

# FATE-X Statements

## Formalization Contribution

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## Mathematical Contribution

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**Exercise (1).** Let  $R$  be a UFD with two nonassociate prime elements  $p$  and  $q$  such that every prime element is an associate of either  $p$  or  $q$ . Prove that  $R$  is a PID.

```
import Mathlib

namespace Problem1

/--
Let $R$ be a UFD with two nonassociate prime elements $p$ and $q$ such that every prime
element is an associate of either $p$ or $q$. Prove that $R$ is a PID.
-/
theorem isPrincipalIdealRing_of_associated_or_associated {R : Type} [CommRing R] [IsDomain R]
  [UniqueFactorizationMonoid R] {p q : R} (hp : Prime p) (hq : Prime q) (hpq : ¬ Associated p q)
  (h : ∀ {x : R}, Prime x → Associated x p ∨ Associated x q) :
  IsPrincipalIdealRing R := by
  sorry

end Problem1
```

**Exercise (2).** Let  $G$  be a finite group and  $L$  a maximal subgroup of  $G$ . Suppose  $L$  is non-Abelian and simple. Then there exist at most two minimal normal subgroups in  $G$ .

```
import Mathlib

namespace Problem2
```

```

/--  

Let $G$ be a finite group and $L$ a maximal subgroup of $G$. Suppose $L$ is non-Abelian and simple.  

Then there exist at most two minimal normal subgroups in $G$.  

-/  

theorem card_minimal_normal_subgroup_le_2 (G : Type) [Group G] [Finite G]  

  (L : Subgroup G) (h_ne_top : L ≠ τ) (h_maximal : IsMax ⟨L, h_ne_top⟩ : {H : Subgroup G // H ≠ τ})  

  (h_simple : IsSimpleGroup L) (h_non_comm : ∃ (x y : L), x * y ≠ y * x) :  

  {H : {H : Subgroup G // H.Normal} | IsMin H}.ncard ≤ 2 := by  

  sorry  

end Problem2

```

**Exercise (3).** Let  $H$  be a subgroup of finite index of a group  $G$ . Show that there exists a subset  $S$  of  $G$ , such that  $S$  is both a set of representatives of the left and the right cosets of  $H$  in  $G$ .

```

import Mathlib  

namespace Problem3  

  

/--  

Let $H$ be a subgroup of finite index of a group $G$. Show that there exists a subset $S$ of $G$,  

such that $S$ is both a set of representatives of the left and the right cosets of $H$ in $G$.  

-/  

theorem exists_leftCoset_rightCosetRepresentative  

  (G : Type) [Group G] (H : Subgroup G) [H.FiniteIndex] :  

  ∃ S : Set G, Subgroup.IsComplement S H ∧ Subgroup.IsComplement H S := by  

  sorry  

end Problem3

```

**Exercise (4).** Let  $p$  be an odd prime number, and let  $G$  be a finite group of order  $p(p+1)$ . Assume that  $G$  does not have a normal Sylow  $p$ -subgroup. Prove that  $p+1$  is a power of 2.

```

import Mathlib  

namespace Problem4  

  

/--  

Let $p$ be an odd prime number, and let $G$ be a finite group of order $p(p+1)$. Assume that $G$ does not have a normal Sylow $p$-subgroup. Prove that $p+1$ is a power of 2.  

-/  

theorem add_one_eq_two_pow_of_sylow_subgroup_not_normal (p : N) (h_odd : Odd p) (G : Type)  

  (hp : p.Prime) [Finite G] [Group G] (h_card : Nat.card G = p * (p + 1))  

  (h_sylow : ∀ (H : Sylow p G), ¬ H.Normal) : ∃ (n : N), p + 1 = 2 ^ n := by  

  sorry  

end Problem4

```

**Exercise (5).** Let  $p$  be a prime, let  $G$  be a finite  $p$ -group. Let  $A$  be a maximal normal abelian subgroup of  $G$ . Prove that  $A$  is also a maximal abelian subgroup of  $G$ .

```
import Mathlib

namespace Problem5

/-
Let  $\{p\}$  be a prime, let  $\{G\}$  be a finite  $p$ -group. Let  $A$  be a maximal normal abelian subgroup of  $\{G\}$ . Prove that  $A$  is also a maximal abelian subgroup of  $\{G\}$ .
-/
theorem maximal_abelian_normal_subgroup_of_p_group_is_maximal_abelian_subgroup
  (p : ℕ) (hp : p.Prime) (G : Type) [Group G] [Finite G] (h_pgrou : IsPGroup p G)
  (H : Subgroup G) (h_normal : H.Normal) (h_comm : IsMulCommutative H)
  (h_maximal_normal_abelian : ∀ (K : Subgroup G), K.Normal → IsMulCommutative K → H ≤ K → H = K) :
  ∀ (K : Subgroup G), IsMulCommutative K → H ≤ K → H = K := by
  sorry

end Problem5
```

**Exercise (6).** Prove that if  $\#G = 396$  then  $G$  is not simple.

```
import Mathlib

namespace Problem6

/-
Prove that if  $\#\mathbb{G} = 396$  then  $\mathbb{G}$  is not simple.
-/
theorem not_isSimpleGroup_of_card_eq_396 (G : Type) [Group G]
  [Finite G] (h_card : Nat.card G = 396) : ¬ IsSimpleGroup G := by
  sorry

end Problem6
```

**Exercise (7).** Prove that if  $\#G = 1785$  then  $G$  is not simple.

```
import Mathlib

namespace Problem7

/-
Prove that if  $\#\mathbb{G} = 1785$  then  $\mathbb{G}$  is not simple.
-/
theorem not_isSimpleGroup_of_card_eq_1785 (G : Type) [Group G]
  [Finite G] (h_card : Nat.card G = 1785) : ¬ IsSimpleGroup G := by
  sorry

end Problem7
```

**Exercise (8).** Let  $A, B \in \mathbb{Q}^\times$  be rational numbers. Consider the quaternion ring

$$D_{A,B,\mathbb{R}} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$$

in which the multiplication satisfies relations:  $i^2 = A$ ,  $j^2 = B$ , and  $ij = -ji = k$ .

Show that  $D_{A,B,\mathbb{R}}$  is either isomorphic to  $\mathbb{H}$  (Hamilton quaternion) or isomorphic to  $\text{Mat}_{2\times 2}(\mathbb{R})$  as  $\mathbb{R}$ -algebras.

```
import Mathlib

namespace Problem8

open Quaternion

/--
Let $A, B \in \mathbb{Q}^\times$ be rational numbers. Consider the quaternion ring
$D_{A,B,\mathbb{R}} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$
in which the multiplication satisfies relations: $i^2 = A$, $j^2 = B$, and $ij = -ji = k$.
Show that $D_{A,B,\mathbb{R}}$ is either isomorphic to $\mathbb{H}$ (Hamilton quaternion) or
isomorphic to $\text{Mat}_{2\times 2}(\mathbb{R})$ as $\mathbb{R}$-algebras.
-/
theorem quaternionAlgebra_isomorphic_to_matrix_ring_or_quaternion_ring
  (A B : ℚ) (ha : A ≠ 0) (hb : B ≠ 0) :
  ((Nonempty (H[R, A, B] ≃ₐ[R] H[R, -1, -1])) ∨ (Nonempty (H[R, A, B] ≃ₐ[R] Matrix (Fin 2) (Fin 2)
    R))) ∧ IsEmpty (Matrix (Fin 2) (Fin 2) R ≃ₐ[R] H[R, -1, -1]) := by
  sorry

end Problem8
```

**Exercise (9).** Let  $G$  be a finite group and let  $\text{Syl}_p(G)$  denote its set of Sylow  $p$ -subgroups. Suppose that  $S$  and  $T$  are distinct members of  $\text{Syl}_p(G)$  chosen so that  $\#(S \cap T)$  is maximal among all such intersections. Prove that the normalizer  $N_G(S \cap T)$  does not admit normal Sylow  $p$ -subgroup.

```
import Mathlib

namespace Problem9

/--
Let $G$ be a finite group and let $\text{Syl}_p(G)$ denote its set of Sylow $p$-subgroups.
Suppose that $S$ and $T$ are distinct members of
$\text{Syl}_p(G)$ chosen so that $\#(S \cap T)$ is maximal
among all such intersections. Prove that the normalizer $N_G(S \cap T)$ does not admit normal
Sylow $p$-subgroup.
-/
theorem sylow_subgroup_not_normal_of_maximal_intersection (G : Type) [Finite G] [Group G]
```

```

(p : ℕ) [Fact (Nat.Prime p)] (S T : Sylow p G) (h_ne : S ≠ T)
(h_maximal : ∀ (S' T' : Sylow p G), S' ≠ T' →
((S' : Set G) ∩ T').ncard ≤ ((S : Set G) ∩ T).ncard) :
  ∀ (P : Sylow p ((S : Subgroup G) ∩ T).normalizer), P.Normal := by
  sorry

end Problem9

```

**Exercise (10).** Let  $A = \mathbb{R}[X, Y]/(X^2 + Y^2 + 1)$ . Then it is a principal ideal domain.

```

import Mathlib

namespace Problem10

/-
Let \(\mathbb{R}[X, Y]/(X^2 + Y^2 + 1)\). Then it is a principal ideal domain.
-/
theorem isPrincipalIdealRing_quot_X_pow_two_plus_Y_pow_two_plus_one :
  IsPrincipalIdealRing ((MvPolynomial (Fin 2) ℝ) /
  Ideal.span {(.X 0 ^ 2 + .X 1 ^ 2 + .C 1 : MvPolynomial (Fin 2) ℝ)}) := by
  sorry

end Problem10

```

**Exercise (11).** Let  $A = \mathbb{R}[X, Y]/(X^2 + Y^2 + 1)$ . Then it is not a Euclidean domain.

```

import Mathlib

namespace Problem11

/-
Definition of a Euclidean norm taking value in \(\mathbb{N}\).
-/
class EuclideanNormNat (R : Type) [CommRing R] extends Nontrivial R where
  quotient : R → R → R
  quotient_zero : ∀ a, quotient a 0 = 0
  remainder : R → R → R
  quotient_mul_add_remainder_eq : ∀ a b, b * quotient a b + remainder a b = a
  norm : R → ℕ
  remainder_lt : ∀ (a) {b}, b ≠ 0 → norm (remainder a b) < norm b
  mul_left_not_lt : ∀ (a) {b}, b ≠ 0 → ¬ norm (a * b) < norm a

/-
Let \(\mathbb{R}[X, Y]/(X^2 + Y^2 + 1)\). Then it is not a Euclidean domain.
-/
theorem not_isomorphic_euclideanDomain : IsEmpty <| EuclideanNormNat (((MvPolynomial (Fin 2) ℝ) /
  Ideal.span {(.X 0 ^ 2 + .X 1 ^ 2 + .C 1 : MvPolynomial (Fin 2) ℝ)})) := by
  sorry

end Problem11

```

**Exercise (12).** Prove that the ring  $\mathbb{Z}[\frac{1+\sqrt{-19}}{2}]$  is a principal ideal domain.

```
import Mathlib

namespace Problem12

/--
Prove that the ring  $\mathbb{Z}[\frac{1+\sqrt{-19}}{2}]$  is a principal ideal domain.
-/
theorem isPrincipalIdealRing_of_quadratic_integer_19 :
  IsPrincipalIdealRing (Algebra.adjoin  $\mathbb{Z} \{ (1 + (\text{Real.sqrt } 19) * \text{Complex.I}) / 2 \}$ ) ∧ IsDomain
    (Algebra.adjoin  $\mathbb{Z} \{ (1 + (\text{Real.sqrt } 19) * \text{Complex.I}) / 2 \}$ ) := by
  sorry

end Problem12
```

**Exercise (13).** Let  $(R, +, \cdot)$  be a (not necessarily commutative) ring. If we know that  $R$  is not a field and  $x^2 = x$  for any  $x \in R$ , where  $x$  is not invertible. Prove that  $x^2 = x$  for any  $x$ .

```
import Mathlib

namespace Problem13

/--
Let  $(R, +, \cdot)$  be a (not necessarily commutative) ring.
If we know that  $R$  is not a field and  $x^2 = x$  for any  $x \in R$ ,
where  $x$  is not invertible. Prove that  $x^2 = x$  for any  $x$ .
-/
theorem sq_eq_self_of_not_unit {R : Type} [Ring R] (h : ¬ IsField R)
  (h2 : ∀ x : R, ¬ IsUnit x → x^2 = x) (x : R) : x^2 = x := by
  sorry

end Problem13
```

**Exercise (14).** Show that if  $R$  is a unique factorization domain such that the quotient field of  $R$  is isomorphic to  $\mathbb{R}$ , then  $R$  is isomorphic to  $\mathbb{R}$ .

```
import Mathlib

namespace Problem14

/--
Show that if  $R$  is a unique factorization domain such that the quotient field of  $R$  is isomorphic to  $\mathbb{R}$ , then  $R$  is isomorphic to  $\mathbb{R}$ .
-/
theorem isomorphic_real_of_fractionRing_isomorphic_real_of_UFD (R : Type) [CommRing R] [IsDomain R]
  [UniqueFactorizationMonoid R] (h : Nonempty ((FractionRing R) ≃+* R)) :
  Nonempty (R ≃+* R) := by
  sorry
```

```
end Problem14
```

**Exercise (15).** Let  $p, q, r$  be three distinct prime numbers,  $t$  a positive integer. Let  $G$  be a finite group,  $H$  a normal subgroup of  $G$  such that the cardinality of  $G/H$  is  $r^t$ . Suppose that there exists a composition series

$$\{e\} = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_n = H,$$

of  $H$  that satisfies  $n = 2$ ,  $H_1/H_0 = \mathbb{Z}/p\mathbb{Z}$ ,  $H_2/H_1 = \mathbb{Z}/q\mathbb{Z}$ . Further suppose that there exists a composition series

$$\{e\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G,$$

and positive integers  $i < j \leq n$  such that  $G_i/G_{i-1} = \mathbb{Z}/q\mathbb{Z}$ ,  $G_j/G_{j-1} = \mathbb{Z}/p\mathbb{Z}$ . Show that there exists a composition series

$$\{e\} = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_n = H,$$

of  $H$  that satisfies  $n = 2$ ,  $H_1/H_0 = \mathbb{Z}/q\mathbb{Z}$ ,  $H_2/H_1 = \mathbb{Z}/p\mathbb{Z}$ .

```
import Mathlib

namespace Problem15

/--
A subgroup `H₁` is a maximal normal subgroup of `H₂` if it is contained in `H₂`,
and `H₁` is maximal normal in `H₂`.
 -/
structure Subgroup.IsMaximalNormal {G : Type} [Group G] (H₁ H₂ : Subgroup G) : Prop where
  le : H₁ ≤ H₂
  subgroupOf_normal : (H₁.subgroupOf H₂).Normal
  is_maximal : ∀ H : Subgroup G, H₁ ≤ H → H ≤ H₂ → (H.subgroupOf H₂).Normal → (H = H₁ ∨ H = H₂)

/--
A normal subgroup composition series of a group `G` is a *maximal* finite chain of normal subgroups
`[ \{e\} = G₀ \trianglelefteq G₁ \trianglelefteq \cdots \trianglelefteq Gₙ = G ]`
such that each quotient `G_{i+1}/G_i` is a simple group.
 -/
structure NormalSubgroupCompositionSeries (G : Type) [Group G] : Type where
  toRelSeries : RelSeries (Subgroup.IsMaximalNormal (G := G))
  maximal : ∀ s : RelSeries (Subgroup.IsMaximalNormal (G := G)), s.length ≤ toRelSeries.length

/--
The `(i)`-th factor of a normal subgroup composition series, which is the quotient of the `(i + 1)`-th
subgroup by the previous one.
 -/
def StepwiseQuotient {G : Type} [Group G] (s : NormalSubgroupCompositionSeries G) (i : Fin s.toRelSeries.length) :
```

```

Type :=
s.toRelSeries i.succ / (s.toRelSeries i.castSucc).subgroupOf _

/-
The  $\langle i \rangle$ -th factor of a normal subgroup composition series is a group.
 -/
instance {G : Type} [Group G] (s : NormalSubgroupCompositionSeries G) (i : Fin s.toRelSeries.length) :
Group (StepwiseQuotient s i) := QuotientGroup.Quotient.group _ (nN := (s.toRelSeries.step
i).subgroupOf_normal)

/-
Let  $p, q, r$  be three distinct prime numbers,  $t$  a positive integer. Let  $G$  be a finite group,
 $H$  a normal subgroup of  $G$  such that the cardinality of  $G/H$  is  $r^{t+1}$ .
Suppose that there exists a composition series
\[
\{e\} = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_n = H,
\]
of  $H$  that satisfies  $n=2$ ,  $H_1/H_0 = \mathbb{Z}/p\mathbb{Z}$ ,
 $H_2/H_1 = \mathbb{Z}/q\mathbb{Z}$ . Further suppose that there exists a composition series
\[
\{e\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G,
\]
and positive integers  $i < j \leq n$  such that  $G_i/G_{i-1} = \mathbb{Z}/q\mathbb{Z}$ ,
 $G_j/G_{j-1} = \mathbb{Z}/p\mathbb{Z}$ . Show that there exists a composition series
\[
\{e\} = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_n = H,
\]
of  $H$  that satisfies  $n=2$ ,  $H_1/H_0 = \mathbb{Z}/q\mathbb{Z}$ ,
 $H_2/H_1 = \mathbb{Z}/p\mathbb{Z}$ .
/-
theorem exists_swap_stepwiseQuotient {p q r t : ℕ} (hp : p.Prime) (hq : q.Prime) (hr : r.Prime)
(ht : 0 < t) (G : Type) [Group G] [Fintype G] (H : Subgroup G) [H.Normal]
(hH : Nat.card (G / H) = r ^ t) (Hs : NormalSubgroupCompositionSeries H)
(hHs : Hs.toRelSeries.length = 2) (hHs0 : StepwiseQuotient Hs ⟨0, by omega⟩ ≅ ZMod p)
(hHs1 : StepwiseQuotient Hs ⟨1, by omega⟩ ≅ ZMod q)
(Gs : NormalSubgroupCompositionSeries G) (i j : Fin Gs.toRelSeries.length) (hij : i < j)
(hGi : StepwiseQuotient Gs i ≅ ZMod q) (hGj : StepwiseQuotient Gs j ≅ ZMod p) :
∃ (Hs' : NormalSubgroupCompositionSeries H) (hlen : Hs'.toRelSeries.length = 2),
Nonempty (StepwiseQuotient Hs' ⟨0, by omega⟩ ≅ ZMod q) ∧
Nonempty (StepwiseQuotient Hs' ⟨1, by omega⟩ ≅ ZMod p) := by
sorry

end Problem15

```

**Exercise (16).** Let  $p$  be a prime and let  $F$  be a field. Let  $K$  be a finite Galois extension of  $F$  whose Galois group is a  $p$ -group (i.e., the degree  $[K : F]$  is a power of  $p$ ). Such an extension is called a  $p$ -extension (note that  $p$ -extensions are Galois by definition). Let  $L$  be a  $p$ -extension of  $K$ . Prove that the Galois closure of  $L$  over  $F$  is a  $p$ -extension of  $F$ .

```
import Mathlib
```

```

namespace Problem16

/--
A Galois extension such that the degree of the extension is a power of a prime  $\langle p \rangle$  is
called a  $p$ -extension.
 -/
class IsPExtension (F E : Type) [Field F] [Field E] [Algebra F E]
  (p : N) : Prop extends IsGalois F E where
  rank_eq_pow :  $\exists (n : N), \text{Module.rank } F E = p^n$ 

/--
Let  $p$  be a prime and let  $F$  be a field.
Let  $K$  be a finite Galois extension of  $F$  whose Galois group is a  $p$ -group (i.e., the degree
 $[K : F]$  is a power of  $p$ ). Such an extension is called a  $p$ -extension (note that
 $p$ -extensions are Galois by definition). Let  $L$  be a  $p$ -extension of  $K$ . Prove that the
Galois closure of  $L$  over  $F$  is a  $p$ -extension of  $F$ .
 -/
theorem normalClosure_isPExtension_of_isPExtension (F E : Type) [Field F] [Field E]
  [Algebra F E] (L : IntermediateField F E) (K : IntermediateField F L) (p : N) (hp : p.Prime)
  [IsPExtension F K p] [IsGalois K L] [IsPExtension K L p]
  (h_normalClosure : IsNormalClosure F L E) : IsPExtension F E p := by
  sorry

end Problem16

```

**Exercise (17).** Let  $K$  be a subfield of  $\mathbb{C}$  maximal with respect to the property that  $\sqrt{2} \notin K$ . Deduce that  $[\mathbb{C} : K]$  is countable (and not finite).

```

import Mathlib

namespace Problem17

/--
Let  $K$  be a subfield of  $\mathbb{C}$  maximal with respect to the property that  $\sqrt{2} \notin K$ .
Deduce that  $[\mathbb{C} : K]$  is countable (and not finite).
 -/
theorem countable_index_of_maximal_subfield_sqrt_2_nmem
  (K : Subfield C) (h_nmem : (Real.sqrt 2 : C)  $\notin K$ )
  (h :  $\forall (L : \text{Subfield } C), K \leq L \rightarrow (\text{Real.sqrt } 2 : C) \notin L \rightarrow K = L$ ) :
  Module.rank K C = Cardinal.aleph0 := by
  sorry

end Problem17

```

**Exercise (18).** Let  $E$  be a subfield of  $\mathbb{R}$  and let  $K/E$  be a finite Galois extension of odd degree  $> 1$ . Prove that  $K$  cannot be  $E$ -embedded into a radical tower that is a subfield of  $\mathbb{R}$ .

```
import Mathlib
```

```

namespace Problem18

/--
Let  $(E)$  be a commutative ring,  $(F)$  be an  $(E)$ -algebra, then we say  $(F)$  is
a radical extension over  $(E)$ , if  $(F)$  is generated by a single element  $(x \in F)$ 
over  $(E)$  such that  $(x^n - e = 0)$  for some  $(e \in E)$ .
 -/
def IsRadicalExtension (E F : Type) [CommRing E] [CommRing F] [Algebra E F] : Prop :=
  ∃ (x : F), Algebra.adjoin E {x} = F ∧ (∃ (n : N) (e : E), n ≥ 1 ∧ x^n - (algebraMap E F) e = 0)

/--
An algebra is said to be a radical tower over the base ring if it can be written as
composition of radical extensions.
 -/
inductive IsRadicalTower : ∀ (E : Type) (F : Type) [CommRing E] [CommRing F] [Algebra E F], Prop
| of_isRadicalExtension (E : Type) (F : Type)
  [CommRing E] [CommRing F] [Algebra E F] : IsRadicalExtension E F → IsRadicalTower E F
| of_composition (E : Type) (F : Type) [CommRing E] [CommRing F] [Algebra E F] (F' : Subalgebra E F) :
  IsRadicalExtension F' F → IsRadicalTower E F' → IsRadicalTower E F

/--
Let  $(E)$  be a subfield of  $(\mathbb{R})$  and let  $(K/E)$  be a finite Galois extension of
odd degree  $(> 1)$ . Prove that  $(K)$  cannot be  $(E)$ -embedded into a radical tower that is
a subfield of  $(\mathbb{R})$ .
 -/
theorem isEmpty_embedding_intermediateField_of_odd_degree_galois (E : Subfield R) (K : Type)
  [Field K] [Algebra E K] [IsGalois E K] (n : N) (h_odd : Odd n) (hn : n > 1) (h_deg_eq :
  Module.rank E K = n)
  (K' : IntermediateField E R) (h_radical : IsRadicalTower E K') :
  IsEmpty (K →ₐ[E] K') := by
  sorry

end Problem18

```

**Exercise (19).** Let  $\alpha = \sqrt{(2 + \sqrt{2})(3 + \sqrt{3})}$  and consider the extension  $E = \mathbb{Q}(\alpha)$ . Show that  $\text{Gal}(E/\mathbb{Q}) \cong Q_8$ , the quaternion group of order 8.

```

import Mathlib

namespace Problem19

/--
Let $E$ denote the algebra $\mathbb{Q}(\sqrt{(2+\sqrt{2})(3+\sqrt{3})})$
 -/
abbrev E : Type := (Algebra.adjoin ℚ {Real.sqrt ((2 + Real.sqrt 2) * (3 + Real.sqrt 3))})

/--
Let $\alpha = \sqrt{(2+\sqrt{2})(3+\sqrt{3})}$ and consider the extension $E = \mathbb{Q}(\alpha)$.
Show that $\text{Gal}(E/\mathbb{Q}) \cong Q_8$, the quaternion group of order $8$.

```

```

 -/
theorem galoisGroup_iso_quaternion_group : Nonempty ((E ≃a[Q] E) ≃* (QuaternionGroup 2)) := by
  sorry

end Problem19

```

**Exercise (20).** Let  $p$  be a prime number. Let  $L/K$  be a finite extension of fields of characteristic  $p$ , and let  $\sigma : x \mapsto x^p$  denote the  $p$ -Frobenius endomorphism on  $L$ , which of course stabilizes  $K$ . Prove that if  $[L : K\sigma(L)] \leq p$ , then  $L/K$  can be generated by one element.

```

import Mathlib

namespace Problem20

/--
Let  $p$  be a prime number. Let  $L/K$  be a finite extension of fields of characteristic  $p$ , and let  $\sigma : x \mapsto x^p$  denote the  $p$ -Frobenius endomorphism on  $L$ , which of course stabilizes  $K$ . Prove that if  $[L : K\sigma(L)] \leq p$ , then  $L/K$  can be generated by one element.
-/
theorem generated_single_elem_of_degree_le_p (p : ℕ) [Fact (Nat.Prime p)]
  (K L : Type) [Field K] [Field L] [CharP L p] [Algebra K L] [FiniteDimensional K L]
  (h : Module.rank (IntermediateField.adjoin K ((frobenius L p).range : Set L)) L ≤ p) :
  ∃ (x : L), IntermediateField.adjoin K {x} = τ := by
  sorry

end Problem20

```

**Exercise (21).** Let  $F$  be a field and let  $f(x) \in F[x]$  be an irreducible polynomial. Suppose that  $K$  is a splitting field for  $f(x)$  over  $F$  and assume that there exists an element  $\alpha \in K$  such that both  $\alpha$  and  $\alpha + 1$  are roots of  $f(x)$ . Prove that there exists an intermediate field  $E$  between  $K$  and  $F$  such that  $[K : E]$  is equal to the characteristic of  $F$ . (In particular, the characteristic of  $F$  is not zero)

```

import Mathlib

namespace Problem21

open Polynomial

/--
Let  $F$  be a field and let  $f(x) \in F[x]$  be an irreducible polynomial.
Suppose that  $K$  is a splitting field for  $f(x)$  over  $F$  and assume that there exists an element  $\alpha \in K$  such that both  $\alpha$  and  $\alpha + 1$  are roots of  $f(x)$ .
Prove that there exists an intermediate field  $E$  between  $K$  and  $F$  such that  $[K : E]$  is equal to the characteristic of  $F$ . (In particular, the characteristic of  $F$  is not zero)
-/
theorem intermediateField_rank_eq_ringChar (F : Type) [Field F] (f : Polynomial F) (hf : Irreducible f)
  (K : Type) [Field K] [Algebra F K] (hK : f.IsSplittingField F K) (α : K)

```

```

(hα : f.aeval α = 0) (hα1 : f.aeval (α + 1) = 0) :
  ∃ (E : IntermediateField F K), Module.rank E K = ringChar F := by
  sorry

end Problem21

```

**Exercise (22).** Let  $F$  be a field with  $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$ , where  $F/\mathbb{Q}$  is a finite abelian Galois extension. Prove that  $F$  contains only finitely many algebraic integers (i.e. elements in  $F$  whose minimal polynomial over  $\mathbb{Q}$  have coefficients in  $\mathbb{Z}$ ) having absolute value 1, and each of the algebraic integers is a root of unity.

```

import Mathlib

namespace Problem22

/--
Let  $F$  be a field with  $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$ , where  $F/\mathbb{Q}$  is a finite abelian Galois extension. Prove that  $F$  contains only finitely many algebraic integers (i.e. elements in  $F$  whose minimal polynomial over  $\mathbb{Q}$  have coefficients in  $\mathbb{Z}$ ) having absolute value 1, and each of the algebraic integers is a root of unity.
 -/
theorem finite_algebraic_integers_of_finite_module
  (F : IntermediateField Q C) (h_fin : Module.Finite Q F) [IsGalois Q F]
  (h : IsMulCommutative (F ≃ₐ[Q] F)) : {x : F | IsIntegral ℤ x ∧ ||(x : C)|| = 1}.Finite ∧
  (∀ x : F, IsIntegral ℤ x → ||(x : C)|| = 1 → ∃ n, x ^ n = 1) := by
  sorry

end Problem22

```

**Exercise (23).** Let  $f(X) \in \mathbb{Z}[X]$  be an irreducible polynomial,  $n_p$  is the number of solutions of  $f(X)$  in  $\mathbb{F}_p$ , show that

$$\lim_{s \rightarrow 1^+} \frac{\sum_{p \text{ prime}} \frac{n_p}{p^s}}{\sum_{p \text{ prime}} \frac{1}{p^s}} = 1$$

```

import Mathlib

namespace Problem23

local instance (p : Nat.Primes) : NeZero p.1 := <(p.2.ne_zero)>
local instance (p : Nat.Primes) : IsDomain (ZMod p) := @ZMod.instIsDomain p <(p.2)>

/--
Let  $f(X) \in \mathbb{Z}[X]$  be an irreducible polynomial,  $n_p$  is the number of solutions of  $f(X)$  in  $\mathbb{F}_p$ , show that  $\lim_{s \rightarrow 1^+} \frac{\sum_{p \text{ prime}} \frac{n_p}{p^s}}{\sum_{p \text{ prime}} \frac{1}{p^s}} = 1$ .

```

```

-/
theorem ratio_tendsto_one_of_irreducible (f : Polynomial ℤ) (h_irr : Irreducible f) :
  Function.rightLim
  (fun (s : ℝ) ↪
    (tsum (fun p : Nat.Primes ↪ (f.rootSet (ZMod p)).ncard * ((p : ℝ) ^ (-s)))) /
    (tsum (fun p : Nat.Primes ↪ (p : ℝ) ^ (-s)))) 1 = 1 := by
  sorry
end Problem23

```

**Exercise (24).** Let  $p_1, \dots, p_r$  be  $r$  different prime numbers. Prove that the Galois group of  $K = \mathbb{Q}(\sqrt{p_1}, \dots, \sqrt{p_r})$  over  $\mathbb{Q}$  is  $(\mathbb{Z}/2\mathbb{Z})^r$ , here  $\mathbb{Z}/2\mathbb{Z}$  is the cyclic group of order 2.

```

import Mathlib

namespace Problem24

/--
The field $K = \mathbb{Q}(\sqrt{p_1}, \dots, \sqrt{p_r})$ for a finite list of integers $p_1, \dots, p_r$.
-/
abbrev RatAdjoinSqrt {I : Type} (p : I → ℕ) : Type :=
Algebra.adjoin ℚ (Set.range (fun i ↪ Real.sqrt (p i)))

/--
Let $p_1, \dots, p_r$ be $r$ different prime numbers. Prove that the Galois group of $K = \mathbb{Q}(\sqrt{p_1}, \dots, \sqrt{p_r})$ over $\mathbb{Q}$ is $(\mathbb{Z}/2\mathbb{Z})^r$, here $\mathbb{Z}/2\mathbb{Z}$ is the cyclic group of order 2.
-/
theorem galoisGroup_iso_of_distinct_primes {I : Type} [Finite I] (p : I → ℕ)
  (hp : ∀ (i : I), (p i).Prime) (h_inj : p.Injective) :
  Nonempty ((RatAdjoinSqrt p ≃₉ RatAdjoinSqrt p) ≃* (Multiplicative (I → (ZMod 2)))) := by
  sorry
end Problem24

```

**Exercise (25).** Prove that the automorphism group of  $\mathbb{F}_2(t)$  is isomorphic to  $S_3$ , and its fixed field is  $\mathbb{F}_2(u)$  with

$$u = \frac{(t^4 - t)^3}{(t^2 - t)^5} = \frac{(t^2 + t + 1)^3}{(t^2 - t)^2}$$

```

import Mathlib

namespace Problem25

/--
Prove that the automorphism group of $\mathbb{F}_2(t)$ is isomorphic to $S_3$, and its fixed field is $\mathbb{F}_2(u)$ with $u = \frac{(t^4 - t)^3}{(t^2 - t)^5} = \frac{(t^2 + t + 1)^3}{(t^2 - t)^2}$.

```

```


$$\begin{aligned} & \text{Nonempty } ((\text{RatFunc } (\text{ZMod } 2) \simeq^{**} \text{RatFunc } (\text{ZMod } 2)) \simeq^{*} (\text{Equiv.Perm } (\text{Fin } 3))) \wedge \\ & \text{IntermediateField.fixedField } (F := \text{ZMod } 2) (E := \text{RatFunc } (\text{ZMod } 2)) \tau = \\ & \text{IntermediateField.adjoin } (\text{ZMod } 2) \{((x^4 - x)^3 / (x^2 - x)^5 : (\text{RatFunc } (\text{ZMod } 2)))\} \\ & := \text{by} \\ & \text{sorry} \end{aligned}$$


```

`end Problem25`

**Exercise (26).** Let  $K/\mathbb{Q}$  be a finite extension. Let  $H$  be a closed subgroup of the absolute Galois group  $G(K)$  of  $K$ . If  $H$  is finite, then the cardinality of  $H$  is either one or two.

```

import Mathlib

namespace Problem26

/--
Let  $\mathbb{K}/\mathbb{Q}$  be a finite extension.
Let  $H$  be a closed subgroup of the absolute Galois group  $G(K)$  of  $\mathbb{K}$ .
If  $H$  is finite, then the cardinality of  $H$  is either one or two.
-/
```

`theorem card_one_or_two_of_finite_closed_subgroup_of_absoluteGaloisGroup`

( $K : \text{Type}$ ) [Field K] [Algebra  $\mathbb{Q}$  K] [Module.Finite  $\mathbb{Q}$  K]  
( $H : \text{Subgroup } (\text{Field.absoluteGaloisGroup } K)$ )  
( $h_{\text{closed}} : \text{IsClosed } (H : \text{Set } (\text{Field.absoluteGaloisGroup } K))$ )  
( $h_{\text{fin}} : \text{Finite } H : \text{Nat.card } H = 1 \vee \text{Nat.card } H = 2 := \text{by}$   
`sorry`

`end Problem26`

**Exercise (27).** Let  $p$  be a prime number. Let  $K/\mathbb{Q}$  be a finite extension, such that the  $p^2$ th root of unity is contained in  $K$ . Let  $L/K$  be a Galois extension of degree  $p$ , show that there exists a Galois extension  $L'/L$  of degree  $p$ , such that the extension  $L'/K$  is Galois.

```

import Mathlib

namespace Problem27

/--
Let  $p$  be a prime number. Let  $\mathbb{K}/\mathbb{Q}$  be a finite extension, such that the  $p^2$ th root of unity is contained in  $\mathbb{K}$ . Let  $L/K$  be a Galois extension of degree  $p$ , show that there exists a Galois extension  $L'/L$  of degree  $p$ , such that the extension  $L'/K$  is Galois.
-/
```

`theorem isGalois_and_rank_eq_of_isPrimitiveRoot_sq` ( $p : \mathbb{N}$ ) ( $hp : p.\text{Prime}$ ) { $K : \text{Type}$ } [Field K]  
[NumberField K] { $\zeta : K$ } ( $h : \text{IsPrimitiveRoot } \zeta (p^2)$ )  
{ $L : \text{IntermediateField } K (\text{AlgebraicClosure } K)$ } [IsGalois K L]  
( $hdeg : \text{Module.rank } K L = p$ ) :  
 $\exists (L' : \text{Type}) (_ : \text{Field } L') (_ : \text{Algebra } K L')$

```

(_ : Algebra L L') (_ : IsScalarTower K L L'),
IsGalois K L' ∧ IsGalois L L' ∧ Module.rank L L' = p := by
sorry

end Problem27

```

**Exercise (28).** Let  $K/\mathbb{Q}$  be a finite extension. Let  $g$  be a nontrivial element of the absolute Galois group  $G(K)$  of  $K$ . Show that  $g$  admits an infinite number of conjugates.

```

import Mathlib

namespace Problem28

/--
Let  $K/\mathbb{Q}$  be a finite extension.
Let  $g$  be a nontrivial element of the absolute Galois group  $G(K)$  of  $K$ .
Show that  $g$  admits an infinite number of conjugates.
 -/
theorem infinite_conj_of_ne_1_absoluteGaloisGroup (K : Type)
  [Field K] [Algebra ℚ K] [Module.Finite ℚ K] (g : Field.absoluteGaloisGroup K) (h : g ≠ 1) :
  {g' : Field.absoluteGaloisGroup K | IsConj g g'}.Infinite := by
sorry

end Problem28

```

**Exercise (29).** Let  $K/\mathbb{Q}$  be a finite extension. Let  $g$  be an element of the absolute Galois group  $G(K)$  of  $K$ . Show that the subgroup generated by  $g$  is closed in  $G(K)$  if and only if  $g$  is torsion.

```

import Mathlib

namespace Problem29

/--
Let  $K/\mathbb{Q}$  be a finite extension. Let  $g$  be an element of the absolute Galois group  $G(K)$  of  $K$ . Show that the subgroup generated by  $g$  is closed in  $G(K)$  if and only if  $g$  is torsion.
 -/
theorem isClosed_zpowers_iff_isOffFinOrder (K : Type)
  [Field K] [Algebra ℚ K] [Module.Finite ℚ K] (g : Field.absoluteGaloisGroup K) :
  IsClosed ((Subgroup.zpowers g) : Set (Field.absoluteGaloisGroup K)) ↔ IsOffFinOrder g := by
sorry

end Problem29

```

**Exercise (30).** Let  $A$  be a subring of a ring  $B$ , such that the set  $B \setminus A$  is closed under multiplication. Show that  $A$  is integrally closed in  $B$ .

```

import Mathlib

namespace Problem30

/--
Let  $\langle A \rangle$  be a subring of a ring  $\langle B \rangle$ , such that the set  $\langle B \setminus A \rangle$  is closed under multiplication. Show that  $\langle A \rangle$  is integrally closed in  $\langle B \rangle$ .
 -/
theorem integrallyClosedIn_of_complement_multiplicatively_closed (B : Type) [CommRing B] (A : Subring B)
  (h : ∀ (x y : B), x ∉ A → y ∉ A → x * y ∉ A) : IsIntegrallyClosedIn A B := by
  sorry

end Problem30

```

**Exercise (31).** Let  $R = \mathbb{C}[x_1, \dots, x_n]/(x_1^2 + x_2^2 + \dots + x_n^2)$ . Then  $R$  is a unique factorization domain for  $n \geq 5$ .

```

import Mathlib

namespace Problem31

open MvPolynomial

/--
Let  $\langle R = \mathbb{C}[x_1, \dots, x_n]/(x_1^2 + x_2^2 + \dots + x_n^2) \rangle$ .
 -/
abbrev R (n : ℕ) : Type :=
  MvPolynomial (Fin n) ℂ / Ideal.span {(\sum i : Fin n, X i ^ 2 : MvPolynomial (Fin n) ℂ)}

/--
Let  $\langle R = \mathbb{C}[x_1, \dots, x_n]/(x_1^2 + x_2^2 + \dots + x_n^2) \rangle$ .
Then  $\langle R \rangle$  is a unique factorization domain for  $\langle n \geq 5 \rangle$ .
 -/
theorem UFD_of_ge_5 (n : ℕ) (h : n ≥ 5) :
  ∃ (h : IsDomain (R n)), UniqueFactorizationMonoid (R n) := by
  sorry

end Problem31

```

**Exercise (32).** Let  $A$  be a Noetherian local ring such that its completion  $\hat{A}$  is a unique factorization domain. Then  $A$  is a unique factorization domain.

```

import Mathlib

namespace Problem32

open IsLocalRing

```

```

/--
Let  $\hat{A}$  be a Noetherian local ring such that its completion  $\hat{\hat{A}}$  is a unique
factorization domain. Then  $\hat{A}$  is a unique factorization domain.
 -/
theorem UFD_of_adicCompletion_UFD (R : Type) [CommRing R] [IsLocalRing R] [IsNoetherianRing R]
  [IsDomain (AdicCompletion (maximalIdeal R) R)]
  [UniqueFactorizationMonoid (AdicCompletion (maximalIdeal R) R)] :
  ∃ (h : IsDomain R), UniqueFactorizationMonoid R := by
  sorry

end Problem32

```

**Exercise (33).** Let  $A \subset B$  be commutative rings such that  $B$  is finitely generated as a module over  $A$ . If  $B$  is a noetherian ring, show that  $A$  is also a noetherian ring.

```

import Mathlib

namespace Problem33

/--
Let  $A \subset B$  be commutative rings such that  $B$  is finitely generated as a module over  $A$ .
If  $B$  is a noetherian ring, show that  $A$  is also a noetherian ring.
 -/
theorem isNoetherianRing_of_fg_of_isNoetherianRing (B : Type) [CommRing B] [IsNoetherianRing B]
  (A : Subring B) (h : Module.Finite A B) : IsNoetherianRing A := by
  sorry

end Problem33

```

**Exercise (34).** If  $R$  is a valuation ring of Krull dimension  $\geq 2$ , then the formal power series ring  $R[[X]]$  is not integrally closed.

```

import Mathlib

namespace Problem34

open PowerSeries

/-
If  $R$  is a valuation ring of Krull dimension  $\geq 2$ ,
then the formal power series ring  $R[[X]]$  is not integrally closed.
 -/
theorem powerSeries_not_integrallyClosed_of_two_lt_ringKrullDim (R : Type) [CommRing R]
  [IsDomain R] [ValuationRing R] (two_lt : 2 ≤ ringKrullDim R) :
  ¬ (IsIntegrallyClosed R[[X]]) := by
  sorry

end Problem34

```

**Exercise (35).** A commutative ring whose prime ideals are finitely generated is Noetherian.

```
import Mathlib

namespace Problem35

/--
A commutative ring whose prime ideals are finitely generated is Noetherian.
 -/
theorem noetherian_of_prime_ideals_fg (R : Type) [CommRing R]
  (h_fg : ∀ (p : Ideal R), p.IsPrime → p.FG) : IsNoetherianRing R := by
  sorry

end Problem35
```

**Exercise (36).** If  $R$  is Noetherian and  $M$  and  $N$  are finitely generated  $R$ -modules, show that

$$\text{Ass Hom}_R(M, N) = \text{Supp } M \cap \text{Ass } N,$$

where  $\text{Supp } M$  is the set of all primes containing the annihilator of  $M$ .

```
import Mathlib

namespace Problem36

/-
If  $(R)$  is Noetherian and  $(M)$  and  $(N)$  are finitely generated  $(R)$ -modules, show that
[
\operatorname{Ass} \operatorname{Hom}_R(M, N) = \operatorname{Supp} M \cap \operatorname{Ass} N,
]
where  $(\operatorname{Supp} M)$  is the set of all primes containing the annihilator of  $(M)$ .
 -/
theorem associatedPrimes_hom_eq_support_inter_associatedPrimes (R : Type) [CommRing R]
  [IsNoetherianRing R] (M N : Type) [AddCommGroup M] [AddCommGroup N] [Module R M] [Module R N]
  [Module.Finite R M] [Module.Finite R N] : associatedPrimes R (M ↦[R] N) =
  {p | p ∈ associatedPrimes R N ∧ Module.annihilator R M ≤ p} := by
  sorry

end Problem36
```

**Exercise (37).** Let  $R = \mathbb{C}[x_{11}, x_{12}, \dots, x_{1n}, x_{21}, x_{22}, \dots, x_{2n}, \dots, x_{n1}, x_{n2}, \dots, x_{nn}] / (\det(x_{ij}) - 1)$ , show that  $R$  is a unique factorization domain.

```
import Mathlib

namespace Problem37

/-
Let $R=\mathbb{C}[x_{11}, x_{12}, \dots, x_{1n}, x_{21}, x_{22}, \dots, x_{2n}, \dots, x_{n1}, x_{n2}, \dots, x_{nn}] / (\det(x_{ij}) - 1)
```

```

x_{2n},\dots,x_{n1},x_{n2},\dots,x_{nn}] / (\det(x_{ij}) - 1)$.

 -/
abbrev QuotDetSubOne (n : ℕ) : Type := MvPolynomial ((Fin n) × (Fin n)) ℂ / Ideal.span {
    Matrix.det (fun (i : Fin n) ↦ (fun (j : Fin n) ↦ (.X ⟨i, j⟩ : (MvPolynomial ((Fin n) × (Fin n)) ℂ)))) - .C 1}

/-
Let $R=\mathbb{C}[x_{11},x_{12},\dots,x_{1n},x_{21},x_{22},\dots,x_{2n},\dots,x_{n1},x_{n2},\dots,x_{nn}] / (\det(x_{ij}) - 1)$,
show that $R$ is a unique factorization domain.
-/
theorem ufd_quotDetSubOne (n : ℕ) (h : n ≥ 1) : ∃ (h : IsDomain (QuotDetSubOne n)),
    UniqueFactorizationMonoid (QuotDetSubOne n) := by
    sorry

end Problem37

```

**Exercise (38).** Let  $k$  be a field, and let  $R = k[t]/(t^2)$ . Set

$$p(x) = tx^3 + tx^2 - x^2 - x \in R[x].$$

Show that  $S = R[x]/(p)$  is a free  $R$ -module of rank 2.

```

import Mathlib

namespace Problem38

open Polynomial DualNumber

/-
Let `(k)` be a field, and let `(R = k[t]/(t^2))`. Set
`[`
p(x) = tx^3 + tx^2 - x^2 - x `in R[x].
`]`
Let `(S = R[x]/(p))`.

/-
abbrev S (k : Type) [Field k] : Type := ((DualNumber k)[X] / Ideal.span {((C ε) * X^3 + (C ε) * X^2 - X^2 - X : (DualNumber k)[X])})

/-
`(`S`)` has a `(R)` module structure inherited from `R[x]`.
/-
noncomputable instance (k : Type) [Field k] : Module (DualNumber k) (S k) := Module.compHom _ C

/-
Let `(k)` be a field, and let `(R = k[t]/(t^2))`. Set
`[`
p(x) = tx^3 + tx^2 - x^2 - x `in R[x].
`]`
Show that `(S = R[x]/(p))` is a free `(R)`-module of rank `(2)`.
-/
```

```

theorem free_dualNumber_and_rank_eq_2 (k : Type) [Field k] :
  Module.Free (DualNumber k) (S k) ∧ Module.rank (DualNumber k) (S k) = 2 := by
  sorry

end Problem38

```

**Exercise (39).** Let  $R$  be a normal Noetherian domain,  $K$  its fraction field,  $L/K$  a finite field extension, and  $\bar{R}$  the integral closure of  $R$  in  $L$ . Prove that only finitely many primes  $\mathfrak{P}$  of  $\bar{R}$  lie over a given prime  $\mathfrak{p}$  of  $R$ .

```

import Mathlib

namespace Problem39

/--
Let  $(R)$  be a normal Noetherian domain,  $(K)$  its fraction field,  $(L/K)$  a finite field extension, and  $(\bar{R})$  the integral closure of  $(R)$  in  $(L)$ .  

Prove that only finitely many primes  $(\mathfrak{P})$  of  $(\bar{R})$  lie over a given prime  $(\mathfrak{p})$  of  $(R)$ .
-/
theorem finite_primes_lies_over_of_finite_extension (R : Type) [CommRing R] [IsDomain R]
  [IsNoetherianRing R] [IsIntegrallyClosed R] (L : Type) [Field L] [Algebra R L]
  [Algebra (FractionRing R) L] [IsScalarTower R (FractionRing R) L]
  [FiniteDimensional (FractionRing R) L] (p : Ideal R) [p.IsPrime] :
  (p.primesOver (integralClosure R L)).Finite := by
  sorry

end Problem39

```

**Exercise (40).** Let  $A$  be a reduced local ring with residue field  $k$  and finite set  $\Sigma$  of minimal primes. For each  $\mathfrak{p} \in \Sigma$ , set  $K(\mathfrak{p}) = \text{Frac}(A/\mathfrak{p})$ . Let  $P$  be a finitely generated module. Show that  $P$  is free of rank  $r$  if and only if  $\dim_k(P \otimes_A k) = r$  and  $\dim_{K(\mathfrak{p})}(P \otimes_A K(\mathfrak{p})) = r$  for each  $\mathfrak{p} \in \Sigma$ .

```

import Mathlib

namespace Problem40

open TensorProduct

/--
Let  $\$A\$$  be a reduced local ring with residue field  $\$k\$$  and finite set  $\$\Sigma\$$  of minimal primes.  

For each  $\$p \in \Sigma$ , set  $\$K(p) = \text{Frac}(A/p)\$$ .  

Let  $\$P\$$  be a finitely generated module. Show that  $\$P\$$  is free of rank  $\$r\$$  if and only if  

 $\$ \dim_k(P \otimes_A k) = r \$$  and  $\$ \dim_{K(p)}(P \otimes_A K(p)) = r \$$   

for each  $\$p \in \Sigma\$$ .
-/
theorem free_of_rank_iff (R : Type) [CommRing R] [IsLocalRing R] [IsReduced R]
  (h : (minimalPrimes R).Finite) (r : ℕ) (M : Type) [AddCommGroup M] [Module R M] [Module.Finite R M] :

```

```

Module.Free R M ∧ Module.rank R M = r ↔
(Module.rank (IsLocalRing.ResidueField R) ((IsLocalRing.ResidueField R) ⊗[R] M) = r ∧
∀ p ∈ minimalPrimes R,
Module.rank (FractionRing (R / p)) ((FractionRing (R / p)) ⊗[R] M) = r) := by
sorry

end Problem40

```

**Exercise (41).** Let  $k$  be a field,  $A := k[X_1, X_2, \dots]$  a polynomial ring,  $m_1 < m_2 < \dots$  positive integers with  $m_{i+1} - m_i > m_i - m_{i-1}$  for  $i > 1$ . Set

$$\mathfrak{p}_i := (X_{m_i+1}, \dots, X_{m_{i+1}})$$

and  $S := A - \bigcup_{i \geq 1} \mathfrak{p}_i$ . Show that  $S^{-1}A$  is noetherian with infinite Krull dimension.

```

import Mathlib

namespace Problem41

/--
The multiplicative subset generated by elements
not in a given family of ideals.
 -/
def compl_all {α : Type} [CommRing R] (I : α → Ideal R) : Submonoid R :=
Submonoid.closure (U (i : α), (I i : Set R))ᶜ

/--
The ideal generated by a set of single
variables in a multivariate polynomial ring.
 -/
def ideal_x {α : Type} (R : Type) [CommRing R] (J : Set α) : Ideal (MvPolynomial α R) :=
Ideal.span ((MvPolynomial.X)'' J)

/--
Let  $\langle A := k[X_1, X_2, \dots] \rangle$ .
Set  $\langle [\mathfrak{p}_i := (X_{m_i+1}, \dots, X_{m_{i+1}})] \rangle$  and
 $\langle S := A - \bigcup_{i \geq 1} \mathfrak{p}_i \rangle$ .
This is the ring  $\langle S^{-1}A \rangle$ .
 -/
abbrev SInvA (k : Type) [Field k] (m : ℕ → ℕ) : Type := (Localization (compl_all fun (n : ℕ) ↦
ideal_x k (Set.Ioc (m n) (m (n + 1)))))

/--
Let  $\langle k \rangle$  be a field,  $\langle A := k[X_1, X_2, \dots] \rangle$  a polynomial ring,  $\langle m_1 < m_2 < \dots \rangle$ 
positive integers with  $\langle m_{i+1} - m_i > m_i - m_{i-1} \rangle$  for  $\langle i > 1 \rangle$ . Set
 $\langle [\mathfrak{p}_i := (X_{m_i+1}, \dots, X_{m_{i+1}})] \rangle$ 
and  $\langle S := A - \bigcup_{i \geq 1} \mathfrak{p}_i \rangle$ .
Show that  $\langle S^{-1}A \rangle$  is noetherian with infinite Krull dimension.
 -/
theorem isNoetherianRing_and_krullDim_eq_top (k : Type) [Field k] (m : ℕ → ℕ) (h : StrictMono m)
(h_diff_mono : StrictMono (fun (i : ℕ) ↦ m (i + 1) - m i)) :

```

```

IsNoetherianRing (SInvA k m) ∧
ringKrullDim (SInvA k m) = τ := by
sorry

end Problem41

```

**Exercise (42).** Let  $k$  be any field. Suppose that  $A = k[[x,y]]/(f)$  and  $B = k[[u,v]]/(g)$ , where  $f = xy$  and  $g = uv + \delta$  with  $\delta \in (u,v)^3$ . Show that  $A$  and  $B$  are isomorphic.

```

import Mathlib

namespace Problem42

/--
Let  $\langle k \rangle$  be any field. Suppose that  $\langle A = k[[x,y]]/(f) \rangle$  and  $\langle B = k[[u,v]]/(g) \rangle$ ,  

where  $\langle f = xy \rangle$  and  $\langle g = uv + \delta \rangle$  with  $\langle \delta \in (u,v)^3 \rangle$ . Show that  $\langle A \rangle$  and  $\langle B \rangle$   

are isomorphic.
 -/
theorem nonEmpty_ringEquiv_of_sub_in_cube (k : Type) [Field k]
  (g : MvPowerSeries (Fin 2) k) (hg : g - .X 0 * .X 1 ∈ (Ideal.span {MvPowerSeries.X 0, .X 1}) ^ 3)
  :
  Nonempty ((MvPowerSeries (Fin 2) k) / Ideal.span {(.X 0 * .X 1 : (MvPowerSeries (Fin 2) k))}) ≅+
  *
  ((MvPowerSeries (Fin 2) k) / Ideal.span {g})) := by
sorry

end Problem42

```

**Exercise (43).** Let  $A$  be a reduced Noetherian local ring,  $\text{Char } A = p$ . Show that the absolute Frobenius  $F_A : A \rightarrow A, a \mapsto a^p$  is flat if and only if  $A$  is regular.

```

import Mathlib

namespace Problem43

open IsLocalRing

/--
A commutative local noetherian ring $R$ is regular if $\dim m/m^2 = \dim R$.
 -/
class IsRegularLocalRing (R : Type) [CommRing R] : Prop extends
  IsLocalRing R, IsNoetherianRing R where
  reg : Module.finrank (ResidueField R) (CotangentSpace R) = ringKrullDim R

/--
Let $A$ be a reduced Noetherian local ring, $\text{Char } A = p$.
Show that the absolute Frobenius $F_A : A \rightarrow A, a \mapsto a^p$ is flat if and only if $A$ is regular.
 -/

```

```

theorem IsRegularLocalRing.frobenius_flat {A : Type} [CommRing A] [IsNoetherianRing A]
[IsLocalRing A] [IsReduced A] (p : ℕ) [Fact p.Prime] [CharP A p] :
(frobenius A p).Flat ↔ IsRegularLocalRing A := by
sorry

end Problem43

```

**Exercise (44).** Let  $k$  be a field, and set  $A = k[X, Y, Z]/(X^2 - Y^2, Y^2 - Z^2, XY, YZ, ZX)$ . Show that  $A$  is not a global complete intersection.

```

import Mathlib

namespace Problem44

open MvPolynomial

/-
Let  $\$k\$$  be a field. Let  $\$S\$$  be a finite type  $\$k\$$ -algebra. We say that  $\$S\$$  is a
\textit{global complete intersection over  $\$k\$$ } if there exists a presentation
 $\$S = k[x_1, \dots, x_n]/(f_1, \dots, f_c)\$$  such that  $\$dim(S) = n - c\$$ .
-/
class IsGlobalCompleteIntersection (k : Type) [Field k] (S : Type) [CommRing S] [Algebra k S] :
Prop extends Algebra.FiniteType k S where
isGlobalCompleteIntersection : ∃ n : ℕ, ∃ rs : List (MvPolynomial (Fin n) k),
Nonempty (S ≈ₐ[k] (MvPolynomial (Fin n) k) / Ideal.ofList rs) ∧ ringKrullDim S + rs.length = n

/-
Let  $\backslash(k \backslash)$  be a field, and set  $\backslash(A = k[X, Y, Z]/(X^2 - Y^2, Y^2 - Z^2, XY, YZ, ZX) \backslash)$ .
Show that  $\backslash(A \backslash)$  is not a global complete intersection.
-/
theorem quot_x2_sub_y2_sub_z2_xy_yz_zx_not_global_complete_intersection (k : Type) [Field k] :
¬ IsGlobalCompleteIntersection k (MvPolynomial (Fin 3) k) / Ideal.span
{ {(X 0)^2 - (X 1)^2, (X 1)^2 - (X 2)^2, (X 0) * (X 1), (X 1) * (X 2), (X 2) * (X 0)} :
Set (MvPolynomial (Fin 3) k))} := by
sorry

end Problem44

```

**Exercise (45).** Let  $k$  be a field and  $A = k[x_1, \dots, x_r]$  the polynomial ring in  $r$  variables. Let  $M$  be a graded module over  $A$ , and let

$$0 \rightarrow K \rightarrow L_{r-1} \rightarrow \cdots \rightarrow L_0 \rightarrow M \rightarrow 0$$

be an exact sequence of graded homomorphisms of graded modules, such that  $L_0, \dots, L_{r-1}$  are free. Then  $K$  is free. Gradings of modules are by  $\mathbb{Z}_{\geq 0}$ .

```

import Mathlib

```

```

namespace Problem45

/--
A linear map `f` between graded modules is a graded homomorphism if it respects the
grading structure.
 -/
def IsGradedHom {R M N : Type} [CommRing R] [AddCommGroup M] [AddCommGroup N]
  [Module R M] [Module R N] (ᾰ : i → Submodule R M) (ᾰ' : i → Submodule R N)
  (f : M ↪[R] N) : Prop := ∀ (i : i) (x : ᾰ i), f x ∈ ᾰ' i

/--
Let  $k$  be a field and  $A = k[x_1, \dots, x_r]$  the polynomial ring in  $r$  variables. Let  $M$  be
a graded module over  $A$ , and let
\[
0 \rightarrow L_{r-1} \rightarrow \dots \rightarrow L_0 \rightarrow M \rightarrow 0
\]
be an exact sequence of graded homomorphisms of graded modules, such that  $L_0, \dots, L_{r-1}$ 
are free. Then  $K$  is free. {Gradings of modules are by  $\mathbb{Z}_{\geq 0}$ .}
 -/
theorem free_of_free_resolution {k : Type} [Field k] {r : ℕ}
  (C : ChainComplex (ModuleCat.{0}) (MvPolynomial (Fin r) k)) N
  (hC : ∀ (n : ℕ), n > (r + 1) → CategoryTheory.Limits.IsZero (C.X n))
  (ᾰ : ∀ (n : ℕ), (N → Submodule (MvPolynomial (Fin r) k) (C.X n)))
  [hM : ∀ (n : ℕ), DirectSum.Decomposition (ᾰ n)]
  [hM' : ∀ (n : ℕ), SetLike.GradedSMul (MvPolynomial.homogeneousSubmodule (Fin r) k) (ᾰ n)]
  (h_exact : C.Acyclic)
  (h_gr : ∀ (i j : ℕ), IsGradedHom (ᾰ i) (ᾰ j) (C.d i j).hom)
  (h_free : ∀ (n : ℕ), 1 ≤ n ∧ n ≤ r → Module.Free (MvPolynomial (Fin r) k) (C.X n)) :
  Module.Free (MvPolynomial (Fin r) k) (C.X (r + 1)) := by
  sorry

end Problem45

```

**Exercise (46).** Let  $M$  be an  $R$ -module. Then  $M$  is flat if and only if the following condition holds: if  $P$  is a finitely presented  $R$ -module and  $f : P \rightarrow M$  a  $R$ -linear map, then there is a free finite  $R$ -module  $F$  and module maps  $h : P \rightarrow F$  and  $g : F \rightarrow M$  such that  $f = g \circ h$ .

```

import Mathlib

namespace Problem46

/-
Let  $(M)$  be an  $(R)$ -module. Then  $(M)$  is flat if and only if the following condition holds:
if  $(P)$  is a finitely presented  $(R)$ -module and  $(f: P \rightarrow M)$  a  $(R)$ -linear map,
then there is a free finite  $(R)$ -module  $(F)$  and module maps  $(h: P \rightarrow F)$  and  $(g: F \rightarrow M)$ 
such that  $(f = g \circ h)$ .
 -/
theorem module_flat_iff (R : Type) [CommRing R] (M : Type) [AddCommGroup M] [Module R M] :
  Module.Flat R M ↔
  ∀ P : Type, ∀ (_ : AddCommGroup P), ∀ (_ : Module R P), ∀ f : P ↪[R] M,
  Module.FinitePresentation R P →

```

```

 $\exists (F : \text{Type}) (\_ : \text{AddCommGroup } F) (\_ : \text{Module } R F), \text{Module.Finite } R F \wedge \text{Module.Free } R F \wedge$ 
 $\exists h : P \rightarrow[R] F, \exists g : F \rightarrow[R] M, f = g \circ h := \text{by}$ 
sorry
end Problem46

```

**Exercise (47).** Show that the ring  $A = k[x,y]/(y^2 - f(x))$  is a Dedekind domain and the class group of the ring  $A$  is not trivial, where  $k$  is a field of characteristic not 2,  $f(x) = (x-t_1)\dots(x-t_n)$  with  $t_1, \dots, t_n \in k$  distinct and  $n \geq 3$  is an odd integer.

```

import Mathlib

namespace Problem47

/--
The ring  $(A = k[x,y]/(y^2 - f(x)))$ ,
where  $(k)$  is a field and  $(f(x) = (x - t_1)\dots(x - t_n))$ .
 -/
abbrev A {k : Type} [Field k] {n : ℕ} (t : (Fin n) → k) : Type := (MvPolynomial (Fin 2) k) /
Ideal.span {(.X 1 ^ 2) - Π (m : Fin n), (.X 0 - .C (t m)) : (MvPolynomial (Fin 2) k) }

/--
Show that the ring  $(A = k[x,y]/(y^2 - f(x)))$  is a Dedekind domain and the class group of the
ring  $(A)$  is not trivial, where  $(k)$  is a field of characteristic not 2,
 $(f(x) = (x - t_1)\dots(x - t_n))$  with  $(t_1, \dots, t_n)$  in  $k$  distinct and
 $(n \geq 3)$  is an odd integer.
 -/
theorem isEmpty_isomorphism_UFD_of_quotient (k : Type) [Field k] (h_char : ¬ CharP k 2)
  (n : ℕ) (h_ge : n ≥ 3) (h_odd : Odd n) (t : (Fin n) → k) (h_inj : Function.Injective t) :
   $\exists \_ : \text{IsDedekindDomain } (A t), \text{Nontrivial } (\text{ClassGroup } (A t)) := \text{by}$ 
sorry
end Problem47

```

**Exercise (48).** A commutative ring  $A$  is absolutely flat if every  $A$ -module is flat. Prove that  $A$  is absolutely flat if and only if every principal ideal is idempotent.

```

import Mathlib

namespace Problem48

/--
A commutative ring  $(A)$  is absolutely flat if every  $(A)$ -module is flat.
 -/
class IsAbsolutelyFlat (R : Type) [CommRing R] : Prop where
  out {P : Type} [AddCommGroup P] [Module R P] : Module.Flat R P
  /-

```

```

Prove that  $\langle A \rangle$  is absolutely flat if and only if every principal ideal is idempotent.
 -/
theorem isAbsolutelyFlat_iff_principal_ideal_idempotent (R : Type) [CommRing R] :
  IsAbsolutelyFlat R ↔ (∀ I : Ideal R, I.IsPrincipal → I ^ 2 = I) := by
  sorry

end Problem48

```

**Exercise (49).** Let  $A$  be a commutative ring. Prove that every principal ideal of  $A$  is idempotent if and only if every finitely generated ideal is a direct summand of  $A$ .

```

import Mathlib

namespace Problem49

/-
Let  $\langle A \rangle$  be a commutative ring. Prove that every principal ideal of  $\langle A \rangle$  is idempotent if and only if every finitely generated ideal is a direct summand of  $\langle A \rangle$ .
-/
theorem principal_ideal_idempotent_iff_fg_ideal_is_direct_summand (A : Type) [CommRing A] :
  (∀ I : Ideal A, I.IsPrincipal → I ^ 2 = I) ↔
  (∀ I : Ideal A, I.FG → (∃ J : Ideal A, I ∪ J = τ ∧ I ∩ J = ⊥)) := by
  sorry

end Problem49

```

**Exercise (50).** Let  $(A, \mathfrak{m}, K)$  be a complete local ring containing a field, and suppose that  $\mathfrak{m}$  is finitely generated over  $A$ . Then  $A$  is Noetherian.

```

import Mathlib

namespace Problem50

/-
Let  $\langle (A, \mathfrak{m}, K) \rangle$  be a complete local ring containing a field, and suppose that  $\langle \mathfrak{m} \rangle$  is finitely generated over  $\langle A \rangle$ . Then  $\langle A \rangle$  is Noetherian.
-/
theorem isNoetherianRing_of_isLocalRing_of_field_inj_of_adicComplete_of_maximalIdeal_finite
  (R : Type) [CommRing R] [IsLocalRing R] [IsAdicComplete (IsLocalRing.maximalIdeal R) R]
  (k : Type) [Field k] [Algebra k R] [NoZeroSMulDivisors k R]
  (hfg : (IsLocalRing.maximalIdeal R).FG) : IsNoetherianRing R := by
  sorry

end Problem50

```

**Exercise (51).** A Noetherian topological ring in which the topology is defined by an ideal contained in the Jacobson radical is called a Zariski ring. Let  $A$  be a Noetherian ring,  $\mathfrak{a}$  an ideal of  $A$ , and  $\widehat{A}$  the  $\mathfrak{a}$ -adic completion of  $A$ . Prove that  $\widehat{A}$  is faithfully flat over  $A$  if and only if  $A$  is a Zariski ring for the  $\mathfrak{a}$ -topology.

```

import Mathlib

namespace Problem51

/--
A Noetherian topological ring in which the topology is defined by an ideal contained in the Jacobson radical is called a Zariski ring.
Let  $(A)$  be a Noetherian ring,  $(\mathfrak{a})$  an ideal of  $(A)$ , and  $(\widehat{A})$  the  $(\mathfrak{a})$ -adic completion of  $A$ .
Prove that  $(\widehat{A})$  is faithfully flat over  $(A)$  if and only if  $(A)$  is a Zariski ring for the  $(\mathfrak{a})$ -topology.
 -/
theorem adicCompletion_faithfullyFlat_iff (A : Type) [CommRing A] [IsNoetherianRing A]
  (I : Ideal A) : Module.FaithfullyFlat A (AdicCompletion I A) ↔ I ≤ Ring.jacobson A := by
  sorry

end Problem51

```

**Exercise (52).** Let  $R$  be a ring,  $\mathfrak{m}$  is an ideal in the Jacobson radical of  $R$ , and  $G_1, G_2 \in R[x]$  are polynomials such that  $G_1$  is monic. If  $G_i \bmod \mathfrak{m}$  generate the unit ideal of  $R/\mathfrak{m}[x]$ , then  $G_1, G_2$  together generate the unit ideal of  $R[x]$ .

```

import Mathlib

namespace Problem52

/-
Let  $R$  be a ring,  $(\mathfrak{m})$  is an ideal in the Jacobson radical of  $(R)$ , and  $(G_1, G_2) \in R[x]$  are polynomials such that  $G_1$  is monic. If  $G_i \bmod \mathfrak{m}$  generate the unit ideal of  $R/\mathfrak{m}[x]$ , then  $(G_1, G_2)$  together generate the unit ideal of  $(R[x])$ .
 -/
theorem generate_unit_ideal_of_quotient (R : Type) [CommRing R] (m : Ideal R)
  (h_le_jac : m ≤ Ring.jacobson R) (G1 G2 : Polynomial R) (h_monic : G1.Monic)
  (h_gen : Ideal.span {G1.map (Ideal.Quotient.mk m), G2.map (Ideal.Quotient.mk m)} = 1) :
  Ideal.span {G1, G2} = 1 := by
  sorry

end Problem52

```

**Exercise (53).** Let  $k$  be a field, and set  $A = k[X, Y, Z]/(X^2 - Y^2, Y^2 - Z^2, XY, YZ, ZX)$ . Show that  $A$  is Gorenstein.

```

import Mathlib

namespace Problem53

open IsLocalRing ModuleCat CategoryTheory MvPolynomial

```

```

instance (R : Type) [CommRing R] : CategoryTheory.HasExt.{0} (ModuleCat.{0} R) :=
CategoryTheory.hasExt_of_enoughProjectives.{0} (ModuleCat.{0} R)

/--
A Noetherian local ring $R$ is a Gorenstein ring if $\mathrm{injDim}_R < +\infty$.
 -/
class IsGorensteinLocalRing (R : Type) [CommRing R] : Prop extends
  IsLocalRing R, IsNoetherianRing R where
  injDim_le_infinity :
    ∃ n : N, ∀ i : N, n ≤ i →
    Subsingleton (Abelian.Ext.{0} (of.{0} R (ResidueField R)) (of.{0} R R) i)

/--
A Noetherian ring is a Gorenstein ring if its localization at every maximal ideal is a
Gorenstein local ring.
 -/
class IsGorensteinRing (R : Type) [CommRing R] : Prop extends IsNoetherianRing R where
  localization_maximal_isGorensteinLocalRing :
    ∀ m : Ideal R, (_ : m.IsMaximal) → IsGorensteinLocalRing (Localization.AtPrime m)

/--
Let $(k)$ be a field, and set $(A = k[X, Y, Z]/(X^2 - Y^2, Y^2 - Z^2, XY, YZ, ZX))$.  

Show that $(A)$ is Gorenstein.
 -/
theorem isGorensteinRing_quot_x2_sub_y2_sub_z2_xy_yz_zx (k : Type) [Field k] :
  IsGorensteinRing <| MvPolynomial (Fin 3) k / Ideal.span {((X 0)^2 - (X 1)^2, (X 1)^2 - (X 2)^2,
  (X 0) * (X 1), (X 1) * (X 2), (X 2) * (X 0))} : Set (MvPolynomial (Fin 3) k)) := by
  sorry

end Problem53

```

**Exercise (54).** Let \$A\$ be a \$\mathbb{Q}\$-algebra. Suppose that \$x \in A\$ and \$D \in \mathrm{Der}(A)\$ are such that \$Dx = 1\$ and \$\bigcap\_{n=1}^{\infty} x^n A = (0)\$. Show that \$x\$ is a non-zero-divisor of \$A\$.

```

import Mathlib

namespace Problem54

/--
Let $(A)$ be a $\mathbb{Q}$-algebra.  

Suppose that $(x \in A)$ and $(D \in \mathrm{Der}(A))$ are such that $(Dx = 1)$ and  

$(\bigcap_{n=1}^{\infty} x^n A = (0))$.  

Show that $(x)$ is a non-zero-divisor of $(A)$.
 -/
theorem not_zero_divisor_of_hausdorff_of_der_eq_one (A : Type) [CommRing A] [Algebra Q A]
  (x : A) (D : Derivation Z A A) (h_dx : D x = 1) (h_hausdorff : IsHausdorff (Ideal.span {x}) A) :
  x ∈ nonZeroDivisors A := by
  sorry

end Problem54

```

**Exercise (55).** A module  $M$  over a ring  $R$  is stably free if there exists a free finitely generated module  $F$  over  $R$  such that

$$M \oplus F$$

is a free module. Prove that if  $M$  is stably free and not finitely generated then  $M$  is free.

```
import Mathlib

namespace Problem55

/-
A module \(( M \) \) over a ring \(( R \) \) is \text{stably free} if there exists a free finitely
generated module \(( F \) \) over \(( R \) \) such that
\[
M \oplus F
\]
is a free module.
-/
def IsStablyFree (R : Type) (M : Type) [CommRing R] [AddCommGroup M] [Module R M] : Prop :=
  ∃ (N : Type) (_ : AddCommGroup N) (_ : Module R N),
    Module.Finite R N ∧ Module.Free R N ∧ Module.Free R (M × N)

/-
Prove that if  $\$M\$$  is stably free and not finitely generated then  $\$M\$$  is free.
-/
theorem stablyFree_iff_free_of_not_fg (R : Type) (M : Type) [CommRing R] [AddCommGroup M]
  [Module R M] (h : ¬ Module.Finite R M) : Module.Free R M ↔ IsStablyFree R M := by
  sorry

end Problem55
```

**Exercise (56).** Let  $R \rightarrow S$  be a faithfully flat ring map. Let  $M$  be an  $R$ -module. If the  $S$ -module  $S \otimes_R M$  is projective, then  $M$  is projective.

```
import Mathlib

namespace Problem56

/-
Let \(( R \rightarrow S \) \) be a faithfully flat ring map. Let \(( M \) \) be an \(( R \) \)-module.
If the \(( S \) \)-module \(( S \otimes_R M \) \) is projective, then \(( M \) \) is projective.
-/
theorem projective_of_faithfullyFlat_base_change (R S M : Type) [CommRing R] [CommRing S]
  [Algebra R S] [Module.FaithfullyFlat R S] [AddCommGroup M] [Module R M]
  [Module.Projective S (TensorProduct R S M)] : Module.Projective R M := by
  sorry

end Problem56
```

**Exercise (57).** Let  $A$  be a domain and  $K$  its field of fractions.  $x \in K$  is called almost integral if there exists an element  $r \in A, r \neq 0$  such that  $rx^n \in A$  for all  $n \geq 0$ .  $A$  is called completely integrally closed if every almost integral element of  $K$  is contained in  $A$ . Show that if  $A$  is completely integrally closed, so is  $A[X]$ .

```
import Mathlib

namespace Problem57

/-
Let  $\langle A \rangle$  be a domain and  $\langle K \rangle$  its field of fractions.
 $\langle K \rangle$  is called almost integral if there exists an element  $\langle r \in A, r \neq 0 \rangle$ 
such that  $\langle rx^n \in A \rangle$  for all  $\langle n \geq 0 \rangle$ .
-/
def IsAlmostIntegral {A : Type} [CommRing A] [IsDomain A] (x : FractionRing A) : Prop :=
  ∃ r : A, r ≠ 0 ∧ ∀ n : ℕ, ∃ y : A, r • (x ^ n) = algebraMap A (FractionRing A) y

/-
 $\langle A \rangle$  is called completely integrally closed if every almost integral element
of  $\langle K \rangle$  is contained in  $\langle A \rangle$ .
-/
def IsCompletelyIntegrallyClosed (A : Type) [CommRing A] [IsDomain A] : Prop :=
  ∀ x : FractionRing A, IsAlmostIntegral x → ∃ y : A, x = algebraMap A (FractionRing A) y

/-
Let  $\langle A \rangle$  be a domain. Show that if  $\langle A \rangle$  is completely integrally closed, so is  $\langle A[X] \rangle$ .
-/
theorem completely_integrally_closed_polynomial_ring {A : Type} [CommRing A] [IsDomain A]
  (h : IsCompletelyIntegrallyClosed A) : IsCompletelyIntegrallyClosed (Polynomial A) := by
  sorry

end Problem57
```

**Exercise (58).** Suppose that  $(R, \mathfrak{P})$  is a local Noetherian ring, and let  $(S, \mathfrak{Q})$  be a local Noetherian  $R$ -algebra such that  $\mathfrak{P}S \subseteq \mathfrak{Q}$ . If  $M$  is a finitely generated  $S$ -module, show that  $M$  is flat as an  $R$ -module if  $M/\mathfrak{P}^nM$  is flat as an  $R/\mathfrak{P}^n$ -module for every  $n$ .

```
import Mathlib

namespace Problem58

open TensorProduct

/-
Suppose that  $(R, \mathfrak{P})$  is a local Noetherian ring,
and let  $(S, \mathfrak{Q})$  be a local Noetherian  $R$ -algebra such that
 $\mathfrak{P}S \subseteq \mathfrak{Q}$ .
If  $M$  is a finitely generated  $S$ -module, show that  $M$  is flat as an  $R$ -module
if  $M / \mathfrak{P}^n M$  is flat as an  $R / \mathfrak{P}^n$ -module for every  $n$ .
-/

```

```

-/
theorem flat_of_flat_over_quotient (R S : Type) [CommRing R] [CommRing S]
  [IsLocalRing R] [IsLocalRing S] [IsNoetherianRing R] [IsNoetherianRing S] [Algebra R S]
  (h_map : Ideal.map (algebraMap R S) (IsLocalRing.maximalIdeal R) ≤ IsLocalRing.maximalIdeal S)
  (M : Type) [AddCommGroup M] [Module S M] [Module R M] [IsScalarTower R S M] [Module.Finite S M]
  (h_flat_quotient : ∀ (n : ℕ), Module.Flat (R / (IsLocalRing.maximalIdeal R) ^ n) ((R /
    (IsLocalRing.maximalIdeal R) ^ n) ⊗[R] M)) :
  Module.Flat R M := by
  sorry
end Problem58

```

**Exercise (59).** Let  $k$  be a field,  $X$  and  $Y$  indeterminates, and suppose that  $\alpha$  is a positive irrational number. Show the map  $v : k[X, Y] \rightarrow \mathbb{R} \cup \{\infty\}$  defined by

$$v \left( \sum c_{n,m} X^n Y^m \right) = \min \{n + m\alpha \mid c_{n,m} \neq 0\}$$

determines a valuation of  $k(X, Y)$  with value group  $\mathbb{Z} + \mathbb{Z}\alpha$ .

```

import Mathlib

namespace Problem59

/--
Let  $k$  be a field,  $X$  and  $Y$  indeterminates, and suppose that  $\alpha$  is a positive irrational number. Show the map  $v : k[X, Y] \rightarrow \mathbb{R} \cup \{\infty\}$  defined by
\[
v \left( \sum c_{n,m} X^n Y^m \right) = \min \{n + m\alpha \mid c_{n,m} \neq 0\}
\]
determines a valuation of  $k(X, Y)$  with value group  $\mathbb{Z} + \mathbb{Z}\alpha$ .
 -/
theorem exists_unique_valuation_eq (α : ℝ) (h_pos : α > 0) (h_irr : Irrational α)
  (k : Type) [Field k] : ∃! (v : AddValuation (FractionRing (MvPolynomial (Fin 2) k)) (WithTop ℝ)),
  ∀ (f : MvPolynomial (Fin 2) k), v (algebraMap _ _ f) = Finset.inf (Finset.image (fun s ↦ ((s 0 +
    α * s 1) : WithTop ℝ)) f.support) id := by
  sorry
end Problem59

```

**Exercise (60).** Let  $R$  be a Noetherian domain, and suppose that for every maximal ideal  $P$  of  $R$  the ring  $R_P$  is factorial. Let  $I \subset R$  be an ideal. Prove that  $I$  is an invertible module iff  $I$  has pure codimension 1. (We say that an ideal  $I$  in a ring  $R$  has pure codimension 1 if every associated prime ideal of  $I$  has codimension 1. We include the case when  $I$  has no associated primes at all—that is, when  $I = R$ .)

```

import Mathlib

namespace Problem60

open Problem60

/--
For a Noetherian domain  $\langle R \rangle$ , we say that an ideal  $\langle I \subset R \rangle$  is invertible if it is not the zero ideal and there exists an ideal  $\langle N \rangle$  such that  $\langle N \cdot I \rangle$  is principal and  $\langle N \rangle$  is not the zero ideal.
 -/
def Ideal.Invertible {R : Type} [CommRing R] [IsDomain R] (I : Ideal R) : Prop :=
  I ≠ ⊥ ∧ ∃ (N : Ideal R), (N * I).IsPrincipal ∧ N ≠ ⊥

/--
Let  $\$R\$$  be a Noetherian domain, and suppose that for every maximal ideal  $\$P\$$  of  $\$R\$$  the ring  $\$R_P\$$  is factorial. Let  $\$I \subset R\$$  be an ideal. Prove that  $\$I\$$  is an invertible module iff  $\$I\$$  has pure codimension  $\$1\$$ . (We say that an ideal  $\$I\$$  in a ring  $\$R\$$  has pure codimension  $\$1\$$  if every associated prime ideal of  $\$I\$$  has codimension  $\$1\$$ . We include the case when  $\$I\$$  has no associated primes at all---that is, when  $\$I = R\$$ .)
 -/
theorem invertible_iff_codimension_one (R : Type) [CommRing R] [IsDomain R] [IsNoetherianRing R]
  (h_ufd : ∀ (p : Ideal R), (h : p.IsMaximal) → UniqueFactorizationMonoid (Localization.AtPrime p))
  (I : Ideal R) : I.Invertible ↔ ∀ (p : associatedPrimes R I), ringKrullDim (R / p.1) = 1 := by
  sorry

end Problem60

```

**Exercise (61).** Let  $R \rightarrow S$  be a ring map. Let  $I \subset R$  be an ideal. Assume

1.  $I^2 = 0$ ,
2.  $R \rightarrow S$  is flat, and
3.  $R/I \rightarrow S/IS$  is formally smooth.

Show  $R \rightarrow S$  is formally smooth.

```

import Mathlib

namespace Problem61

/--
Let  $\langle R \rightarrow S \rangle$  be a ring map. Let  $\langle I \subset R \rangle$  be an ideal. Assume
\begin{enumerate}
  \item  $\langle I^2 = 0 \rangle$ ,
  \item  $\langle R \rightarrow S \rangle$  is flat, and
  \item  $\langle R/I \rightarrow S/IS \rangle$  is formally smooth.
\end{enumerate}
Show  $\langle R \rightarrow S \rangle$  is formally smooth.

```

```

-/
theorem formallySmooth_of_formallySmooth_quotient (R S : Type) [CommRing R] [CommRing S]
[Algebra R S] [Module.Flat R S] (I : Ideal R) (h : I ^ 2 = 0)
[Algebra.FormalySmooth (R / I) (S / (I.map (algebraMap R S)))] :
Algebra.FormalySmooth R S := by
sorry

end Problem61

```

**Exercise (62).** Let  $\varphi : R \rightarrow S$  be a smooth ring map. Let  $\sigma : S \rightarrow R$  be a left inverse to  $\varphi$ . Set  $I = \text{Ker}(\sigma)$ . If  $I/I^2$  is free, show  $S^\wedge \cong R[[t_1, \dots, t_d]]$  as  $R$ -algebras, where  $S^\wedge$  is the  $I$ -adic completion of  $S$ .

```

import Mathlib

namespace Problem62

/-
Let  $\varphi : R \rightarrow S$  be a smooth ring map. Let  $\sigma : S \rightarrow R$  be a left inverse to  $\varphi$ . Set  $I = \text{Ker}(\sigma)$ . If  $I/I^2$  is free, show  $S^\wedge \cong R[[t_1, \dots, t_d]]$  as  $R$ -algebras, where  $S^\wedge$  is the  $I$ -adic completion of  $S$ .
-/
theorem adicCompletion_equiv_of_smooth (R S : Type) [CommRing R] [CommRing S]
[Algebra R S] [Algebra.Smooth R S] ( $\sigma : S \rightarrow R$ )
(h : Function.LeftInverse  $\sigma$  (algebraMap R S)) (hf : Module.Free R  $\sigma.\ker.\text{Cotangent}$ ) :
 $\exists d : \mathbb{N}, \text{Nonempty } (\text{AdicCompletion} (\text{RingHom}.ker \sigma) S \simeq_a R) \text{ MvPowerSeries } (\text{Fin } d) R$  := by
sorry

end Problem62

```

**Exercise (63).** Let  $R \rightarrow S$  be a formally unramified ring map. Show there exists a surjection of  $R$ -algebras  $S' \rightarrow S$  whose kernel is an ideal of square zero with the following universal property: Given any commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{a} & A/I \\ \uparrow & & \uparrow \\ R & \xrightarrow{b} & A \end{array}$$

where  $I \subset A$  is an ideal of square zero, there is a unique  $R$ -algebra map  $\alpha' : S' \rightarrow A$  such that  $S' \rightarrow A \rightarrow A/I$  is equal to  $S' \rightarrow S \rightarrow A/I$ .

```

import Mathlib

namespace Problem63

/-

```

```

The universal property:
Given any commutative diagram
\[
\begin{tikzcd}
S \arrow[r, "a"] & A/I \\
R \arrow[u] \arrow[r, "b"] & A \arrow[u]
\end{tikzcd}
\]
where  $\{ I \subset A \}$  is an ideal of square zero, there is a unique  $\{ R \}$ -algebra map
 $\{ \alpha: S' \rightarrow A \}$  such that  $\{ S' \rightarrow A \rightarrow A/I \}$  is equal to  $\{ S' \rightarrow S \rightarrow A/I \}$ .
-/
def UniversalProperty.liftOfSqZeroIdeal {R S S' : Type} [CommRing R] [CommRing S] [CommRing S']
  [Algebra R S] [Algebra R S'] (f : S' →a[R] S) :=
  ∀ (A : Type) [CommRing A] [Algebra R A] (I : Ideal A) (g : S →a[R] /AI),
  I^2 = 0 → (g.toRingHom.comp (algebraMap R S) = (Ideal.Quotient.mk I).comp (algebraMap R A)) →
  ∃! (g' : S' →a[R] A), (Ideal.Quotient.mk I).comp g'.toRingHom = g.comp f

/-
Let  $\{ R \rightarrow S \}$  be a formally unramified ring map. Show there exists a surjection of
 $\{ R \}$ -algebras  $\{ S' \rightarrow S \}$  whose kernel is an ideal of square zero with the following
universal property:
Given any commutative diagram
\[
\begin{tikzcd}
S \arrow[r, "a"] & A/I \\
R \arrow[u] \arrow[r, "b"] & A \arrow[u]
\end{tikzcd}
\]
where  $\{ I \subset A \}$  is an ideal of square zero, there is a unique  $\{ R \}$ -algebra map
 $\{ \alpha: S' \rightarrow A \}$  such that  $\{ S' \rightarrow A \rightarrow A/I \}$  is equal to  $\{ S' \rightarrow S \rightarrow A/I \}$ .
-/
theorem surjection_of_formally_unramified (R S : Type) [CommRing R] [CommRing S]
  [Algebra R S] [Algebra.FormallyUnramified R S] :
  ∃ (S' : Type) (A : CommRing S') (f : S' →a[R] S), (RingHom.ker f) ^ 2 = 0 ∧
  UniversalProperty.liftOfSqZeroIdeal f := by
  sorry

end Problem63

```

**Exercise (64).** Prove that the homogeneous coordinate ring of a smooth rational quartic in three-space

$$R = k[s^4, s^3t, st^3, t^4] \subset k[s, t]$$

is not Cohen-Macaulay.

```

import Mathlib

namespace Problem64

section

```

```

open CategoryTheory Abelian Problem64

variable {R : Type} [CommRing R]

instance : CategoryTheory.HasExt.{0} (ModuleCat.{0} R) :=  

  CategoryTheory.hasExt_of_enoughProjectives (ModuleCat R)

noncomputable def moduleDepth (N M : ModuleCat.{0} R) : N∞ :=  

  sSup {n : N∞ | ∀ i : N, i < n → Subsingleton (CategoryTheory.Abelian.Ext.{0} N M i)}

noncomputable def Ideal.depth (I : Ideal R) (M : ModuleCat.{0} R) : N∞ :=  

  moduleDepth (ModuleCat.of R (R / I)) M

noncomputable def IsLocalRing.depth [IsLocalRing R] (M : ModuleCat.{0} R) : N∞ :=  

  (IsLocalRing.maximalIdeal R).depth M

variable (R)

class IsCohenMacaulayLocalRing : Prop extends IsLocalRing R where
  depth_eq_dim : ringKrullDim R = IsLocalRing.depth (ModuleCat.of R R)

class IsCohenMacaulayRing : Prop where
  CM_localize : ∀ p : Ideal R, ∀ (_ : p.IsPrime), IsCohenMacaulayLocalRing (Localization.AtPrime p)

end

open MvPolynomial

/--
Prove that the homogeneous coordinate ring of a smooth rational quartic in three-space
\[
R=k[s^4, s^3t, st^3, t^4] \subset k[s,t]
\]
is not Cohen-Macaulay.
-/
theorem homogeneous_coordinate_ring_not_isCohenMacaulayRing (k : Type) [Field k] :  

  ¬ IsCohenMacaulayRing (Algebra.adjoin k ({(X 0) ^ 4, (X 0) ^ 3 * X 1,  

    X 0 * (X 1) ^ 3, (X 1) ^ 4} : Set (MvPolynomial (Fin 2) k))) := by
  sorry

end Problem64

```

**Exercise (65).** If  $A$  is a Neotherian Gorenstein ring, then so is the polynomial ring  $A[X]$ .

```

import Mathlib

namespace Problem65

open IsLocalRing ModuleCat CategoryTheory Polynomial

instance (R : Type) [CommRing R] : CategoryTheory.HasExt.{0} (ModuleCat.{0} R) :=  

  CategoryTheory.hasExt_of_enoughProjectives.{0} (ModuleCat.{0} R)

```

```

/--
A Noetherian local ring  $\$R\$$  is a Gorenstein ring if  $\dim_R R < +\infty$ .
 -/
class IsGorensteinLocalRing (R : Type) [CommRing R] : Prop extends
  IsLocalRing R, IsNoetherianRing R where
  injDim_le_infty :
     $\exists n : \mathbb{N}, \forall i : \mathbb{N}, n \leq i \rightarrow$ 
    Subsingleton (Abelian.Ext.{0} (of.{0} R (ResidueField R)) (of.{0} R R) i)

/--
A Noetherian ring is a Gorenstein ring if its localization at every maximal ideal is a
Gorenstein local ring.
 -/
class IsGorensteinRing (R : Type) [CommRing R] : Prop extends IsNoetherianRing R where
  localization_maximal_isGorensteinLocalRing :
     $\forall m : \text{Ideal } R, (\_ : m.\text{IsMaximal}) \rightarrow \text{IsGorensteinLocalRing} (\text{Localization}.AtPrime m)$ 

/--
If  $(A)$  is a Noetherian Gorenstein ring, then so is the polynomial ring  $(A[X])$ .
 -/
theorem Polynomial.isGorensteinRing {R : Type} [CommRing R] [IsGorensteinRing R] :
  IsGorensteinRing R[X] := by
  sorry

end Problem65

```

**Exercise (66).** Show that if an ideal  $I$  in a Noetherian ring  $R$  can be generated by a regular sequence, then it can be generated by a set of elements that is a regular sequence in any order.

```

import Mathlib

namespace Problem66

open RingTheory

/-
Show that if an ideal  $\$I\$$  in a Noetherian ring  $\$R\$$  can be generated by a regular sequence,
then it can be generated by a set of elements that is a regular sequence in any order.
 -/
theorem exists_eq_ofList_and_isRegular_of_perm {R : Type} [CommRing R] [IsNoetherianRing R] (I :
  Ideal R) (rs : List R)
  (gen : I = Ideal.ofList rs) (h2 : Sequence.IsRegular R rs) :  $\exists rs' : \text{List } R,$ 
  I = Ideal.ofList rs'  $\wedge (\forall l : \text{List } R, (l.\text{Perm } rs') \rightarrow \text{Sequence}.IsRegular R l) := \text{by}$ 
  sorry

end Problem66

```

**Exercise (67).** Let  $A$  be the ring  $k[[x_1, \dots, x_n]]$ , where  $k$  is a field,  $n \in \mathbb{N}$ ,  $n \neq 0$ . Show that there is no isomorphism

$$A \otimes_k A \cong k[[x_1, \dots, x_n, y_1, \dots, y_n]].$$

```

import Mathlib

namespace Problem67

open scoped TensorProduct

/--
Let  $A$  be the ring  $k[[x_1, \dots, x_n]]$ , where  $k$  is a field,  $n \in \mathbb{N}$ ,  $n \neq 0$ .
Show that there is an isomorphism

$$A \otimes_k A \cong k[[x_1, \dots, x_n, y_1, \dots, y_n]].$$

 -/
theorem isEmpty_mvPowerSeries_tensor_mvPowerSeries_algEquiv
  {k : Type} [Field k] (n : ℕ) (hn : n ≠ 0) :
  IsEmpty ((MvPowerSeries (Fin n) k) ⊗[k] (MvPowerSeries (Fin n) k)) ≈[k]
  (MvPowerSeries (Fin (n + n)) k) := by
  sorry

end Problem67

```

**Exercise (68).** Let  $A$  be a Noetherian local ring with maximal ideal  $\mathfrak{m}$ . For any  $f \in \mathfrak{m}$  such that  $f$  is not nilpotent,  $A_f$  is Jacobson.

```

import Mathlib

namespace Problem68

/--
Let  $A$  be a Noetherian local ring with maximal ideal  $\mathfrak{m}$ .
For any  $f \in \mathfrak{m}$  such that  $f$  is not nilpotent,  $A_f$  is Jacobson.
 -/
theorem localization_jacobson_of_one_lt_ringKrullDim (R : Type) [CommRing R] [IsLocalRing R]
  [IsNoetherianRing R] (f : R) (hf : f ∈ IsLocalRing.maximalIdeal R) (ne0 : ¬ IsNilpotent f) :
  IsJacobsonRing (Localization.Away f) := by
  sorry

end Problem68

```

**Exercise (69).** If  $R$  is a regular local ring with maximal ideal  $\mathfrak{m}$  and  $P \in \text{Spec}(R[x])$  is a prime ideal with  $\mathfrak{m} = P \cap R$ , then  $R[x]_P$  is regular.

```

import Mathlib

namespace Problem69

open IsLocalRing Polynomial

/-

```

```

A commutative local noetherian ring $R$ is regular if $\dim m/m^2 = \dim R$.
 -/
class IsRegularLocalRing (R : Type) [CommRing R] : Prop extends
  IsLocalRing R, IsNoetherianRing R where
  reg : Module.firrank (ResidueField R) (CotangentSpace R) = ringKrullDim R

/--
Let $(A)$ be a Noetherian ring.
If $(R)$ is a regular local ring with maximal ideal $(\mathfrak{m})$ and
$(P \in \operatorname{Spec}(R[x]))$ is a prime ideal with $(\mathfrak{m} = P \cap R)$,
then $(R[x]_P)$ is regular.
-/
theorem IsRegularLocalRing.regularAtPrime {R : Type} [CommRing R] [IsRegularLocalRing R]
  (P : Ideal R[X]) [P.IsPrime] [P.LiesOver (maximalIdeal R)] :
  IsRegularLocalRing (Localization.AtPrime P) := by
  sorry

end Problem69

```

**Exercise (70).** All rings considered are noetherian. Show that if \$R\$ is an integral domain contained in the local ring \$(S, Q)\$, then there is a minimal prime of \$S\$ contracting to 0 in \$R\$.

```

import Mathlib

namespace Problem70

/-
All rings considered are noetherian.
Show that if $(R)$ is an integral domain contained in the local ring $(S, Q)$,
then there is a minimal prime of $(S)$ contracting to $(0)$ in $(R)$.
-/
theorem exists_minimalPrime_map_zero (R S : Type) [CommRing R] [IsDomain R] [IsNoetherianRing R]
  [CommRing S] [IsNoetherianRing S] [IsLocalRing S] [Algebra R S] [NoZeroSMulDivisors R S] :
  ∃ (p : minimalPrimes S), Ideal.comap (algebraMap R S) p.1 = 0 := by
  sorry

end Problem70

```

**Exercise (71).** Let \$G\$ be a finite group acting as automorphisms of an algebra \$R\$ over a field of characteristic 0. Show that if \$R\$ is Cohen-Macaulay, then the ring of invariants \$R^G\$ is Cohen-Macaulay.

```

import Mathlib

namespace Problem71

section

variable (A B : Type) [CommRing A] [CommRing B] [Algebra A B]

```

```

variable (G : Type) [Monoid G] [MulSemiringAction G B] [SMulCommClass G A B]

/--
The set of fixed points under a group action, as a subring.
 -/
def FixedPoints.subring : Subring B where
  .. := FixedPoints.addSubgroup G B
  .. := FixedPoints.submonoid G B

/--
The set of fixed points under a group action, as a subalgebra.
 -/
def FixedPoints.subalgebra : Subalgebra A B where
  .. := FixedPoints.addSubgroup G B
  .. := FixedPoints.submonoid G B
  algebraMap_mem' r := by simp

end

section

open CategoryTheory Abelian Problem71

variable {R : Type} [CommRing R]

instance : CategoryTheory.HasExt.{0} (ModuleCat.{0} R) :=
  CategoryTheory.hasExt_of_enoughProjectives (ModuleCat R)

noncomputable def moduleDepth (N M : ModuleCat.{0} R) : N∞ :=
  sSup {n : N∞ | ∀ i : N, i < n → Subsingleton (CategoryTheory.Abelian.Ext.{0} N M i)}

noncomputable def Ideal.depth (I : Ideal R) (M : ModuleCat.{0} R) : N∞ :=
  moduleDepth (ModuleCat.of R (R / I)) M

noncomputable def IsLocalRing.depth [IsLocalRing R] (M : ModuleCat.{0} R) : N∞ :=
  (IsLocalRing.maximalIdeal R).depth M

variable (R)

class IsCohenMacaulayLocalRing : Prop extends IsLocalRing R where
  depth_eq_dim : ringKrullDim R = IsLocalRing.depth (ModuleCat.of R R)

class IsCohenMacaulayRing : Prop where
  CM_localize : ∀ p : Ideal R, ∀ (_ : p.IsPrime), IsCohenMacaulayLocalRing (Localization.AtPrime p)

end

/--
Let  $\langle G \rangle$  be a finite group acting as automorphisms of an algebra  $\langle R \rangle$  over a field of characteristic  $\langle 0 \rangle$ . Show that if  $\langle R \rangle$  is Cohen-Macaulay, then the ring of invariants  $\langle R^G \rangle$  is Cohen-Macaulay.
 -/
theorem fixedPoints_isCohenMacaulayRing {R : Type} [CommRing R] (k : Type) [Field k]

```

```

[CharZero k] [Algebra k R] [IsNoetherianRing R] [IsCohenMacaulayRing R]
(G : Subgroup (R ≃ₐ[k] R)) [Finite G] :
  IsCohenMacaulayRing (FixedPoints.subalgebra k R G) := by
  sorry

end Problem71

```

**Exercise (72).** Let  $R$  be a Noetherian ring. Let  $M$  be a Cohen-Macaulay module over  $R$ . Then  $M \otimes_R R[x_1, \dots, x_n]$  is a Cohen-Macaulay module over  $R[x_1, \dots, x_n]$ .

```

import Mathlib

namespace Problem72

/--
The krull dimension of module, defined as `krullDim` of its support.
-/
noncomputable def Module.supportDim (R : Type) [CommRing R] (M : Type) [AddCommGroup M]
  [Module R M] : WithBot N∞ :=
  Order.krullDim (Module.support R M)

section

open CategoryTheory Abelian Problem72

variable {R : Type} [CommRing R]

instance : CategoryTheory.HasExt.{0} (ModuleCat.{0} R) :=
  CategoryTheory.hasExt_of_enoughProjectives (ModuleCat R)

noncomputable def moduleDepth (N M : ModuleCat.{0} R) : N∞ :=
  sSup {n : N∞ | ∀ i : N, i < n → Subsingleton (CategoryTheory.Abelian.Ext.{0} N M i)}

noncomputable def Ideal.depth (I : Ideal R) (M : ModuleCat.{0} R) : N∞ :=
  moduleDepth (ModuleCat.of R (R / I)) M

noncomputable def IsLocalRing.depth [IsLocalRing R] (M : ModuleCat.{0} R) : N∞ :=
  (IsLocalRing.maximalIdeal R).depth M

class ModuleCat.IsCohenMacaulay [IsLocalRing R] (M : ModuleCat.{0} R) : Prop where
  depth_eq_dim : Subsingleton M ∨ Module.supportDim R M = IsLocalRing.depth M

variable (R)

class Module.IsCohenMacaulay (M : Type) [AddCommGroup M] [Module R M] : Prop where
  depth_eq_dim : ∀ p : Ideal R, ∀ (_ : p.IsPrime), (ModuleCat.of (Localization.AtPrime p)
    (LocalizedModule.AtPrime p M)).IsCohenMacaulay

end

open TensorProduct

```

```

/--  

Let  $\langle R \rangle$  be a Noetherian ring. Let  $\langle M \rangle$  be a Cohen-Macaulay module over  $\langle R \rangle$ .  

Then  $\langle M \otimes_R R[x_1, \dots, x_n] \rangle$  is a Cohen-Macaulay module over  $\langle R[x_1, \dots, x_n] \rangle$ .  

-/  

theorem isCohenMacaulay_extendScalars_over_mvPolynomial_of_isCohenMacaulay  

  (R : Type) [CommRing R] (M : Type) [AddCommGroup M] [Module R M]  

  [IsNoetherianRing R] [Module.IsCohenMacaulay R M] (n : ℕ) :  

  Module.IsCohenMacaulay (MvPolynomial (Fin n) R) ((MvPolynomial (Fin n) R) ⊗[R] M) := by  

  sorry  

end Problem72

```

**Exercise (73).** If  $I$  is an homogeneous ideal of  $k[x_0, \dots, x_n]$ ,  $R = k[x_0, \dots, x_n]/I$ , then  $R$  is Cohen-Macaulay if and only if  $R_P$  is Cohen-Macaulay, where  $P = (x_0, \dots, x_n)$ .

```

import Mathlib

namespace Problem73

section

open CategoryTheory Abelian Problem73

variable {R : Type} [CommRing R]

instance : CategoryTheory.HasExt.{0} (ModuleCat.{0} R) :=  

  CategoryTheory.hasExt_of_enoughProjectives (ModuleCat R)

noncomputable def moduleDepth (N M : ModuleCat.{0} R) : N∞ :=  

  sSup {n : N∞ | ∀ i : N, i < n → Subsingleton (CategoryTheory.Abelian.Ext.{0} N M i)}

noncomputable def Ideal.depth (I : Ideal R) (M : ModuleCat.{0} R) : N∞ :=  

  moduleDepth (ModuleCat.of R (R / I)) M

noncomputable def IsLocalRing.depth [IsLocalRing R] (M : ModuleCat.{0} R) : N∞ :=  

  (IsLocalRing.maximalIdeal R).depth M

variable (R)

class IsCohenMacaulayLocalRing : Prop extends IsLocalRing R where
  depth_eq_dim : ringKrullDim R = IsLocalRing.depth (ModuleCat.of R R)

class IsCohenMacaulayRing : Prop where
  CM_localize : ∀ p : Ideal R, ∀ (_ : p.IsPrime), IsCohenMacaulayLocalRing (Localization.AtPrime p)

end

attribute [local instance] MvPolynomial.gradedAlgebra

/--
If  $\$I\$$  is an homogeneous ideal of  $\$k[x_0, \dots, x_n]\$, \langle R = k[x_0, \dots, x_n]/I \rangle$ ,  

then  $\langle R \rangle$  is Cohen-Macaulay if and only if  $\langle R_P \rangle$  is Cohen-Macaulay, where

```

```

\(
P = (x_0, \dots, x_n) \).
 -/
theorem mvPolynomial_quotient_isCohenMacaulayRing_iff (k : Type) [Field k] (n : ℕ)
  (R : Type) [CommRing R] (f : (MvPolynomial (Fin n) k) →+* R) (surj : Function.Surjective f)
  (homo : (RingHom.ker f).IsHomogeneous (MvPolynomial.homogeneousSubmodule (Fin n) k))
  (le : RingHom.ker f ≤ RingHom.ker MvPolynomial.constantCoeff) :
  IsCohenMacaulayRing R ↔
  IsCohenMacaulayRing (Localization.AtPrime ((RingHom.ker MvPolynomial.constantCoeff).map f))
    (hp := Ideal.map_isPrime_of_surjective surj le (H := RingHom.ker_isPrime _)) := by
  sorry
end Problem73

```

**Exercise (74).** Let  $R$  be a regular local ring and let  $x_1, \dots, x_c$  be a regular sequence in  $R$ . Let  $y \in R$ ,  $y \notin (x_1, \dots, x_c)$ , and set  $J := ((x_1, \dots, x_c) : y)$ . Prove that  $R/J$  is Gorenstein.

```

import Mathlib

namespace Problem74

open IsLocalRing ModuleCat CategoryTheory

instance (R : Type) [CommRing R] : CategoryTheory.HasExt.{0} (ModuleCat.{0} R) :=
  CategoryTheory.hasExt_of_enoughProjectives.{0} (ModuleCat.{0} R)

/-
A commutative local noetherian ring  $\$R\$$  is regular if  $\dim m/m^2 = \dim R$ .
-/
class IsRegularLocalRing (R : Type) [CommRing R] : Prop extends
  IsLocalRing R, IsNoetherianRing R where
  reg : Module.firrank (ResidueField R) (CotangentSpace R) = ringKrullDim R

/-
A Noetherian local ring  $\$R\$$  is a Gorenstein ring if  $\dim_R R < +\infty$ .
-/
class IsGorensteinLocalRing (R : Type) [CommRing R] : Prop extends
  IsLocalRing R, IsNoetherianRing R where
  injDim_le_infty :
    ∃ n : ℕ, ∀ i : ℕ, n ≤ i →
    Subsingleton (Abelian.Ext.{0} (of.{0} R (ResidueField R)) (of.{0} R R) i)

/-
A Noetherian ring is a Gorenstein ring if its localization at every maximal ideal is a
Gorenstein local ring.
-/
class IsGorensteinRing (R : Type) [CommRing R] : Prop extends IsNoetherianRing R where
  localization_maximal_isGorensteinLocalRing :
    ∀ m : Ideal R, (_ : m.IsMaximal) → IsGorensteinLocalRing (Localization.AtPrime m)
  variable {R : Type} [CommRing R]

/-
Let  $\$R\$$  be a regular local ring and let  $\$x_1, \dots, x_c\$$  be a regular sequence in  $\$R\$$ .

```

```

Let $y \in R$, $y \notin \{x_1, \dots, x_c\}$, and set $J := ((x_1, \dots, x_c) : y)$. Prove that $R/J$ is Gorenstein.
 -/
theorem IsRegularLocalRing.gorensteinAtRegularSequence {R : Type} [CommRing R]
  [IsRegularLocalRing R] {rs : List R} (reg : RingTheory.Sequence.IsRegular R rs) (y : R)
  (h : y ∉ Ideal.ofList rs) : IsGorensteinRing (R / (Ideal.ofList rs / Ideal.span {y})) := by
  sorry
end Problem74

```

**Exercise (75).** Let  $A$  be a graded Noetherian ring, with  $A_0$  a field and  $A$  generated by  $A_1$ . Show that  $A$  is Cohen-Macaulay if and only if for all homogeneously prime  $\mathfrak{p}$ ,  $(A_{\mathfrak{p}})_0$  is Cohen-Macaulay.

```

import Mathlib

namespace Problem75

open IsLocalRing ModuleCat CategoryTheory Problem75

section

variable {R : Type} [CommRing R]

instance : CategoryTheory.HasExt.{0} (ModuleCat.{0} R) :=
  CategoryTheory.hasExt_of_enoughProjectives (ModuleCat R)

noncomputable def moduleDepth (N M : ModuleCat.{0} R) : N∞ :=
  sSup {n : N∞ | ∀ i : N, i < n → Subsingleton (CategoryTheory.Abelian.Ext.{0} N M i)}

noncomputable def Ideal.depth (I : Ideal R) (M : ModuleCat.{0} R) : N∞ :=
  moduleDepth (ModuleCat.of R (R / I)) M

noncomputable def IsLocalRing.depth [IsLocalRing R] (M : ModuleCat.{0} R) : N∞ :=
  (IsLocalRing.maximalIdeal R).depth M

variable (R)

class IsCohenMacaulayLocalRing : Prop extends IsLocalRing R where
  depth_eq_dim : ringKrullDim R = IsLocalRing.depth (ModuleCat.of R R)

class IsCohenMacaulayRing : Prop where
  CM_localize : ∀ p : Ideal R, ∀ (l : p.IsPrime), IsCohenMacaulayLocalRing (Localization.AtPrime p)

end

/--
Let $A$ be a graded Noetherian ring, with $A_0$ a field and $A$ generated by $A_1$. Show that $A$ is Cohen-Macaulay if and only if for all homogeneously prime $\mathfrak{p}$, $(A_{\mathfrak{p}})_0$ is Cohen-Macaulay.
 -/
theorem gradedAlgebra_isCohenMacaulay_iff_homogeneously_localize {A : Type} [CommRing A]
  [IsNoetherianRing A]

```

```


$$(\text{r} : \mathbb{N} \rightarrow \text{Submodule } \mathbb{Z} A) [\text{GradedAlgebra } \text{r}] (h : \text{IsField } (\text{r } 0)) (h1 : \text{Algebra.adjoin } (\text{r } 0) (\text{r } 1) = (\tau : \text{Subalgebra } (\text{r } 0) A)) : \text{IsCohenMacaulayRing } A \leftrightarrow$$


$$\forall p : \text{Ideal } A, (- : p.\text{IsPrime}) \rightarrow p.\text{IsHomogeneous } \text{r} \rightarrow$$


$$\text{IsCohenMacaulayLocalRing } (\text{HomogeneousLocalization.AtPrime } \text{r } p) := \text{by}$$


$$\text{sorry}$$


```

**end Problem75**

**Exercise (76).** Let  $A$  be a Noetherian UFD of dimension  $d \leq 3$ . Prove that  $A$  is catenary.

```

import Mathlib

namespace Problem76

open List

/-
A ring  $\$R\$$  is said to be  $\text{catenary}$  if for any pair of prime ideals  $\mathfrak{p} \subsetneq \mathfrak{q}$ , there exists an integer bounding the lengths of all finite chains of prime ideals  $\mathfrak{p} = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \dots \subsetneq \mathfrak{p}_e = \mathfrak{q}$  and all maximal such chains have the same length.
-/
def IsCatenary (R : Type) [CommRing R] : Prop :=

$$\forall p q : \text{PrimeSpectrum } R, p \leq q \rightarrow$$


$$\exists n : \mathbb{N}, \forall (l : \text{LTSeries } (\text{PrimeSpectrum } R)), l.\text{head} = p \rightarrow l.\text{last} = q \rightarrow$$


$$(\forall l' : \text{LTSeries } (\text{PrimeSpectrum } R), l'.\text{head} = p \rightarrow l'.\text{last} = q \rightarrow l.\text{toList} <+ l'.\text{toList} \rightarrow l' = l) \rightarrow$$


$$l.\text{toList.length} = n$$


/-
Let  $\$A\$$  be a Noetherian UFD of dimension  $d \leq 3$ . Prove that  $\$A\$$  is catenary.
-/
theorem IsCatenary.of_noetherian_ufd_of_dim_le_three {A : Type} [CommRing A] [IsNoetherianRing A]
  [IsDomain A] [UniqueFactorizationMonoid A] (h : ringKrullDim A  $\leq 3$ ) : IsCatenary A := by
  sorry

end Problem76

```

**Exercise (77).** Let  $A$  be a Noetherian ring,  $P \subset Q$  prime ideals such that  $\text{ht } P = h$ ,  $\text{ht } Q/P = d$ , where  $d > 1$ . Prove that there exist infinitely many intermediate primes  $P'$ ,  $P \subset P' \subset Q$  such that  $\text{ht } P' = h + 1$  and  $\text{ht } Q/P' = d - 1$ .

```

import Mathlib

namespace Problem77

/-
Let  $\$A\$$  be a Noetherian ring,  $\$P \subset Q\$$  prime ideals such that
 $\text{ht } P = h$ ,  $\text{ht } Q/P = d$ , where  $d > 1$ .
Prove that there exist infinitely many intermediate primes  $\$P' \$,  $P \subset P' \subset Q\$$$ 
```

```

such that  $\operatorname{ht} P' = h + 1$  and  $\operatorname{ht} Q/P' = d - 1$ .
 -/
theorem infinite_intermediate_primes (R : Type) [CommRing R] [IsNoetherianRing R] (P Q : Ideal R)
  (le : P ≤ Q) [P.IsPrime] [Q.IsPrime] (h d : ℕ) (lt : 1 < d) (ht1 : P.height = h)
  (ht2 : (Q.map (Ideal.quotient.mk P)).height = d) :
  {P' : Ideal R | P ≤ P' ∧ P' ≤ Q ∧ P'.IsPrime ∧ P'.height = h + 1 ∧
    (Q.map (Ideal.quotient.mk P')).height = d - 1}.Infinite := by
  sorry

end Problem77

```

**Exercise (78).** Let  $A$  be a local Cohen–Macaulay (CM) ring that is a quotient of a regular local ring. If  $A$  is a UFD, then  $A$  is Gorenstein.

```

import Mathlib

namespace Problem78

open IsLocalRing ModuleCat CategoryTheory Problem78

section

variable {R : Type} [CommRing R]

instance : CategoryTheory.HasExt.{0} (ModuleCat.{0} R) :=
  CategoryTheory.hasExt_of_enoughProjectives (ModuleCat R)

noncomputable def moduleDepth (N M : ModuleCat.{0} R) : ℕ∞ :=
  sSup {n : ℕ∞ | ∀ i : ℕ, i < n → Subsingleton (CategoryTheory.Abelian.Ext.{0} N M i)}

noncomputable def Ideal.depth (I : Ideal R) (M : ModuleCat.{0} R) : ℕ∞ :=
  moduleDepth (ModuleCat.of R (R / I)) M

noncomputable def IsLocalRing.depth [IsLocalRing R] (M : ModuleCat.{0} R) : ℕ∞ :=
  (IsLocalRing.maximalIdeal R).depth M

variable (R)

class IsCohenMacaulayLocalRing : Prop extends IsLocalRing R where
  depth_eq_dim : ringKrullDim R = IsLocalRing.depth (ModuleCat.of R R)

class IsCohenMacaulayRing : Prop where
  CM_localize : ∀ p : Ideal R, ∀ (_ : p.IsPrime), IsCohenMacaulayLocalRing (Localization.AtPrime p)

end

/--
A commutative local noetherian ring  $R$  is regular if  $\dim \mathfrak{m}/\mathfrak{m}^2 = \dim R$ .
 -/
class IsRegularLocalRing (R : Type) [CommRing R] : Prop extends
  IsLocalRing R, IsNoetherianRing R where
  reg : Module.firrank (ResidueField R) (CotangentSpace R) = ringKrullDim R

```

```

/--
A Noetherian local ring  $\$R\$$  is a Gorenstein ring if  $\dim_R R < +\infty$ .
 -/
class IsGorensteinLocalRing (R : Type) [CommRing R] : Prop extends
  IsLocalRing R, IsNoetherianRing R where
  injDim_le_infty :
     $\exists n : \mathbb{N}, \forall i : \mathbb{N}, n \leq i \rightarrow$ 
    Subsingleton (Abelian.Ext.{0} (of.{0} R (ResidueField R)) (of.{0} R R) i)

/--
A Noetherian ring is a Gorenstein ring if its localization at every maximal ideal is a
Gorenstein local ring.
 -/
class IsGorensteinRing (R : Type) [CommRing R] : Prop extends IsNoetherianRing R where
  localization_maximal_isGorensteinLocalRing :
     $\forall m : \text{Ideal } R, (\_ : m.\text{IsMaximal}) \rightarrow \text{IsGorensteinLocalRing}(\text{Localization.AtPrime } m)$ 

/--
Let  $\$A\$$  be a local Cohen–Macaulay (CM) ring that is a quotient of a regular local ring.
If  $\$A\$$  is a UFD, then  $\$A\$$  is Gorenstein.
 -/
theorem IsCohenMacaulayLocalRing.isGorensteinRing_of_ufd {A B : Type} [CommRing A]
  [IsCohenMacaulayLocalRing A] [IsDomain A] [UniqueFactorizationMonoid A] [CommRing B]
  [IsRegularLocalRing B] {f : B  $\rightarrow^{**} A$ } (hf : Function.Surjective f) :
  IsGorensteinRing A := by
  sorry

end Problem78

```

**Exercise (79).** Let  $B$  be a regular local ring and  $I \subset B$  an ideal such that  $B/I$  is Gorenstein but not a complete intersection. Show that  $I$  cannot have height 0 or 1.

```

import Mathlib

namespace Problem79

open IsLocalRing ModuleCat CategoryTheory

instance (R : Type) [CommRing R] : CategoryTheory.HasExt.{0} (ModuleCat.{0} R) :=
  CategoryTheory.hasExt_of_enoughProjectives.{0} (ModuleCat.{0} R)

/--
A commutative local noetherian ring  $\$R\$$  is regular if  $\dim m/m^2 = \dim R$ .
 -/
class IsRegularLocalRing (R : Type) [CommRing R] : Prop extends
  IsLocalRing R, IsNoetherianRing R where
  reg : Module.firrank (ResidueField R) (CotangentSpace R) = ringKrullDim R

/--
A Noetherian local ring  $\$R\$$  is a Gorenstein ring if  $\dim_R R < +\infty$ .
 -/

```

```

class IsGorensteinLocalRing (R : Type) [CommRing R] : Prop extends
  IsLocalRing R, IsNoetherianRing R where
  injDim_le_infty :
    ∃ n : N, ∀ i : N, n ≤ i →
    Subsingleton (Abelian.Ext.{0} (of.{0} R (ResidueField R)) (of.{0} R R) i)

/--
A Noetherian ring is a Gorenstein ring if its localization at every maximal ideal is a
Gorenstein local ring.
 -/
class IsGorensteinRing (R : Type) [CommRing R] : Prop extends IsNoetherianRing R where
  localization_maximal_isGorensteinLocalRing :
    ∀ m : Ideal R, (_ : m.IsMaximal) → IsGorensteinLocalRing (Localization.AtPrime m)

/--
A Noetherian local ring $A$ is a local complete intersection if every surjection of local rings
$R \rightarrow \widehat{A}$ with $R$ a regular local ring, the kernel of $R \rightarrow \widehat{A}$ is generated by a
regular sequence.
 -/
@[stacks 09Q3]
class IsLocalCompleteIntersectionRing (A : Type) [CommRing A] : Prop extends
  IsLocalRing A, IsNoetherianRing A where
  out (R : Type) [CommRing R] [IsRegularLocalRing R]
    (f : R →+* (AdicCompletion (maximalIdeal A) A)) (_ : IsLocalHom f) (_ : Function.Surjective f) :
      ∃ (rs : List R), RingTheory.Sequence.IsRegular R rs ∧ RingHom.ker f = Ideal.ofList rs

/--
Let $B$ be a regular local ring and $I \subset B$ an ideal such that
$B/I$ is Gorenstein but not a local complete intersection.
Show that $I$ cannot have height $0$ or $1$.
 -/
theorem IsLocalRing.not_isCompleteIntersection.height_not_zero_and_not_one (B : Type) [CommRing B]
  [IsRegularLocalRing B] (I : Ideal B) [IsGorensteinRing (B / I)]
  (hc : ¬ IsLocalCompleteIntersectionRing (B / I)) : I.height ≠ 0 ∧ I.height ≠ 1 := by
  sorry

end Problem79

```

**Exercise (80).** Consider the ideal  $I \subset k[x_1, \dots, x_6]$  generated by the following polynomials:

$$\begin{aligned}
f_1 &= x_2x_4 + x_3x_6, \\
f_2 &= x_3x_5 + x_1x_6, \\
f_3 &= x_1x_2 - x_2x_5 + x_3x_5 - x_5x_6, \\
f_4 &= x_2x_3 + x_2x_4 + x_2x_6 + x_6^2, \\
f_5 &= x_3^2 + x_3x_4 + x_3x_6 - x_4x_6, \\
f_6 &= x_1x_3 + x_1x_4 + x_4x_5 + x_1x_6.
\end{aligned}$$

Prove that  $R/I$  is Cohen–Macaulay of dimension 3.

```

import Mathlib

namespace Problem80

section

open CategoryTheory Abelian Problem80

variable {R : Type} [CommRing R]

instance : CategoryTheory.HasExt.{0} (ModuleCat.{0} R) :=
  CategoryTheory.hasExt_of_enoughProjectives (ModuleCat R)

noncomputable def moduleDepth (N M : ModuleCat.{0} R) : N∞ :=
  sSup {n : N∞ | ∀ i : N, i < n → Subsingleton (CategoryTheory.Abelian.Ext.{0} N M i)}

noncomputable def Ideal.depth (I : Ideal R) (M : ModuleCat.{0} R) : N∞ :=
  moduleDepth (ModuleCat.of R (R / I)) M

noncomputable def IsLocalRing.depth [IsLocalRing R] (M : ModuleCat.{0} R) : N∞ :=
  (IsLocalRing.maximalIdeal R).depth M

variable (R)

class IsCohenMacaulayLocalRing : Prop extends IsLocalRing R where
  depth_eq_dim : ringKrullDim R = IsLocalRing.depth (ModuleCat.of R R)

class IsCohenMacaulayRing : Prop where
  CM_localize : ∀ p : Ideal R, ∀ (_ : p.IsPrime), IsCohenMacaulayLocalRing (Localization.AtPrime p)

end

open MvPolynomial

abbrev target_ring_aux (k : Type) [Field k] :=
  (MvPolynomial (Fin 6) k) / Ideal.span ({ X 1 * X 3 + X 2 * X 5, X 2 * X 4 + X 0 * X 5, X 0 * X 1 - X 1 * X 4 + X 2 * X 4 - X 4 * X 5,
    X 1 * X 2 + X 1 * X 3 + X 1 * X 5 + (X 5)^2, (X 2)^2 + X 2 * X 3 + X 2 * X 5 - X 3 * X 5,
    X 0 * X 2 + X 0 * X 3 + X 3 * X 4 + X 0 * X 5 } : Set (MvPolynomial (Fin 6) k))

/--
Consider the ideal \(\langle I \subset k[x_1, \dots, x_6] \rangle\) generated by the following polynomials:
\[
\begin{aligned}
f_1 &= x_2x_4 + x_3x_6, \\
f_2 &= x_3x_5 + x_1x_6, \\
f_3 &= x_1x_2 - x_2x_5 + x_3x_5 - x_5x_6, \\
f_4 &= x_2x_3 + x_2x_4 + x_2x_6 + x_6^2, \\
f_5 &= x_3^2 + x_3x_4 + x_3x_6 - x_4x_6, \\
f_6 &= x_1x_3 + x_1x_4 + x_4x_5 + x_1x_6.
\end{aligned}
\]
\]
```

```

Prove that  $\langle R/I \rangle$  is Cohen–Macaulay of dimension  $\langle 3 \rangle$ .
 -/
theorem isCohenMacaulayRing_of_dimension_three (k : Type) [Field k] :
  IsCohenMacaulayRing (target_ring_aux k) ∧ (ringKrullDim (target_ring_aux k) = 3) := by
  sorry

end Problem80

```

**Exercise (81).** Let  $A$  be a local Noetherian ring,  $I \subset A$  an ideal. Show that  $I$  is generated by a regular sequence if and only if  $I/I^2$  is free over  $A/I$  and  $\text{pd}_A I < \infty$ .

```

import Mathlib

namespace Problem81

/--
Let  $\langle A \rangle$  be a local Noetherian ring,  $\langle I \subset A \rangle$  an ideal. Show that
 $\langle I \rangle$  is generated by a regular sequence if and only if  $\langle I/I^2 \rangle$  is free over  $\langle A/I \rangle$  and
 $\text{pd}_A I < \infty$ .
 -/
theorem generated_by_regular_sequence_iff (R : Type) [CommRing R] [IsLocalRing R]
  [IsNoetherianRing R] (I : Ideal R) (netop : I ≠ 0) :
  ∃ (rs : List R), (RingTheory.Sequence.IsRegular R rs) ∧ Ideal.ofList rs = I ↔
  Module.Free (R / I) I.Cotangent ∧
  (∃ n, CategoryTheory.HasProjectiveDimensionLE (ModuleCat.of R I) n) := by
  sorry

end Problem81

```

**Exercise (82).** Let  $A$  be a Noetherian complete local ring of dimension  $d$ , of mixed characteristic (i.e.,  $\text{Char}A = 0$  and  $\text{Char}A/\mathfrak{m}$ ), and let  $p = \text{char}(A/\mathfrak{m})$ . Assume that  $\text{ht}(p \cdot A) = 1$ . Prove that  $A$  is a finitely generated module over a subring  $B \subset A$  such that

$$B \cong C[[x_1, \dots, x_{d-1}]],$$

where  $C$  is a discrete valuation ring (DVR).

```

import Mathlib

namespace Problem82

open IsLocalRing

/--
Let  $\langle A \rangle$  be a Noetherian complete local ring of dimension  $\langle d \rangle$ , of mixed characteristic
(i.e.,  $\text{Char}A = 0$  and  $\text{Char}A / \mathfrak{m}$ ), and let
 $\langle p = \text{char}(A/\mathfrak{m}) \rangle$ . Assume that  $\langle \text{ht}(p \cdot A) = 1 \rangle$ .
Prove that  $\langle A \rangle$  is a finitely generated module over a subring  $\langle B \subset A \rangle$  such that

```

```

\[
B \cong C[[x_1, \dots, x_{d-1}]],
\]
where  $(C)$  is a discrete valuation ring (DVR).
 -/
theorem subring_iso_mvPowerSeries_over_DVR (d : ℕ) (A : Type) [CommRing A] [IsLocalRing A]
[IsNoetherianRing A] [IsAdicComplete (maximalIdeal A) A] (dim : ringKrullDim A = d)
{p : ℕ} (hp : p.Prime) [CharZero A] [CharP (ResidueField A) p]
(ht : (Ideal.span {p}).height = 1) :
 $\exists B : \text{Subring } A, \text{Module.Finite } B A \wedge$ 
 $\exists (C : Type) (\_ : \text{CommRing } C) (\_ : \text{IsDomain } C), \text{IsDiscreteValuationRing } C \wedge$ 
Nonempty (B  $\simeq^{**}$  MvPowerSeries (Fin (d - 1)) C) := by
sorry

end Problem82

```

**Exercise (83).** Let  $f: A \rightarrow B$  be a flat local homomorphism of Noetherian rings, having maximal ideals  $\mathfrak{m}_A$  and  $\mathfrak{m}_B$  respectively. Prove that if  $A$  and  $B/\mathfrak{m}_A B$  are regular, then  $B$  is regular.

```

import Mathlib

namespace Problem83

open IsLocalRing

/-
A commutative local noetherian ring  $R$  is regular if  $\dim m/m^2 = \dim R$ .
-/
class IsRegularLocalRing (R : Type) [CommRing R] : Prop extends
  IsLocalRing R, IsNoetherianRing R where
  reg : Module.firrank (ResidueField R) (CotangentSpace R) = ringKrullDim R

/-
Let  $f: A \rightarrow B$  be a flat local homomorphism of Noetherian rings,
having maximal ideals  $\mathfrak{m}_A$  and  $\mathfrak{m}_B$  respectively.
Prove that if  $A$  and  $B/\mathfrak{m}_A B$  are regular, then  $B$  is regular.
-/
theorem IsRegularLocalRing.flat_local_of_regular {A B : Type} [CommRing A] [CommRing B]
  [IsRegularLocalRing A] [IsNoetherianRing B] [IsLocalRing B] {f : A  $\rightarrow^{**}$  B} (hfl : IsLocalHom f)
  (hff : f.Flat) [IsRegularLocalRing (B / (maximalIdeal A).map f)] : IsRegularLocalRing B := by
sorry

end Problem83

```

**Exercise (84).** For a projective module  $M$  over a commutative ring  $R$ , there exists a free  $R$ -module  $N$ , such that  $M \oplus N$  is free.

```
import Mathlib
```

```

namespace Problem84

/--
For a projective module  $(M)$  over a commutative ring  $(R)$ ,
there exists a free  $(R)$ -module  $(N)$ , such that  $(M \oplus N)$  is free.
 -/
theorem exists_directSum_free_free_of_projective (R M : Type) [CommRing R] [AddCommGroup M]
    [Module R M] [Module.Projective R M] : ∃ (N : Type) (_ : AddCommGroup N) (_ : Module R N),
    Module.Free R N ∧ Module.Free R (N × M) := by
  sorry

end Problem84

```

**Exercise (85).** *There exists a transfinite Euclidean domain such that it cannot be given a Euclidean norm taking value in  $\mathbb{N}$ .*

```

import Mathlib

namespace Problem85

/--
Definition of a Euclidean norm taking value in  $(\mathbb{N})$ .
 -/
class EuclideanNormNat (R : Type) [CommRing R] extends Nontrivial R where
  quotient : R → R → R
  quotient_zero : ∀ a, quotient a 0 = 0
  remainder : R → R → R
  quotient_mul_remainder_eq : ∀ a b, b * quotient a b + remainder a b = a
  norm : R → ℕ
  remainder_lt : ∀ (a) {b}, b ≠ 0 → norm (remainder a b) < norm b
  mul_left_not_lt : ∀ (a) {b}, b ≠ 0 → ¬ norm (a * b) < norm a

/--
There exists a transfinite Euclidean domain such that it cannot be given a Euclidean norm taking
value in  $(\mathbb{N})$ .
 -/
theorem exist_euclideanDomain_not_norm_nat :
    ∃ (R : Type) (_ : EuclideanDomain R), IsEmpty (EuclideanNormNat R) := by
  sorry

end Problem85

```

**Exercise (86).** *For a commutative ring  $A$ ,  $\dim A[x, y] + \dim A \leq 2 * \dim A[x]$ .*

```

import Mathlib

namespace Problem86

/--
For a commutative ring  $(A)$ ,  $(\dim A[x, y] + \dim A \leq 2 * \dim A[x])$ .

```

```

-/
theorem dimension_convex (A : Type) [CommRing A] :
  ringKrullDim (MvPolynomial (Fin 2) A) + ringKrullDim A ≤ 2 * ringKrullDim (Polynomial A) := by
  sorry

end Problem86

```

**Exercise (87).** There exists two commutative rings  $R, S$ , such that  $R[x]$  is isomorphic to  $S[x]$  but  $R$  is not isomorphic to  $S$ .

```

import Mathlib

namespace Problem87

/-
There exists two commutative rings  $\langle R, S \rangle$ , such that  $\langle R[x] \rangle$  is isomorphic to  $\langle S[x] \rangle$  but  $\langle R \rangle$  is not isomorphic to  $\langle S \rangle$ .
-/
theorem exists_polynomial_ringEquiv_isEmpty_ringEquiv :
  ∃ (R S : Type) (_ : CommRing R) (_ : CommRing S),
  Nonempty ((Polynomial R) ≃+* (Polynomial S)) ∧ IsEmpty (R ≃+* S) := by
  sorry

end Problem87

```

**Exercise (88).**  $\mathbb{C}[x, y, z]/(x^2 + y^3 + z^7)$  is a UFD.

```

import Mathlib

namespace Problem88

/-
The ring $R = \mathbb{C}[x, y, z] / (x^2 + y^3 + z^7)$.
-/
abbrev R : Type := (MvPolynomial (Fin 3) ℂ) / Ideal.span {(.X 0 ^ 2 + .X 1 ^ 3 + .X 2 ^ 7 :
  MvPolynomial (Fin 3) ℂ)}

/-
$ \mathbb{C}[x, y, z] / (x^2 + y^3 + z^7)$ is a UFD.
-/
theorem quotient_not_UFD :
  ∃ (h : IsDomain R),
  (UniqueFactorizationMonoid R) := by
  sorry

end Problem88

```

**Exercise (89).** Prove that if  $\#G = 336$  then  $G$  is not simple.

```

import Mathlib

namespace Problem89

/--
Prove that if  $\#G = 336$  then  $G$  is not simple.
 -/
theorem not_isSimpleGroup_of_card_eq_336 (G : Type) [Group G]
  [Finite G] (h_card : Nat.card G = 336) : ¬ IsSimpleGroup G := by
  sorry

end Problem89

```

**Exercise (90).** Given a field  $k$ , there exists some  $n > 0$ , there exists some subfield  $K \subseteq k(x_1, \dots, x_n)$ , such that  $K \cap k[X_1, \dots, x_n]$  is not a finitely generated  $k$ -algebra.

```

import Mathlib

namespace Problem90

/--
Given a field  $k$ , there exists some  $n > 0$ , there exists some subfield
 $K \subsetneq k(x_1, \dots, x_n)$ , such that  $K \cap k[X_1, \dots, x_n]$  is not a finitely
generated  $k$ -algebra.
 -/
theorem not_finiteType_inf_algebraMap_range (k : Type) [Field k] :
  ∃ (n : ℕ) (K : IntermediateField k (FractionRing (MvPolynomial (Fin n) k))),
    ¬ Algebra.FiniteType k (K.toSubalgebra n (Algebra.algHom k (MvPolynomial (Fin n) k)
      (FractionRing (MvPolynomial (Fin n) k))).range :
      Subalgebra k (FractionRing (MvPolynomial (Fin n) k))) := by
  sorry

end Problem90

```

**Exercise (91).** Let  $k$  be a field,  $A := k[x, y]/(xy(x + y - 1))$ , then  $\text{Pic } A \cong k^\times$ .

```

import Mathlib

namespace Problem91

open CategoryTheory MvPolynomial

/--
The Picard group of a commutative ring  $R$  consists of the invertible  $R$ -modules,
up to isomorphism.
 -/
abbrev CommRing.Pic (R : Type) [CommRing R] : Type 1 := (Skeleton <| ModuleCat.{0} R)ˣ
/-

```

```

Let $ k $ be a field, $ A := k[x, y]/(xy(x + y - 1)) $, then $ \mathrm{Pic}(A) \cong k^{\times} $.  

-/  

theorem pic_three_lines {k : Type} [Field k] : Nonempty <  

  CommRing.Pic (MvPolynomial (Fin 2) k / Ideal.span {((x 0) * (x 1) * (x 0 + x 1 - 1)) :  

    Set (MvPolynomial (Fin 2) k)}) ≈ kx := by  

  sorry  

end Problem91

```

**Exercise (92).** Let  $A$  be a commutative ring with identity,  $\dim A = 1$ . Then all possible sequences for  $a_n = \dim A[x_1, \dots, x_n]$  ( $n \in \mathbb{N}$ ) are exactly the sequences of the form:  $a_n = 2n + 1$  if  $n \leq k$  else  $a_n = n + k + 1$ , for some  $k \in \mathbb{N} \cup \{+\infty\}$ .

```

import Mathlib

namespace Problem92

/--
\(\text{a}_n = 2n+1\) if \(n \leq k\) else \(\text{a}_n = n + k + 1\), for some \(k \in \mathbb{N} \cup \{+\infty\}\).
-/
def a (k : N $\omega$ ) (n : N) :=
  if h : n ≤ k then 2 * n + 1
  else n + WithTop.untop k (by rintro rfl; exact h le_top) + 1

/--
Let $ A $ be a commutative ring with identity, $\dim A = 1$.
Then all possible sequences for $(a_n = \dim A[x_1, \dots, x_n] \mid n \in \mathbb{N})$ are exactly
the sequences of the form: $(a_n = 2n+1)$ if $(n \leq k)$ else $(a_n = n + k + 1)$, for some
$(k \in \mathbb{N} \cup \{+\infty\})$.
-/
theorem dimension_sequences_of_one_dimensional_rings :
  (V (A : Type) [CommRing A] (h : ringKrullDim A = 1),
   ∃ (k : N $\omega$ ), (V (n : N), ringKrullDim (MvPolynomial (Fin n) A) = a k n)) ∧
  (V (k : N), ∃ (A : Type) (_ : CommRing A) (h : ringKrullDim A = 1),
   (V (n : N), ringKrullDim (MvPolynomial (Fin n) A) = a k n)) := by
  sorry

end Problem92

```

**Exercise (93).** There exists a field  $k$  and a (not necessarily commutative) ring  $A$  such that  $A$  is integral and finitely generated over  $k$  but  $\dim_k A$  is not finite.

```

import Mathlib

namespace Problem93

/--
There exists a field $k$ and a (not necessarily commutative) ring $A$
such that $A$ is integral and finitely generated over $k$ but $\dim_k A$ is not finite.

```

```

-/
theorem exists_integral_finiteType_not_finiteDimensional : ∃ (k : Type) (A : Field k)
  (R : Ring A) (A : Algebra k A),
  Algebra.IsIntegral k A ∧ Algebra.FiniteType k A ∧ ¬ FiniteDimensional k A := by
  sorry

end Problem93

```

**Exercise (94).** Let  $k$  be field,  $\text{char } k = 0$ ,  $A$  be a finite-type  $k$ -algebra,  $f : A \rightarrow A$  be an étale endomorphism,  $\varphi : A \rightarrow k$ ,  $I \subset A$  be a ideal. If  $A$  is a domain, then

$$\{n \in \mathbb{N} \mid \varphi \circ f^n|_I = 0\}$$

is either finite or contains an arithmetic progression with a positive common difference.

```

import Mathlib

namespace Problem94

variable {k A : Type} [Field k] [CharZero k] [CommRing A] [IsDomain A] [Algebra k A]
[Algebra.FiniteType k A] (f : A →ₐ[k] A) (φ : A →ₐ[k] k) (I : Ideal A)

/-
The set  $\{n \in \mathbb{N} \mid \varphi \circ f^n|_I = 0\}$ .
-/
def zeroSet : Set N := {n | ∀ x : I, (φ.comp (f ^ n)) (x : A) = 0}

/-
Let  $k$  be field,  $A$  be a finite-type  $k$ -algebra,  $f : A \rightarrow A$  be an étale endomorphism,  $\varphi : A \rightarrow k$ ,  $I \subset A$  be a ideal. If  $A$  is a domain, then  $\{n \in \mathbb{N} \mid \varphi \circ f^n|_I = 0\}$  is either finite or contains an arithmetic progression with a positive common difference.
-/
theorem zeroSet_finite_or_contain_arithmetic_progression (hf : f.FormallyEtale) :
  (zeroSet f φ I).Finite ∨ ∃ (d : N+) (a : N), ∀ n : N, a + d * n ∈ zeroSet f φ I := by
  sorry

end Problem94

```

**Exercise (95).** Let  $f : \mathbb{C}[x, y] \rightarrow \mathbb{C}[x, y]$ ,  $x \mapsto p(x) + ay, y \mapsto x$ , where  $a \in \mathbb{C}$ ,  $a \neq 0$ ,  $p(x) \in \mathbb{C}[x]$  have degree  $> 1$ ,  $\mathfrak{p} \subset \mathbb{C}[x, y]$  be a prime ideal. If height  $\mathfrak{p} = 1$ , then  $f(\mathfrak{p}) \neq \mathfrak{p}$ .

```

import Mathlib

namespace Problem95

open Polynomial Bivariate

/-

```

```

Let $f : \mathbb{C}[x, y] \rightarrow \mathbb{C}[x, y]$, $x \mapsto p(x) + ay$, $y \mapsto x$,
where $a \in \mathbb{C}$, $p(x) \in \mathbb{C}[x]$.
 -/
noncomputable
def f (a : ℂ) (p : ℂ[X]): ℂ[X][Y] →+* ℂ[X][Y] :=
  eval₂RingHom (aeval (a • Y + C p)).toRingHom (C X)

/--
Let $f : \mathbb{C}[x, y] \rightarrow \mathbb{C}[x, y]$, $x \mapsto p(x) + ay$, $y \mapsto x$,
where $a \in \mathbb{C}$, $a \neq 0$, $p(x) \in \mathbb{C}[x]$ have degree $>1$, $\mathfrak{p} \subset \mathbb{C}[x, y]$ be a prime ideal. If $\mathrm{height}\ \mathfrak{p} = 1$, then
$f(\mathfrak{p}) \neq \mathfrak{p}$.
 -/
theorem p_map_ne_p (p : ℂ[X]) (h : p.natDegree > 1) {a : ℂ} (ha : a ≠ 0)
  (hp : Ideal ℂ[X][Y]) (hp_is_prime : hp.IsPrime) (h : hp.height = 1) :
  hp.map (f a p) ≠ hp := by
  sorry

end Problem95

```

**Exercise (96).** Let  $f(x) \in \mathbb{Q}(x)$  be a rational function of degree at least 2,  $\alpha \in \mathbb{Q}$ . If the orbit  $\mathcal{O}_f(\alpha)$  contains infinitely many integers, then  $f^2(x)$  is a polynomial.

```

import Mathlib

namespace Problem96

open RatFunc

/--
Let $f(x) \in \mathbb{Q}(x)$ be a rational function of degree at least 2, $\alpha \in \mathbb{Q}$.
If the orbit $f(\alpha)$ contains infinitely many integers, then $f^2(x)$ is
a polynomial.
 -/
theorem ratFunc_square_is_poly_of_orbit_contain_infinite_integer
  {f : RatFunc ℚ} (hf : f.num.natDegree ≥ 2 ∨ f.denom.natDegree ≥ 2) {a : ℚ}
  (h : ∀ n : ℕ, (f.eval (RingHom.id ℚ))^[n] a ≠ 0) -- exclude the case that the `denom` is zero
  (ha : {m : ℤ | ∃ n : ℕ, m = (f.eval (RingHom.id ℚ))^[n] a}.Infinite) :
  ∃ g : Polynomial ℚ, g = f.eval C f := by
  sorry

end Problem96

```

**Exercise (97).** If  $k$  is a field of characteristic zero,  $n \in \mathbb{N}$ ,  $n \neq 0$ , and  $\phi: k[x_1, \dots, x_n] \rightarrow k[x_1, \dots, x_n]$  is given by  $(x_1, \dots, x_n) \mapsto (f_1(x_1), \dots, f_n(x_n))$ , where  $f_i(x_i) \in k[x_i]$  having degree at least two, then there is a point  $a \in k^n$  such that for any non-zero polyminal  $p \in k[x_1, \dots, x_n]$ , there exists  $m \in \mathbb{N}$  such that  $p(\phi^m(a)) \neq 0$ .

```

import Mathlib

namespace Problem97

open scoped Polynomial

/-
If  $k$  is a field of characteristic zero,  $\phi : k[x_1, \dots, x_n] \rightarrow k[x_1, \dots, x_n]$  is given by  $\phi(x_1, \dots, x_n) = f_1(x_1), \dots, f_n(x_n)$ , where  $f_i : k[x_i]$  having degree at least two, then there is a point  $a \in k^n$  such that for any non-zero polynimal  $p \in k[x_1, \dots, x_n]$ , there exists  $m \in \mathbb{N}$  such that  $p(\phi^m(a)) \neq 0$ .
-/
theorem exists_point_not_in_zero_set {τ k : Type} [Finite τ] [Nonempty τ] [Field k] [CharZero k]
  {f : τ → k[X]} (hfd : ∀ i : τ, (f i).natDegree ≥ 2) : ∃ a : τ → k,
  ∀ p : MvPolynomial τ k, p ≠ 0 →
  ∃ m : ℕ, ((MvPolynomial.eval (fun i ↦ (f i).toMvPolynomial i)) ^ m) p.eval a ≠ 0 := by
  sorry

end Problem97

```

**Exercise (98).** If  $K$  be a number field,  $A$  be a finite-type  $K$ -algebra,  $f : A \rightarrow A$  be an endomorphism. If  $A$  is a domain and  $f$  is not of finite order, then there exists a maximal ideal  $m \subset A$  such that for all  $n \in \mathbb{N}_+$ ,  $f^{-n}(m) \neq m$ .

```

import Mathlib

namespace Problem98

/-
If  $K$  be a number field,  $A$  be a finite-type  $K$ -algebra,  $f : A \rightarrow A$  be an endomorphism.
If  $A$  is a domain and  $f$  is not of finite order, then there exists a maximal ideal  $m \subset A$  such that for all  $n \in \mathbb{N}_+$ ,  $f^{-n}(m) \neq m$ .
-/
theorem exists_maximal_ideal_not_in_finite_order {K A : Type} [Field K] [NumberField K] [CommRing A]
  [IsDomain A] [Algebra K A] [Algebra.FiniteType K A] {f : A →ₐ[K] A} (hf : ∀ n > 0, f ^ n ≠ 1) :
  ∃ m : Ideal A, m.IsMaximal ∧ ∀ n > 0, m.comap (f ^ n) ≠ m := by
  sorry

end Problem98

```

**Exercise (99).** Let  $A$  be a finite-type  $\mathbb{C}$ -algebra,  $n \in \mathbb{N}$ ,  $n \geq 1$ . If  $A$  is a domain, and  $\text{Aut}_{\mathbb{C}} A$  is isomorphic to  $\text{Aut}_{\mathbb{C}} \mathbb{C}[x_1, \dots, x_n]$ , then  $A$  is isomorphic to  $\mathbb{C}[x_1, \dots, x_n]$  as  $\mathbb{C}$ -algebras.

```

import Mathlib

namespace Problem99

```

```

/--
Let  $A$  be a finite-type  $\mathbb{C}$ -algebra,  $n \in \mathbb{N}$ ,  $n \geq 1$ . If  $A$  is a domain, and  $\mathrm{Aut}_\mathbb{C} A$  is isomorphic to  $\mathrm{Aut}_\mathbb{C}[\mathbb{C}[x_1, \dots, x_n]]$ , then  $A$  is isomorphic to  $\mathbb{C}[x_1, \dots, x_n]$  as  $\mathbb{C}$ -algebras.
 -/
theorem equiv_of_aut_equiv {A : Type} [CommRing A] [IsDomain A] [Algebra C A]
  [Algebra.FiniteType C A] {n : ℕ} (hn : n ≥ 1)
  (e : (A ≃₉ C) ≃* (MvPolynomial (Fin n) C) ≃₉ (MvPolynomial (Fin n) C)) :
  Nonempty (A ≃₉ (MvPolynomial (Fin n) C)) := by
  sorry
end Problem99

```

**Exercise (100).** Let  $R$  be a Noetherian ring,  $P$  be a countably generated projective  $R$ -module such that  $P_{\mathfrak{m}}$  has infinite rank for all maximal ideals  $\mathfrak{m}$  of  $R$ . Then  $P$  is free.

```

import Mathlib

namespace Problem100

open Module

/--
Let  $R$  be a Noetherian ring,  $P$  be a countably generated projective  $R$ -module such that  $P_{\mathfrak{m}}$  has infinite rank for all maximal ideals  $\mathfrak{m}$  of  $R$ . Then  $P$  is free.
 -/
theorem free_of_countably_generated_projective_of_local_infinite_rank {R : Type} [CommRing R]
  [IsNoetherianRing R] (P : Type) [AddCommGroup P] [Module R P] [Projective R P]
  (hcg : ∃ s : Set P, s.Countable ∧ Submodule.span R s = ⊤)
  (hm : ∀ m : Ideal R, (m : m.IsMaximal) →
    Module.Finite (Localization.AtPrime m) (LocalizedModule.AtPrime m P)) : Free R P := by
  sorry
end Problem100

```