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9.18. Beta and Gamma functions.

In many problems in the applications of Integral Calculus, the use of the Beta and Gamma functions often facilitates calculations. So we give below an account of those functions – their definitions and important properties, some of which are, however, mentioned without any proof.*

Definitions:

(A) It can be shown that the integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ is

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* Results (v), (vi) and (vii) are given without any proof here. The proofs are based on "double integration" which is treated in chapter 21 of the present book. Nevertheless, the results are extremely important in applications and are to be carefully remembered.

convergent if and only if $m>0$, $n>0$. Then if $m>0$, $n>0$, the above integral has a definite value which is denoted by $B(m, n)$. Thus we get a function given by $B(m, n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, $m>0$, $n>0$.

This function is called *Beta function* and the integral is called **First Eulerian integral.**

(B) It can be shown that the integral $\int_0^\infty e^{-x}x^{n-1}\,dx$ is convergent if and only if $n>0$. Then if $n>0$, the above integral has a definite value which is denoted by $\Gamma(n)$. Thus we get a function given by $\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}\,dx$, $n>0$. This function is called *gamma function* and the integral is called **Second Eulerian integral.**

Here m and n are positive but they need not be integers.

Properties:

(i) $B(m, n)=B(n, m)$

We see that, for $m>0$, $n>0$,

$$B(m, n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$=\lim_{\substack{\epsilon\to 0+\\ \delta\to 0+}}\int_\epsilon^{1-\delta} x^{m-1}(1-x)^{n-1}\,dx$$

$$=\lim_{\substack{\delta\to 0+\\ \epsilon\to 0+}}\int_\delta^{1-\epsilon}(1-y)^{m-1}y^{n-1}\,dy,\quad \text{where } 1-x=y$$

$$=\int_0^1 y^{n-1}(1-y)^{m-1}\,dy=B(n, m).$$

$\therefore\quad \boldsymbol{B(m,n)=B(n,m)}.$

(ii) $\Gamma(1)=\int_0^\infty e^{-x}\,dx=1$. [*See Ex.* 1 *of Art* 9.14]

$\therefore\quad \boldsymbol{\Gamma(1)=1}.$

(iii) $\Gamma(n+1)=n\Gamma(n)$ where $n>0$.

Since $n>0$, $\Gamma(n)$ and $\Gamma(n+1)$ are both defined.

$$\text{Now } \Gamma(n+1)=\int_0^\infty e^{-x}x^n dx$$

$$= \lim_{\substack{\epsilon\to 0+\\ B\to\infty}} = \int_\epsilon^B e^{-x}x^n dx, \quad (B>\epsilon>0)$$

$$= \lim_{\substack{\epsilon\to 0+\\ B\to\infty}} \left[-e^{-x}x^n\Big|_\epsilon^B + \int_\epsilon^B nx^{n-1}e^{-x}dx \right]$$

$$= \lim_{\substack{\epsilon\to 0+\\ B\to\infty}} \left[-e^{-B}B^n + e^{-\epsilon}\epsilon^n + n\int_\epsilon^B e^{-x}x^{n-1}dx \right]$$

$$=0+0+n \lim_{\substack{\epsilon\to 0+\\ B\to\infty}} \int_\epsilon^B e^{-x}x^{n-1}dx$$

$$\left[\because \lim_{B\to\infty} e^{-B}B^n=0 \text{ and } \lim_{\epsilon\to 0+}\epsilon^n e^{-\epsilon}=0 \text{ if } n>0\right]$$

$$=n\int_0^\infty e^{-x}x^{n-1}dx$$

$$=n\Gamma(n).$$

$\therefore \quad \Gamma(n+1)=n\Gamma(n)$ if $n>0$.

When n is a positive integer,

$\Gamma(n+1)=n!$ [C. P. '85, '88]

(iv) Writing kx for x in (B), we easily get

$$\int_0^\infty e^{-kx}x^{n-1}dx=\frac{\Gamma(n)}{k^n}. \quad [k>0, n>0]$$ [C.P. '83]

(v) $B(m,n)=\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$. [C. H. '86]

(vi) $\Gamma(m)\,\Gamma(1-m)=\dfrac{\pi}{\sin m\pi}\quad (0<m<1)$.

(vii) Putting $m=\dfrac{1}{2}$ in (vi), we get

$$\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)=\frac{\pi}{\sin\frac{1}{2}\pi}=\pi.$$ [C. H. '86]

$$\therefore\quad \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}.$$ [C. P. '82]

Alternatively, we can deduce the value of $\Gamma\left(\frac{1}{2}\right)$ in the following way.

Putting $m=n=\dfrac{1}{2}$ in (v),

$$\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)}=B\left(\frac{1}{2},\frac{1}{2}\right)=\int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}\,dx$$

$$=2\int_0^{\frac{1}{2}\pi} d\theta\ \text{[on putting } x=\sin^2\theta\text{]}$$

$$=\pi.$$ [C. P. '81]

Hence the result.

(viii) $B(m,n)=\displaystyle\int_0^\infty \frac{x^{m-1}\,dx}{(1+x)^{m+n}}=\int_0^\infty \frac{x^{n-1}\,dx}{(1+x)^{m+n}}$.

9.19 Standard Integrals.

(1) $\displaystyle\int_0^{\frac{1}{2}\pi}\sin^p\theta\cos^q\theta\,d\theta=\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)},\ \begin{bmatrix}p>-1\\ q>-1\end{bmatrix}$.

$$\text{Left side} = \int_0^{\frac{1}{2}\pi} (\sin^2\theta)^{p/2}(1-\sin^2\theta)^{q/2}\,d\theta$$

$$= \frac{1}{2}\int_0^1 x^{\frac{p+1}{2}-1}(1-x)^{\frac{q+1}{2}-1}\,dx \quad \text{[on putting } x=\sin^2\theta\text{]}$$

$$= \frac{1}{2}B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \text{Right side by §9.18 (v).}$$

[Compare §*6.23 B*.]

(2) $$\int_0^{\frac{1}{2}\pi} \sin^p\theta d\theta = \int_0^{\frac{1}{2}\pi} \cos^p\theta\, d\theta = \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)}.$$

The proof is similar to (**1**). [Compare §*6.23 B*.]

(3) $$\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}.$$ [C. P. '83]

$$\text{Left side} = \frac{1}{2}\int_0^\infty e^{-z} z^{\frac{1}{2}-1}\,dz \quad \text{[on putting } x^2=z\text{]}$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right) \text{ by (B)} = \frac{1}{2}\sqrt{\pi} \text{ by §9.18 (vii).} \quad \text{[Compare } \textit{Art.}\ 9.15\text{]}$$

9.20 Illustrative Examples.

Ex. 1 Obtain a reduction formula for $\int_0^\infty e^{-ax}\cos^n x\,dx$, $(a>0)$ and hence find the value of $\int_0^\infty e^{-4x}\cos^5 x\,dx$.

From §7.6, replacing a by $-a$,

$$I_n = \int_0^\infty e^{-ax}\cos^n x\,dx$$

$$=\left[\frac{e^{-ax}\cos^{n-1}x(-a\cos x+n\sin x)}{n^2+a^2}\right]_0^{\infty}+\frac{n(n-1)}{n^2+a^2}I_{n-2}$$

$$=\frac{a}{n^2+a^2}\frac{n(n-1)}{n^2+a^2}I_{n-2}\ \left[\text{since } \underset{x\to\infty}{Lt}\ e^{-ax}\to 0 \text{ for } a>0\right]$$

is the required reduction formula.

$$\therefore\quad I_5=\frac{4}{5^2+4^2}+\frac{5.4}{5^2+4^2}I_3=\frac{4}{41}+\frac{20}{41}I_3;$$

$$I_3=\frac{4}{3^2+4^2}+\frac{3.2}{3^2+4^2}I_1=\frac{4}{25}+\frac{6}{25}I_1;$$

$$I_1=\frac{4}{1^2+4^2}=\frac{4}{17}.\qquad\therefore\quad I_5=\frac{708}{3485}.$$

Ex. 2. *Show that*

(*i*) $\Gamma\left(\frac{7}{2}\right)=\frac{15}{8}\sqrt{\pi}$. (*ii*) $\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)=\frac{2}{\sqrt{3}}\pi;$

(*iii*) $\int_0^{\frac{1}{2}\pi}\sin^4\theta\cos^6\theta\,d\theta=\int_0^{\frac{1}{2}\pi}\sin^6\theta\cos^4\theta\,d\theta=\frac{3}{512}\pi$.

(i) $\Gamma\left(\frac{7}{2}\right)=\Gamma\left(\frac{5}{2}+1\right)=\frac{5}{2}\Gamma\left(\frac{5}{2}\right)$

[since $\Gamma(n+1)=n\Gamma(n)$, *Art. 9.18 (iii)*]

$$=\frac{5}{2}\Gamma\left(\frac{3}{2}+1\right)=\frac{5}{2}\cdot\frac{3}{2}\Gamma\left(\frac{1}{2}+1\right)=\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$=\frac{15}{8}\sqrt{\pi}.$$ [*by Art. 9.21* (vii)].

(ii) Left side $=\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right)=\dfrac{\pi}{\sin\frac{1}{3}\pi}$ [*by Art. 9.18* (vi)].

(iii) **By** Art. 9.19 (1), $=\frac{2}{\sqrt{3}}\pi$.

$$\text{First Integral} = \frac{1}{2}\cdot\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{7}{2}\right)}{\Gamma(6)} = \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{5!} = \frac{3}{512}\pi.$$

By Art. 6.19 (iv), Second Integral = First Integral.

Ex. 3. *Show that* $\Gamma\left(n+\frac{1}{2}\right) = \frac{\Gamma(2n+1)\sqrt{\pi}}{2^{2n}\Gamma(n+1)}$. [*C. H.* '85]

$$\Gamma\left(n+\frac{1}{2}\right) = \Gamma\left(\frac{2n+1}{2}\right) = \Gamma\left(\frac{2n-1}{2}+1\right)$$

$$= \frac{2n-1}{2}\Gamma\left(\frac{2n-1}{2}\right) \qquad [\text{ by Art. 9.18 (iii) }]$$

$$= \frac{2n-1}{2}\Gamma\left(\frac{2n-3}{2}+1\right)$$

$$= \frac{2n-1}{2}\cdot\frac{2n-3}{2}\Gamma\left(\frac{2n-3}{2}\right)$$

$$= \frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdot\frac{2n-5}{2}\cdots\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

[*By repeated application of the result of the above Article.*]

$$= \frac{(2n-1)(2n-3)(2n-5)\ldots5.3.1}{2^n}\sqrt{\pi} \qquad \ldots \quad (1)$$

Now multiply the numerator and denominator of (1) by

$$2n(2n-2)(2n-4)\ldots4.2.$$

$$\therefore\ \Gamma\left(n+\frac{1}{2}\right) = \frac{2n(2n-1)(2n-2)(2n-3)\ldots5.4.3.2.1}{2^n.2.n.2(n-1).2(n-2)\ldots2.2.2.1}\sqrt{\pi}.$$

$$= \frac{\Gamma(2n+1)}{2^n.2^n.n(n-1)(n-2)\ldots2.1}\sqrt{\pi}$$

$$= \frac{\Gamma(2n+1)\sqrt{\pi}}{2^{2n}.\Gamma(n+1)}.$$

Note 1. The above result can be written in the form

$$\Gamma\left(\frac{1}{2}\right)\Gamma(2n) = 2^{2n-1}\Gamma(n)\Gamma\left(n+\frac{1}{2}\right).$$

It is an important result often used in Higher Mathematics.

Note 2. The right side of (1) can be written as $\left(\frac{1}{2}\right)_n \cdot \Gamma\left(\frac{1}{2}\right)$ where the notation $(a)_n$ denotes $a(a+1)(a+2)\ldots(a+n-1)$.

$$\therefore\ \Gamma\left(n+\frac{1}{2}\right)=\left(\frac{1}{2}\right)_n \Gamma\left(\frac{1}{2}\right).$$

Ex. 4. *Show that* $B(m,n)\,B(m+n,l)=B(n,l)\,B(n+l,m)$.

$$\text{Left side} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\,\frac{\Gamma(m+n)\Gamma(l)}{\Gamma(l+m+n)}=\frac{\Gamma(l)\,\Gamma(m)\Gamma(n)}{\Gamma(l+m+n)}.$$

$$\text{Similarly, right side} = \frac{\Gamma(l)\,\Gamma(m)\Gamma(n)}{\Gamma(l+m+n)}.$$

Hence the result.

Ex. 5. *Evaluate* : $\int_0^t x^{\alpha+k-1}(t-k)^{\beta+k-1}dx$

and find its value when $\alpha=\beta=\frac{1}{2}$.

Put $x=ty$. $\therefore$ $dx=t\,dy$; when $x=0$, $y=0$; $x=t$, $y=1$.

$$\therefore\ I=\int_0^1 t^{\alpha+\beta+2k-1}y^{\alpha+k-1}(1-y)^{\beta+k-1}dy$$

$$=t^{\alpha+\beta+2k-1}\frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{\Gamma(\alpha+\beta+2k)}.$$

$$=t^{2k}\frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(2k+1)} \qquad \left[\text{putting } \alpha=\beta=\frac{1}{2}\right]$$

$$=t^{2k}\frac{\Gamma(2k+1)\sqrt{\pi}\left(\frac{1}{2}\right)_k\Gamma\left(\frac{1}{2}\right)}{2^{2k}\Gamma(2k+1)\Gamma(k+1)}$$

[by Ex. 3 & *Note* (2) *of Art.* 9.20]

$$=t^{2k}\frac{\left(\frac{1}{2}\right)_k\pi}{2^{2k}k!}.$$

Ex. 5. (*i*) From the recurrence relation $\Gamma(n+1) = n\,\Gamma(n)$, calculate $\Gamma(5)$.

[*C. P. 1984*]

(*ii*) Find the value of $\int_0^\infty e^{-x^2}\,dx$, assuming $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

[*C. P. 1983, 86, 88, 95*]

(*iii*) Express $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ as Gamma function and then evaluate it.

[*C. P. 1995*]

(*iv*) Show that $\int_0^\infty e^{-4x} \cdot x^{\frac{2}{3}}\,dx = \frac{3}{128}\sqrt{\pi}$. [*C. P. 1993, 98*]

(v) Find the value of $\int_0^\infty e^{-5x^2}\,dx$ [C. P. 1990, 97]

Solution : (i) $\Gamma(5)=\Gamma(4+1)=4\,\Gamma(4)=4\cdot3\cdot\Gamma(3)$

$$=4\cdot3\cdot2\cdot\Gamma(2)=4\cdot3\cdot2\cdot1\cdot\Gamma(1)$$

$$=24,\qquad \because\ \Gamma(1)=1.$$

(ii) Let $I=\int_0^\infty e^{-x^2}dx$

We put $x^2=z,\Rightarrow\quad dx=\dfrac{dz}{2x}=\dfrac{dz}{2\sqrt{z}}$

When $x\to0,\ z\to0$ and when $x\to\infty,\ z\to\infty$

$$I=\int_0^\infty e^{-z}\frac{dz}{2\sqrt{z}}=\frac{1}{2}\int_0^\infty e^{-z}\cdot z^{-\frac{1}{2}}dz$$

$$=\frac{1}{2}\int_0^\infty e^{-z}z^{\frac{1}{2}-1}dz=\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{1}{2}\sqrt{\pi}$$

(iii) $I=\int_{-\infty}^\infty e^{-x^2}dx$

Let, $f(x)=e^{-x^2}$, then $f(-x)=e^{-x^2}=f(x)$

So, $f(x)$ is an even function.

$$\therefore\quad I=\int_0^\alpha e^{-x^2}dx$$

$$=2\cdot\frac{1}{2}\cdot\sqrt{\pi}=\sqrt{\pi}$$ [Vide (ii) above]

(iv) Here, let us put, $4x=z\Rightarrow 4dx=dz$

When $x\to0,\ z\to0$, and when $x\to\infty,\ z\to\infty$.

$$I=\int_0^\infty e^{-4x}\cdot x^{\frac{3}{2}}\,dx=\int_0^\infty e^{-z}\cdot\left(\frac{1}{4}z\right)^{\frac{3}{2}}\cdot\frac{dz}{4}$$

$$= \frac{1}{4\sqrt{4}\cdot 4}\int_0^{\infty} e^{-z}\cdot z^{\frac{3}{2}}\,dz$$

$$= \frac{1}{32}\int_0^{\infty} e^{-z}\cdot z^{\frac{5}{2}-1}\,dz$$

$$= \frac{1}{32}\Gamma\left(\frac{5}{2}\right)$$

$$= \frac{1}{32}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{3}{128}\sqrt{\pi}\,.$$

(v) Here, we substitute, $5x^2 = z \Rightarrow 10x\,dx = dz$

$$\therefore \quad dx = \frac{\sqrt{5}\,dz}{10\sqrt{z}}\,.$$

When $x \to 0$, $z \to 0$, and when $x \to \infty$, $z \to \infty$.

$$I = \int_0^{\infty} e^{-5x^2}\,dx = \int_0^{\infty} e^{-z}\,\frac{\sqrt{5}}{10}\cdot z^{-\frac{1}{2}}\,dz$$

$$= \frac{\sqrt{5}}{10}\int_0^{\infty} e^{-z}\cdot z^{\frac{1}{2}-1}\,dz$$

$$= \frac{\sqrt{5}}{10}\cdot\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{5}}{10}\cdot\sqrt{\pi} = \frac{\sqrt{5\pi}}{10}\,.$$

Ex. 6. (i) Prove that $\int_0^{\infty} e^{-kx}\cdot x^{n-1} = \frac{n-1}{k^n}$, where $k > 0$ and n is a positive integer.

[C. P. 1983]

(ii) Show that $\int_0^{\infty} e^{-x^2}\cdot x^9\,dx = 12$, [C. P. 1992]

(iii) Prove that $\int_{-\infty}^{\infty} 5^{-x^2}\,dx = \frac{1}{2\sqrt{\log 5}}\cdot\sqrt{\pi}\,.$

(*iv*) Show that $\int_0^\infty x^3 \cdot e^{-x^2}\, dx = \frac{1}{2}$ [C. P. 1982]

Solution : (*i*) Let, $kx = z \Rightarrow dx = \frac{dz}{k}$

As $x \to 0$, $z \to 0$ and as $x \to \infty$, $z \to \infty$

$$I = \int_0^\infty e^{-kx} \cdot x^{n-1} dx$$

$$= \int_0^\infty e^{-z} \cdot \frac{z^{n-1}}{k^{n-1}} \cdot \frac{dz}{k}$$

$$= \frac{1}{k^n} \int_0^\infty e^{-z} \cdot z^{n-1} \cdot dz$$

$$= \frac{1}{k^n} \Gamma(n) = \frac{(n-1)!}{k^n}.$$ ($\because$ n is a positive integer).

(*ii*) We substitute, $x^2 = z \Rightarrow dx = \frac{dz}{2x} = \frac{dz}{2\sqrt{z}}$

As $x \to 0$, $z \to 0$, and as $x \to \infty$, $z \to \infty$

$$I = \int_0^\infty e^{-x^2} \cdot x^9\, dx = \int_0^\infty e^{-z} \cdot \left(z^{\frac{1}{2}}\right)^9 \cdot \frac{dz}{2\sqrt{z}}$$

$$= \frac{1}{2}\int_0^\infty e^{-z} \cdot z^4 = \frac{1}{2}\int_0^\infty e^{-z} \cdot z^{5-1} dz$$

$$= \frac{1}{2}\Gamma(5) = \frac{1}{2} \cdot 4! = 12.$$

(*iii*) $I = \int_{-\infty}^{\infty} 5^{-x^2} dx$

$$= \int_{-\infty}^{\infty} e^{\log 5^{-x^2}} dx = \int_0^\infty e^{-x^2 \cdot \log 5}\, dx$$

Let, $x^2 \cdot \log 5 = z \Rightarrow 2x \cdot \log 5\, dx = dz$

$$\Rightarrow \quad dx = \frac{dz}{2x\ \log 5} = \frac{dz}{2\sqrt{\frac{z}{\log 5}} \cdot \log 5} \cdot$$

$$\Rightarrow \quad dx = \frac{dz}{2\sqrt{\log 5} \cdot \sqrt{z}}$$

$$\Rightarrow \quad dx = \frac{dz}{2\sqrt{\log 5} \cdot \sqrt{z}}$$

As $x \to 0,\ z \to 0$ and as $x \to \infty,\ z \to \infty$.

$$\therefore \quad I = \frac{1}{2\sqrt{\log 5}} \int_0^\infty e^{-z} z^{-\frac{1}{2}} dz$$

$$= \frac{1}{2\sqrt{\log 5}} \int_0^\infty e^{-z} \cdot z^{\frac{1}{2}-1} dz$$

$$= \frac{1}{2\sqrt{\log 5}}\ \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2\sqrt{\log 5}}.$$

(*iv*) Here, we put $x^2 = z \Rightarrow dx = \frac{dz}{2\sqrt{z}}$

As $x \to 0,\ z \to 0$ and as $x \to \infty,\ z \to \infty$.

$$I = \int_0^\infty e^{-x^2} \cdot x^3 dx = \frac{1}{2}\int_0^\infty e^{-z} \cdot z^{\frac{3}{2}} \cdot z^{-\frac{1}{2}} dz$$

$$= \frac{1}{2}\int_0^\infty e^{-z} \cdot z^{2-1} dz = \frac{1}{2}\Gamma(1) = \frac{1}{2} \cdot 1 = \frac{1}{2}.$$

Ex. 7. (*i*) Assuming $\Gamma(m)\,\Gamma(1-m) = \pi \operatorname{cosec} m\pi,\ 0 < m < 1$, show that

(*a*) $\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{2\pi}{\sqrt{3}}$ [*C. P. 1997*]

(*b*) $\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right) \cdots \cdots \Gamma\left(\frac{8}{9}\right) = \frac{16}{3}\pi^4$ [*C. P. 2001*]

(*ii*) $\int_0^1 \frac{dx}{(1-x^6)^{\frac{1}{6}}} = \frac{\pi}{3}$ [*C. P. 1996*]

(iii) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\sin\theta}} \times \int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\, d\theta = \pi$ [C. P. 1991]

(iv) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\cos\theta}} \times \int_0^{\frac{\pi}{2}} \sqrt{\cos\theta}\, d\theta = \pi$ [C. P. 1994]

Solution : (i) (a) $\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right)$

$$= \pi \operatorname{cosec}\left(\frac{1}{3}\pi\right) = \pi \cdot \frac{2}{\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$$

(b) $\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{3}{9}\right)\Gamma\left(\frac{4}{9}\right)\Gamma\left(\frac{5}{9}\right)\Gamma\left(\frac{6}{9}\right)\Gamma\left(\frac{7}{9}\right)\Gamma\left(\frac{8}{9}\right)$

$$= \left\{\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{8}{9}\right)\right\}\left\{\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{7}{9}\right)\right\}\left\{\Gamma\left(\frac{3}{9}\right)\Gamma\left(\frac{6}{9}\right)\right\}\left\{\Gamma\left(\frac{4}{9}\right)\Gamma\left(\frac{5}{9}\right)\right\}$$

$$= \left\{\Gamma\left(\frac{1}{9}\right)\Gamma\left(1-\frac{1}{9}\right)\right\}\left\{\Gamma\left(\frac{2}{9}\right)\Gamma\left(1-\frac{2}{9}\right)\right\}\left\{\Gamma\left(\frac{3}{9}\right)\Gamma\left(1-\frac{3}{9}\right)\right\}$$
$$\times\left\{\Gamma\left(\frac{4}{9}\right)\Gamma\left(1-\frac{4}{9}\right)\right\}$$

$$= \pi \operatorname{cosec}\frac{1}{9}\pi \times \pi \operatorname{cosec}\frac{2}{9}\pi \times \pi \operatorname{cosec}\frac{3}{9}\pi \times \pi \operatorname{cosec}\frac{4}{9}\pi$$

$$= \pi^4 \times \left(\operatorname{cosec}\frac{\pi}{9} \cdot \operatorname{cosec}\frac{2\pi}{9} \cdot \operatorname{cosec}\frac{3\pi}{9} \cdot \operatorname{cosec}\frac{4\pi}{9}\right)$$

$$= \pi^4 \times \frac{16}{3} = \frac{16\pi^4}{3}.$$

(ii) We put, $x^6 = \sin^2\theta$

$$\Rightarrow \quad 6x^5 dx = 2\sin\theta\cos\theta\, d\theta$$

$$\Rightarrow \quad dx = \frac{1}{3}\frac{\sin\theta\cos\theta\, d\theta}{\sin^{\frac{5}{3}}\theta}$$

When $x \to 0$, $\theta \to 0$, and when $x \to 1$, $\theta \to \frac{\pi}{2}$.

$$I = \int_0^1 \frac{dx}{\left(1-x^6\right)^{\frac{1}{6}}}$$

$$= \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{\sin^{-\frac{2}{3}} \theta \cdot \cos\theta}{\cos^{\frac{1}{3}} \theta}\, d\theta$$

$$= \frac{1}{3} \int_0^{\frac{\pi}{2}} \sin^{-\frac{2}{3}} \theta \cdot \cos^{\frac{2}{3}} \theta \, d\theta$$

$$= \frac{1}{3} \int_0^{\frac{\pi}{2}} \sin^{\left(2\times\frac{1}{6}-1\right)} \theta \cdot \cos^{\left(2\times\frac{5}{6}-1\right)} \theta \, d\theta$$

$$= \frac{1}{3} \cdot \frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)}{2\Gamma\left(\frac{1}{6}+\frac{5}{6}\right)}$$

$$= \frac{1}{3} \cdot \frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(1-\frac{1}{6}\right)}{2\Gamma(1)}$$

$$= \frac{1}{6} \cdot \pi \operatorname{cosec} \frac{\pi}{6} \qquad [\because \Gamma(1) = 1]$$

$$= \frac{1}{6} \cdot \pi \times 2 = \frac{\pi}{3}.$$

(*iii*) Let, $I_1 = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\sin\theta}}$ and $I_2 = \int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\, d\theta$

Now, $I_1 = \int_0^{\frac{\pi}{2}} \sin^{-\frac{1}{2}} \theta \cos^0 \theta \, d\theta$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\dfrac{-\frac{1}{2}+1}{2}\right) \cdot \Gamma\left(\dfrac{0+1}{2}\right)}{\Gamma\left(\dfrac{-\frac{1}{2}+0+2}{2}\right)}$$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{1}{4}\right) \cdot \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} \qquad \ldots \qquad (1)$$

$$I_2 = \int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}} \theta \, d\theta = \int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}} \theta \cos^0 \theta \, d\theta$$

$$= \frac{1}{2} \cdot \frac{\Gamma \frac{1}{2}\left(\frac{1}{2}+1\right) \Gamma \frac{1}{2}(0+1)}{\Gamma \frac{1}{2}\left(\frac{1}{2}+0+2\right)} = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)} \qquad \ldots \qquad (2)$$

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\sin\theta}} \times \int_0^{\frac{\pi}{2}} \sqrt{\sin\theta} \, d\theta = I_1 \times I_2$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} \cdot \frac{1}{2} \frac{\Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)} \qquad \text{[From (1) and (2)]}$$

$$= \frac{1}{4} \cdot \frac{\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 \Gamma\left(\frac{1}{4}\right)}{\frac{1}{4} \cdot \Gamma\left(\frac{1}{4}\right)} = \left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = \pi.$$

(*iv*) Let, $$I = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\cos\theta}} \times \int_0^{\frac{\pi}{2}} \sqrt{\cos\theta}\; d\theta = I_1 \times I_2$$

$$I_1 = \int_0^{\frac{\pi}{2}} \sin^0\theta \cdot \cos^{-\frac{1}{2}}\theta\, d\theta$$

$$= \frac{1}{2} \cdot \frac{\Gamma\frac{1}{2}(0+1)\,\Gamma\frac{1}{2}\left(-\frac{1}{2}+1\right)}{\Gamma\frac{1}{2}\left(0-\frac{1}{2}+2\right)}$$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$I_2 = \int_0^{\frac{\pi}{2}} \sin^0\theta \cdot \cos^{\frac{1}{2}}\theta\, d\theta$$

$$= \frac{1}{2} \cdot \frac{\Gamma\frac{1}{2}(0+1)\,\Gamma\frac{1}{2}\left(\frac{1}{2}+1\right)}{\Gamma\frac{1}{2}\left(0+\frac{1}{2}+2\right)}$$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)}$$

$$I = I_1 \times I_2 = \frac{1}{4} \cdot \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{5}{4}\right)} \left\{\Gamma\left(\frac{1}{2}\right)\right\}^2$$

$$= \frac{1}{4} \cdot \frac{\Gamma\left(\frac{1}{4}\right)}{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)} \left(\sqrt{\pi}\right)^2 = \pi \quad \left[\because \Gamma(1) = 1,\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$