Hi everyone and welcome back to a brand new post! Today discuss even maps on spheres and their degree. Specifically, we show that they always had even degree.

Recall that a map $f\colon S^n \to S^n$ is called *even* if $f(x) = f(-x)$ for all $x \in S^n$. Its degree is defined as the unique integer m such that the induced map on homology

$$f_*\colon H_n(S^n) \to H_n(S^n)$$

is multiplication by m.

Proposition 0.1. *An even map f has even degree. Moreover, if n is even,* $\deg f = 0$.

Proof. Let us first suppose n is even. Denoting by a the antipodal map on S^n,

$$\deg f = \deg f \circ a = (-1)^{n+1} \deg f = -\deg f$$

Thus, $\deg f = 0$.

Let now n be odd. Since f is even, it factors through $\mathbb{R}P^n$:

$$S^n \xrightarrow{\pi} \mathbb{R}P^n \xrightarrow{\tilde{f}} S^n$$

By functoriality of H_n, it suffices to show that the induced map π_* takes a generator of $H_n(S^n)$ to twice a generator of $H_n(\mathbb{R}P^n)$. To do this, we look at S^n as a cell complex with 2 i-cells for $i = 0, 1, \dots, n$, namely the two hemipheres glued along the equatorial. Thus, the celular chain complex of S^n is

$$0 \longrightarrow \mathbb{Z}^2 \longrightarrow \cdots \longrightarrow \mathbb{Z}^2 \longrightarrow 0$$

Let e_1^i, e_2^i be the two i-cells. Using the cellular boundary formula, we see that

$$\partial e_k^i = e_1^{i-1} + (-1)^i e_2^{i-1}.$$

Thus, $H_n(S^n)$ is generated by the homology class $\langle e_1^n - e_2^n \rangle$ and, denoting the only n-cell of $\mathbb{R}P^n$ by e^n, we have

$$\pi_*(\langle e_1^n - e_2^n \rangle) = 2\langle e^n \rangle.$$

Again by the cellular boundary formula, we see that $\langle e^n \rangle$ generates $H_n(\mathbb{R}P^n)$. This completes the proof. $\square$

Until next time folks, stay fresh!