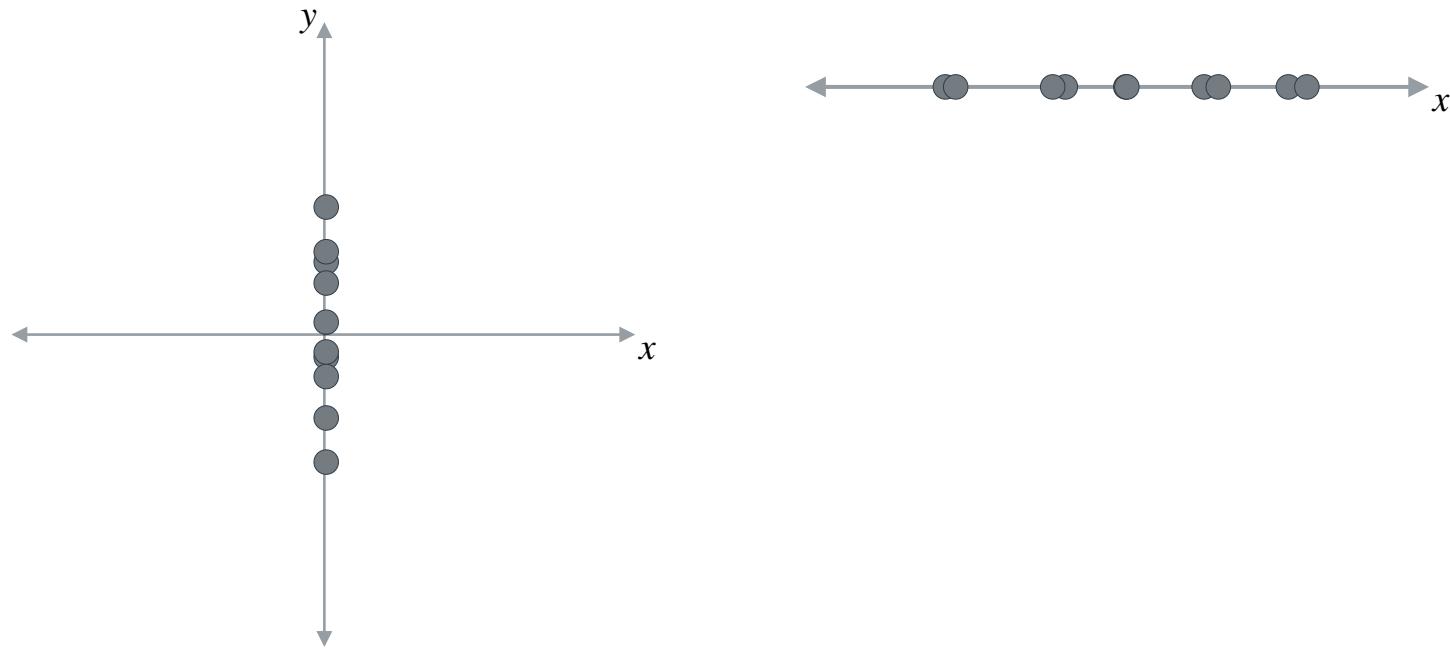
# Principal Component Analysis (PCA)



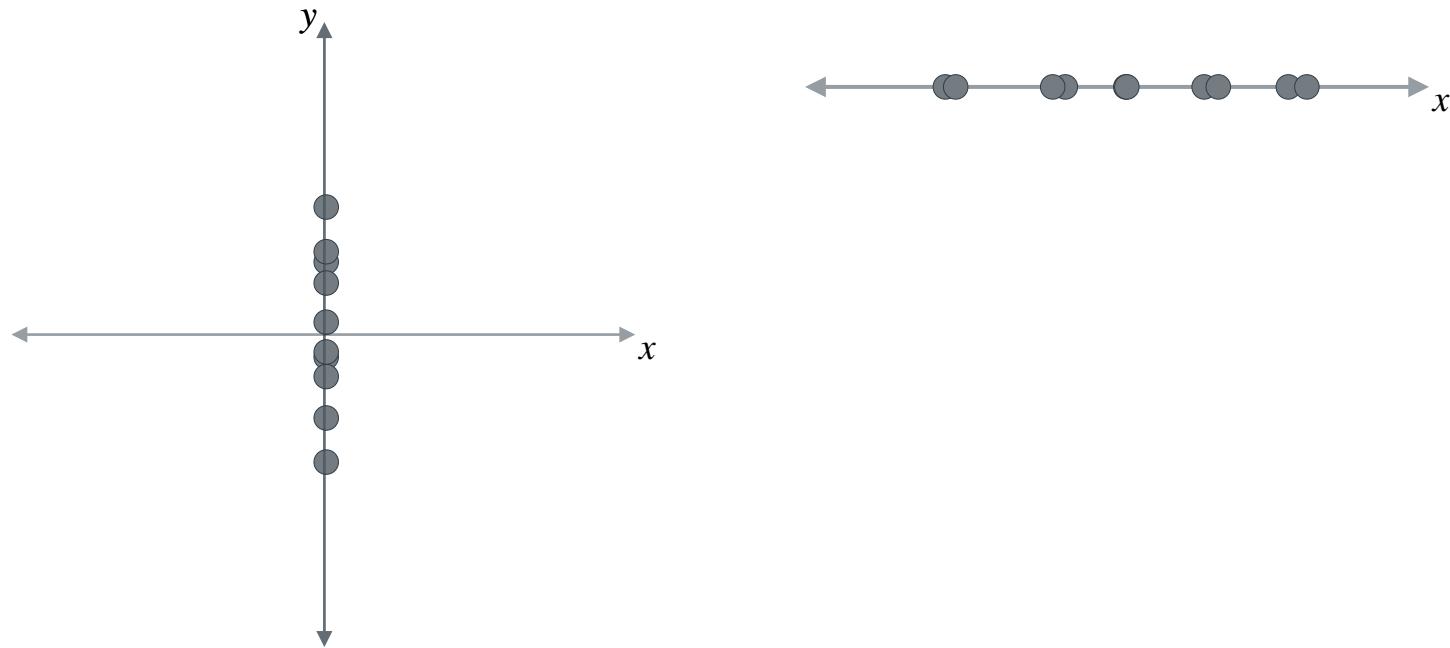
# Principal Component Analysis (PCA)



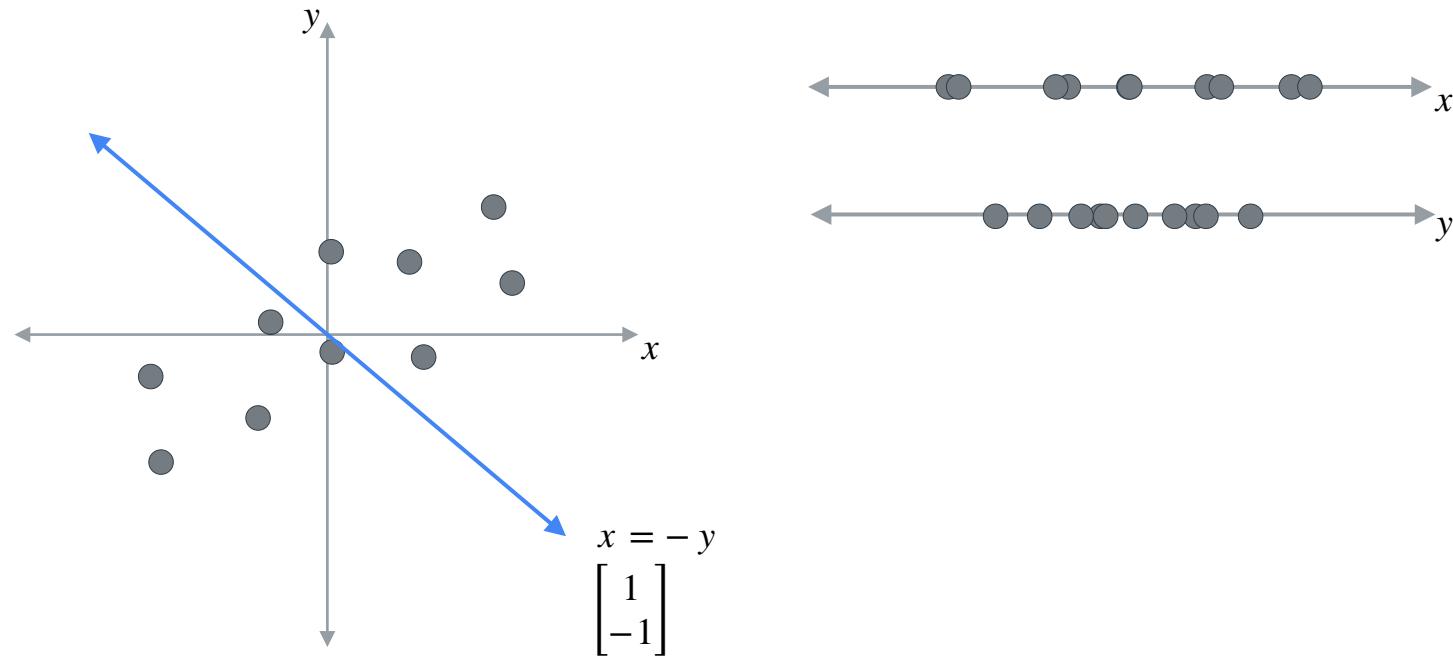
# Principal Component Analysis (PCA)



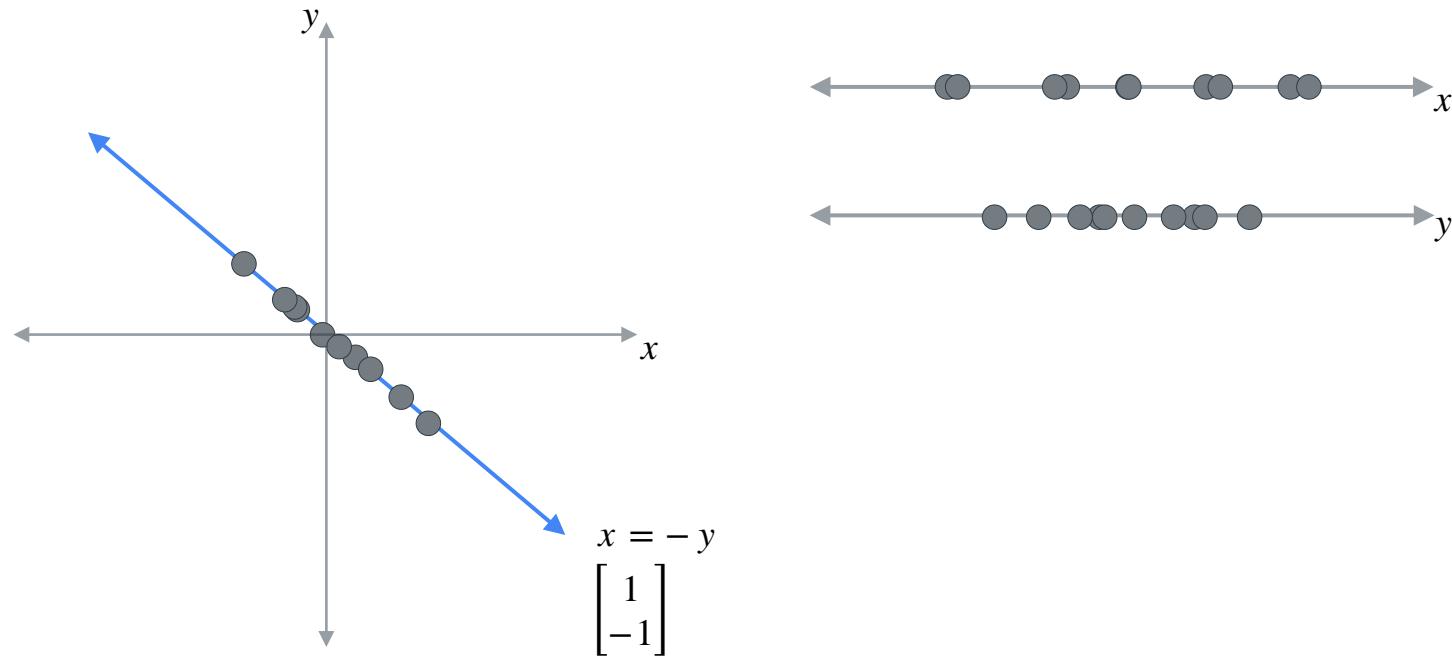
# Principal Component Analysis (PCA)



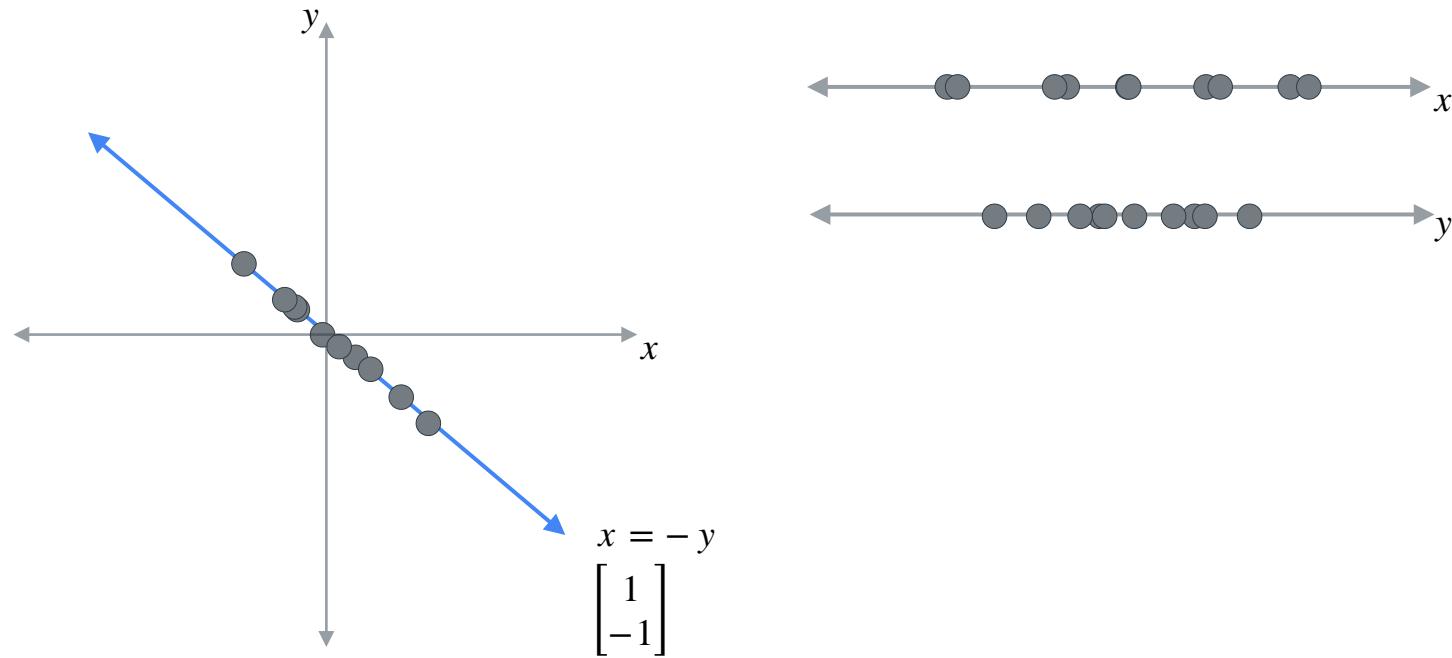
# Principal Component Analysis (PCA)



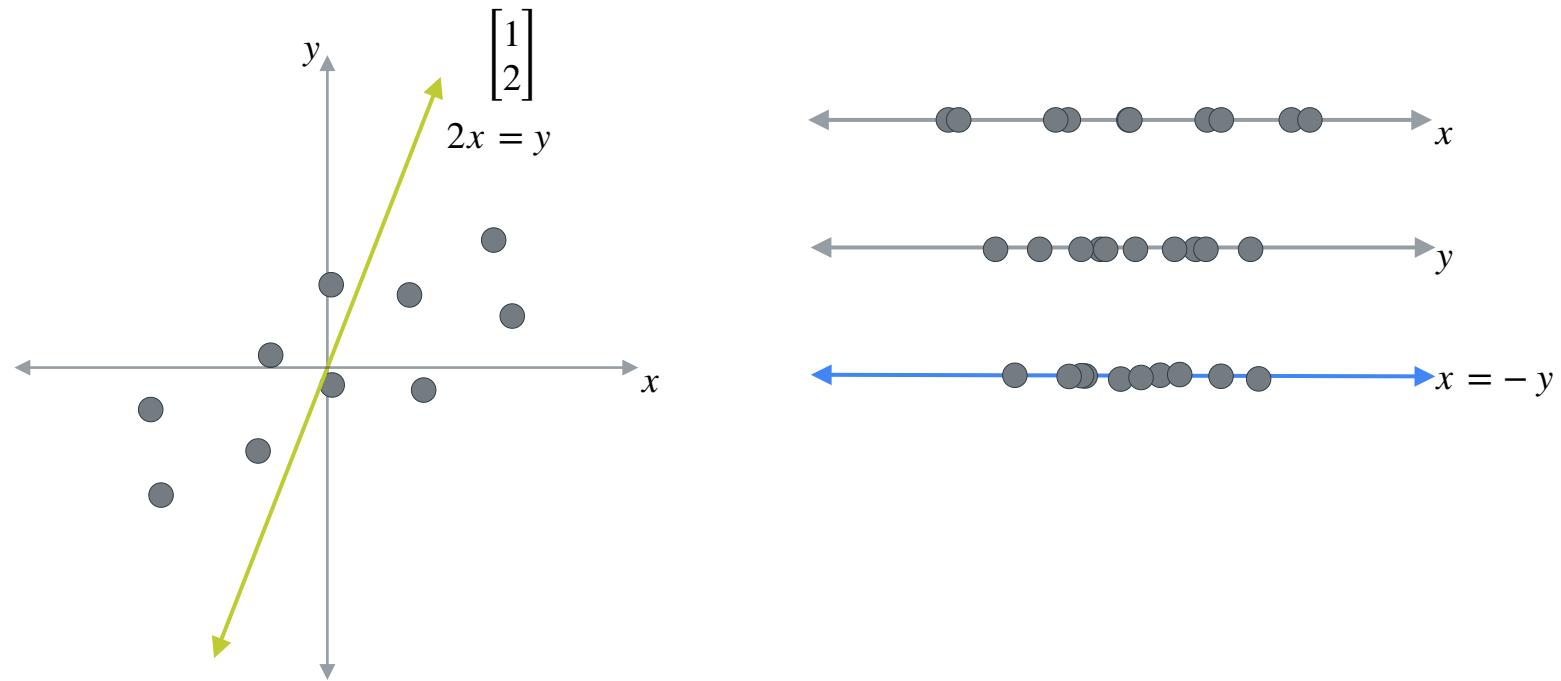
# Principal Component Analysis (PCA)



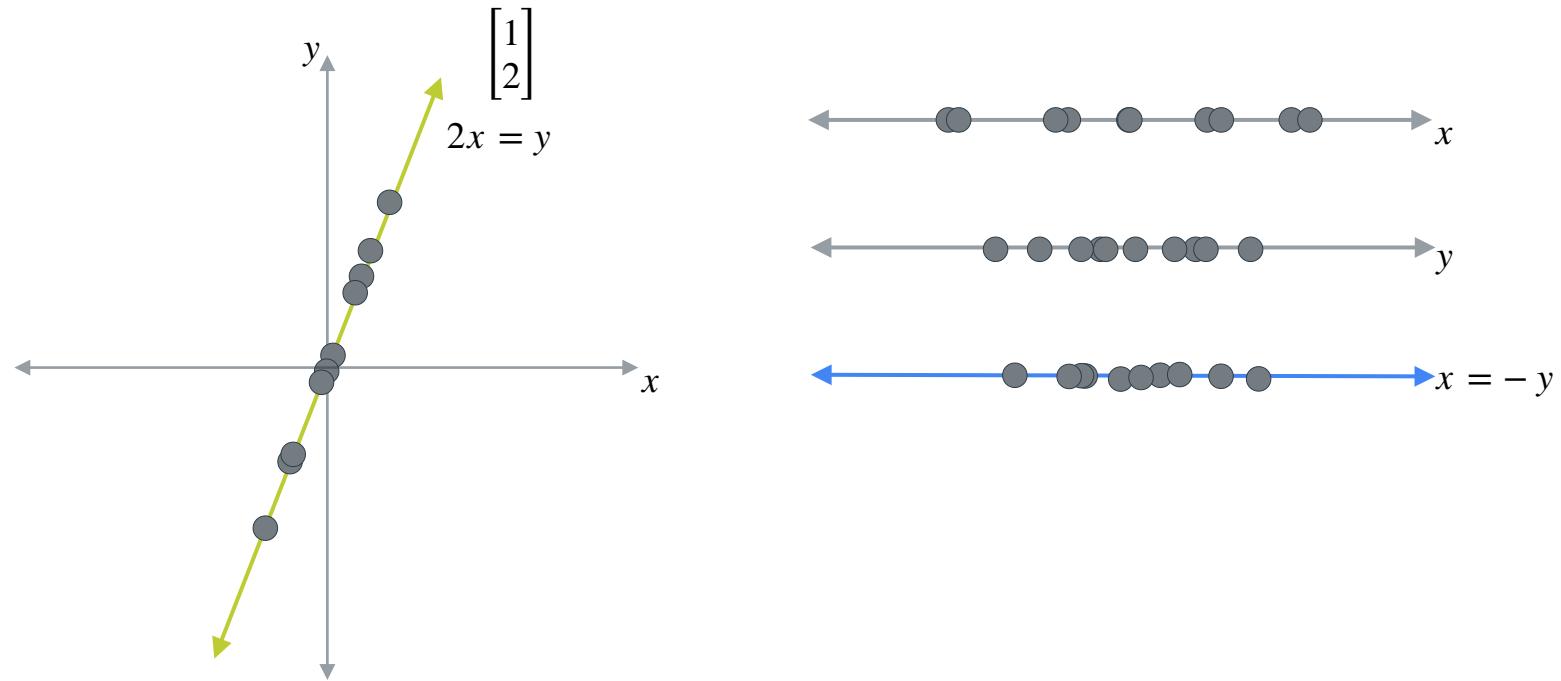
# Principal Component Analysis (PCA)



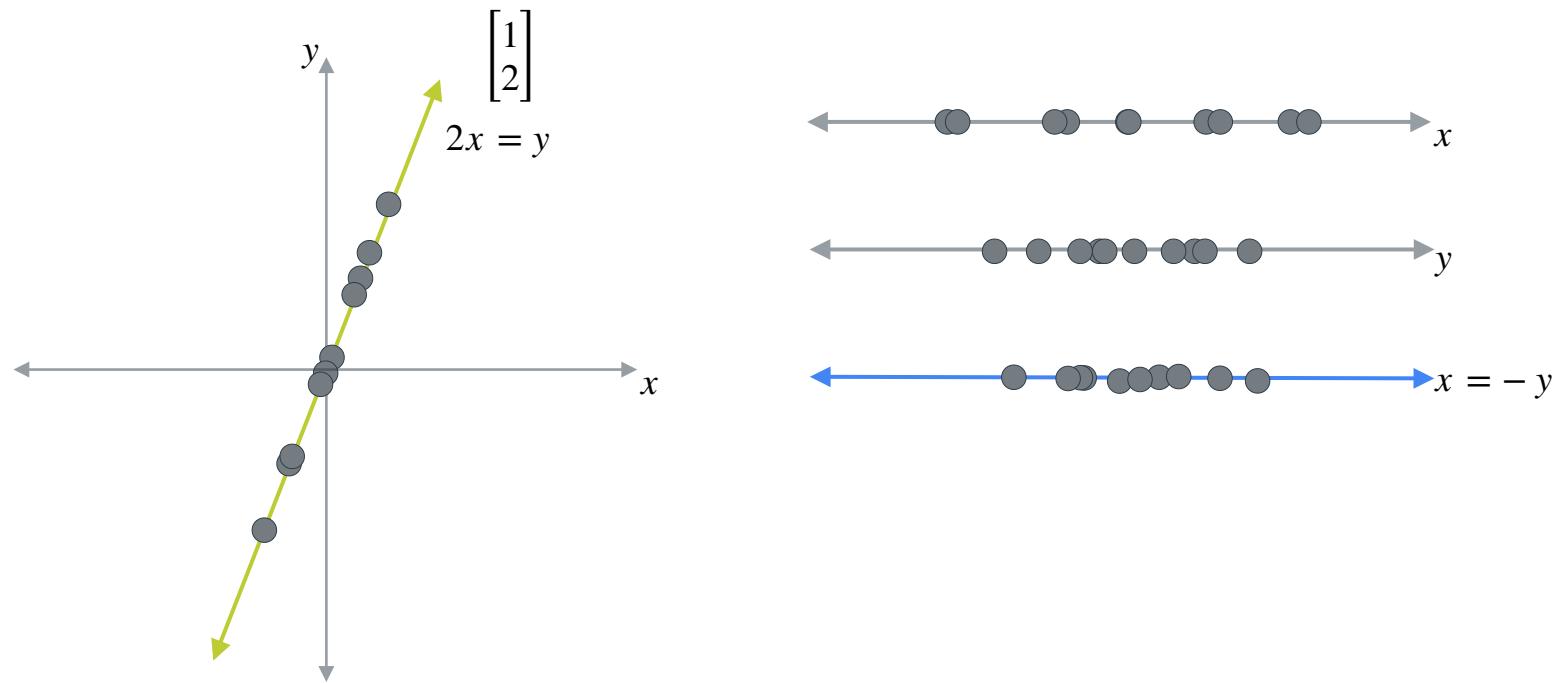
# Principal Component Analysis (PCA)



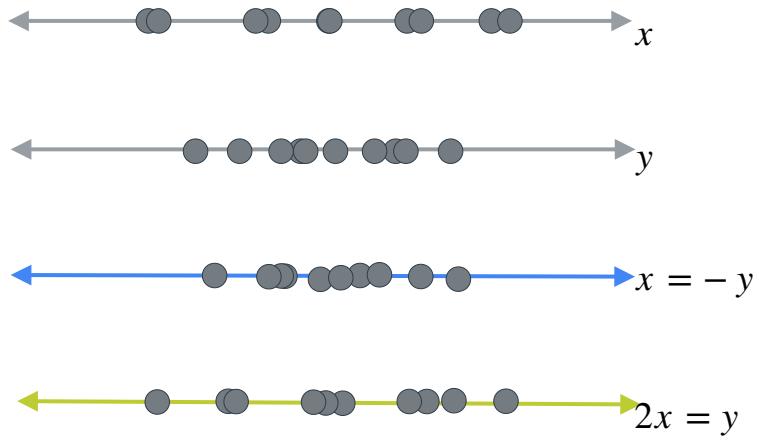
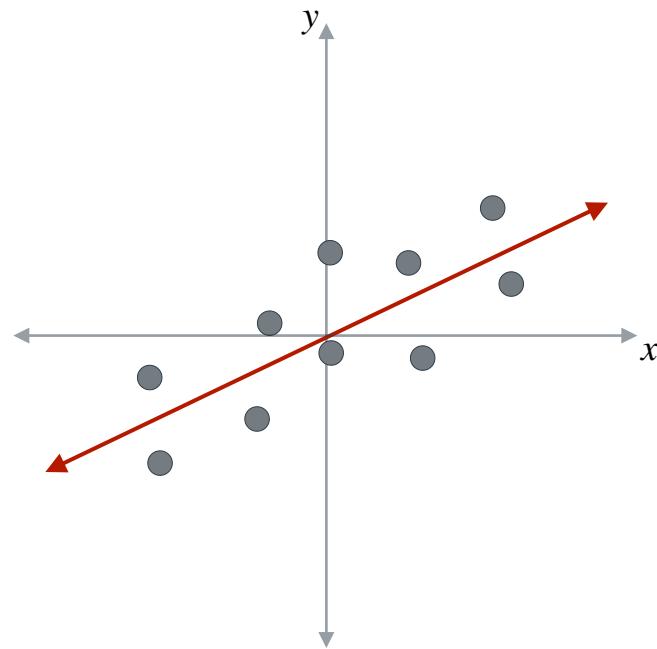
# Principal Component Analysis (PCA)



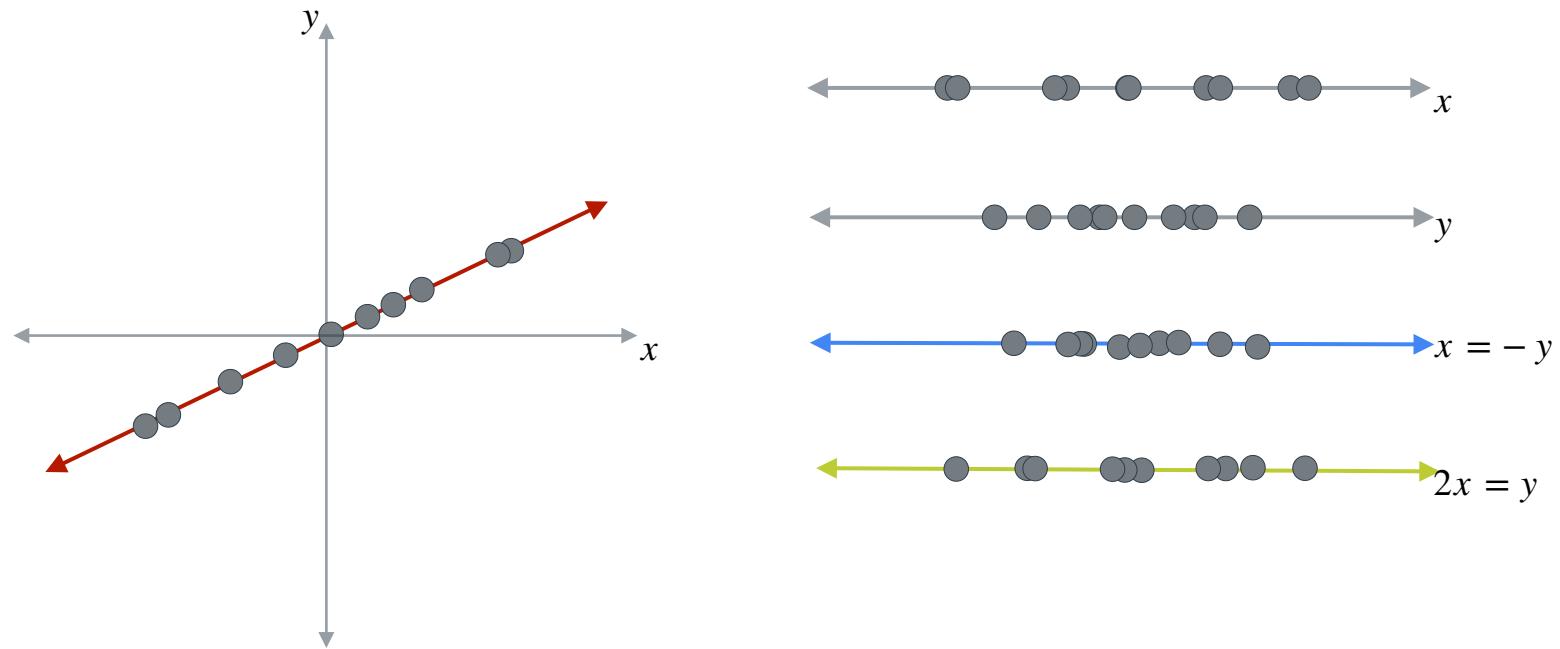
# Principal Component Analysis (PCA)



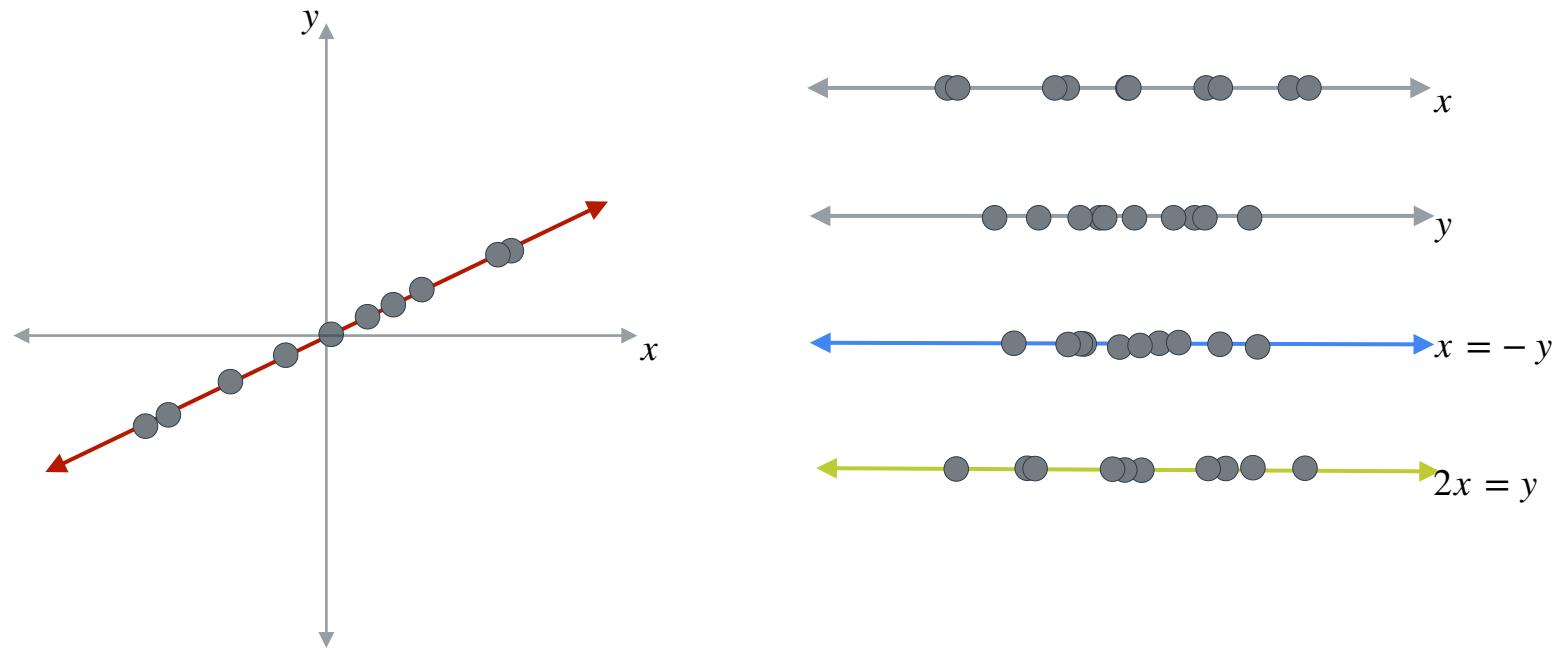
# Principal Component Analysis (PCA)



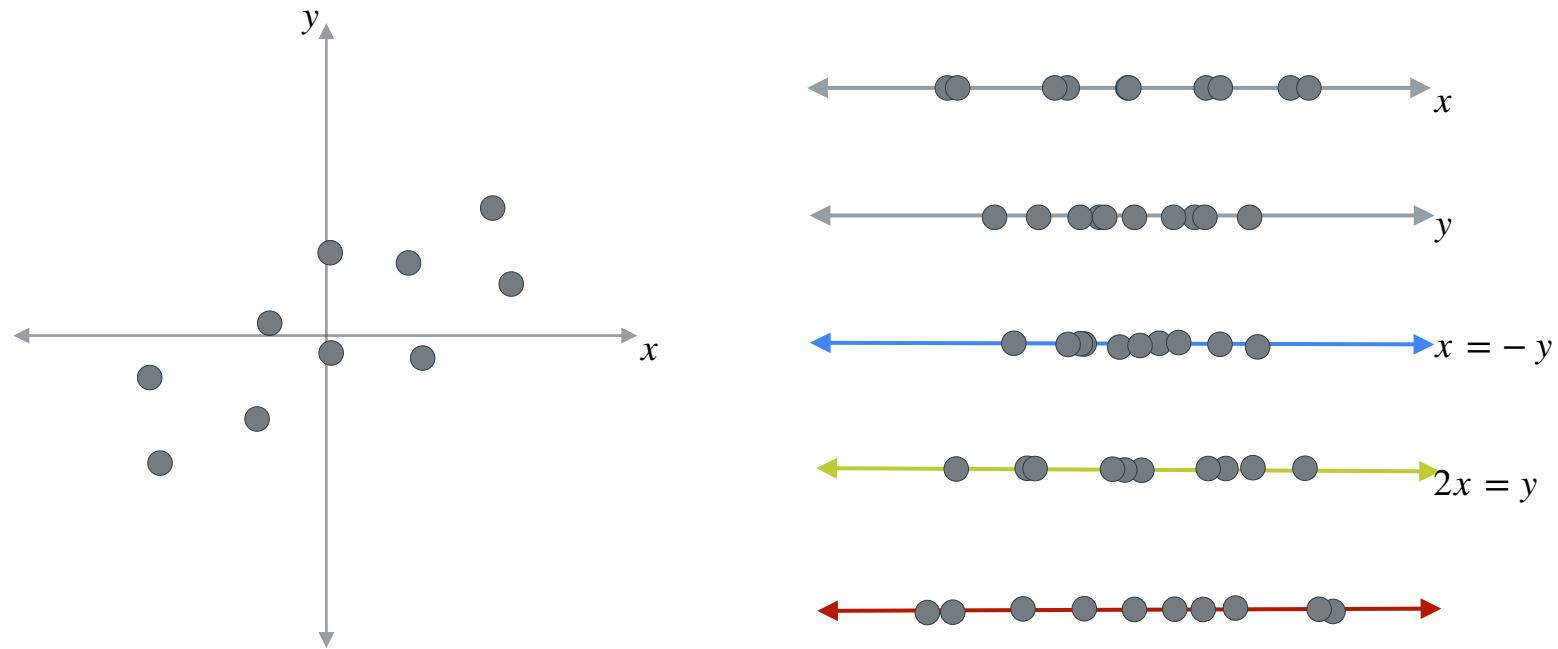
# Principal Component Analysis (PCA)



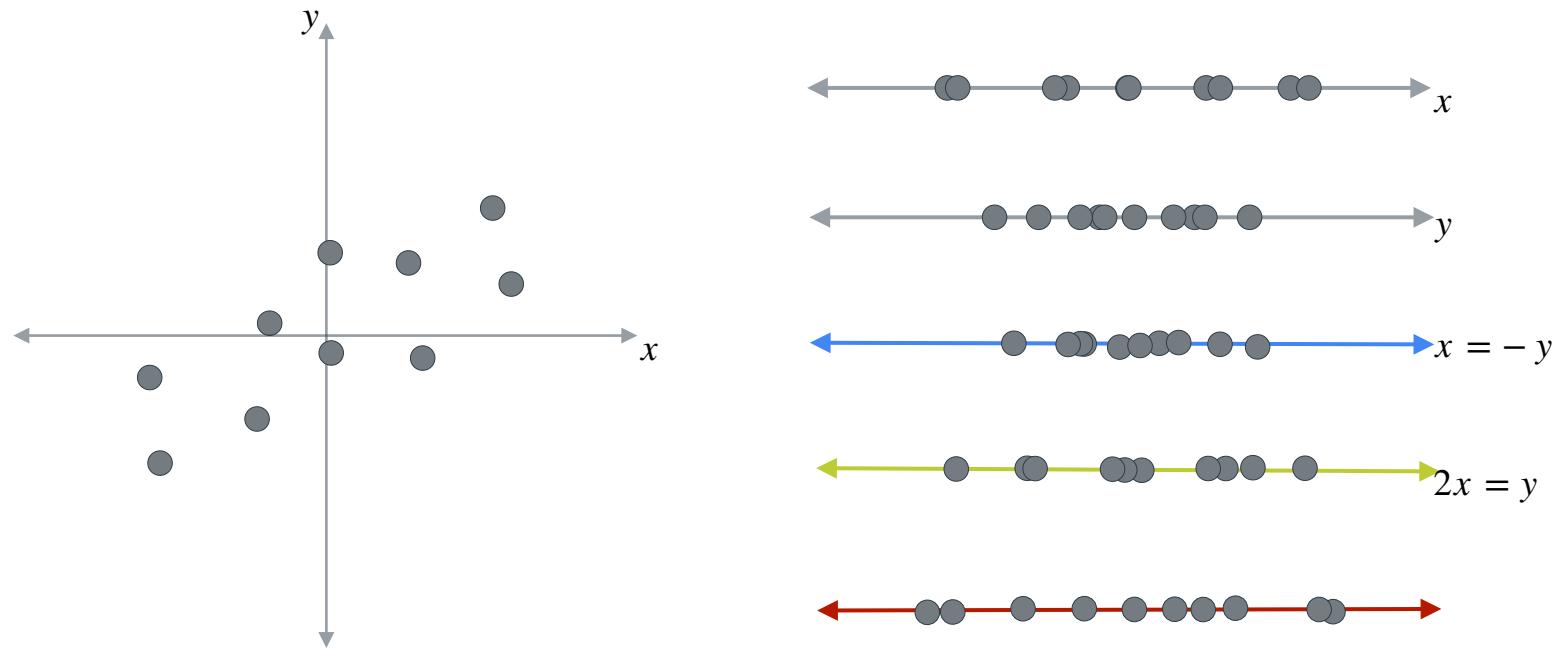
# Principal Component Analysis (PCA)



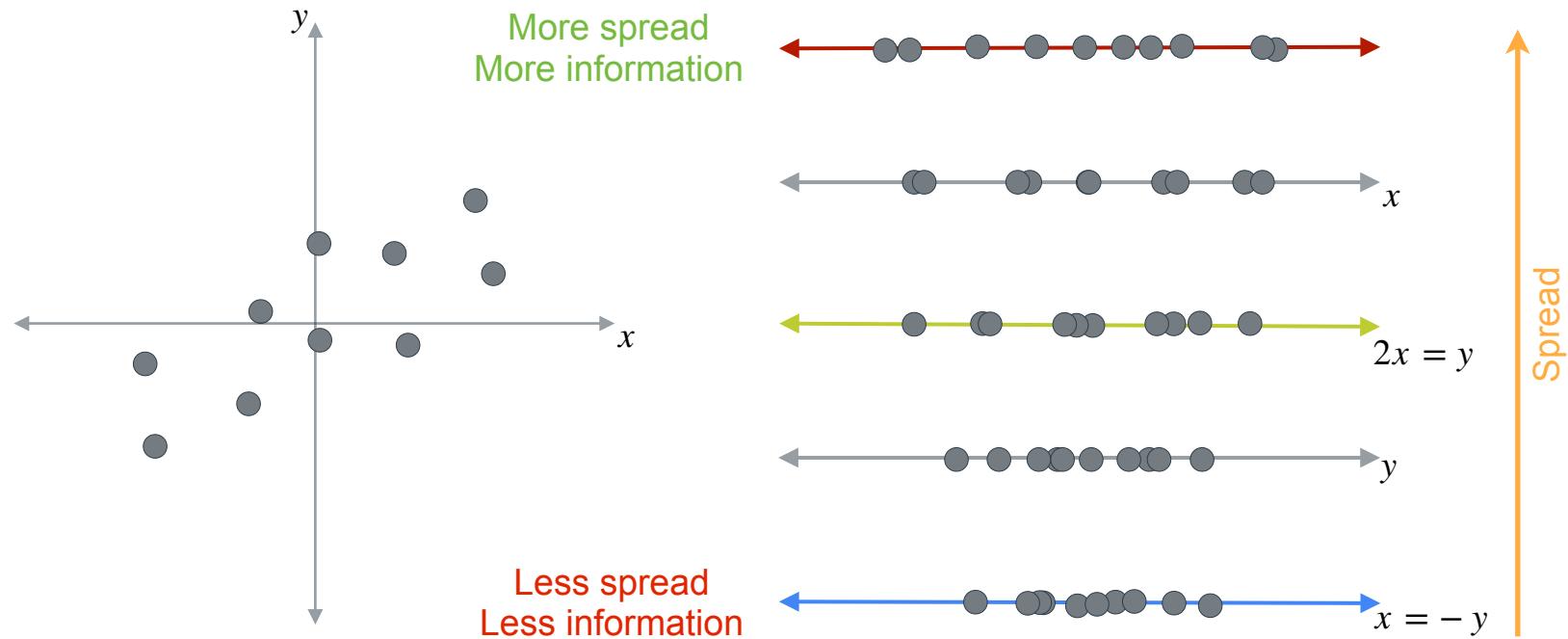
# Principal Component Analysis (PCA)



# Principal Component Analysis (PCA)

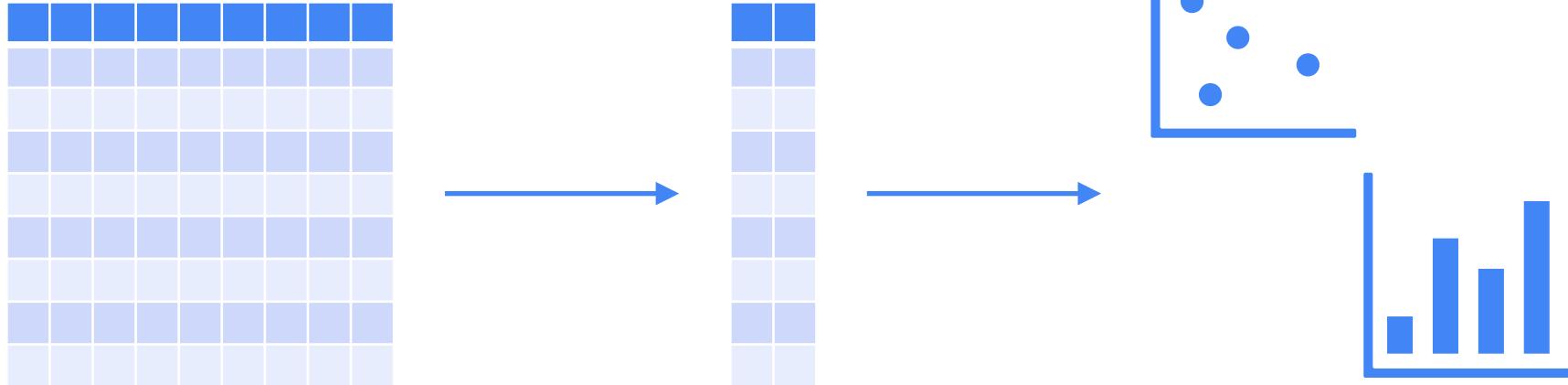


# Principal Component Analysis (PCA)



# Benefits of Dimensionality Reduction

- Easier dataset to manage
- PCA reduces dimensions while minimizing information loss
- Simpler visualization





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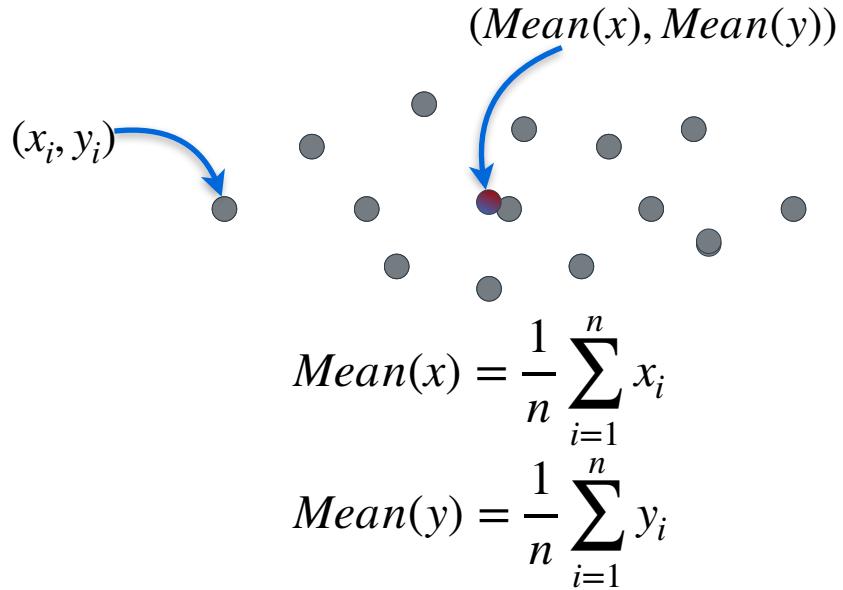
## Determinants and Eigenvectors

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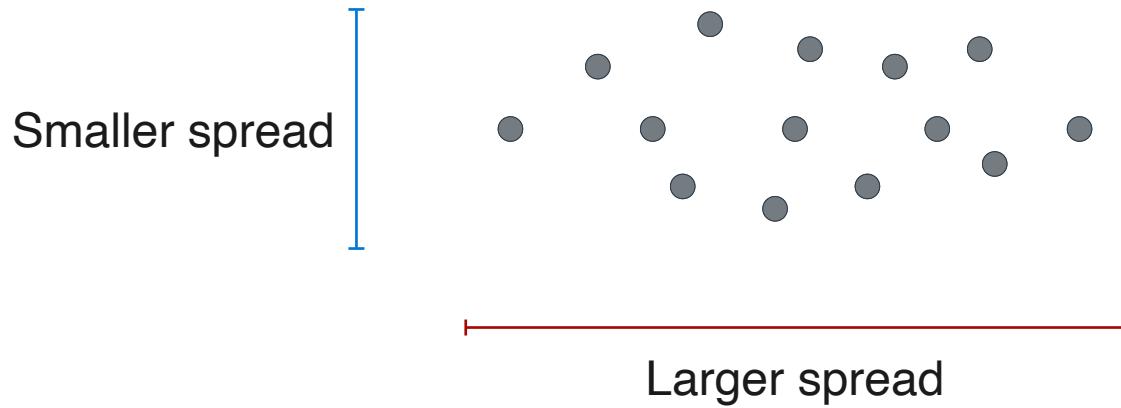
## Variance and covariance

# Mean

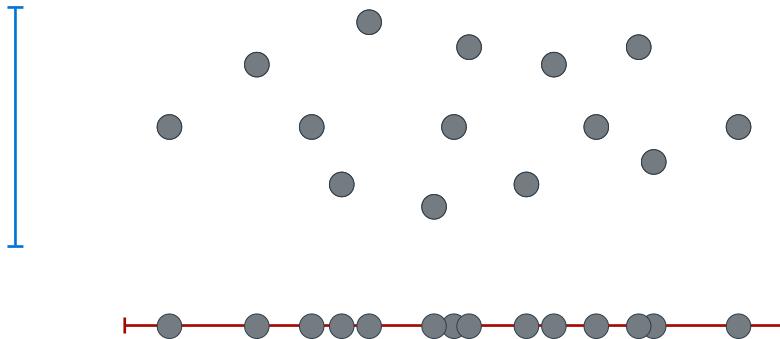
“The average of the data”



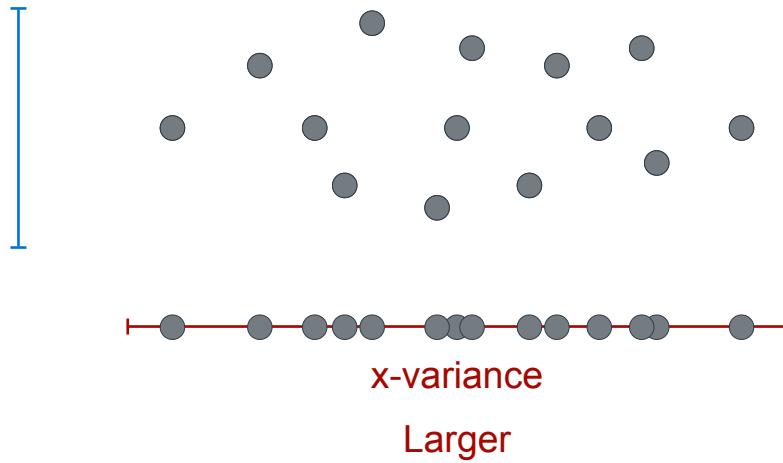
# Variance



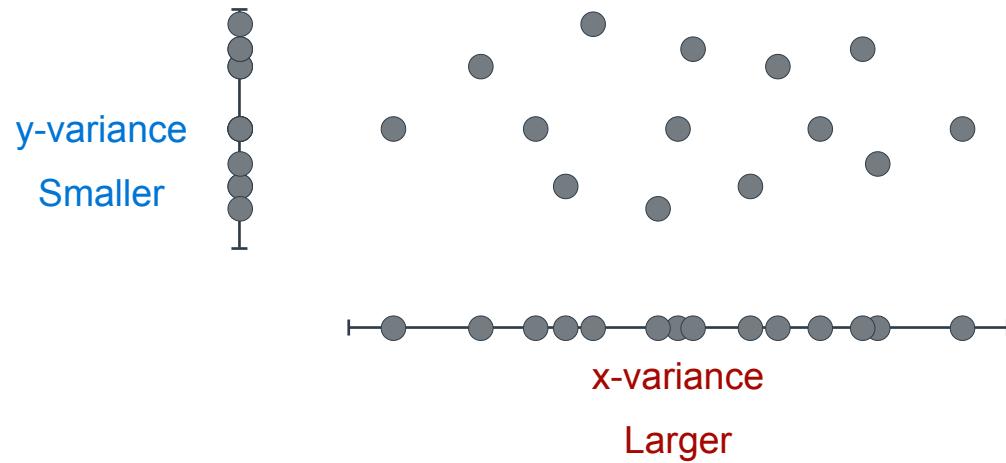
# Variance



# Variance



# Variance



# Variance

$$Variance(x) = \frac{1}{n - 1} \sum_{i=1}^n (x_i - Mean(x))^2 = 16$$

	$x_i$	$x_i - Mean(x)$	$(x_i - Mean(x))^2$
1	10	1	1
2	4	-5	25
3	11	2	4
4	14	5	25
5	6	-3	9

→ 64

$$Mean(x) = 9$$

# Variance

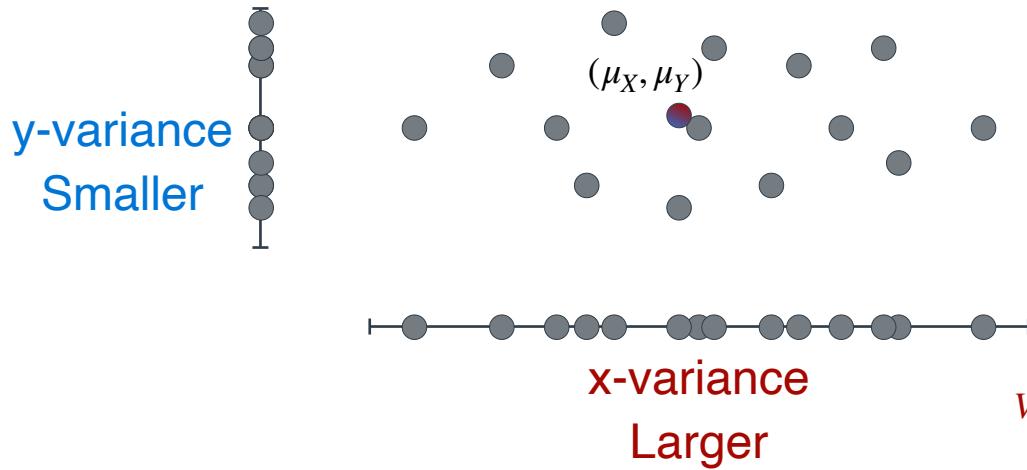
$$Variance(x) = \frac{1}{n-1} \sum_{i=1}^n (x_i - \text{Mean}(x))^2$$

$$Var(x) \quad \mu_x$$

“The average squared distance from the mean”

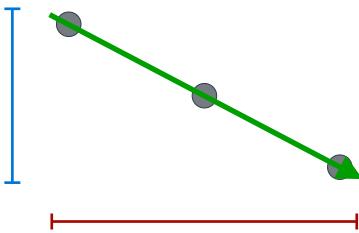
# Variance

$$Var(y) = \frac{1}{n-1} \sum_{i=1}^n (y_i - \mu_Y)^2$$



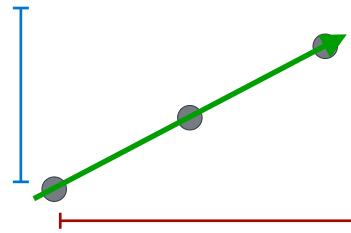
$$Var(x) = \frac{1}{n-1} \sum_{i=1}^n (x_i - \mu_x)^2$$

# Problem



Negative covariance

Solution: Covariance



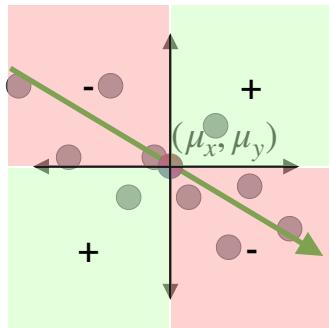
Positive covariance

# Covariance

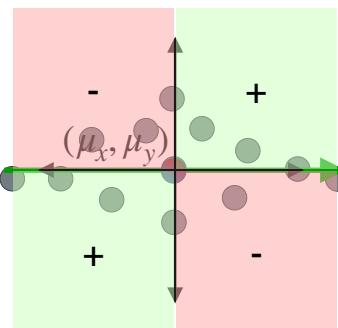
“Take the average”

$$Cov(x, y) = \frac{1}{n-1} \sum_{i=1}^n (x_i - \mu_x)(y_i - \mu_y)$$

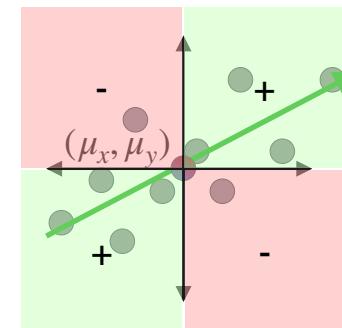
$$Var(x) = \frac{1}{n-1} \sum_{i=1}^n (x_i - \mu_x)^2$$



negative covariance



covariance zero  
(or very small)

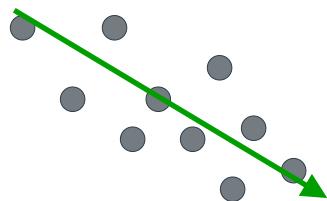


positive covariance

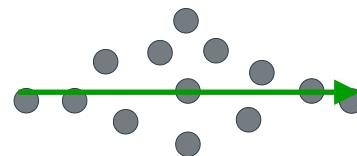
# Covariance

$$Cov(x, y) = \frac{1}{n - 1} \sum_{i=1}^n (x_i - \mu_x)(y_i - \mu_y)$$

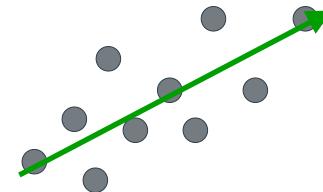
“The direction of the relationship between two variables”



negative covariance



covariance zero  
(or very small)



positive covariance



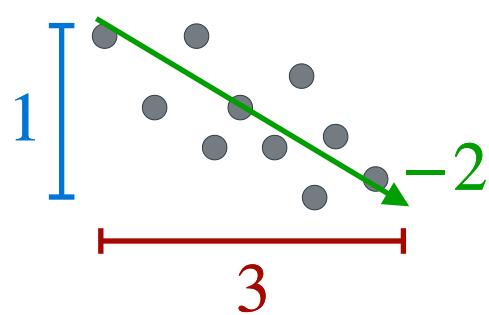
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## Determinants and Eigenvectors

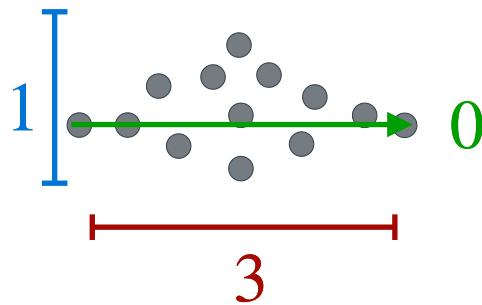
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### The covariance matrix

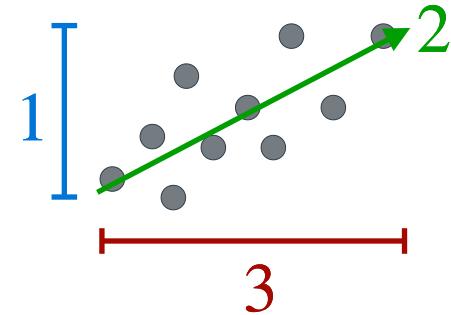
# Covariance matrix



$$\begin{bmatrix} & \\ & \end{bmatrix}$$

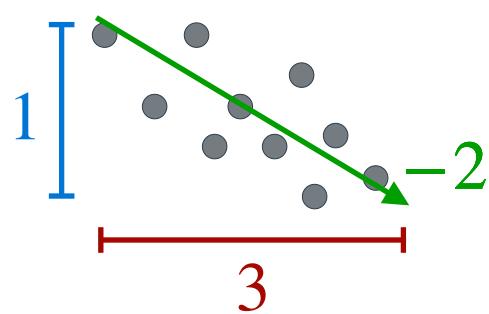


$$\begin{bmatrix} & \\ & \end{bmatrix}$$

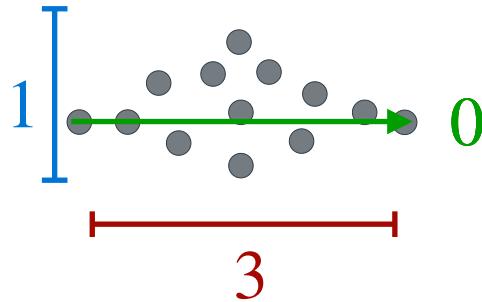


$$\begin{bmatrix} & \\ & \end{bmatrix}$$

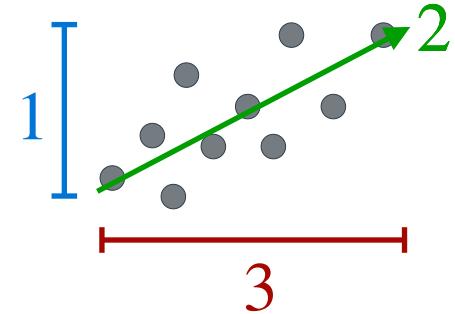
# Covariance matrix



$$\begin{bmatrix} 3 & \\ & 1 \end{bmatrix}$$

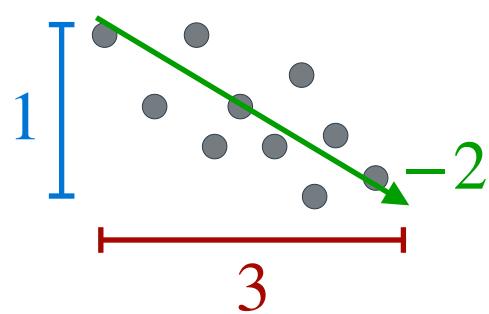


$$\begin{bmatrix} 3 & \\ & 1 \end{bmatrix}$$

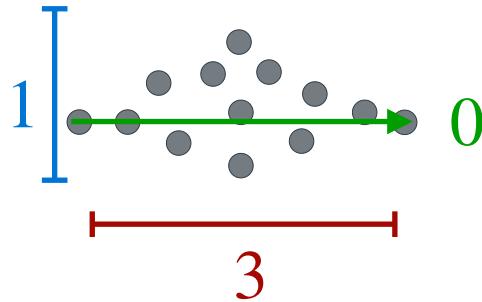


$$\begin{bmatrix} 3 & \\ & 1 \end{bmatrix}$$

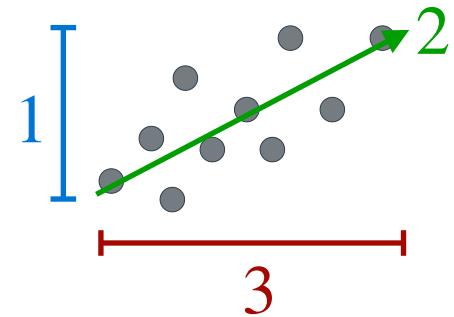
# Covariance matrix



$$\begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}$$

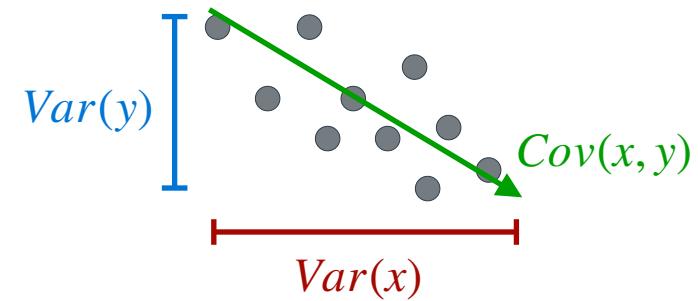


$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$



$$\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

# Covariance matrix



$$C = \begin{bmatrix} x & y \\ Cov(x, x) & Cov(x, y) \\ Cov(y, x) & Cov(y, y) \end{bmatrix}$$

$$Cov(x, x) = Var(x)$$

# Covariance matrix

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix}$$

# Covariance matrix

$$A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} \quad \mu = \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}$$

$$C = \frac{1}{n - 1} (\mathbf{x} - \boldsymbol{\mu})^T (\mathbf{x} - \boldsymbol{\mu})$$

# Covariance matrix

$$A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} \quad \mu = \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}$$

$$C = \frac{1}{n - 1}(A - \mu)^T(A - \mu)$$

# Covariance matrix

$$A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} \quad \mu = \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}$$

$$\begin{aligned} C &= \frac{1}{n-1}(A - \mu)^T(A - \mu) = \frac{1}{n-1} \left( \begin{array}{c|c} & - \\ \hline - & \end{array} \right)^T \left( \begin{array}{c|c} & - \\ \hline - & \end{array} \right) \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \end{aligned}$$

# Covariance matrix

$$A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} \quad \mu = \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}$$

$$\begin{aligned} C = \frac{1}{n-1}(A - \mu)^T(A - \mu) &= \frac{1}{n-1} \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \end{aligned}$$

# Covariance matrix

$$\begin{aligned} C = \frac{1}{n-1}(A - \mu)^T(A - \mu) &= \frac{1}{n-1} \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_n \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\ &\quad \boxed{2} \times n \qquad \qquad \qquad n \times \boxed{2} \end{aligned}$$

# Covariance matrix

$$\begin{aligned} C &= \frac{1}{n-1}(A - \mu)^T(A - \mu) = \frac{1}{n-1} \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_n \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\ &\quad (x_1 - \mu_x)(x_1 - \mu_x) + (x_2 - \mu_x)(x_2 - \mu_x) + \dots + (x_n - \mu_x)(x_n - \mu_x) \end{aligned}$$

# Covariance matrix

$$\begin{aligned} C &= \frac{1}{n-1}(A - \mu)^T(A - \mu) = \frac{1}{n-1} \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_n \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\ &\quad \sum_{i=1}^n (x_i - \mu_x)^2 = \text{Var}(x) \end{aligned}$$

# Covariance matrix

$$\begin{aligned} C &= \frac{1}{n-1}(A - \mu)^T(A - \mu) = \frac{1}{n-1} \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_n \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} = \begin{bmatrix} Var(x) & \\ & \ddots \\ & & Var(y) \end{bmatrix} \\ &\quad \frac{1}{n-1} \sum_{i=1}^n (x_i - \mu_x)^2 = Var(x) \end{aligned}$$

# Covariance matrix

$$\begin{aligned} C = \frac{1}{n-1}(A - \mu)^T(A - \mu) &= \frac{1}{n-1} \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_n \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} = \begin{bmatrix} \textcolor{red}{Var(x)} \\ \vdots \end{bmatrix} \end{aligned}$$

$(x_1 - \mu_x)(y_1 - \mu_y) + (x_2 - \mu_x)(y_2 - \mu_y) + \dots + (x_n - \mu_x)(y_n - \mu_y)$

# Covariance matrix

$$\begin{aligned} C = \frac{1}{n-1}(A - \mu)^T(A - \mu) &= \frac{1}{n-1} \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_n \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} = \begin{bmatrix} \text{Var}(x) \\ \text{Var}(y) \end{bmatrix} \\ &\quad \sum_{i=1}^n (x_i - \mu_x)(y_i - \mu_y) = \text{Cov}(x, y) \end{aligned}$$

# Covariance matrix

$$\begin{aligned} C = \frac{1}{n-1}(A - \mu)^T(A - \mu) &= \frac{1}{n-1} \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_n \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\ &= \frac{1}{n-1} \sum_{i=1}^n (x_i - \mu_x)(y_i - \mu_y) = Cov(x, y) \end{aligned}$$

# Covariance matrix

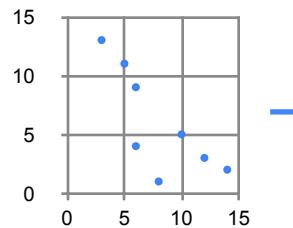
$$\begin{aligned} C = \frac{1}{n-1}(A - \mu)^T(A - \mu) &= \frac{1}{n-1} \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_n \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} = \begin{bmatrix} \textcolor{red}{Var(x)} & \textcolor{green}{Cov(x,y)} \\ \textcolor{green}{Cov(y,x)} & \textcolor{blue}{Var(y)} \end{bmatrix} \end{aligned}$$

# Covariance matrix

$$\begin{aligned} C = \frac{1}{n-1}(A - \mu)^T(A - \mu) &= \frac{1}{n-1} \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_n \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} = \begin{bmatrix} \textcolor{red}{Var(x)} & \textcolor{green}{Cov(x,y)} \\ \textcolor{green}{Cov(y,x)} & \textcolor{blue}{Var(y)} \end{bmatrix} \end{aligned}$$

# Matrix formula

$$A - \mu = \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \quad C = \frac{1}{n-1}(A - \mu)^T(A - \mu)$$



$A$

$x$	$y$
10	5
12	3
6	9
6	4
5	11
14	2
8	1
3	13

$A - \mu$

$x - \mu_x$	$y - \mu_y$
2	-1
4	-3
-2	3
-2	-2
-3	5
6	-4
0	-5
-5	8

$(A - \mu)^T$

$$\frac{1}{7}$$

$x - \mu_x$	2	4	-2	-2	-3	6	0	-5
$y - \mu_y$	-1	-3	3	-2	5	-4	-5	8

$A - \mu$

$x - \mu_x$	$y - \mu_y$
2	-1
4	-3
-2	3
-2	-2
-3	5
6	-4
0	-5
-5	8

$$\mu_x = 8 \quad \mu_y = 6$$

$$C = \begin{bmatrix} 14 & -11.86 \\ -11.86 & 19.71 \end{bmatrix}$$

# Matrix formula

$$A = \begin{bmatrix} x_1 & y_1 & z_1 \\ \vdots & \vdots & \vdots \\ x_n & y_n & z_n \end{bmatrix} \quad C = \frac{1}{n-1} (A - \mu)^T (A - \mu)$$

1. Arrange data with a different feature in each column
2. Calculate column averages
3. Subtract each average from their respective column to generate  $A - \mu$
4.  $\frac{1}{n-1} (A - \mu)^T (A - \mu)$  gives the covariance matrix  $C$



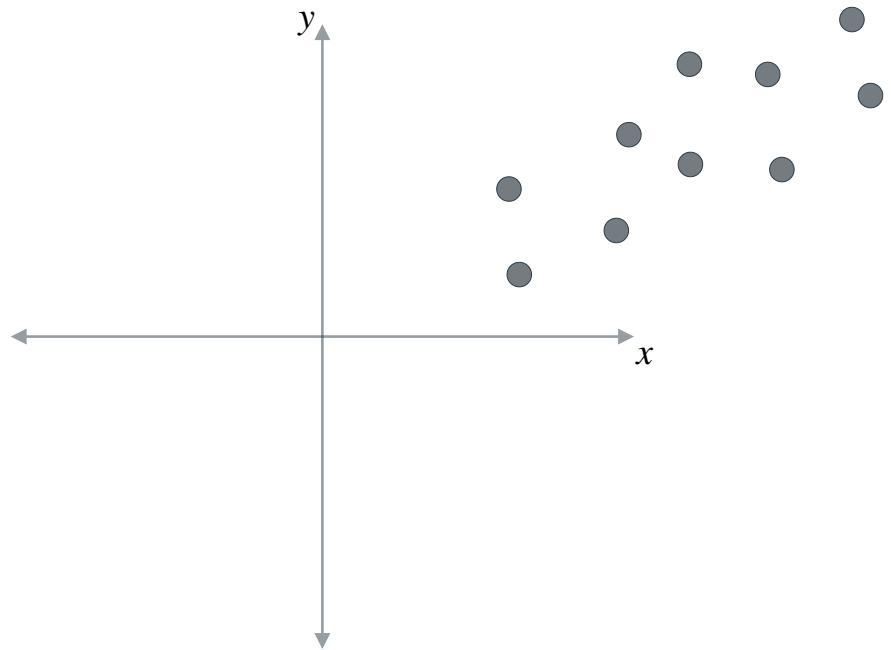
DeepLearning.AI

## Determinants and Eigenvectors

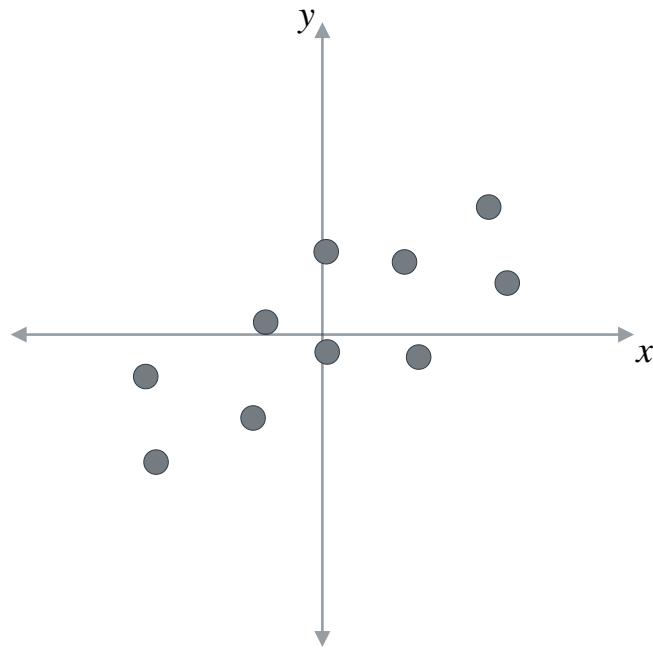
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### PCA - Overview

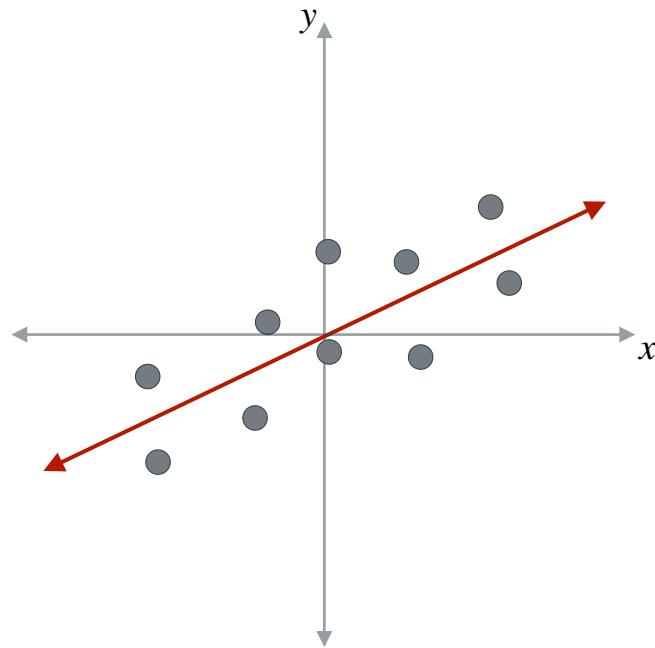
# Principal Component Analysis (PCA)



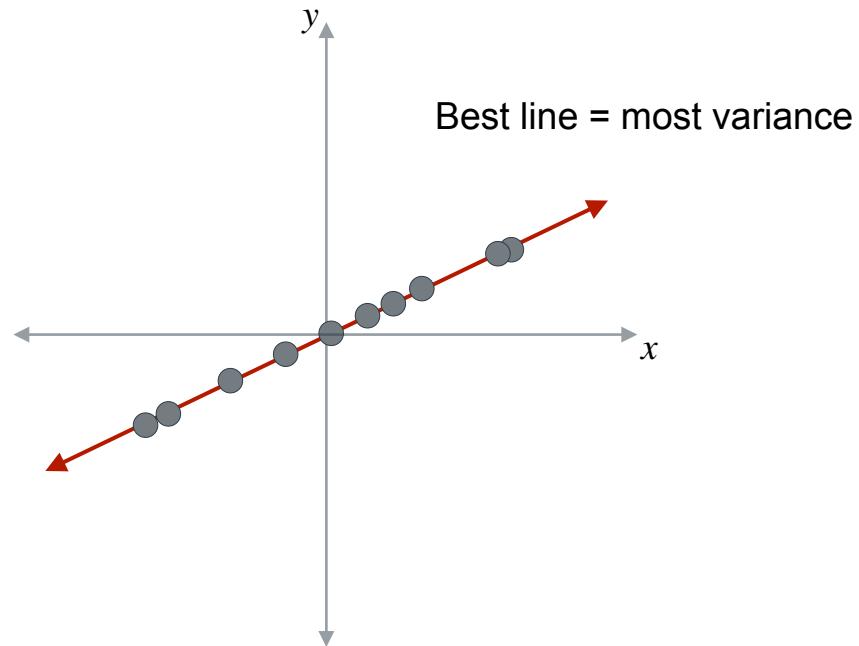
# Principal Component Analysis (PCA)



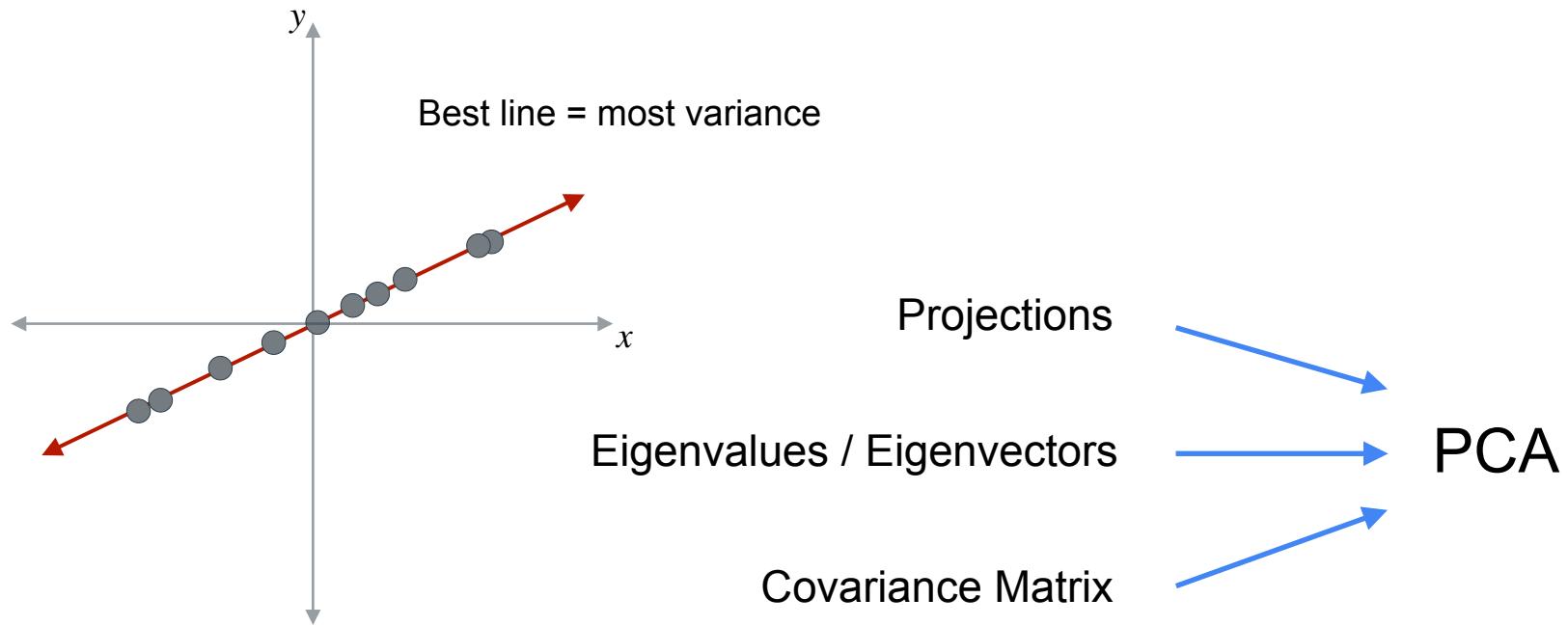
# Principal Component Analysis (PCA)



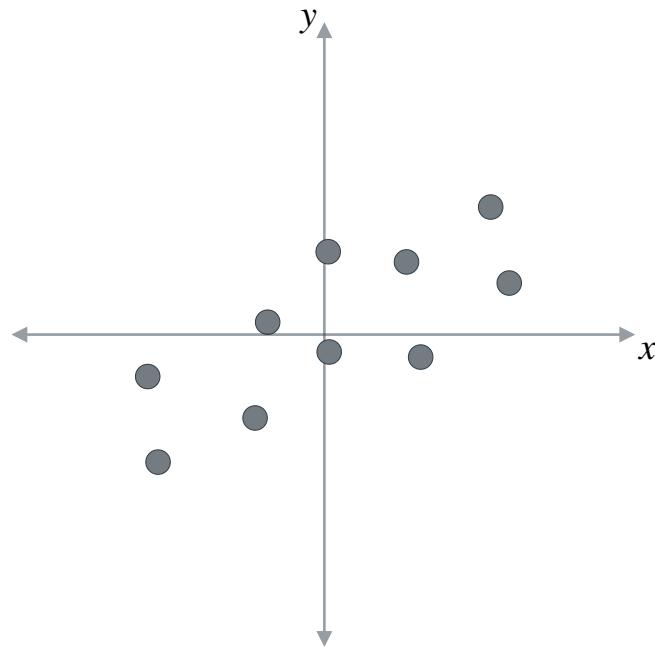
# Principal Component Analysis (PCA)



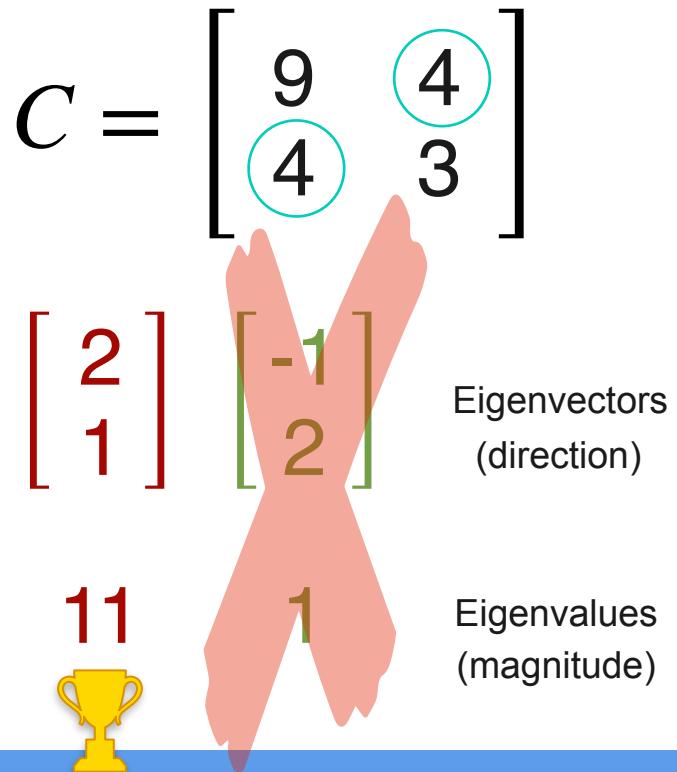
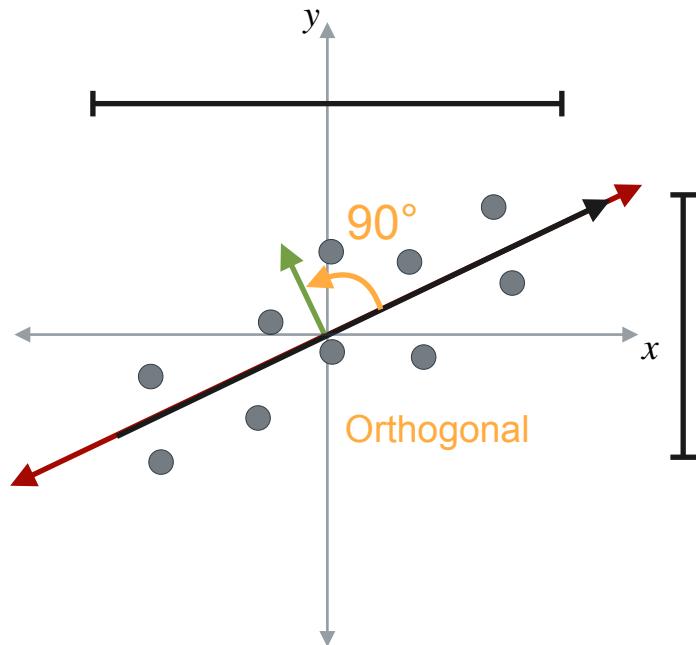
# Principal Component Analysis (PCA)



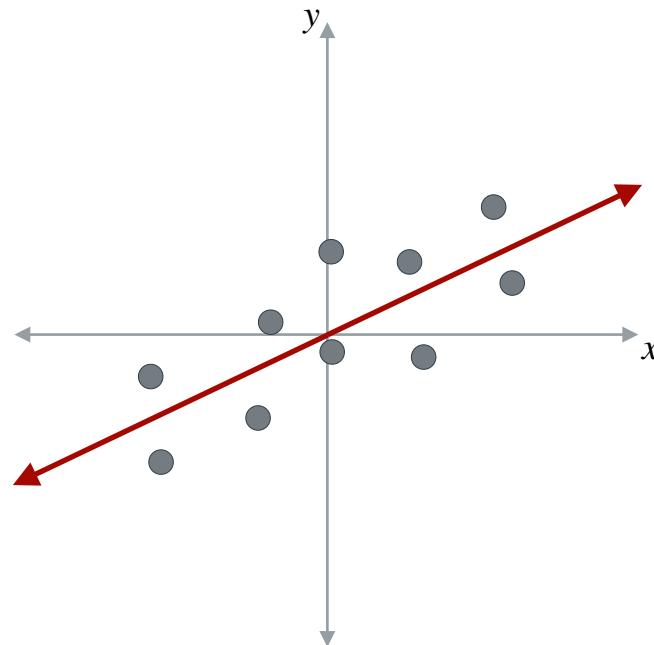
# Principal Component Analysis (PCA)



# Principal Component Analysis (PCA)



# Principal Component Analysis (PCA)



$$C = \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

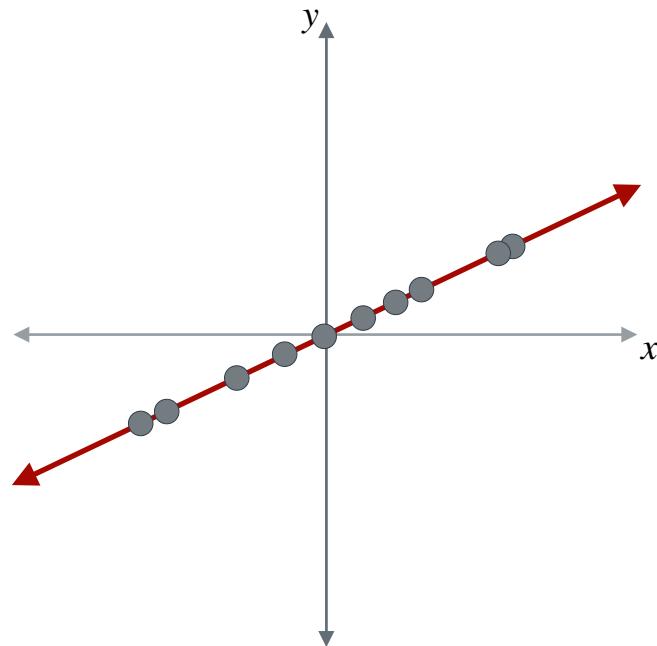
$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Eigenvectors  
(direction)

$$11$$

Eigenvalues  
(magnitude)

# Principal Component Analysis (PCA)



$$C = \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

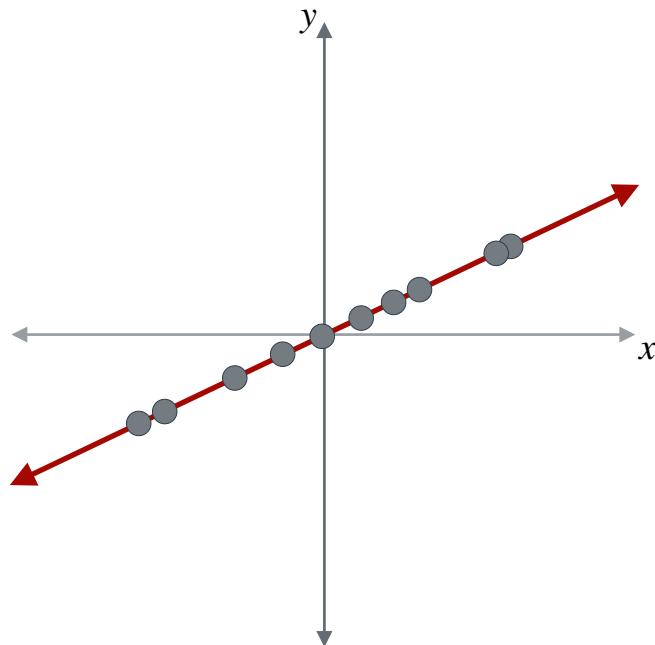
$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Eigenvectors  
(direction)

$$11$$

Eigenvalues  
(magnitude)

# Principal Component Analysis (PCA)



$$C = \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

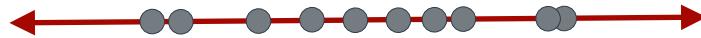
$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Eigenvectors  
(direction)

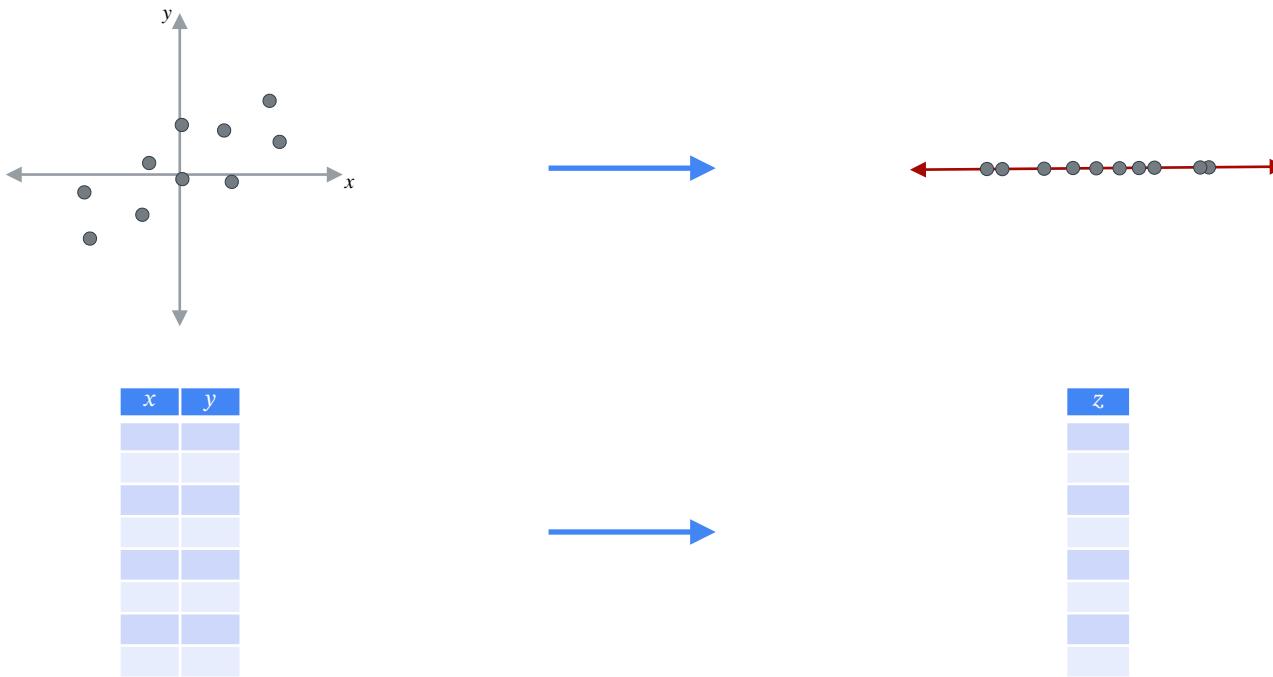
$$11$$

Eigenvalues  
(magnitude)

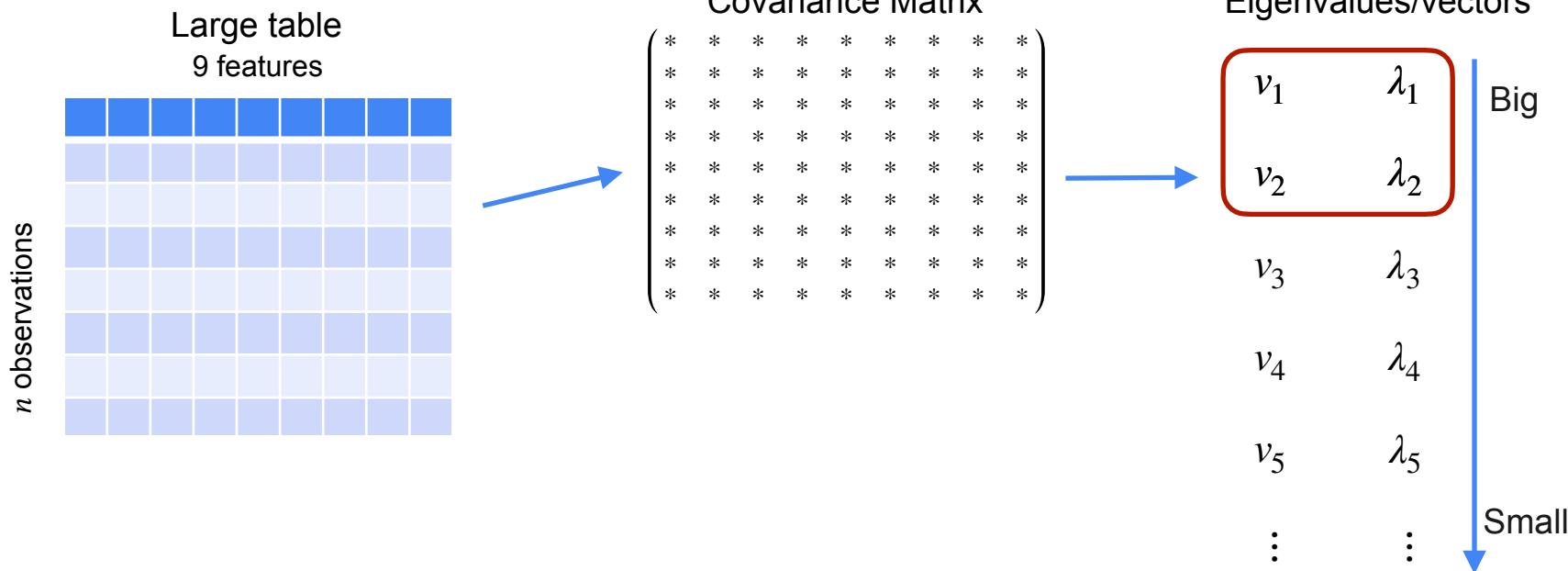
# Principal Component Analysis (PCA)



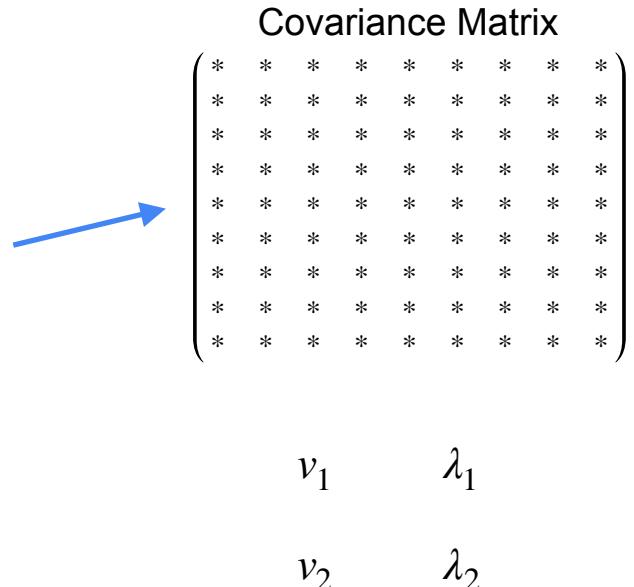
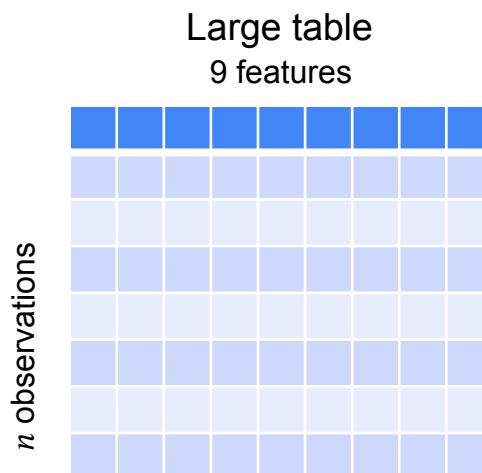
# Principal Component Analysis (PCA)



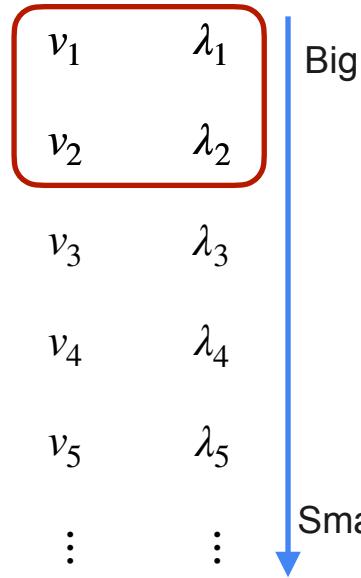
# PCA: Principal Component Analysis



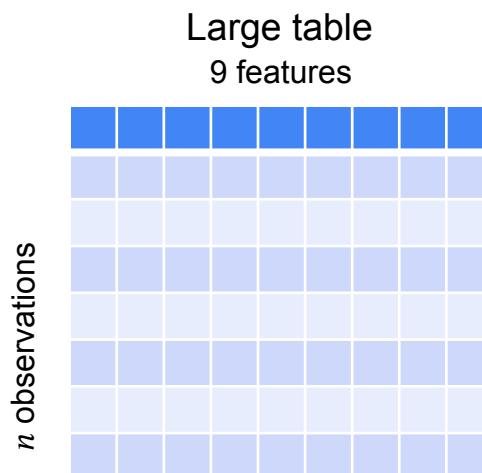
# PCA: Principal Component Analysis



Eigenvalues/vectors



# PCA: Principal Component Analysis



Covariance Matrix

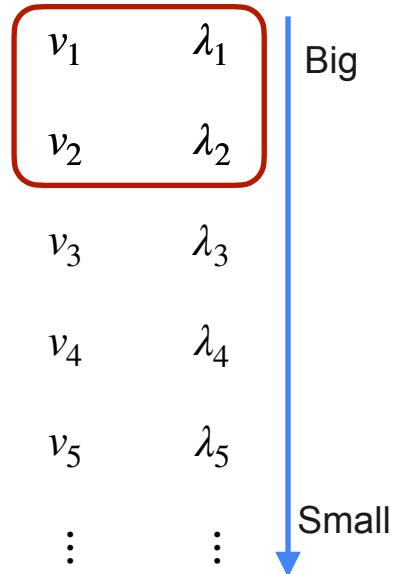
$$\begin{pmatrix} * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \end{pmatrix}$$

$$\begin{array}{ll} v_1 & \lambda_1 \\ v_2 & \lambda_2 \end{array}$$

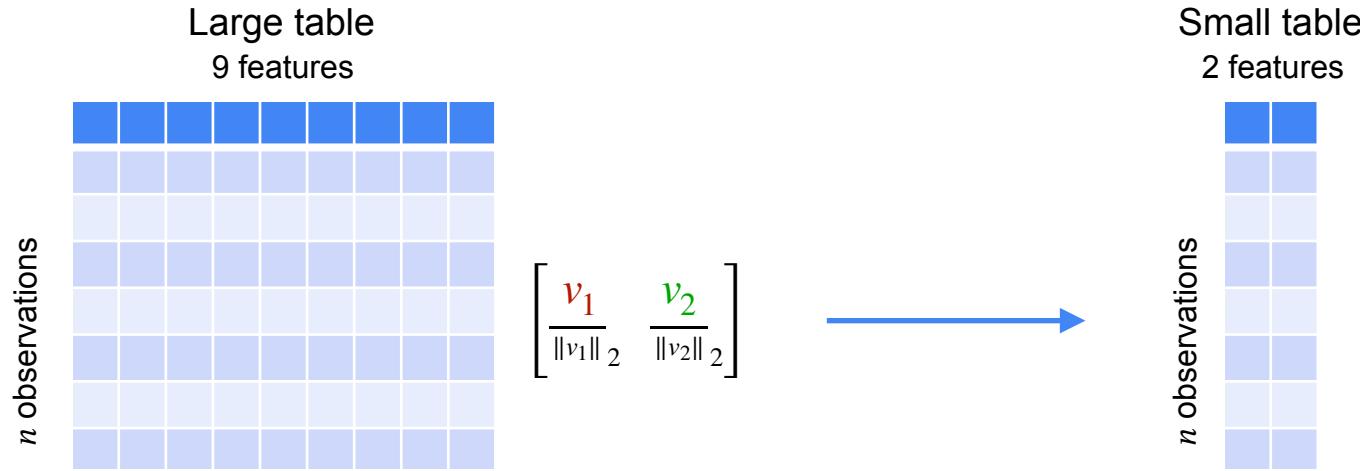
$$\begin{array}{ll} v_3 & \lambda_3 \\ v_4 & \lambda_4 \end{array}$$

$$\begin{array}{ll} v_5 & \lambda_5 \\ \vdots & \vdots \end{array}$$

Eigenvalues/vectors



# PCA: Principal Component Analysis





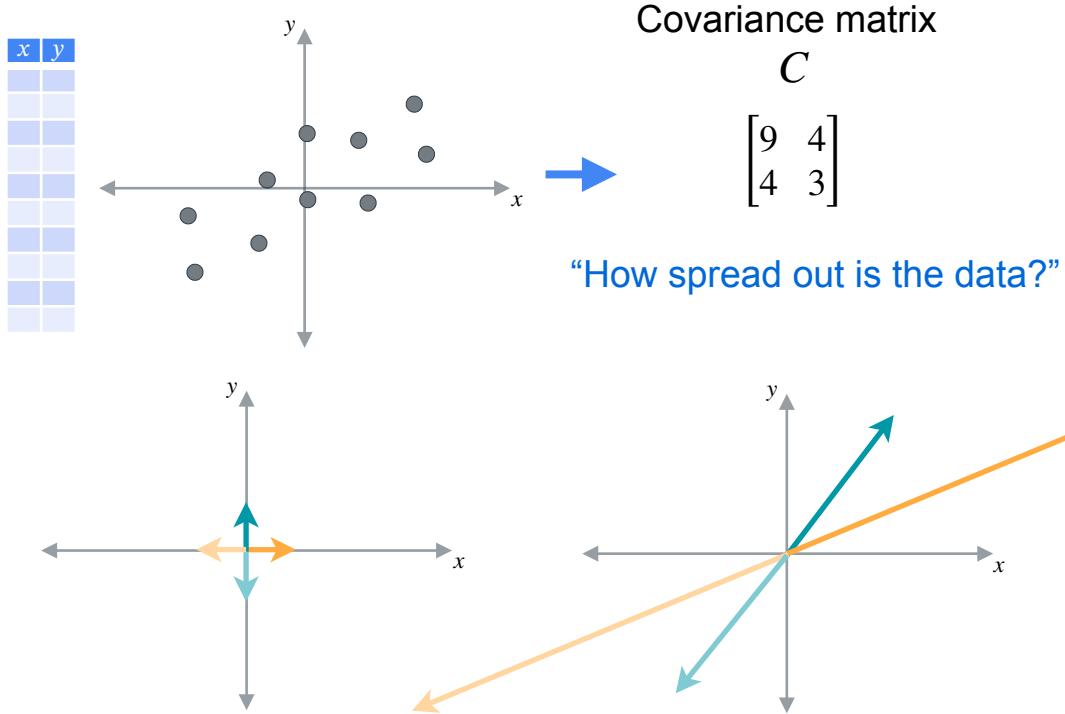
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## Determinants and Eigenvectors

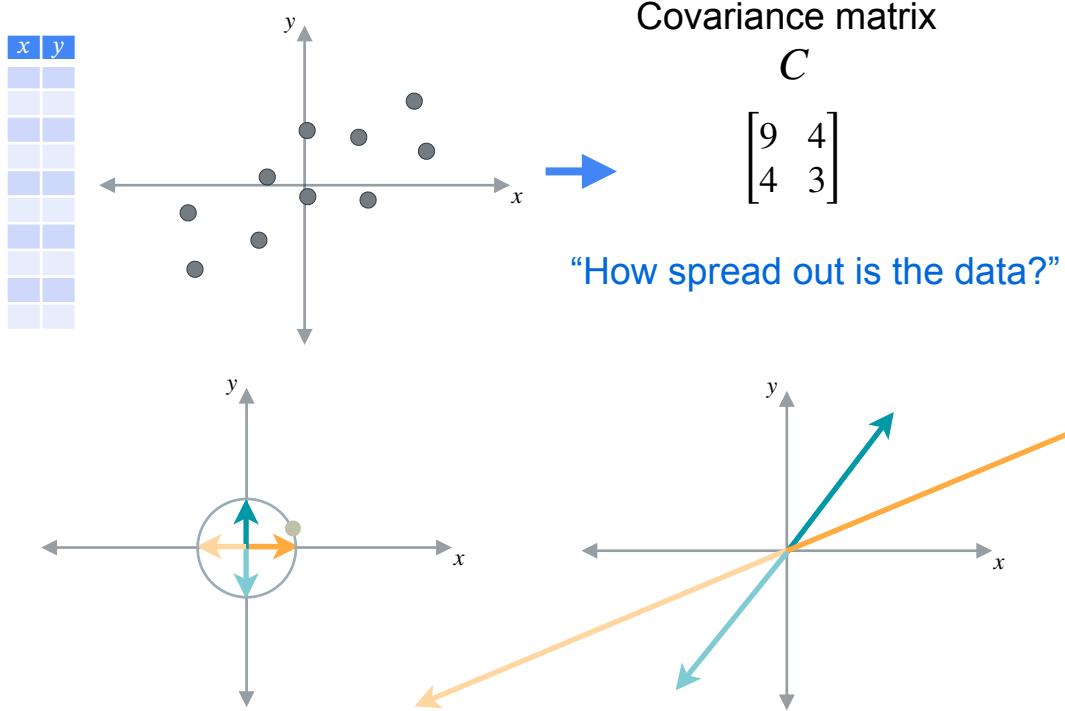
---

### PCA - Why it works

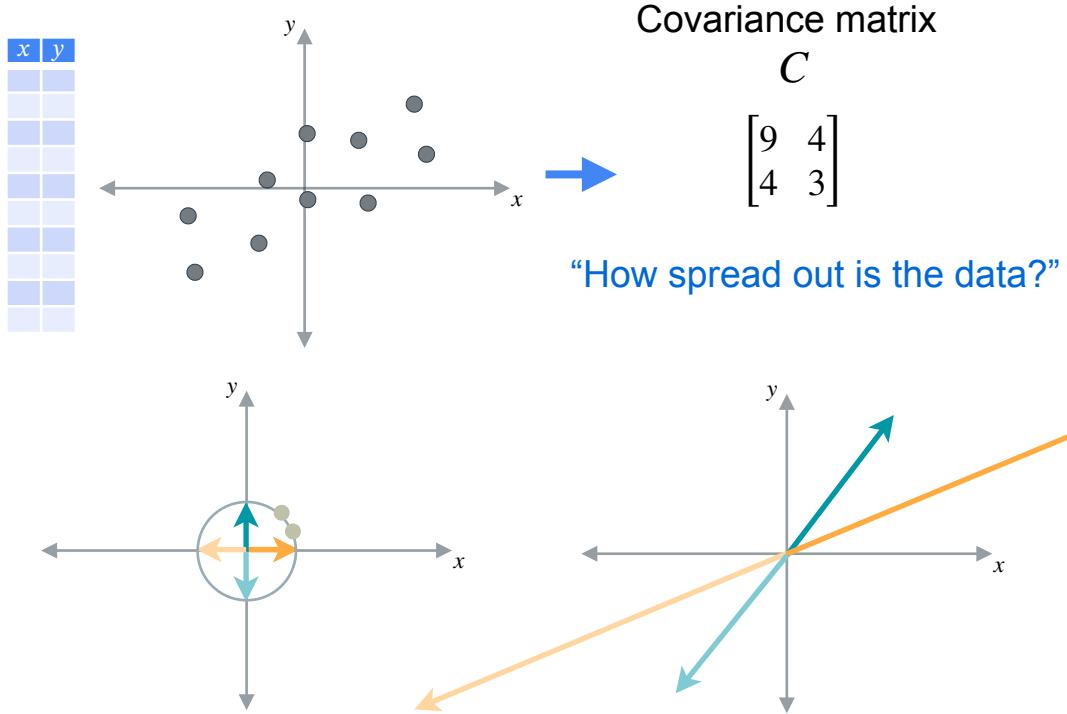
# PCA: Why It Works



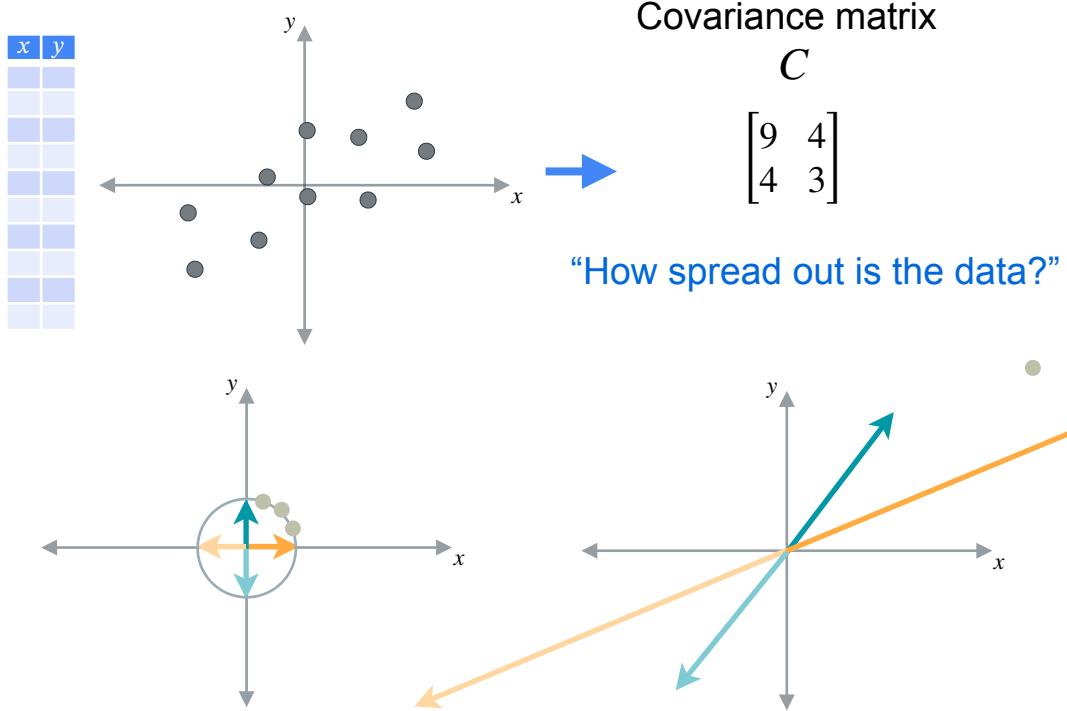
# PCA: Why It Works



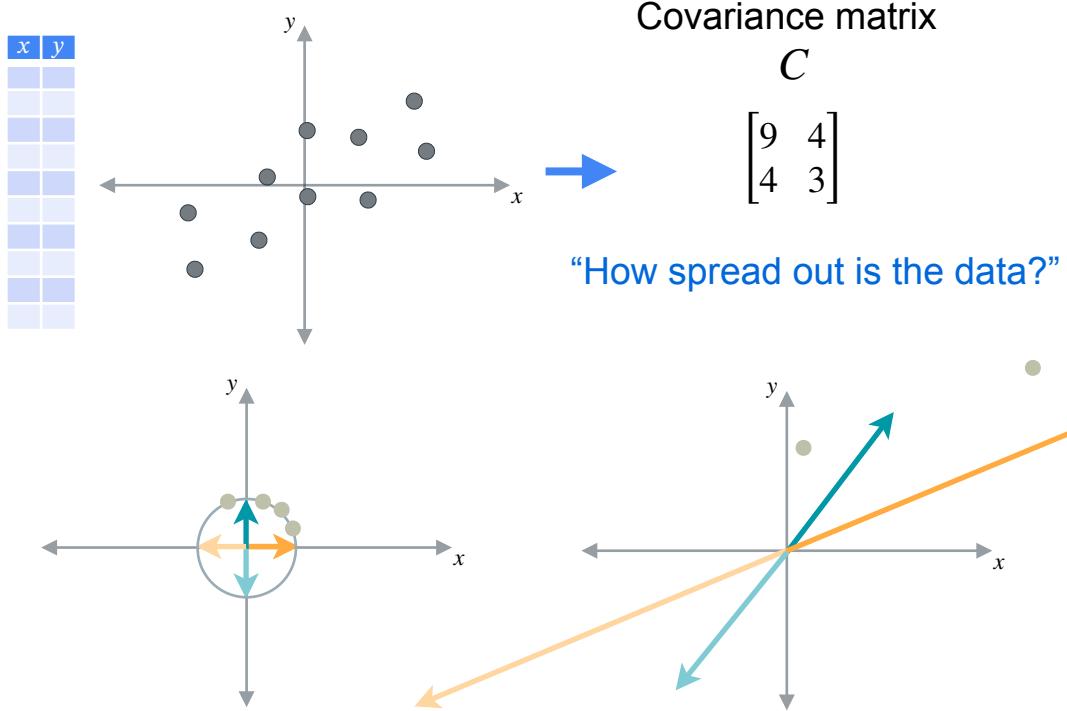
# PCA: Why It Works



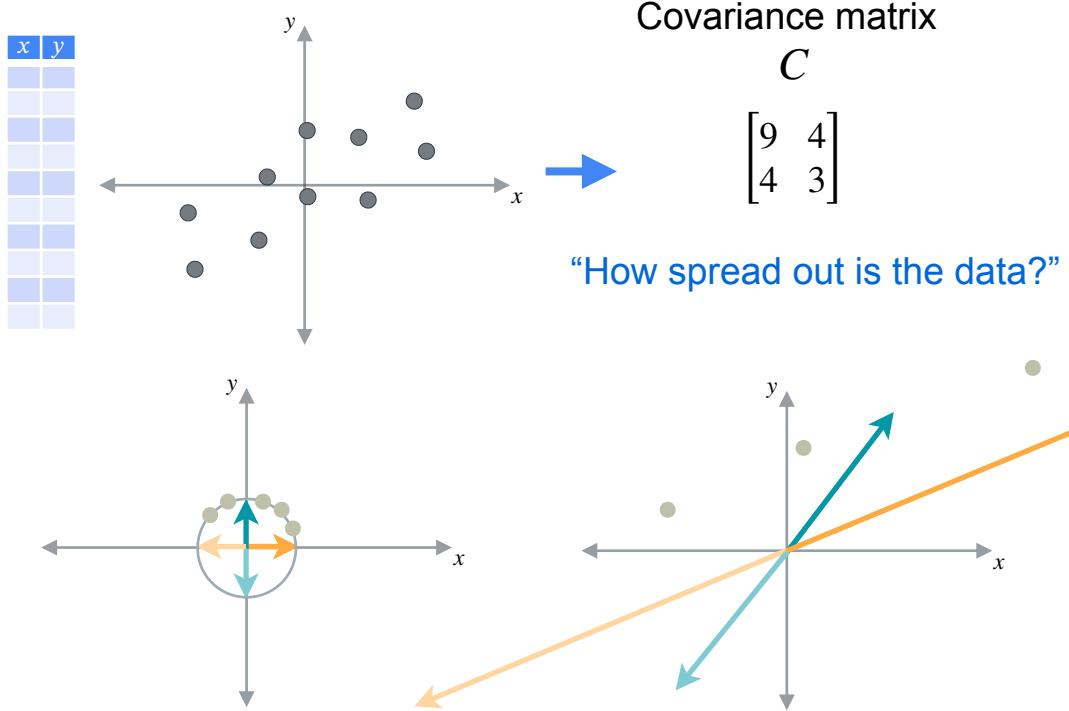
# PCA: Why It Works



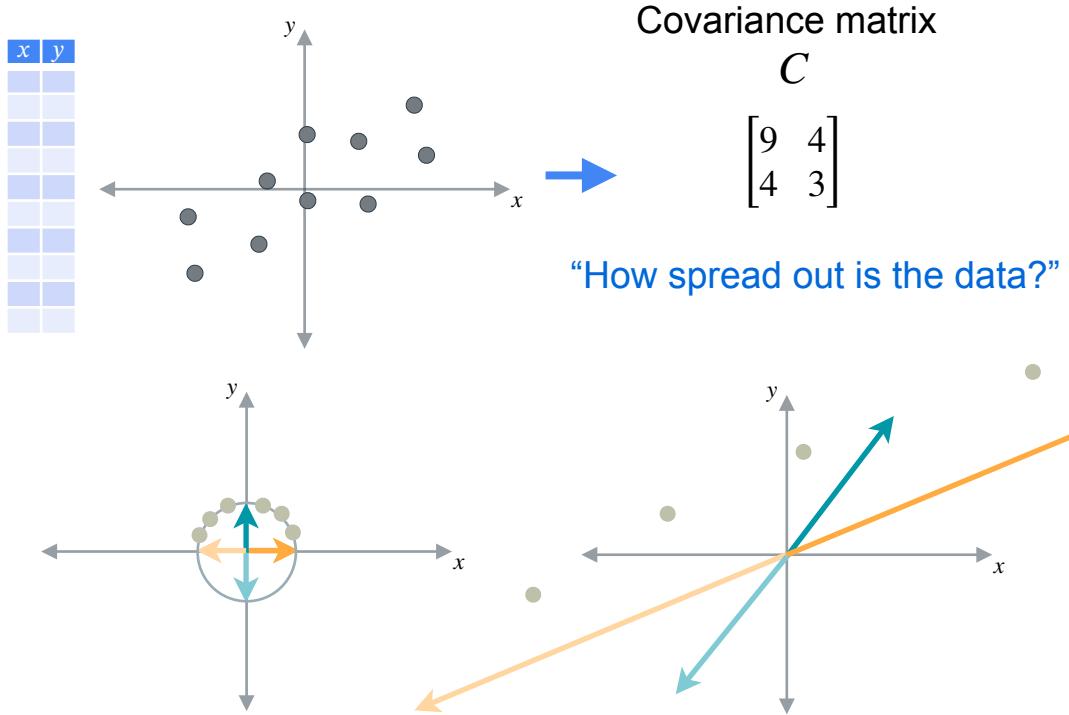
# PCA: Why It Works



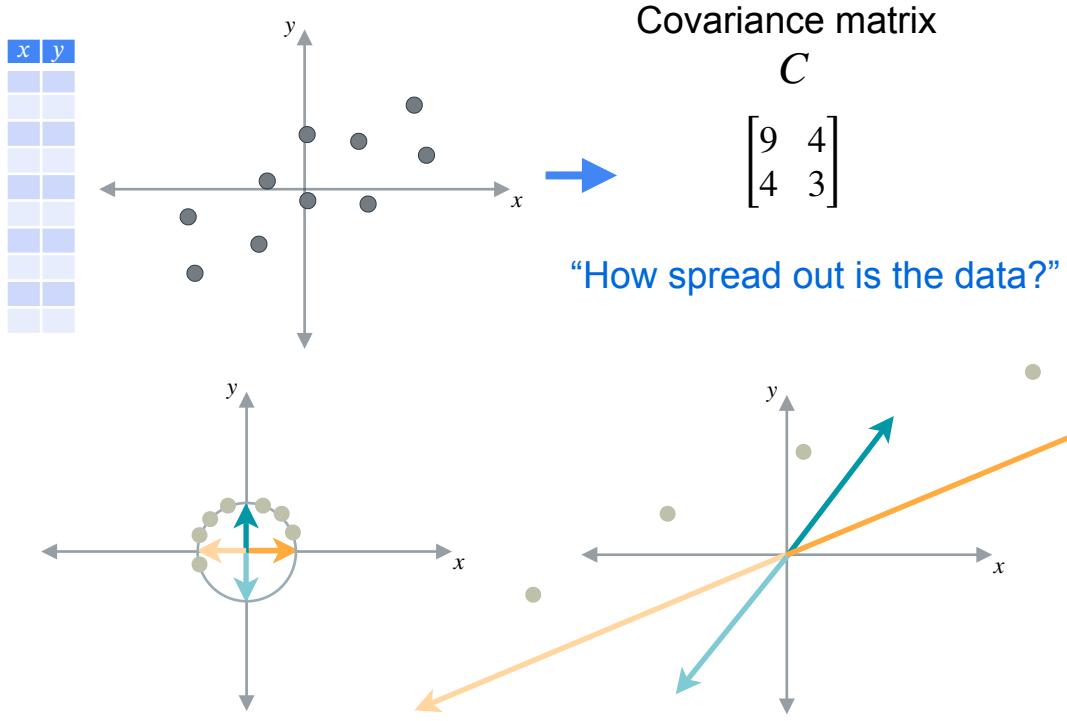
# PCA: Why It Works



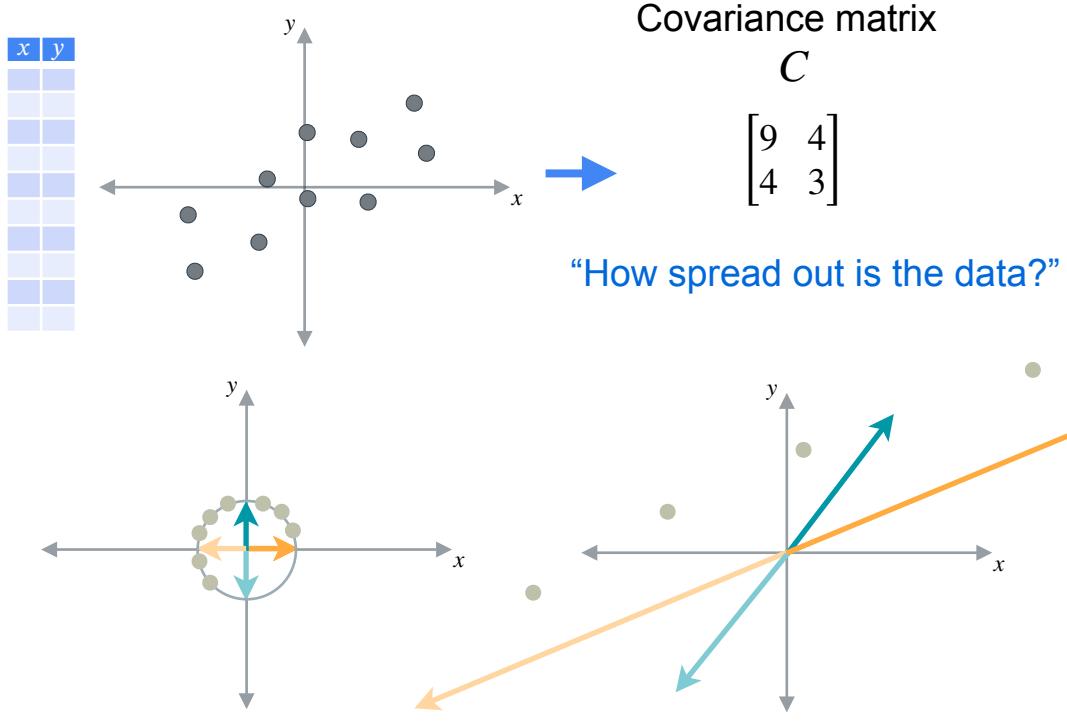
# PCA: Why It Works



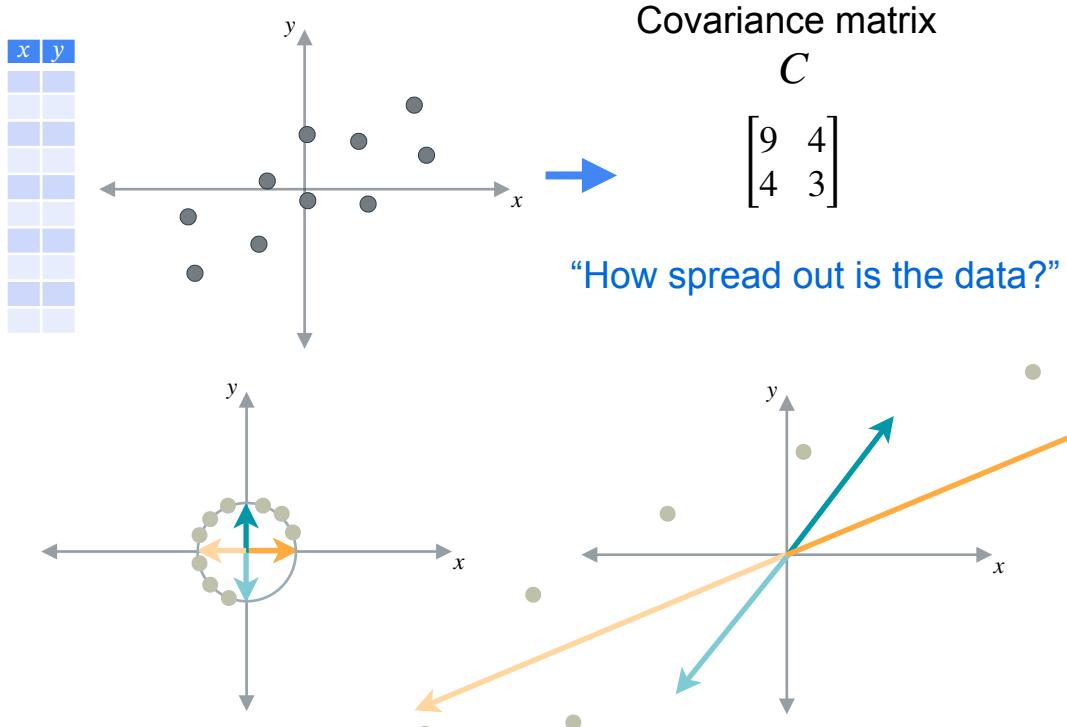
# PCA: Why It Works



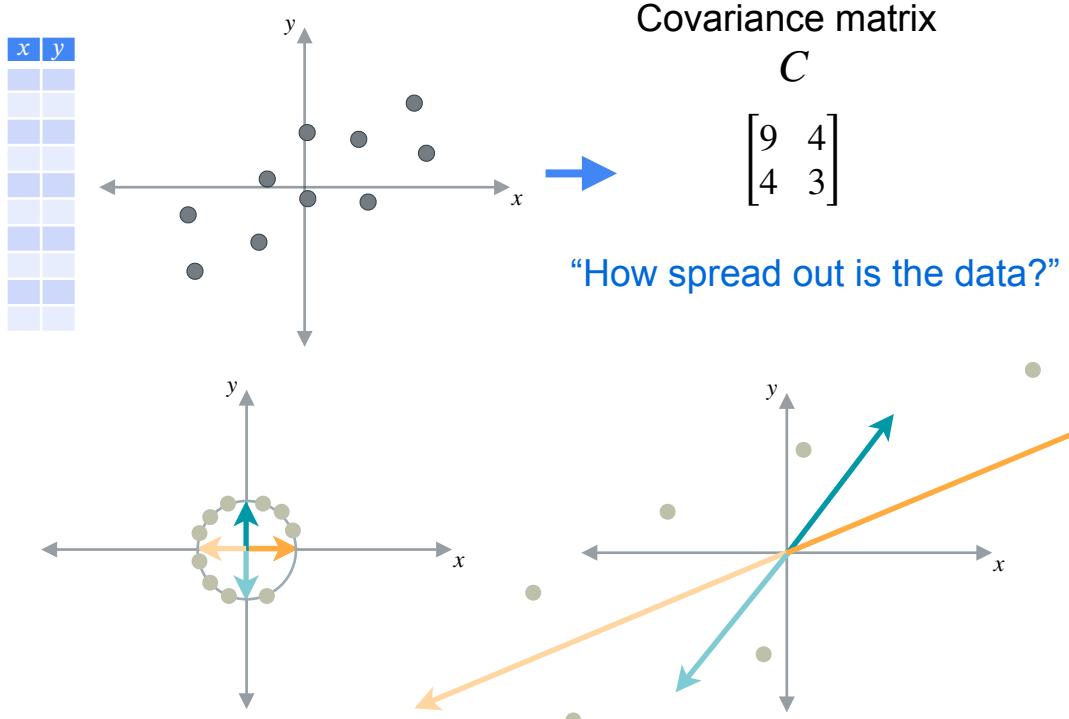
# PCA: Why It Works



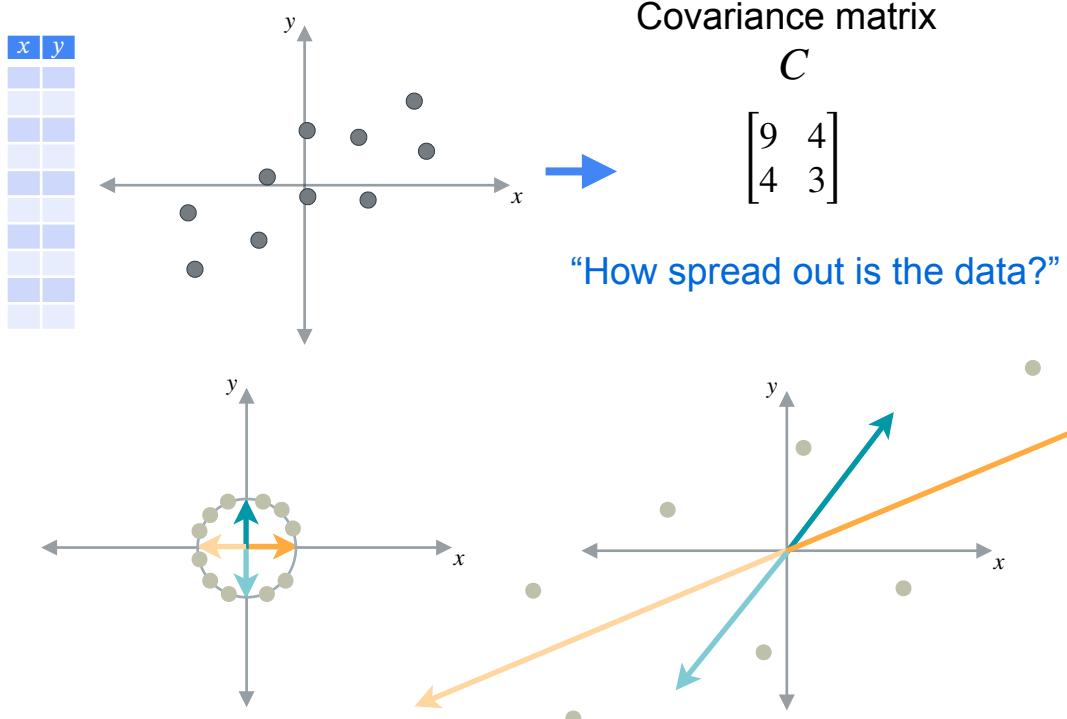
# PCA: Why It Works



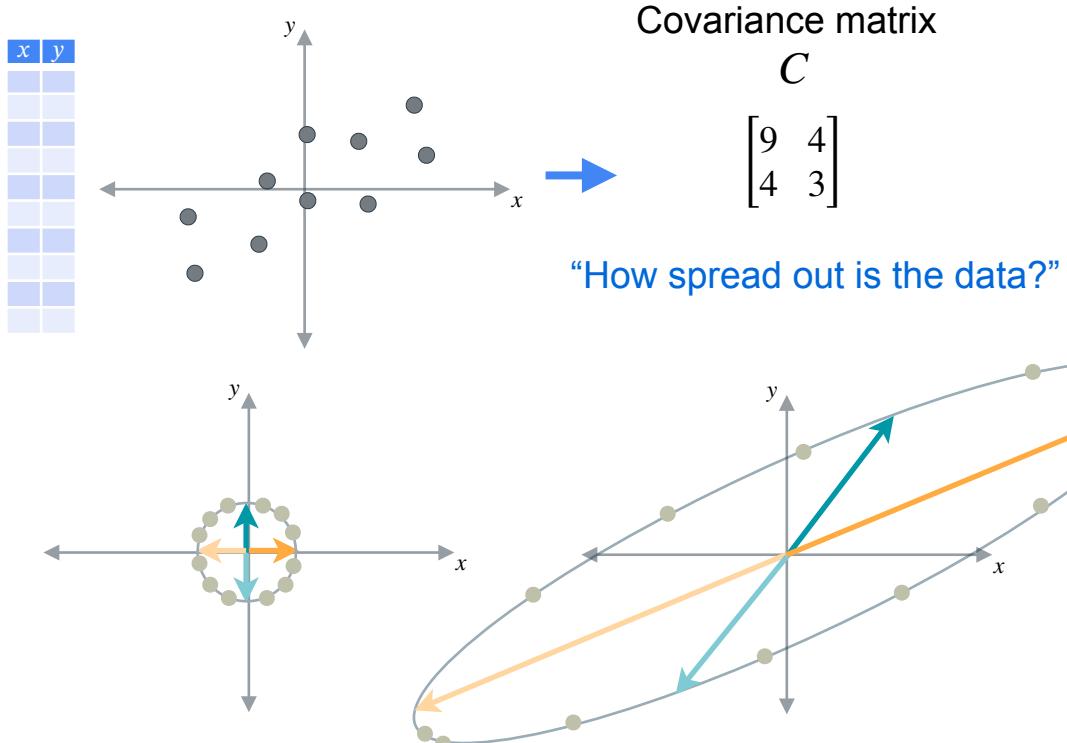
# PCA: Why It Works



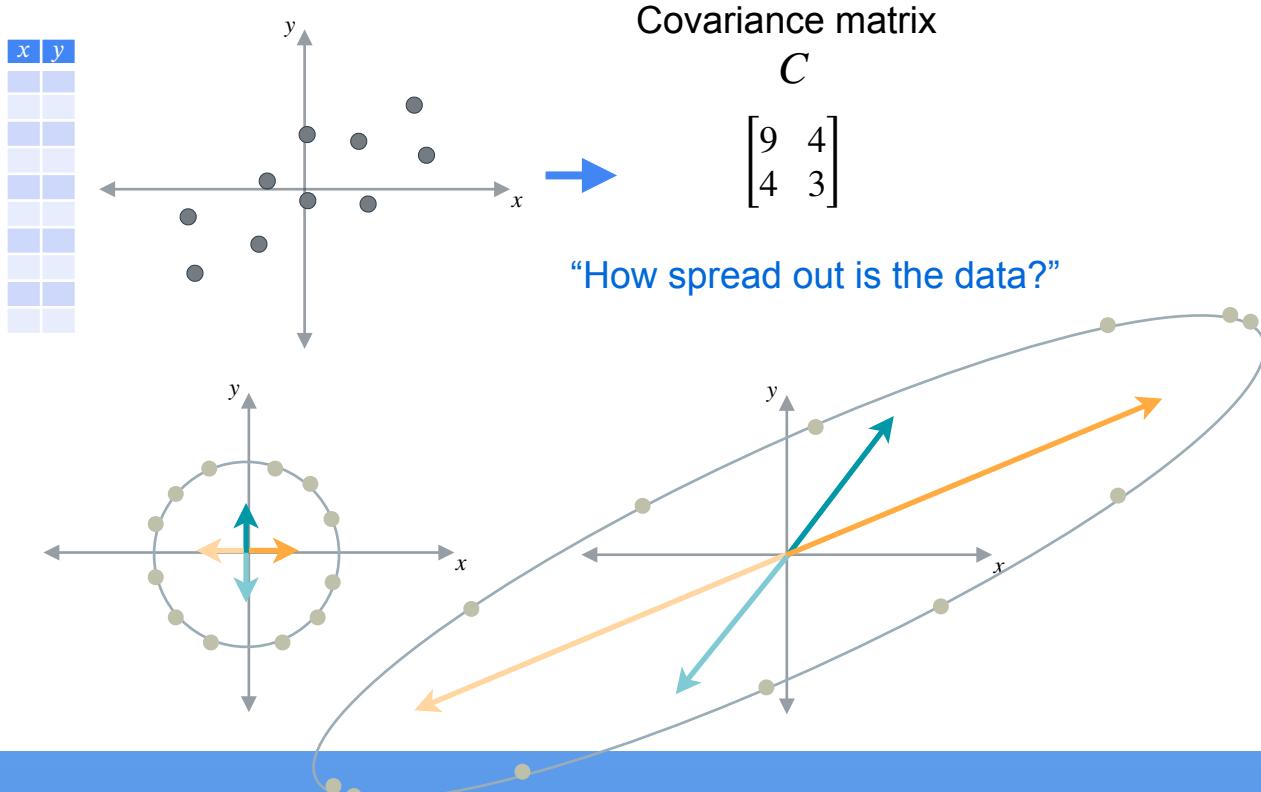
# PCA: Why It Works



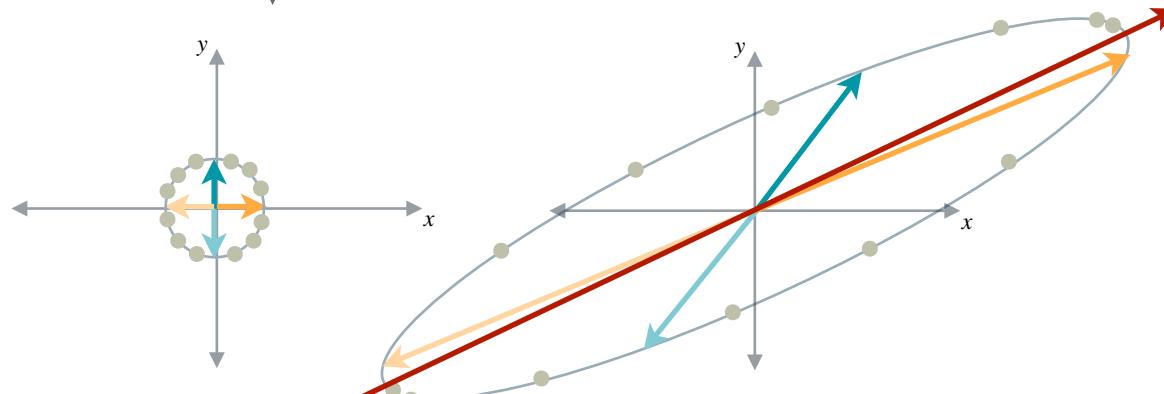
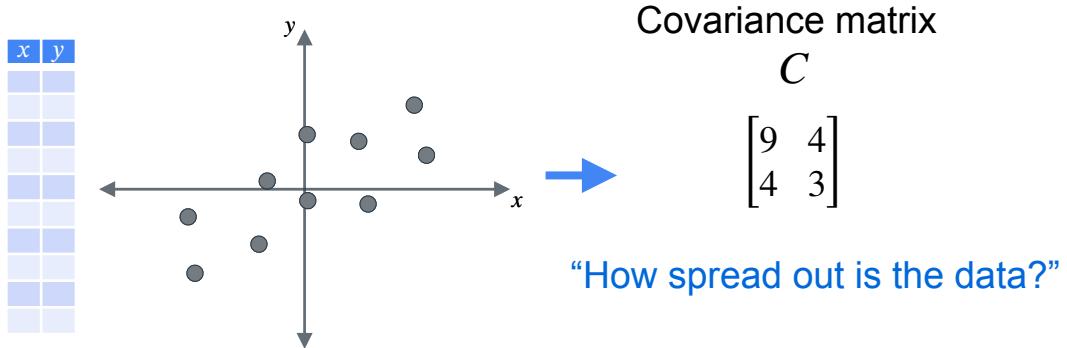
# PCA: Why It Works



# PCA: Why It Works

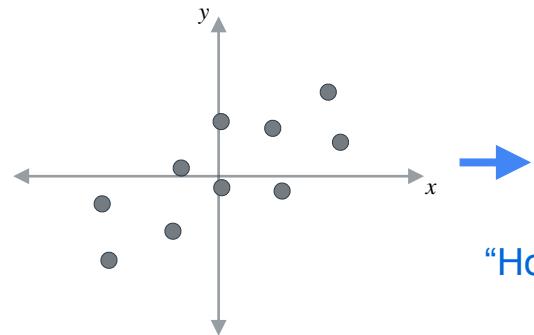


# PCA: Why It Works



# PCA: Why It Works

x	y
1	1
1	2
1	3
1	4
1	5
2	1
2	2
2	3
2	4
2	5
3	1
3	2
3	3
3	4
3	5
4	1
4	2
4	3
4	4
4	5
5	1
5	2
5	3
5	4
5	5



Covariance matrix

$C$

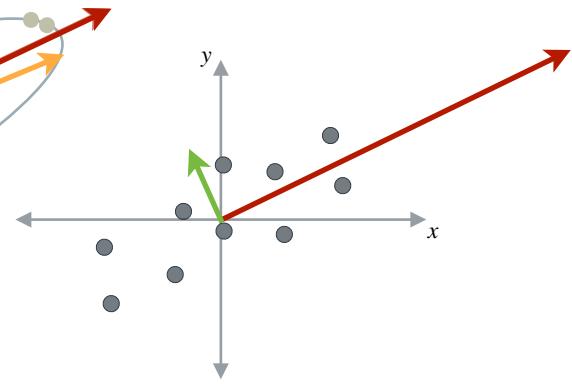
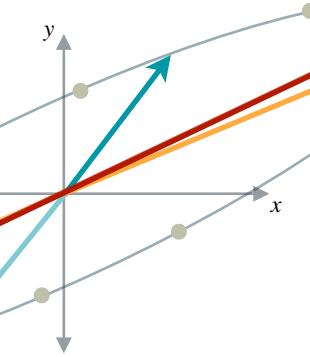
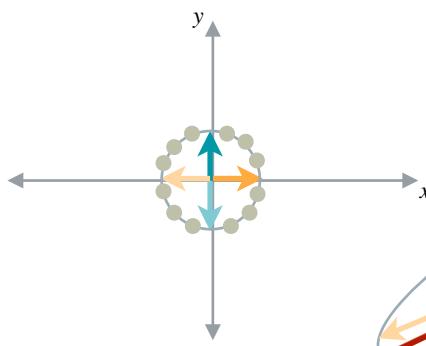
$$\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

“How spread out is the data?”

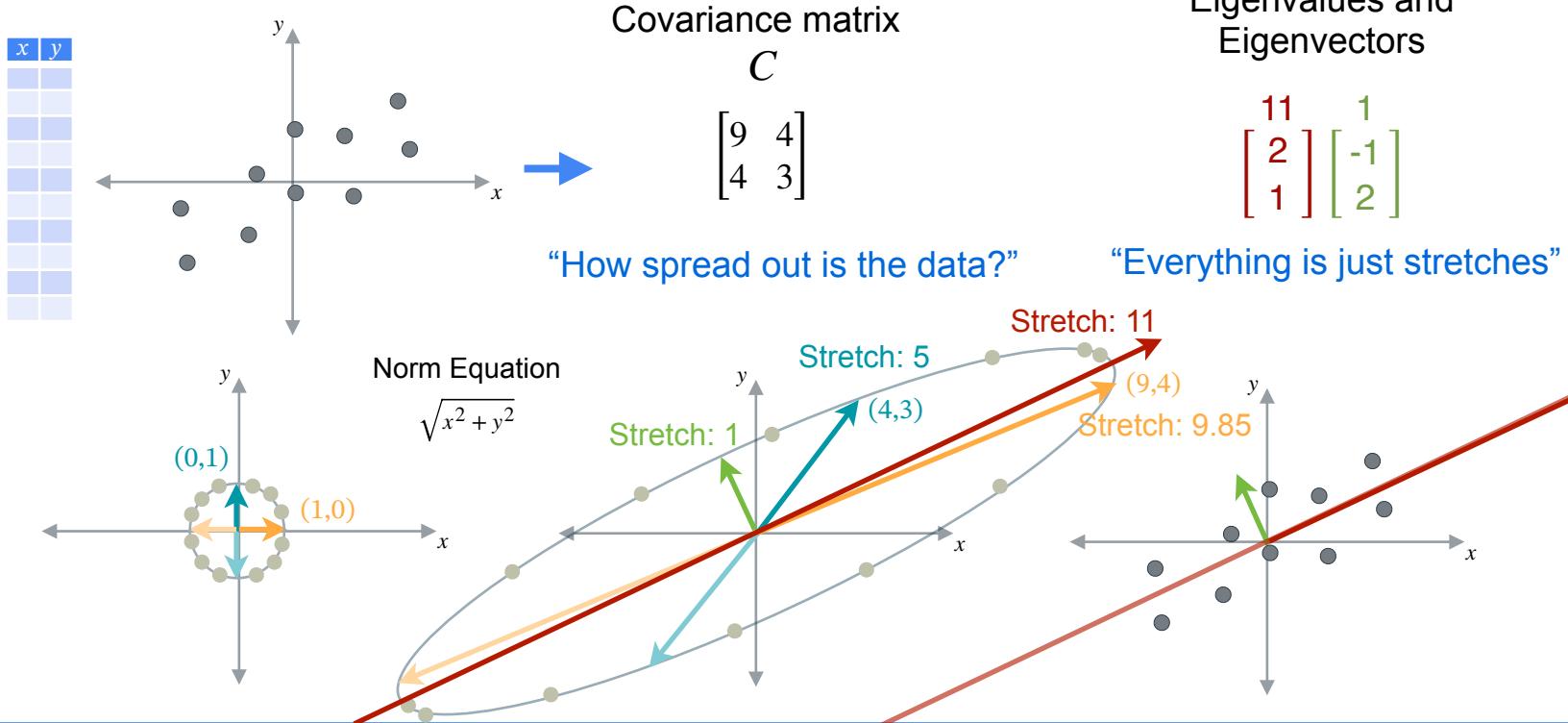
Eigenvalues and  
Eigenvectors

$$\begin{bmatrix} 11 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

“Everything is just stretches”



# PCA: Why It Works





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## Determinants and Eigenvectors

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**PCA - Mathematical  
formulation**

# PCA Mathematical formulation

You have  $n$  observations of 5 variables ( $x_1, x_2, x_3, x_4, x_5$ )

Goal: Reduce to 2 variables

1 Create matrix

$$X = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{15} \\ x_{21} & x_{22} & \dots & x_{25} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{n5} \end{bmatrix}$$

5 variables

*n Observations*

2 Center the data

$$X - \mu = \begin{bmatrix} x_{11} - \mu_1 & x_{12} - \mu_2 & \dots & x_{15} - \mu_5 \\ x_{21} - \mu_1 & x_{22} - \mu_2 & \dots & x_{25} - \mu_5 \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} - \mu_1 & x_{n2} - \mu_2 & \dots & x_{n5} - \mu_5 \end{bmatrix}$$

# PCA Mathematical formulation

You have  $n$  observations of 5 variables  $(x_1, x_2, x_3, x_4, x_5)$

Goal: Reduce to 2 variables

3

Calculate Covariance Matrix

$$C = \frac{1}{n-1}(X - \mu)^T(X - \mu) = \begin{bmatrix} Var(X_1) & Cov(X_1, X_2) & Cov(X_1, X_3) & Cov(X_1, X_4) & Cov(X_1, X_5) \\ Cov(X_1, X_2) & Var(X_2) & Cov(X_2, X_3) & Cov(X_2, X_4) & Cov(X_2, X_5) \\ Cov(X_1, X_3) & Cov(X_2, X_3) & Var(X_3) & Cov(X_3, X_4) & Cov(X_3, X_5) \\ Cov(X_1, X_4) & Cov(X_2, X_4) & Cov(X_3, X_4) & Var(X_4) & Cov(X_4, X_5) \\ Cov(X_1, X_5) & Cov(X_2, X_5) & Cov(X_3, X_5) & Cov(X_4, X_5) & Var(X_5) \end{bmatrix}$$

# PCA Mathematical formulation

You have  $n$  observations of 5 variables ( $x_1, x_2, x_3, x_4, x_5$ )

Goal: Reduce to 2 variables

- 4 Calculate Eigenvectors and Eigenvalues

- 5 Create Projection Matrix

- 6 Project Centered Data

Big	$\lambda_1$	$v_1$
	$\lambda_2$	$v_2$
	$\lambda_3$	$v_3$
	$\lambda_4$	$v_4$
Small	$\lambda_5$	$v_5$

$$V = \begin{bmatrix} & \\ \frac{1}{\|v_1\|_2} & \frac{1}{\|v_2\|_2} \\ & \end{bmatrix}$$

$$X_{PCA} = (X - \mu)V$$



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## Determinants and Eigenvectors

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## Conclusion