CHAPTER 2

Linear Differential Equations of Second Order

2.1 INTRODUCTION

A linear differential equation of the nth order has the form

$$\frac{d^n y}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\ldots+P_{n-1}\frac{dy}{dx}+P_n y = X \qquad \ldots(2.1)$$

where $P_1, P_2, \ldots, P_n, X$ are functions of x only or constants. The general solution of (2.1) contains n arbitrary constants.

Non-homogeneous

Differential equation (2.1) is said to be non-homogeneous if the right hand side of (2.1), $X \neq 0$.

Otherwise

Homogeneous

when $X = 0$.

Linear Differential Equations of First Order with Constant Coefficients

A linear differential equation of first order with constant coefficients is of the form

$$\frac{dy}{dx}+Py = Q \qquad \ldots(2.2)$$

Using the symbol D for the differential operator $\frac{d}{dx}$, (2.2) becomes

$$(D+P)\,y = Q, \quad \text{or} \quad f(D)y = Q, \text{ where } f(D) = D + P \qquad \ldots(2.3)$$

Here P is constant or a function of x only and Q is constant or a function of x only.

$$\therefore \qquad \text{I.F.} = e^{\int P dx}$$

Multiplying both sides of (2.2) by $e^{\int P dx}$, we get

$$e^{\int P dx}\frac{dy}{dx}+Pye^{\int P dx} = Qe^{\int P dx}, \quad \text{or} \quad \frac{d}{dx}\left(ye^{\int P dx}\right) = Qe^{\int P dx}$$

or
$$d\left(ye^{\int P\,dx}\right) = \left(Qe^{\int P\,dx}\right)dx$$

Integrating,
$$\int d\left(ye^{\int P\,dx}\right) = \int\left(Qe^{\int P\,dx}\right)dx$$

$\therefore$
$$ye^{\int P\,dx} = c+\int\left(Qe^{\int P\,dx}\right)dx, \text{ where } c \text{ is an arbitrary constant.}$$

$\therefore$
$$y = cu+v, \text{ where } u = e^{-\int P\,dx}, \quad v = e^{-\int P\,dx}\int\left(Qe^{\int P\,dx}\right)dx \qquad ...(2.4)$$

Now,
$$\frac{du}{dx} = -Pe^{-\int P\,dx} = -Pu, \quad \text{or} \quad \frac{du}{dx}+Pu = 0$$

or
$$\frac{d}{dx}(cu)+P(cu) = 0$$

Therefore cu is the general solution (since it contains one arbitrary constant c) of the corresponding homogeneous equation of (2.2).

Also,
$$\frac{dv}{dx} = -Pe^{-\int P\,dx}\int\left(Qe^{\int P\,dx}\right)dx+e^{-\int P\,dx}Qe^{\int P\,dx}$$

or
$$\frac{dv}{dx} = -Pv+Q, \quad \text{or} \quad \frac{dv}{dx}+Pv = Q$$

Therefore v is a solution of (2.2).

Hence the general solution of (2.2) is (2.4) consisting of two parts, *i.e.*, cu and v, where cu is the general solution of the corresponding homogeneous equation known as *complementary function* and v is known as *particular integral* (since it is free from any arbitrary constant).

$\therefore$ *General solution (or complete solution)*

$$= \textit{Complementary Function} + \textit{Particular Integral}$$

or
$$y = \text{C.F.} + \text{P.I.}$$

Linear Differential Equations of Second Order with Constant Coefficients

The general form of a linear differential equation of second order with constant coefficients is

$$\frac{d^2y}{dx^2}+P_1\frac{dy}{dx}+P_2y = X \qquad ...(2.5)$$

where P_1, P_2 are constants and X is a function of x only or constant.

Using the symbol D for the differential operator $\frac{d}{dx}$, (2.5) becomes

$$(D^2+P_1D+P_2)y = X, \quad \text{or} \quad f(D)y = X, \text{ where } f(D) = D^2+P_1D+P_2 \qquad ...(2.6)$$

When $X = 0$, then $f(D)y = 0$, or $\frac{d^2y}{dx^2}+P_1\frac{dy}{dx}+P_2y = 0$ ...(2.7)

is called the homogeneous equation of (2.5) or (2.6).

Definition: Two solutions $y_1(x)$, $y_2(x)$ of (2.7) are said to be *linearly independent* if $c_1y_1+c_2y_2 = 0$ for arbitrary constants c_1, c_2 implies $c_1 = c_2 = 0$.

Theorem 1: Two solutions $y_1(x)$, $y_2(x)$ of (2.7) are linearly independent if and only if their Wronskian $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$.

Proof: Let us suppose that y_1, y_2 are linearly dependent.

Then by definition there exist constants c_1, c_2 not all zero, such that

$$c_1 y_1 + c_2 y_2 = 0, \text{ therefore, } c_1 y_1' + c_2 y_2' = 0.$$

Since this system of two equations in c_1, c_2 has a nontrivial solution (*i.e.*, c_1, c_2 are not all zero), we must have $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 0$, *i.e.*, $W(y_1, y_2) = 0$.

Conversely, let $W(y_1, y_2) = 0$, then there exists a nontrivial solution (*i.e.*, c_1, c_2 are not all zero) to the following homogeneous equations

$$c_1 y_1 + c_2 y_2 = 0$$
$$c_1 y_1' + c_2 y_2' = 0$$

This implies there exist constants c_1, c_2 not all zero, such that $c_1 y_1 + c_2 y_2 = 0$ and hence y_1, y_2 are dependent.

Theorem 2: If $y_1(x)$ and $y_2(x)$ are any two linearly independent solutions of the homogeneous differential equation (2.7), then the general solution of (2.7) is given by $y(x) = c_1 y_1(x) + c_2 y_2(x)$, where c_1 and c_2 are two arbitrary constants.

Proof: Since $y_1(x)$, $y_2(x)$ are solutions of (2.7), therefore

$$\frac{d^2 y_1}{dx^2} + P_1 \frac{dy_1}{dx} + P_2 y_1 = 0 \qquad ...(2.8)$$

and

$$\frac{d^2 y_2}{dx^2} + P_1 \frac{dy_2}{dx} + P_2 y_2 = 0 \qquad ...(2.9)$$

If c_1, c_2 are two arbitrary constants, then

$$\frac{d^2}{dx^2}(c_1 y_1 + c_2 y_2) + P_1 \frac{d}{dx}(c_1 y_1 + c_2 y_2) + P_2 (c_1 y_1 + c_2 y_2)$$

$$= c_1 \left(\frac{d^2 y_1}{dx^2} + P_1 \frac{dy_1}{dx} + P_2 y_1 \right) + c_2 \left(\frac{d^2 y_2}{dx^2} + P_1 \frac{dy_2}{dx} + P_2 y_2 \right)$$

$$= c_1 \cdot 0 + c_2 \cdot 0 = 0 \qquad \text{[by (2.8) and (2.9)]}$$

Hence $y(x) = c_1 y_1(x) + c_2 y_2(x)$ is the general solution of (2.7), since it contains two arbitrary constants c_1 and c_2.

Note: This superposition principle is not applicable to non-homogeneous and non-linear equations.

Theorem 3: If $u(x)$ be the general solution of the homogeneous differential equation (2.7) and $v(x)$ be a particular solution (*i.e.*, free from any arbitrary constant) of (2.5), then the general solution of the differential equation (2.5) is $y = u(x) + v(x)$.

Proof: Since $u(x)$ is the general solution of (2.7),

therefore, $$\frac{d^2u}{dx^2}+P_1\frac{du}{dx}+P_2u = 0 \quad ...(2.10)$$

Since $v(x)$ is a solution of (2.5),

therefore, $$\frac{d^2v}{dx^2}+P_1\frac{dv}{dx}+P_2v = X \quad ...(2.11)$$

Adding (2.10) and (2.11), we get

$$\frac{d^2}{dx^2}(u+v)+P_1\frac{d}{dx}(u+v)+P_2(u+v) = X, \text{ or } \frac{d^2y}{dx^2}+P_1\frac{dy}{dx}+P_2y = X$$

This shows that $y = u(x) + v(x)$ is the general solution of (2.5).

Note: To obtain the general (or complete) solution of (2.5), we have first to obtain two independent solutions of (2.7) and any solution (free from any arbitrary constant) of (2.5). The sum of the general solution of (2.7) and a particular solution of (2.5) will be the general solution of (2.5).

The expression which is the general (or complete) solution of (2.7) is known as the *Complementary Function* of (2.5) and any particular solution of (2.5) is known as the *Particular Integral* of (2.5).

Thus the process of solving

$$\frac{d^2y}{dx^2}+P_1\frac{dy}{dx}+P_2y = X$$

involves following steps:

(*i*) Replace X by 0 and then find the complete solution $u = c_1u_1 + c_2u_2$ of the reduced equation.

(*ii*) Find any particular solution of the original equation.

(*iii*) Sum of the complementary function found in step (*i*) and the particular integral found in step (*ii*) gives the general solution of the given differential equation.

These conclusions are also valid for general linear differential equations with constant coefficients.

∴ *General Solution (or Complete Solution)*

= *Complementary Function* + *Particular Integral*

or $y =$ C.F. + P.I.

2.2 METHOD FOR FINDING THE COMPLEMENTARY FUNCTION (C.F.)

1. In finding the complementary function, right hand side of the given differential equation is replaced by zero.
2. The C.F. of a linear second order differential equation with constant coefficients can be found from

$$\frac{d^2y}{dx^2}+P_1\frac{dy}{dx}+P_2y = 0, \quad \text{or} \quad (D^2+P_1D+P_2)y = 0 \quad ...(2.12)$$

where $D \equiv \frac{d}{dx}$ and P_1, P_2 are constants.

Let e^{mx}, where m is a constant, be a solution of (2.12).
Substituting $y = e^{mx}$ in (2.12), we get

$$(m^2 + P_1 m + P_2)e^{mx} = 0, \quad \text{or} \quad m^2 + P_1 m + P_2 = 0 \qquad (\because e^{mx} \neq 0) \qquad ...(2.13)$$

The algebraic equation (2.13) is known as the *characteristic* or *auxiliary equation* of (2.12). Also $y = e^{mx}$ is known as the *trial solution* of (2.12).

Note: If we write (2.12) in the form $f(D)y = 0$, then the auxiliary equation is $f(m) = 0$.

3. Solve the auxiliary equation.

Case I: Roots are real and unequal

If m_1, m_2 are two real and distinct roots of (2.13), then C.F. $= c_1 e^{m_1 x} + c_2 e^{m_2 x}$, where c_1, c_2 are two arbitrary constants.

Case II: Roots are real and equal

Let the roots of the auxiliary equation (2.13) be real and equal to α. Then (2.12) can be written as

$$(D-\alpha)^2 y = 0, \quad \text{or} \quad (D^2 - 2\alpha D + \alpha^2)y = 0, \quad \text{or} \quad \frac{d^2y}{dx^2} - 2\alpha\frac{dy}{dx} + \alpha^2 y = 0 \qquad ...(2.14)$$

Let its solution be $y = ve^{\alpha x}$, where v is a function of x.
Substituting $y = ve^{\alpha x}$, in (2.14), we get

$$\alpha^2 ve^{\alpha x} + 2\alpha e^{\alpha x}\frac{dv}{dx} + e^{\alpha x}\frac{d^2v}{dx^2} - 2\alpha\left(\alpha ve^{\alpha x} + e^{\alpha x}\frac{dv}{dx}\right) + \alpha^2 ve^{\alpha x} = 0$$

or
$$\frac{d^2v}{dx^2} = 0 \qquad (\because e^{\alpha x} \neq 0)$$

The solution of this equation is $v = c_1 x + c_2$.

Hence C.F. $= (c_1 x + c_2)e^{\alpha x}$, where c_1, c_2 are arbitrary constants.

Case III: Roots are complex

Let the auxiliary equation (2.13) has complex roots $\alpha \pm i\beta$ (α, β real), then

$$\begin{aligned}
\text{C.F.} &= c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x} = e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x}) \\
&= e^{\alpha x}[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)] \\
&= e^{\alpha x}[(c_1 + c_2)\cos\beta x + i(c_1 - c_2)\sin\beta x] \\
&= e^{\alpha x}(A\cos\beta x + B\sin\beta x).
\end{aligned}$$

ILLUSTRATIVE EXAMPLES

Example 1: Solve: $\dfrac{d^2y}{dx^2} + (a+b)\dfrac{dy}{dx} + aby = 0$, where a, b are unequal.

Solution: The given equation can be written as

$$\{D^2 + (a+b)D + ab\}y = 0, \quad \text{where} \quad D \equiv \frac{d}{dx}.$$

Let $y = e^{mx}$ be a trial solution. Then the auxiliary equation is $m^2 + (a+b)m + ab = 0$, or $(m+a)$ $(m + b) = 0$

$\therefore$ $$m = -a, -b.$$

Hence the general solution is

$$y = c_1 e^{-ax} + c_2 e^{-bx}, \; c_1, c_2 \text{ are arbitrary constants.}$$

Example 2: Solve the equation $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 0$ and find the particular solution if $y = 1$, $\dfrac{dy}{dx} = -2$ when $x = 0$.

Solution: The given equation can be written as

$$(D^2 - 4D + 4)y = 0, \text{ where } D \equiv \frac{d}{dx}.$$

Let $y = e^{mx}$ be a trial solution. Then the auxiliary equation is $m^2 - 4m + 4 = 0$, or $(m-2)^2 = 0$. Therefore, $m = 2, 2$.

Hence the general solution is

$$y = (c_1 x + c_2)e^{2x}, \text{ where } c_1, c_2 \text{ are two arbitrary constants.}$$

$$\therefore \quad \frac{dy}{dx} = c_1 e^{2x} + 2(c_1 x + c_2)e^{2x} = (2c_1 x + c_1 + 2c_2)e^{2x}.$$

By the given condition $y = 1$, $\dfrac{dy}{dx} = -2$ when $x = 0$.

$\therefore$ $c_2 = 1$, $\quad c_1 + 2c_2 = -2$, or $c_1 = -4 \quad (\because c_2 = 1)$.

Therefore the particular solution is

$$y = (1 - 4x)e^{2x}.$$

Example 3: Solve: $\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} + 25y = 0.$

Solution: The given equation can be written as

$$(D^2 + 8D + 25)y = 0, \quad \text{where } D \equiv \frac{d}{dx}.$$

Let $y = e^{mx}$ be a trial solution. Then the auxiliary equation is $m^2 + 8m + 25 = 0$. Thus $m = -4 \pm 3i$ and the general solution is $y = e^{-4x}(c_1 \cos 3x + c_2 \sin 3x)$, where c_1, c_2 are two arbitrary constants.

Example 4: Solve: $\dfrac{d^2s}{dt^2} + n^2 s = 0$; $s = a$, $\dfrac{ds}{dt} = 0$ when $t = 0$.

Solution: The given equation can be written as

$$(D^2+n^2)s = 0, \quad \text{where} \quad D \equiv \frac{d}{dt}.$$

Let $s = e^{mt}$ be a trial solution. Then the auxiliary equation is $m^2 + n^2 = 0$, or $m = \pm\, in$.
Hence the general solution is

$$s = c_1 \cos nt + c_2 \sin nt,$$ where c_1, c_2 are two arbitrary constants.

$$\therefore \quad \frac{ds}{dt} = -c_1 n \sin nt + c_2 n \cos nt.$$

By the given condition $s = a$, $\dfrac{ds}{dt} = 0$ when $t = 0$.

$\therefore \quad c_1 = a, \quad c_2 n = 0, \quad$ or $\quad c_2 = 0 \qquad (\because n \neq 0).$

Therefore, the required solution is $s = a \cos nt$.

2.3 THE OPERATOR D

The operators D, D^2, D^3, ... stand respectively for $\dfrac{d}{dx}, \dfrac{d^2}{dx^2}, \dfrac{d^3}{dx^3}, \ldots$. Also $\dfrac{1}{D}$ or D^{-1}, $\dfrac{1}{D^2}$ or D^{-2}, ... are used to denote the inverse operators, *i.e.*, the operators which integrate respectively once, twice, ...

This symbolic operator D largely satisfies the ordinary laws of algebra. We have $D(u + v) = Du + Dv$, so D satisfies distributive law. Also $(D + a)u = Du + au = au + Du = (a + D)u$ and $Dau = aDu$, so D is commutative with a constant a. It is also noted that $D^m D^n u = D^{m+n} u$ where m, n are integers. If α, β are constants, then

$$\begin{aligned}(D+\alpha)(D+\beta)\, u &= \{D^2 + (\alpha+\beta)D + \alpha\beta\}u \\ &= (D+\beta)(D+\alpha)u.\end{aligned}$$

Thus D obeys the fundamental laws of algebra except that it is not commutative with variables.

Proof: (i) Let $y = \frac{1}{D}X$, therefore, $Dy = D\left(\frac{1}{D}X\right)$

or $$\frac{dy}{dx} = X \qquad \text{(by definition)}$$

$\therefore$ $$dy = X\,dx, \quad \text{by integration,} \quad y = \int X\,dx.$$

(ii) Let $y = \frac{1}{D-a}X$, therefore, $(D-a)y = (D-a)\left(\frac{1}{D-a}X\right)$

or $$Dy - ay = X \qquad \text{(by definition)}$$

$\therefore$ $$\frac{dy}{dx} - ay = X(x), \text{ which is a linear equation in } y \qquad \ldots(2.15)$$

$\therefore$ $$\text{I.F.} = e^{-\int a\,dx} = e^{-ax}$$

Multiplying both sides of (2.15) by e^{-ax}, we get

$$e^{-ax}\frac{dy}{dx} - aye^{-ax} = Xe^{-ax}, \quad \text{or} \quad \frac{d}{dx}(ye^{-ax}) = Xe^{-ax}$$

or $$d(ye^{-ax}) = Xe^{-ax}dx$$

Integrating, $$\int d(ye^{-ax}) = \int Xe^{-ax}dx.$$

or $$ye^{-ax} = \int Xe^{-ax}dx.$$

$\therefore$ $$y = \frac{1}{D-a}X = e^{ax}\int Xe^{-ax}dx.$$

General method of finding particular integral (P.I.)

Let us consider the following linear differential equation of second order with constant coefficients

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = X$$

or $$f(D)y = X, \text{ where } f(D) = D^2 + P_1 D + P_2,\ D \equiv \frac{d}{dx}.$$

Here P_1, P_2 are constants and X is a function of x only or constant.

Let m_1, m_2 be the roots of the auxiliary equation of the corresponding homogeneous equation, then

$$f(D) = (D - m_1)(D - m_2).$$

Let $\frac{1}{f(D)}$ can be resolved into partial fractions, say,

$$\frac{1}{f(D)} = \frac{A_1}{D-m_1} + \frac{A_2}{D-m_2}, \text{ where } A_1, A_2 \text{ are constants.}$$

Then $$\text{P.I.} = \frac{1}{f(D)}X = \frac{A_1}{D-m_1}X + \frac{A_2}{D-m_2}X$$

$$= A_1 e^{m_1 x} \int X e^{-m_1 x} dx + A_2 e^{m_2 x} \int X e^{-m_2 x} dx \quad \text{[by Result } (ii)] \quad ...(2.16)$$

Note: Since (2.16) is a particular solution, it must not contain any arbitrary constant.

ILLUSTRATIVE EXAMPLES

Example 1: Solve: $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = e^{4x}$.

Solution: The given equation can be written as

$$(D^2 - 5D + 6)y = e^{4x}, \quad \text{or} \quad (D-3)(D-2)y = e^{4x}, \text{ where } D \equiv \frac{d}{dx}.$$

First step: Determination of complementary function (C.F.)
The complementary function is found from

$$(D - 3)(D - 2) = 0.$$

Let $y = e^{mx}$ be a trial solution, then the auxiliary equation becomes $(m-3)(m-2) = 0$, the roots of which are 2, 3.

$$\therefore \quad \text{C.F.} = c_1 e^{2x} + c_2 e^{3x}.$$

Second step: Determination of particular integral (P.I.)

$$\text{P.I.} = \frac{1}{(D-3)(D-2)} e^{4x} = \left(\frac{1}{D-3} - \frac{1}{D-2}\right) e^{4x}$$

$$= \frac{1}{D-3} e^{4x} - \frac{1}{D-2} e^{4x} = e^{3x} \int e^{4x} \cdot e^{-3x} dx - e^{2x} \int e^{4x} \cdot e^{-2x} dx$$

$$= e^{3x} \int e^{x} dx - e^{2x} \int e^{2x} dx = e^{3x} \cdot e^{x} - e^{2x} \cdot \frac{e^{2x}}{2}$$

$$= \frac{e^{4x}}{2}.$$

Third step: The general solution is therefore $y = \text{C.F.} + \text{P.I.} = c_1 e^{2x} + c_2 e^{3x} + \dfrac{e^{4x}}{2}$, where c_1, c_2 are arbitrary constants.

Example 2: Solve: $\dfrac{d^2y}{dx^2} - y = 4xe^{x}$.

Solution: The given equation can be written as

$$(D^2 - 1)y = 4xe^{x}, \quad \text{or} \quad (D-1)(D+1)y = 4xe^{x}, \text{ where } D \equiv \frac{d}{dx}.$$

First step: Determination of complementary function (C.F.)
The complementary function is found from

$$(D - 1)(D + 1)y = 0.$$

Let $y = e^{mx}$ be a trial solution, then the auxiliary equation becomes $(m-1)(m+1) = 0$, the roots of which are $-1, 1$.

$$\therefore \quad \text{C.F.} = c_1 e^{-x} + c_2 e^{x}.$$

Second step: Determination of particular integral (P.I.)

$$\text{P.I.} = \frac{1}{(D-1)(D+1)} 4xe^x = \frac{1}{2}\left(\frac{1}{D-1} - \frac{1}{D+1}\right) 4xe^x$$

$$= 2 \cdot \frac{1}{D-1} xe^x - 2 \cdot \frac{1}{D+1} xe^x = 2e^x \int xe^x \cdot e^{-x} dx - 2e^{-x} \int xe^x \cdot e^x dx$$

$$= 2e^x \int x\,dx - 2e^{-x} \int xe^{2x} dx$$

$$= 2e^x \cdot \frac{x^2}{2} - 2e^{-x}\left[x \int e^{2x} dx - \int \left\{\left(\frac{dx}{dx}\right) \int e^{2x} dx\right\} dx\right]$$

$$= x^2 e^x - 2e^{-x}\left[\frac{xe^{2x}}{2} - \frac{e^{2x}}{4}\right] = e^x\left(x^2 - x + \frac{1}{2}\right).$$

Third step: The general solution is therefore

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{-x} + c_2 e^x + e^x\left(x^2 - x + \frac{1}{2}\right),$$

where c_1, c_2 are two arbitrary constants.

Example 3: Solve the following differential equation

$$\frac{d^2y}{dx^2} + a^2 y = \sec ax$$

with the symbolic operator D.

Solution: The given differential equation can be written as $(D^2 + a^2)y = \sec ax$,

or $$(D + ia)(D - ia)y = \sec ax, \text{ where } D \equiv \frac{d}{dx}.$$

First step: Determination of complementary function (C.F.)

The complementary function is found from

$$(D + ia)(D - ia)y = 0.$$

Let $y = e^{mx}$ be a trial solution, then the auxiliary equation becomes $(m + ia)(m - ia) = 0$, the roots of which are $-ia$, ia.

$$\therefore \quad \text{C.F.} = c_1 \cos ax + c_2 \sin ax.$$

Second step: Determination of particular integral (P.I.)

$$\text{P.I.} = \frac{1}{(D+ia)(D-ia)} \sec ax = \frac{1}{2ia}\left(\frac{1}{D-ia} - \frac{1}{D+ia}\right) \sec ax \qquad \text{...(1)}$$

Now, $$\frac{1}{D-ia} \sec ax = e^{iax} \int e^{-iax} \sec ax\, dx$$

$$= e^{iax} \int \frac{(\cos ax - i \sin ax)}{\cos ax} dx \qquad [\because e^{i\theta} = \cos\theta + i \sin\theta]$$

$$= e^{iax}\left(x + \frac{i}{a} \log \cos ax \right)$$

$$= (\cos ax + i \sin ax)\left(x + \frac{i}{a} \log \cos ax \right)$$

$$= \left(x \cos ax - \frac{1}{a} \sin ax \log \cos ax \right)$$

$$+ i\left(x \sin ax + \frac{1}{a} \cos ax \log \cos ax \right)$$

Similarly (replacing i by $-i$), we have

$$\frac{1}{D + ia} \sec ax = \left(x \cos ax - \frac{1}{a} \sin ax \log \cos ax \right)$$

$$- i\left(x \sin ax + \frac{1}{a} \cos ax \log \cos ax \right)$$

Therefore, from (1) we get,

$$\text{P.I.} = \frac{1}{2ia}\left(\frac{1}{D - ia} \sec ax - \frac{1}{D + ia} \sec ax \right)$$

$$= \frac{x \sin ax}{a} + \frac{\cos ax \log \cos ax}{a^2}.$$

Third step: The general solution is therefore $y = \text{C.F.} + \text{P.I.} = c_1 \cos ax + c_2 \sin ax + \frac{x \sin ax}{a}$ $+ \frac{\cos ax \log \cos ax}{a^2}$, where c_1, c_2 are two arbitrary constants.

2.5 SHORT METHODS FOR FINDING PARTICULAR INTEGRALS (P.I.) IN SOME SPECIAL CASES

Expand $\{f(D)\}^{-1}$ in ascending powers of D and operate on x^m. The terms of this expansion beyond mth power of D need not be considered, because the result of their operation on x^m becomes zero.

Note: This result is also valid for general nth $(n > 2)$ order linear differential equations with constant coefficients.

Example: Solve: $\dfrac{d^2y}{dx^2}-4\dfrac{dy}{dx}+4y = x^2+x.$

Solution: The given differential equation can be written as $(D^2-4D+4)y = x^2+x$, or $(D-2)^2y$ $= x^2+x$, where $D \equiv \dfrac{d}{dx}$.

First step: Determination of complementary function (C.F.). The complementary function is found from

$$(D-2)^2y = 0.$$

Let $y = e^{mx}$ be a trial solution, then the auxiliary equation becomes $(m-2)^2 = 0$, the roots of which are 2, 2.

$$\therefore \qquad \text{C.F.} = (c_1x+c_2)e^{2x}.$$

Second step: Determination of particular integral (P.I.)

$$\text{P.I.} = \frac{1}{(D-2)^2}(x^2+x) = \frac{1}{4\left(1-\dfrac{D}{2}\right)^2}(x^2+x) = \frac{1}{4}\left(1-\frac{D}{2}\right)^{-2}(x^2+x)$$

$$= \frac{1}{4}\left\{1+D+\frac{3}{4}D^2+\ldots\right\}(x^2+x)$$

$$= \frac{1}{4}\left\{x^2+x+2x+1+\frac{3}{2}\right\} = \frac{1}{4}\left(x^2+3x+\frac{5}{2}\right)$$

Third step: The general solution is therefore $y = \text{C.F.} + \text{P.I.} = (c_1x+c_2)e^{2x}+\dfrac{1}{4}\left(x^2+3x+\dfrac{5}{2}\right)$ where c_1, c_2 are two arbitrary constants.

Note: Remember

$$(1+x)^{-1} = 1-x+x^2-x^3+\ldots$$
$$(1-x)^{-1} = 1+x+x^2+x^3+\ldots$$
$$(1+x)^{-2} = 1-2x+3x^2-4x^3+\ldots$$
$$(1-x)^{-2} = 1+2x+3x^2+4x^3+\ldots$$

Case II: $X = e^{ax}$, a is a constant. Then

(*i*) $\text{P.I.} = \dfrac{1}{f(D)}e^{ax} = \dfrac{e^{ax}}{f(a)}$, if $f(a) \neq 0$ ...(2.17)

(ii) P.I. $= \dfrac{1}{f(D)}e^{ax} = x\dfrac{e^{ax}}{f'(a)}$, if $f(a) = 0$ but $f'(a) \neq 0$ ...(2.18)

(iii) P.I. $= \dfrac{1}{f(D)}e^{ax} = x^2\dfrac{e^{ax}}{f''(a)}$, if $f(a) = f'(a) = 0$ ...(2.19)

Proof: (i) Here $De^{ax} = ae^{ax}$, $D^2e^{ax} = a^2e^{ax}$.

$\therefore \quad (D^2+P_1D+P_2)e^{ax} = (a^2+P_1a+P_2)e^{ax}$, i.e., $f(D)e^{ax} = f(a)e^{ax}$.

Operating on both sides by $\dfrac{1}{f(D)}$, we have

$$\frac{1}{f(D)}\{f(D)e^{ax}\} = \frac{1}{f(D)}\{f(a)e^{ax}\}, \quad \text{or} \quad e^{ax} = f(a)\frac{1}{f(D)}e^{ax}$$

$$\therefore \quad \frac{1}{f(D)}e^{ax} = \frac{e^{ax}}{f(a)}, \text{ provided } f(a) \neq 0.$$

(ii) If $f(a) = 0$, i.e., a is a root of the auxiliary equation of the corresponding homogeneous differential equation, the above rule fails and we proceed as follows.

Here $D - a$ is a factor of $f(D)$. Suppose

$$f(D) = (D-a)\,\varphi(D), \text{ where } f'(a) = \varphi(a) \neq 0. \text{ Then}$$

$$\frac{1}{f(D)}e^{ax} = \frac{1}{D-a}\cdot\frac{1}{\varphi(D)}e^{ax} = \frac{1}{D-a}\cdot\frac{1}{\varphi(a)}e^{ax} \qquad \text{[by (2.17)]}$$

$$= \frac{1}{f'(a)}\cdot\frac{1}{D-a}e^{ax} = \frac{1}{f'(a)}e^{ax}\int e^{ax}\cdot e^{-ax}dx$$

$$= \frac{1}{f'(a)}e^{ax}\int dx = \frac{xe^{ax}}{f'(a)}, \text{ provided } f(a) = 0,\ f'(a) \neq 0.$$

(iii) If $f(a) = f'(a) = 0$, then $f(D) = (D - a)^2$, for the differential equation under consideration.

$$\therefore \quad \frac{1}{f(D)}e^{ax} = \frac{1}{(D-a)^2}e^{ax} = \frac{1}{D-a}\cdot\frac{1}{D-a}e^{ax} = \frac{1}{D-a}e^{ax}\int e^{ax}\cdot e^{-ax}dx$$

$$= \frac{1}{D-a}xe^{ax} = e^{ax}\int xe^{ax}\cdot e^{-ax}dx = e^{ax}\int x\,dx$$

$$= \frac{x^2}{2}e^{ax} = x^2\frac{e^{ax}}{f''(a)} \qquad [\because f''(D) = 2], \text{ provided } f(a) = f'(a) = 0.$$

Note: These results are also valid for general nth $(n > 2)$ order linear differential equations with constant coefficients provided for (iii) $f''(a) \neq 0$ and so on.

Example: Solve: $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = e^x$, if $y = 3$ and $\dfrac{dy}{dx} = 3$, when $x = 0$.

Solution: The given differential equation can be written as $(D^2 - 3D + 2)y = e^x$, or $(D - 1)(D - 2)y = e^x$, where $D \equiv \dfrac{d}{dx}$.

First step: Determination of complementary function (C.F.)

The complementary function is found from

$$(D-1)(D-2)y = 0.$$

Let $y = e^{mx}$ be a trial solution, then the auxiliary equation becomes $(m-1)(m-2) = 0$, the r of which are 1, 2.

$$\therefore \quad \text{C.F.} = c_1e^x + c_2e^{2x}.$$

Second step: Determination of particular integral (P.I.)

$$\text{P.I.} = \frac{1}{(D-1)(D-2)}e^x = \frac{1}{(D-1)(1-2)}e^x = -\frac{1}{D-1}e^x$$

$$= -e^x\int e^x \cdot e^{-x}dx = -e^x\int dx = -xe^x.$$

Third step: The general solution is therefore $y = \text{C.F.} + \text{P.I.} = c_1e^x + c_2e^{2x} - xe^x$, where c are two arbitrary constants.

$$\therefore \quad \frac{dy}{dx} = c_1e^x + 2c_2e^{2x} - e^x - xe^x = (c_1 - 1 - x)e^x + 2c_2e^{2x}.$$

By question, $y = \dfrac{dy}{dx} = 3$, when $x = 0$.

$$\therefore \quad \left.\begin{array}{r} c_1 + c_2 = 3 \\ c_1 - 1 + 2c_2 = 3 \end{array}\right\} \quad \text{solving, } c_1 = 2,\ c_2 = 1.$$

Hence the required solution is $y = (1-x)\,e^x + 2e^{2x}$.

Case III: $X = e^{ax}V$, where V is any function of x.

Then
$$\text{P.I.} = \frac{1}{f(D)}e^{ax}V = e^{ax}\cdot\frac{1}{f(D+a)}V. \qquad \ldots(2$$

Proof: If u is a function of x, then

$$D(e^{ax}u) = e^{ax}Du + ae^{ax}u = e^{ax}(D+a)u$$

$$D^2(e^{ax}u) = e^{ax}D^2u + 2ae^{ax}Du + a^2e^{ax}u$$
$$= e^{ax}\,(D+a)^2\,u.$$

$$\therefore \quad (D^2 + P_1D + P_2)e^{ax}u = e^{ax}\{(D+a)^2 + P_1(D+a) + P_2\}u$$

$$\therefore \quad f(D)e^{ax}u = e^{ax}f(D+a)u$$

Operating on both sides by $\dfrac{1}{f(D)}$, we have

$$\frac{1}{f(D)}\{f(D)e^{ax}u\} = \frac{1}{f(D)}\{e^{ax}f(D+a)u\}$$

or
$$e^{ax}u = \frac{1}{f(D)}\{e^{ax}f(D+a)u\}$$

Let us put $f(D+a)u = V$, *i.e.*, $u = \dfrac{1}{f(D+a)}V$, therefore

$$e^{ax}\frac{1}{f(D+a)}V = \frac{1}{f(D)}(e^{ax}V), \text{ i.e., } \frac{1}{f(D)}(e^{ax}V) = e^{ax}\frac{1}{f(D+a)}V.$$

Note: This result is also valid for general nth ($n > 2$) order linear differential equations with constant coefficients.

Example: Solve: $\dfrac{d^2y}{dx^2} - 4y = x \sinh x.$

Solution: The given differential equation can be written as $(D^2 - 4)y = x \sinh x$, or $(D-2)(D+2)y = x \sinh x$, where $D \equiv \dfrac{d}{dx}$.

First step: Determination of complementary function (C.F.)

The complementary function is found from

$$(D-2)(D+2)y = 0.$$

Let $y = e^{mx}$ be a trial solution, then the auxiliary equation becomes $(m-2)(m+2) = 0$, the roots of which are $2, -2$.

$$\therefore \qquad \text{C.F.} = c_1e^{2x} + c_2e^{-2x}$$

Second step: Determination of particular integral (P.I.)

$$\text{P.I.} = \frac{1}{D^2-4}x\sinh x = \frac{1}{D^2-4}x\frac{1}{2}(e^x - e^{-x})$$

$$= \frac{1}{2}\left\{\frac{1}{D^2-4}e^x x - \frac{1}{D^2-4}e^{-x}x\right\} = \frac{1}{2}\left\{e^x\frac{1}{(D+1)^2-4}x - e^{-x}\frac{1}{(D-1)^2-4}x\right\}$$

$$= \frac{1}{2}\left\{e^x\frac{1}{D^2+2D-3}x - e^{-x}\frac{1}{D^2-2D-3}x\right\}$$

$$= \frac{1}{2}\left[\frac{e^x}{-3}\left\{1-\left(\frac{2D}{3}+\frac{D^2}{3}\right)\right\}^{-1}x - \frac{e^{-x}}{-3}\left\{1+\left(\frac{2D}{3}-\frac{D^2}{3}\right)\right\}^{-1}x\right]$$

$$= -\frac{1}{6}\left[e^x\left(1+\frac{2D}{3}+\frac{D^2}{3}+\ldots\right)x - e^{-x}\left(1-\frac{2D}{3}+\frac{D^2}{3}+\ldots\right)x\right]$$

$$= -\frac{1}{6}\left[e^x\left(x+\frac{2}{3}\right) - e^{-x}\left(x-\frac{2}{3}\right)\right] = -\frac{x}{3}\left(\frac{e^x-e^{-x}}{2}\right) - \frac{2}{9}\left(\frac{e^x+e^{-x}}{2}\right)$$

$$= -\frac{x}{3}\sinh x - \frac{2}{9}\cosh x.$$

Third step: *The general solution is therefore* $y = \text{C.F.} + \text{P.I.} = c_1 e^{2x} + c_2 e^{-2x} - \frac{x}{3}\sinh x - 2$

where c_1, c_2 *are two arbitrary constants.*

Note: (*i*) $\sinh x = \frac{1}{2}(e^x - e^{-x})$, $\cosh x = \frac{1}{2}(e^x + e^{-x})$,

$$\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{1}{\tanh x}, \quad \text{cosech } x = \frac{1}{\sinh x},$$

$$\text{sech}\, x = \frac{1}{\cosh x}, \quad \cosh^2 x - \sinh^2 x = 1$$

(*ii*) $\frac{d}{dx}\sinh x = \cosh x$, $\frac{d}{dx}\cosh x = \sinh x$.

Case IV: $X = \sin(ax + b)$ or $\cos(ax + b)$. Then

$$(i)\ \text{P.I.} = \frac{1}{f(D)}\sin(ax+b) = \frac{1}{\varphi(D^2)}\sin(ax+b)$$

$$= \frac{1}{\varphi(-a^2)}\sin(ax+b), \text{ provided } \varphi(-a^2) \neq 0, \quad \ldots(1)$$

$$(ii)\ \text{P.I.} = \frac{1}{f(D)}\cos(ax+b) = \frac{1}{\varphi(D^2)}\cos(ax+b)$$

$$= \frac{1}{\varphi(-a^2)}\cos(ax+b), \text{ provided } \varphi(-a^2) \neq 0, \quad \ldots(2)$$

if $f(D) = \varphi(D^2)$ and it is possible to express $f(D)$ in terms of $\varphi(D^2)$.

Proof: (*i*) Now

$$D\sin(ax+b) = a\cos(ax+b)$$

$$D^2\sin(ax+b) = -a^2\sin(ax+b)$$

$$D^3\sin(ax+b) = -a^3\cos(ax+b)$$

$$D^4\sin(ax+b) = a^4\sin(ax+b)$$

$$\therefore \quad D^2\sin(ax+b) = (-a^2)\sin(ax+b),$$

$$(D^2)^2\sin(ax+b) = (-a^2)^2\sin(ax+b).$$

In general $(D^2)^r\sin(ax+b) = (-a^2)^r\sin(ax+b)$, $r = 1, 2, 3, \ldots$

$$\therefore \quad \varphi(D^2)\sin(ax+b) = \varphi(-a^2)\sin(ax+b).$$

Operating on both sides by $\frac{1}{\varphi(D^2)}$, we have

$$\frac{1}{\varphi(D^2)}\{\varphi(D^2)\sin(ax+b)\} = \frac{1}{\varphi(D^2)}\{\varphi(-a^2)\sin(ax+b)\}$$

$$\sin(ax+b) = \varphi(-a^2)\frac{1}{\varphi(D^2)}\sin(ax+b)$$

$$\therefore \quad \frac{1}{\varphi(D^2)}\sin(ax+b) = \frac{1}{\varphi(-a^2)}\sin(ax+b), \text{ provided } \varphi(-a^2) \neq 0.$$

(ii) Proceed as in (*i*).

Note: If $\varphi(-a^2) = 0$, proceed with the method illustrated in Example 2 below.

Example 1: Solve $\dfrac{d^2y}{dx^2} + y = 2\cos^2 x$.

Solution: The given differential equation can be written as $(D^2+1)y = 2\cos^2 x$, where $D \equiv \dfrac{d}{dx}$.

First step: Determination of complementary function (C.F.).

The complementary function is found from

$$(D^2 + 1)\, y = 0.$$

Let $y = e^{mx}$ be a trial solution, then the auxiliary equation becomes $m^2 + 1 = 0$, the roots of which are $i, -i$.

$$\therefore \quad \text{C.F.} = c_1 \cos x + c_2 \sin x.$$

Second step: Determination of particular integral (P.I.).

$$\text{P.I.} = \frac{1}{D^2+1}2\cos^2 x = \frac{1}{D^2+1}(1+\cos 2x) = (1+D^2)^{-1}1 + \frac{1}{D^2+1}\cos 2x$$

$$= (1-D^2+\ldots)1 + \frac{1}{-(2)^2+1}\cos 2x \quad \text{[replacing } D^2 \text{ by } -(2)^2 \text{ in the second term]}$$

$$= 1 - \frac{1}{3}\cos 2x.$$

Third step: The general solution is therefore

$$y = \text{C.F.} + \text{P.I.} = c_1 \cos x + c_2 \sin x + 1 - \frac{1}{3}\cos 2x,$$

where c_1, c_2 are two arbitrary constants.

Example 2: Solve $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = x + e^x \cos x$.

Solution: Let us write the given differential equation as

$$(D^2 - 2D + 2)\, y = x + e^x \cos x, \text{ where } D \equiv \frac{d}{dx}.$$

First step: Determination of complementary function (C.F.).

The complementary function is found from

$$(D^2 - 2D + 2)y = 0$$

Let $y = e^{mx}$ be a trial solution, then the auxiliary equation becomes $m^2 - 2m + 2 = 0$, the roots of which are $1 \pm i$.

$$\therefore \qquad \text{C.F.} = e^{x}(c_1\cos x + c_2\sin x).$$

Second step: Determination of particular integral (P.I.).

$$\text{P.I.} = \frac{1}{D^2-2D+2}(x+e^x\cos x) = \frac{1}{D^2-2D+2}x + \frac{1}{D^2-2D+2}(e^x\cos x)$$

$$= \frac{1}{2}\left\{1-\left(D-\frac{D^2}{2}\right)\right\}^{-1} x + e^x\frac{1}{(D+1)^2-2(D+1)+2}\cos x$$

$$= \frac{1}{2}\left(1+D-\frac{D^2}{2}+\ldots\right)x + e^x\frac{1}{D^2+1}\cos x$$

In $\frac{1}{D^2+1}\cos x$, if we replace D^2 by -1^2, the method fails.

Now, $$\frac{1}{D^2+1}(\cos x + i\sin x) = \frac{1}{D^2+1}e^{ix} = e^{ix}\frac{1}{(D+i)^2+1}1$$

$$= e^{ix}\frac{1}{D^2+2iD}1 = e^{ix}\frac{1}{2iD}\left(1+\frac{D}{2i}\right)^{-1}1$$

$$= e^{ix}\frac{1}{2iD}\left(1-\frac{D}{2i}+\ldots\right)1 = e^{ix}\cdot\frac{1}{2i}\cdot\frac{1}{D}1 = \frac{e^{ix}}{2i}x$$

$$= \frac{(\cos x + i\sin x)x}{2i} = \frac{1}{2}x\sin x - \frac{i}{2}x\cos x$$

$$\therefore \quad \frac{1}{D^2+1}(\cos x + i\sin x) = \frac{1}{2}x\sin x - \frac{i}{2}x\cos x$$

Equating real part from both sides, we get

$$\frac{1}{D^2+1}\cos x = \frac{1}{2}x\sin x$$

Hence, from (1), we get

$$\text{P.I.} = \frac{1}{2}(x+1) + \frac{1}{2}xe^x\sin x.$$

Third step: The general solution is therefore

$$y = \text{C.F.} + \text{P.I.} = e^x(c_1\cos x + c_2\sin x) + \frac{1}{2}(x+1) + \frac{1}{2}xe^x\sin x,$$

where c_1, c_2 are two arbitrary constants.

Example 3: Solve $(D^2-4D+3)y = \sin 3x\cos 2x$, where $D \equiv \frac{d}{dx}$.

Solution:

First step: Determination of complementary function (C.F.).

The complementary function is found from

$$(D^2-4D+3)y = 0.$$

Let $y = e^{mx}$ be a trial solution, then the auxiliary equation becomes $m^2-4m+3 = 0$, or $m^2-3m-m+3 = 0$, or $(m-3)(m-1) = 0$, the roots are $m = 3, 1$.

$$\therefore \quad \text{C.F.} = c_1e^{3x}+c_2e^{x}.$$

Second step: Determination of particular integral (P.I.).

$$\text{P.I.} = \frac{1}{D^2-4D+3}(\sin 3x\cos 2x) = \frac{1}{D^2-4D+3}\left\{\frac{1}{2}(\sin 5x+\sin x)\right\}$$

$$= \frac{1}{2}\frac{D^2+3+4D}{(D^2+3)^2-16D^2}\sin 5x+\frac{1}{2}\frac{D^2+3+4D}{(D^2+3)^2-16D^2}\sin x$$

$$= \frac{1}{2}\frac{D^2+4D+3}{(-5^2+3)^2-16(-5^2)}\sin 5x+\frac{1}{2}\frac{D^2+4D+3}{(-1^2+3)^2-16(-1^2)}\sin x$$

$$= \frac{1}{2}\cdot\frac{1}{884}(-25\sin 5x+20\cos 5x+3\sin 5x)$$

$$+\frac{1}{2}\cdot\frac{1}{20}(-\sin x+4\cos x+3\sin x)$$

$$= \frac{1}{884}(10\cos 5x-11\sin 5x)+\frac{1}{20}(\sin x+2\cos x)$$

Third step: The general solution is therefore

$$y = \text{C.F.} + \text{P.I.} = c_1e^{3x}+c_2e^{x}+\frac{1}{884}(10\cos 5x-11\sin 5x)+\frac{1}{20}(\sin x+2\cos x),$$

where c_1, c_2 are arbitrary constants.

Case V: $X = xV$, where V is any function of x.

Then $$\text{P.I.} = \frac{1}{f(D)}xV = \left\{x-\frac{1}{f(D)}f'(D)\right\}\frac{1}{f(D)}V \qquad \ldots(2.23)$$

Proof: Let $$V_1 = \frac{1}{f(D)}V.$$

Now, $$D(xV_1) = xDV_1 + V_1$$

$$D^2(xV_1) = DV_1+xD^2V_1+DV_1 = xD^2V_1+2DV_1$$

$$\therefore \quad (D^2+P_1D+P_2)xV_1 = xD^2V_1+2DV_1+P_1xDV_1+P_1V_1+P_2xV_1$$

$$= x(D^2+P_1D+P_2)V_1+(2D+P_1)V_1$$

Thus $$f(D)xV_1 = xf(D)V_1+f'(D)V_1$$

or $$f(D)x\left\{\frac{1}{f(D)}V\right\} = xf(D)\left\{\frac{1}{f(D)}V\right\}+f'(D)\left\{\frac{1}{f(D)}V\right\} \qquad \left[\because V_1 = \frac{1}{f(D)}V\right]$$

$$f(D)x\left\{\frac{1}{f(D)}V\right\} = xV + f'(D)\left\{\frac{1}{f(D)}V\right\}$$

or

$$x\left\{\frac{1}{f(D)}V\right\} = \frac{1}{f(D)}xV + \frac{1}{f(D)}f'(D)\left\{\frac{1}{f(D)}V\right\}$$

or

$$\therefore \quad \frac{1}{f(D)}xV = \left\{x - \frac{1}{f(D)}f'(D)\right\}\frac{1}{f(D)}V.$$

Note: (*i*) By successive applications of this method the P.I. of $X = x^m V$, m is a positive integer, may be found.

(*ii*) The result (2.23) is also valid for general nth ($n > 2$) order linear differential equations with constant coefficients.

Example 1: Solve $\dfrac{d^2y}{dx^2} + 4y = x\cos x$.

Solution: Let us write the given differential equation as $(D^2 + 4)y = x\cos x$, where $D \equiv \dfrac{d}{dx}$.

First step: Determination of complementary function (C.F.).

The complementary function is found from

$$(D^2 + 4)y = 0.$$

Let $y = e^{mx}$ be a trial solution, then the auxiliary equation becomes $m^2 + 4 = 0$, the roots of which are $\pm 2i$.

$$\therefore \qquad \text{C.F.} = c_1\cos 2x + c_2 \sin 2x.$$

Second step: Determination of particular integral (P.I.).

$$\text{P.I.} = \frac{1}{D^2+4}x\cos x = \left(x - \frac{2D}{D^2+4}\right)\frac{1}{D^2+4}\cos x$$

$$= \left(x - \frac{2D}{D^2+4}\right)\frac{\cos x}{-1^2+4} \qquad \text{(replacing } D^2 \text{ by } -1^2\text{)}$$

$$= \frac{x}{3}\cos x + \frac{2}{3}\cdot\frac{1}{D^2+4}\sin x = \frac{x}{3}\cos x + \frac{2}{3}\cdot\frac{\sin x}{-1^2+4}$$

$$= \frac{x}{3}\cos x + \frac{2}{9}\sin x.$$

Third step: The general solution is therefore

$$y = \text{C.F.} + \text{P.I.} = c_1\cos 2x + c_2\sin 2x + \frac{x}{3}\cos x + \frac{2}{9}\sin x,$$

c_1, c_2 are two arbitrary constants.

Example 2: Solve $\dfrac{d^2y}{dx^2} - y = x^2 \sin x$.

Solution: Let us write the given differential equation as $(D^2 - 1)y = x^2 \sin x$, where $D \equiv \dfrac{d}{dx}$.

First step: Determination of complementary function (C.F.).

The complementary function is found from

$$(D^2 - 1)y = 0.$$

Let $y = e^{mx}$ be a trial solution, then the auxiliary equation becomes $m^2 - 1 = 0$, the roots of which are ± 1.

$$\therefore \qquad \text{C.F.} = c_1 e^{x} + c_2 e^{-x}.$$

Second step: Determination of particular integral (P.I.).

$$\text{P.I.} = \frac{1}{D^2 - 1} x^2 \sin x = \frac{1}{D^2 - 1}\{x(x \sin x)\}$$

$$= \left(x - \frac{2D}{D^2 - 1}\right)\frac{1}{D^2 - 1}(x \sin x) = \left(x - \frac{2D}{D^2 - 1}\right)\left(x - \frac{2D}{D^2 - 1}\right)\frac{1}{D^2 - 1}\sin x$$

$$= \left(x - \frac{2D}{D^2 - 1}\right)\left(x - \frac{2D}{D^2 - 1}\right)\frac{\sin x}{-1^2 - 1} \qquad \text{(replacing } D^2 \text{ by } -1^2\text{)}$$

$$= -\frac{1}{2}\left(x - \frac{2D}{D^2 - 1}\right)\left(x \sin x - \frac{1}{D^2 - 1} 2\cos x\right) = -\frac{1}{2}\left(x - \frac{2D}{D^2 - 1}\right)\left(x \sin x - \frac{2\cos x}{-1^2 - 1}\right)$$

$$= -\frac{1}{2}\left\{x^2 \sin x + x \cos x - \frac{1}{D^2 - 1} 2(\sin x + x \cos x - \sin x)\right\}$$

$$= -\frac{1}{2}(x^2 \sin x + x \cos x) + \frac{1}{D^2 - 1} x \cos x$$

$$= -\frac{1}{2}(x^2 \sin x + x \cos x) + \left(x - \frac{2D}{D^2 - 1}\right)\frac{1}{D^2 - 1}\cos x$$

$$= -\frac{1}{2}(x^2 \sin x + x \cos x) + \left(x - \frac{2D}{D^2 - 1}\right)\frac{\cos x}{-1^2 - 1}$$

$$= -\frac{1}{2}(x^2 \sin x + x \cos x) - \frac{1}{2}\left(x \cos x + \frac{1}{D^2 - 1} 2 \sin x\right)$$

$$= -\frac{1}{2}(x^2 \sin x + x \cos x) - \frac{1}{2}\left(x \cos x + \frac{2 \sin x}{-1^2 - 1}\right)$$

$$= -\frac{1}{2}x^2 \sin x - x \cos x + \frac{1}{2}\sin x$$

Third step: The general solution is therefore $y = \text{C.F.} + \text{P.I.} = c_1e^x + c_2e^{-x} - \frac{1}{2}x^2 \sin x - x$ $+\frac{1}{2}\sin x$, where c_1, c_2 are two arbitrary constants.

MISCELLANEOUS EXAMPLES

Example 1: Solve the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^x \cos x$$

(W.B.U.T. 2...)

Solution: Let us write the given differential equation as

$$(D^2 - 5D + 6)y = e^x \cos x, \text{ where } D \equiv \frac{d}{dx}.$$

First step: Determination of complementary function (C.F.).

The complementary function is found from

$$(D^2 - 5D + 6)y = 0.$$

Let $y = e^{mx}$ be a trial solution, then the auxiliary equation becomes $m^2 - 5m + 6 = 0$, or $(m-3)$ $(m-2) = 0$, the roots of which are 3, 2.

$\therefore$ $$\text{C.F.} = c_1 e^{3x} + c_2 e^{2x}.$$

Second step: Determination of particular integral (P.I.).

$$\text{P.I.} = \frac{1}{D^2 - 5D + 6} e^x \cos x = e^x \frac{1}{(D+1)^2 - 5(D+1) + 6} \cos x$$

$$= e^x \frac{1}{D^2 - 3D + 2} \cos x = e^x \frac{D^2 + 2 + 3D}{(D^2+2)^2 - 9D^2} \cos x$$

$$= e^x \frac{D^2 + 3D + 2}{(-1^2+2)^2 - 9(-1^2)} \cos x$$

$$= \frac{e^x}{10}(D^2 + 3D + 2)\cos x = \frac{e^x}{10}(-\cos x - 3\sin x + 2\cos x)$$

$$= \frac{e^x}{10}(\cos x - 3\sin x).$$

Third step: The general solution is therefore

$$y = \text{C.F.} + \text{P.I.} = c_1e^{3x} + c_2e^{2x} + \frac{e^x}{10}(\cos x - 3\sin x), \text{ where } c_1, c_2 \text{ are}$$

two arbitrary constants.

Example 2: Solve $(D^2-2D)y = e^x \sin x$, where $D \equiv \frac{d}{dx}$. (W.B.U.T. 2007)

Solution:
First step: Determination of complementary function (C.F.).
The complementary function is found from

$$(D^2-2D)y = 0.$$

Let $y = e^{mx}$ be a trial solution, then the auxiliary equation becomes $m^2 - 2m = 0$, or $m(m-2) = 0$, the roots of which are 0, 2.

$$\therefore \quad \text{C.F.} = c_1 e^{0x} + c_2 e^{2x} = c_1 + c_2 e^{2x}.$$

Second step: Determination of particular integral (P.I.).

$$\text{P.I.} = \frac{1}{D^2-2D} e^x \sin x = e^x \frac{1}{(D+1)^2 - 2(D+1)} \sin x = e^x \frac{1}{D^2-1} \sin x$$

$$= e^x \frac{\sin x}{-1^2-1} = -\frac{1}{2} e^x \sin x.$$

Third step: The general solution is therefore

$$y = \text{C.F.} + \text{P.I.} = c_1 + c_2 e^{2x} - \frac{1}{2} e^x \sin x,$$ where, c_1, c_2 are two arbitrary constants.

Example 3: Solve: $(D^2+4)y = x\sin^2 x$, where $D \equiv \frac{d}{dx}$. (W.B.U.T. 2008, 2010)

Solution:
First step: Determination of complementary function (C.F.).
The complementary function is found from

$$(D^2+4)y = 0.$$

Let $y = e^{mx}$ be a trial solution, then the auxiliary equation becomes $m^2 + 4 = 0$, the roots are $m = \pm 2i$.

$$\therefore \quad \text{C.F.} = c_1 \cos 2x + c_2 \sin 2x.$$

Second step:

$$\text{P.I.} = \frac{1}{D^2+4} x \sin^2 x = \frac{1}{2} \cdot \frac{1}{D^2+4} x(1-\cos 2x) = \frac{1}{2} \cdot \frac{1}{D^2+4} x - \frac{1}{2} \frac{1}{D^2+4} x \cos 2x \quad \text{...(1)}$$

Now,
$$\frac{1}{D^2+4} x = \frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1} x = \frac{1}{4}\left(1-\frac{D^2}{4}+\dots\right)x = \frac{x}{4} \quad \text{...(2)}$$

$$\frac{1}{D^2+4} x \cos 2x = \left(x - \frac{2D}{D^2+4}\right)\frac{1}{D^2+4} \cos 2x \quad \text{...(3)}$$

In $\frac{1}{D^2+4} \cos 2x$, if we replace D^2 by -2^2, the method fails.

$$\frac{1}{D^2+4}(\cos 2x + i\sin 2x) = \frac{1}{D^2+4}e^{2ix} = e^{2ix}\frac{1}{(D+2i)^2+4}1$$

$$= e^{2ix}\frac{1}{D^2+4iD}1 = e^{2ix}\frac{1}{4iD}\left(1+\frac{D}{4i}\right)^{-1}1 = e^{2ix}\frac{1}{4iD}\left(1-\frac{D}{4i}+\dots\right)1$$

$$= e^{2ix}\cdot\frac{1}{4i}\cdot\frac{1}{D}1 = \frac{e^{2ix}}{4i}x = \frac{(\cos 2x + i\sin 2x)x}{4i} = \frac{1}{4}x\sin 2x - \frac{i}{4}x\cos 2x$$

$$\therefore \quad \frac{1}{D^2+4}(\cos 2x + i\sin 2x) = \frac{1}{4}x\sin 2x - \frac{i}{4}x\cos 2x.$$

Equating real and imaginary parts from both sides, we get

$$\frac{1}{D^2+4}\cos 2x = \frac{1}{4}x\sin 2x, \quad \frac{1}{D^2+4}\sin 2x = -\frac{1}{4}x\cos 2x$$

Therefore, from (3), we get

$$\frac{1}{D^2+4}x\cos 2x = \left(x - \frac{2D}{D^2+4}\right)\frac{1}{4}x\sin 2x = \frac{1}{4}x^2\sin 2x - \frac{1}{2}\frac{1}{D^2+4}D(x\sin 2x)$$

$$= \frac{1}{4}x^2\sin 2x - \frac{1}{2}\cdot\frac{1}{D^2+4}(\sin 2x + 2x\cos 2x)$$

$$= \frac{1}{4}x^2\sin 2x - \frac{1}{2}\cdot\frac{1}{D^2+4}\sin 2x - \frac{1}{D^2+4}x\cos 2x$$

$$\therefore \quad 2\cdot\frac{1}{D^2+4}x\cos 2x = \frac{1}{4}x^2\sin 2x - \frac{1}{2}\cdot\frac{1}{D^2+4}\sin 2x$$

$$= \frac{1}{4}x^2\sin 2x + \frac{1}{8}x\cos 2x \qquad \text{[by (4)]}$$

$$\therefore \quad \frac{1}{D^2+4}x\cos 2x = \frac{1}{8}x^2\sin 2x + \frac{1}{16}x\cos 2x \qquad \dots(5)$$

From (1), (2), (3) and (5), we get

$$\text{P.I.} = \frac{x}{8} - \frac{x^2\sin 2x}{16} - \frac{x\cos 2x}{32}.$$

Third step: The general solution is therefore

$$y = \text{C.F.} + \text{P.I.} = c_1\cos 2x + c_2\sin 2x + \frac{x}{8} - \frac{x^2\sin 2x}{16} - \frac{x\cos 2x}{32},$$

where c_1, c_2 are two arbitrary constants.

Example 4: Solve the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2e^{3x}$$

(W.B.U.T. 2009, 2010)

Solution: Let us write the given differential equation as

$$(D^2-5D+6)y = x^2e^{3x}, \text{ where } D \equiv \frac{d}{dx}.$$

First step: Determination of complementary function (C.F.).
The complementary function is found from

$$(D^2-5D+6)y = 0.$$

Let $y = e^{mx}$ be a trial solution, then the auxiliary equation becomes $m^2-5m+6 = 0$, or $(m-3)(m-2) = 0$, the roots are $m = 2, 3$.

$$\therefore \quad \text{C.F.} = c_1e^{2x}+c_2e^{3x}.$$

Second step: Determination of particular integral (P.I.).

$$\text{P.I.} = \frac{1}{D^2-5D+6}(x^2e^{3x}) = e^{3x}\cdot\frac{1}{(D+3)^2-5(D+3)+6}x^2$$

$$= e^{3x}\cdot\frac{1}{D^2+D}x^2 = e^{3x}\cdot\frac{1}{D}(1+D)^{-1}x^2$$

$$= e^{3x}\cdot\frac{1}{D}(1-D+D^2-D^3+\ldots)x^2 = e^{3x}\frac{1}{D}(x^2-2x+2)$$

$$= e^{3x}\left(\frac{1}{3}x^3-x^2+2x\right).$$

Third step: The general solution is therefore

$$y = \text{C.F.} + \text{P.I.} = c_1e^{2x}+c_2e^{3x}+e^{3x}\left(\frac{1}{3}x^3-x^2+2x\right)$$

where c_1, c_2 are two arbitrary constants.

Example 5: Solve $\dfrac{d^2y}{dx^2}+4\dfrac{dy}{dx}+4y = 3\sin x+4\cos x$, $y(0) = 1$ and $y'(0) = 0$.

Solution: Let us write the given differential equation as

$$(D^2+4D+4)y = 3\sin x+4\cos x, \text{ or } (D+2)^2y = 3\sin x+4\cos x,$$

where $D \equiv \dfrac{d}{dx}$.

First step: Determination of complementary function (C.F.).
The complementary function is found from

$$(D+2)^2y = 0.$$

Let $y = e^{mx}$ be a trial solution, then the auxiliary equation becomes $(m+2)^2 = 0$, the roots are $m = -2, -2$.

$$\therefore \quad \text{C.F.} = (c_1+c_2x)e^{-2x}.$$

Second step: Determination of particular integral (P.I.).

$$\text{P.I.} = \frac{1}{(D+2)^2}(3\sin x+4\cos x) = \frac{(D-2)^2}{(D^2-4)^2}(3\sin x+4\cos x)$$

$$= \frac{D^2-4D+4}{(-1^2-4)^2}(3\sin x+4\cos x)$$

$$= \frac{1}{25}[-3\sin x-4\cos x-4(3\cos x-4\sin x)+12\sin x+16\cos x]$$

$$= \frac{1}{25}(25\sin x) = \sin x.$$

Third step: The general solution is therefore

$$y = \text{C.F.} + \text{P.I.} = (c_1+c_2x)e^{-2x}+\sin x, \text{ where } c_1, c_2 \text{ are two arbitrary constants.}$$

$$\therefore \quad y'(x) = \frac{dy}{dx} = c_2e^{-2x}-2(c_1+c_2x)e^{-2x}+\cos x.$$

By question, $y(0) = 1$ and $y'(0) = 0$.

$\therefore$ $c_1 = 1$, $c_2 - 2c_1 + 1 = 0$, or $c_2 = 1$ $(\because c_1 = 1)$

Hence the required solution is

$$y = (1+x)e^{-2x}+\sin x.$$

Example 6: Solve: $\dfrac{d^2y}{dx^2}-3\dfrac{dy}{dx}+2y = xe^{3x}+\sin 2x.$

Solution: Let us write the given differential equation as

$$(D^2-3D+2)y = xe^{3x}+\sin 2x, \text{ where } D \equiv \frac{d}{dx}.$$

First step: Determination of complementary function (C.F.).
The complementary function is found from

$$(D^2-3D+2)y = 0.$$

Let $y = e^{mx}$ be a trial solution, then the auxiliary equation becomes $m^2 - 3m + 2 = 0$, or $(m-2)(m-1) = 0$, the roots are $m = 1, 2$.

$$\therefore \quad \text{C.F.} = c_1e^x+c_2e^{2x}.$$

Second step: Determination of particular integral (P.I.).

$$\text{P.I.} = \frac{1}{D^2-3D+2}(xe^{3x}+\sin 2x)$$

$$= \frac{1}{D^2-3D+2}(e^{3x}x)+\frac{1}{D^2-3D+2}\sin 2x$$

$$= e^{3x}\frac{1}{(D+3)^2-3(D+3)+2}x+\frac{D^2+2+3D}{(D^2+2)^2-9D^2}\sin 2x$$

$$= e^{3x}\frac{1}{D^2+3D+2}x+\frac{D^2+3D+2}{(-2^2+2)^2-9(-2^2)}\sin 2x$$

$$= \frac{e^{3x}}{2}\left\{1+\left(\frac{3D}{2}+\frac{D^2}{2}\right)\right\}^{-1} x + \frac{1}{40}(-4\sin 2x + 6\cos 2x + 2\sin 2x)$$

$$= \frac{e^{3x}}{2}\left\{1-\frac{3D}{2}-\frac{D^2}{2}+\ldots\right\}x + \frac{1}{20}(3\cos 2x - \sin 2x)$$

$$= \frac{e^{3x}}{2}\left(x-\frac{3}{2}\right)+\frac{1}{20}(3\cos 2x - \sin 2x).$$

Third step: The general solution is therefore

$$y = \text{C.F.} + \text{P.I.} = c_1e^x + c_2e^{2x} + \frac{e^{3x}}{2}\left(x-\frac{3}{2}\right)+\frac{1}{20}(3\cos 2x - \sin 2x),$$ where c_1, c_2 are two arbitrary constants.

Example 7: Solve $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = xe^x \sin x$.

Solution: Let us write the given differential equation as

$(D^2-2D+1)y = xe^x \sin x$, or $(D-1)^2 y = xe^x \sin x$, where $D \equiv \dfrac{d}{dx}$.

First step: Determination of complementary function (C.F.).

The complementary function is found from

$$(D-1)^2 y = 0.$$

Let $y = e^{mx}$ be a trial solution, then the auxiliary equation becomes $(m-1)^2 = 0$, the roots are $m = 1, 1$.

$$\therefore \qquad \text{C.F.} = (c_1 + c_2 x)e^x.$$

Second step: Determination of particular integral (P.I.).

$$\text{P.I.} = \frac{1}{(D-1)^2} e^x x\sin x = e^x \frac{1}{(D+1-1)^2} x\sin x = e^x \frac{1}{D^2} x\sin x$$

$$= e^x \frac{1}{D}\int x\sin x\, dx = e^x \frac{1}{D}\{x(-\cos x) - \int 1\cdot(-\cos x)\, dx\}$$

$$= e^x \int(-x\cos x + \sin x)dx$$

$$= e^x[-\{x\sin x - \int 1\cdot \sin x\, dx\} - \cos x] = e^x(-x\sin x - \cos x - \cos x)$$

$$= -e^x(x\sin x + 2\cos x).$$

Third step: The general solution is therefore

$$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2 x)e^x - e^x(x\sin x + 2\cos x),$$ where c_1, c_2 are two arbitrary

Example 8: Solve $\dfrac{d^2y}{dx^2}-4\dfrac{dy}{dx}+4y = 8x^2e^{2x}\sin 2x.$

Solution: Let us write the given differential equation as
$(D^2-4D+4)y = 8x^2e^{2x}\sin 2x$, or $(D-2)^2y = 8x^2e^{2x}\sin 2x$,

where $D \equiv \dfrac{d}{dx}$.

First step: Determination of complementary function (C.F.).
The complementary function is found from

$$(D-2)^2y = 0.$$

Let $y = e^{mx}$ be a trial solution, then the auxiliary equation becomes $(m-2)^2 = 0$, the roots
$m = 2, 2.$

$$\therefore \qquad \text{C.F.} = (c_1+c_2x)e^{2x}.$$

Second step: Determination of particular integral (P.I.)

$$\text{P.I.} = \frac{1}{(D-2)^2}(8x^2e^{2x}\sin 2x) = 8e^{2x}\frac{1}{(D+2-2)^2}(x^2\sin 2x)$$

$$= 8e^{2x}\frac{1}{D^2}(x^2\sin 2x) = 8e^{2x}\frac{1}{D}\int x^2\sin 2x\,dx$$

$$= 8e^{2x}\frac{1}{D}\left\{x^2\left(-\frac{\cos 2x}{2}\right)-\int 2x\left(-\frac{\cos 2x}{2}\right)dx\right\}$$

$$= 8e^{2x}\frac{1}{D}\left\{-\frac{x^2}{2}\cos 2x + x\frac{\sin 2x}{2}-\int 1\cdot\frac{\sin 2x}{2}dx\right\}$$

$$= 8e^{2x}\int\left(-\frac{x^2}{2}\cos 2x+\frac{x}{2}\sin 2x+\frac{\cos 2x}{4}\right)dx$$

$$= 8e^{2x}\left[\left\{-\frac{x^2}{2}\frac{\sin 2x}{2}-\int(-x)\frac{\sin 2x}{2}dx\right\}+\left\{\int\frac{x}{2}\sin 2x\,dx\right\}+\frac{\sin 2x}{8}\right]$$

$$= 8e^{2x}\left\{\left(-\frac{x^2}{4}+\frac{1}{8}\right)\sin 2x+\int x\sin 2x\,dx\right\}$$

$$= 8e^{2x}\left\{\left(\frac{1}{8}-\frac{x^2}{4}\right)\sin 2x + x\left(-\frac{\cos 2x}{2}\right)-\int 1\cdot\left(-\frac{\cos 2x}{2}\right)dx\right\}$$

$$= 8e^{2x}\left\{\left(\frac{1}{8}-\frac{x^2}{4}\right)\sin 2x-\frac{x}{2}\cos 2x+\frac{\sin 2x}{4}\right\}$$

$$= e^{2x}\{(1-2x^2)\sin 2x-4x\cos 2x+2\sin 2x\}$$

$$= e^{2x}\{(3-2x^2)\sin 2x-4x\cos 2x\}.$$

Third step: The general solutio

y = C.F. + P.I. =

arbitrary constants.

Example 9: Solve $\dfrac{d^2y}{dx^2}-6\dfrac{dy}{dx}$

Solution: Let us write the giv

$(D^2-6D+13)y =$

First step: Determination of
The complementary function

$(D^2-6D+13)y$

Let $y = e^{mx}$ be a trial solution
$m = 3 \pm 2i.$

$\therefore$ C.F

Second step: Determinatio

P.

Third step: The gener

y = C.F.

where c_1, c_2 are two arbitrar

Example 10: Solve (

Solution:

First step: Determin

Third step: The general solution is therefore

y = C.F. + P.I. = $e^{2x}\{c_1 + c_2 x + (3-2x^2)\sin 2x - 4x\cos 2x\}$, where c_1, c_2 are two arbitrary constants.

Example 9: Solve $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 13y = 8e^{3x}\sin 4x + 2^x$.

Solution: Let us write the given differential equation as

$$(D^2 - 6D + 13)y = 8e^{3x}\sin 4x + 2^x, \quad \text{where } D \equiv \frac{d}{dx}.$$

First step: Determination of complementary function (C.F.).
The complementary function is found from

$$(D^2 - 6D + 13)y = 0.$$

Let $y = e^{mx}$ be a trial solution, then the auxiliary equation becomes $m^2 - 6m + 13 = 0$, the roots are $m = 3 \pm 2i$.

$$\therefore \quad \text{C.F.} = e^{3x}(c_1\cos 2x + c_2\sin 2x)$$

Second step: Determination of particular integral (P.I.)

$$\text{P.I.} = \frac{1}{D^2 - 6D + 13}(8e^{3x}\sin 4x + 2^x)$$

$$= \frac{1}{D^2 - 6D + 13}(8e^{3x}\sin 4x) + \frac{1}{D^2 - 6D + 13}e^{x\log 2}$$

$$= 8e^{3x}\frac{1}{(D+3)^2 - 6(D+3) + 13}\sin 4x + \frac{e^{x\log 2}}{(\log 2)^2 - 6\log 2 + 13}$$

$$= 8e^{3x}\cdot\frac{1}{D^2 + 4}\sin 4x + \frac{2^x}{(\log 2)^2 - 6\log 2 + 13}$$

$$= 8e^{3x}\frac{1}{-4^2 + 4}\sin 4x + \frac{2^x}{(\log 2)^2 - 6\log 2 + 13}$$

$$= -\frac{2}{3}e^{3x}\sin 4x + \frac{2^x}{(\log 2)^2 - 6\log 2 + 13}.$$

Third step: The general solution is therefore

$$y = \text{C.F.} + \text{P.I.} = e^{3x}(c_1\cos 2x + c_2\sin 2x) - \frac{2}{3}e^{3x}\sin 4x + \frac{2^x}{(\log 2)^2 - 6\log 2 + 13},$$

where c_1, c_2 are two arbitrary constants.

Example 10: Solve $(D^2 + 1)y = 3\cos^2 x + 2\sin^3 x$.

Solution:

First step: Determination of complementary function (C.F.).

The complementary function is found from

$$(D^2+1)y = 0.$$

Let $y = e^{mx}$ be a trial solution, then the auxiliary equation becomes $m^2 + 1 = 0$, the roots $m = \pm i$.

$\therefore$ $$\text{C.F.} = c_1 \cos x + c_2 \sin x.$$

Second step: Determination of particular integral (P.I.)

Now,
$$3\cos^2 x = \frac{3}{2}(1+\cos 2x).$$

$$\sin 3x = 3\sin x - 4\sin^3 x. \quad \therefore\ 2\sin^3 x = \frac{1}{2}(3\sin x - \sin 3x).$$

$\therefore$
$$\text{P.I.} = \frac{1}{D^2+1}(3\cos^2 x + 2\sin^3 x)$$

$$= \frac{3}{2}\frac{1}{D^2+1}(1+\cos 2x) + \frac{1}{2}\cdot\frac{1}{D^2+1}(3\sin x - \sin 3x)$$

$$= \frac{3}{2}(1+D^2)^{-1}1 + \frac{3}{2}\cdot\frac{1}{D^2+1}\cos 2x + \frac{3}{2}\frac{1}{D^2+1}\sin x - \frac{1}{2}\frac{1}{D^2+1}\sin 3x$$

$$= \frac{3}{2}(1-D^2+\ldots)1 + \frac{3}{2}\cdot\frac{1}{-2^2+1}\cos 2x + \frac{3}{2}\cdot\frac{1}{D^2+1}\sin x - \frac{1}{2}\cdot\frac{\sin 3x}{-3^2+1}$$

$$= \frac{3}{2} - \frac{1}{2}\cos 2x + \frac{3}{2}\frac{1}{D^2+1}\sin x + \frac{1}{16}\sin 3x \quad \ldots(1)$$

In $\frac{1}{D^2+1}\sin x$, if we replace D^2 by -1^2, the method fails.

$$\frac{1}{D^2+1}(\cos x + i\sin x) = \frac{1}{D^2+1}e^{ix} = e^{ix}\frac{1}{(D+i)^2+1}1 = e^{ix}\frac{1}{D^2+2iD}1.$$

$$= e^{ix}\frac{1}{2iD}\left(1+\frac{D}{2i}\right)^{-1}1 = e^{ix}\frac{1}{2iD}\left(1-\frac{D}{2i}+\ldots\right)1 = \frac{e^{ix}}{2i}\frac{1}{D}1$$

$$= \frac{e^{ix}}{2i}x = \frac{x(\cos x + i\sin x)}{2i} = \frac{1}{2}x\sin x - i\frac{x}{2}\cos x$$

$\therefore$
$$\frac{1}{D^2+1}(\cos x + i\sin x) = \frac{1}{2}x\sin x - i\frac{x}{2}\cos x.$$

Equating imaginary part from both sides, we get

$$\frac{1}{D^2+1}\sin x = -\frac{x}{2}\cos x.$$

Therefore, from (1), we have

$$\text{P.I.} = \frac{3}{2} - \frac{1}{2}\cos 2x - \frac{3}{4}x\cos x + \frac{1}{16}\sin 3x.$$

Third step: The general solution is therefore

$$y = \text{C.F.} + \text{P.I.} = c_1 \cos x + c_2 \sin x + \frac{3}{2} - \frac{1}{2}\cos 2x - \frac{3}{4} x \cos x + \frac{1}{16}\sin 3x,$$

where c_1, c_2 are two arbitrary constants.

Example 11: Solve $\dfrac{d^2y}{dx^2} + a^2 y = \tan ax \ (a \neq 0)$

with the symbolic operator D, where $D \equiv \dfrac{d}{dx}$.

Solution: The given equation can be written as $(D^2 + a^2)\, y = \tan ax$, or $(D+ia)(D-ia)y = \tan ax$, where $D \equiv \dfrac{d}{dx}$.

First step: Determination of complementary function (C.F.).

The complementary function is found from

$$(D+ia)(D-ia)y = 0.$$

Let $y = e^{mx}$ be a trial solution, then the auxiliary equation becomes $(m+ia)(m-ia) = 0$, the roots are $m = \pm\, ia$.

$$\therefore \quad \text{C.F.} = c_1 \cos ax + c_2 \sin ax.$$

Second step: Determination of particular integral (P.I.).

$$\text{P.I.} = \frac{1}{(D+ia)(D-ia)}\tan ax = \frac{1}{2ia}\left(\frac{1}{D-ia} - \frac{1}{D+ia}\right)\tan ax \qquad \ldots(1)$$

Now,

$$\frac{1}{D-ia}\tan ax = e^{iax}\int e^{-iax}\tan ax\, dx \qquad \left[\because \frac{1}{D-a}X = e^{ax}\int X e^{-ax}\, dx\right]$$

$$= e^{iax}\int(\cos ax - i\sin ax)\tan ax\, dx \qquad [\because e^{i\theta} = \cos\theta + i\sin\theta]$$

$$= e^{iax}\left[\int \sin ax\, dx - i\int \frac{\sin^2 ax}{\cos ax}\, dx\right]$$

$$= e^{iax}\left[-\frac{1}{a}\cos ax - i\int \frac{(1-\cos^2 ax)}{\cos ax}\, dx\right]$$

$$= e^{iax}\left[-\frac{1}{a}\cos ax - i\int \sec ax\, dx + i\int \cos ax\, dx\right]$$

$$= \frac{e^{iax}}{a}(-\cos ax - i\log|\sec ax + \tan ax| + i\sin ax)$$

$$= \frac{1}{a}(\cos ax + i\sin ax)\{-\cos ax - i(\log|\sec ax + \tan ax| - \sin ax)\}$$

$$= \frac{1}{a}(\sin ax \log|\sec ax + \tan ax| - 1) - \frac{i}{a}\cos ax \log|\sec ax + \tan ax|$$

Similarly (replacing i by $-i$), we get

$$\frac{1}{D+ia}\tan ax = \frac{1}{a}(\sin ax \log|\sec ax + \tan ax| - 1) + \frac{i}{a}\cos ax \log|\sec ax + \tan ax|$$

Therefore, from (1):

$$\text{P.I.} = \frac{1}{2ia}\left(\frac{1}{D-ia}\tan ax - \frac{1}{D+ia}\tan ax\right)$$

$$= -\frac{1}{a^2}\cos ax \log|\sec ax + \tan ax|.$$

Third step: The general solution is therefore

$$y = \text{C.F.} + \text{P.I.} = c_1 \cos ax + c_2 \sin ax - \frac{1}{a^2}\cos ax \log|\sec ax + \tan ax|,$$

where c_1, c_2 are two arbitrary constants.

2.6 METHOD OF VARIATION OF PARAMETERS

It is a more powerful method of finding a particular solution of any linear non-homogeneous differential equation of second order even with variable coefficients also provided its complementary function is known. We state and prove the result in the following theorem.

Theorem

Consider the linear differential equation of second order of the form

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = X \qquad \text{...(2.24)}$$

where P_1, P_2 and X are functions of x or constants, then

$$\text{P.I.} = -y_1\int\frac{y_2 X}{W}dx + y_2\int\frac{y_1 X}{W}dx,$$

where y_1 and y_2 are two linearly independent solutions of

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = 0 \qquad \text{...(2.25)}$$

and $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is called the Wronskian of y_1, y_2.

Proof: Let the general solution of (2.25) is

$$y = c_1 y_1 + c_2 y_2$$

c_1, c_2 are two arbitrary constants and y_1, y_2 are two linearly independent solutions of (2.25).

us assume that P.I. of (2.24) is

$$y_p = u_1 y_1 + u_2 y_2 \qquad \text{...(2.26)}$$

where u_1, u_2 are unknown functions of x.

Differentiating both sides of (2.26) w.r.t. x, we get

$$y_p' = u_1 y_1' + u_2 y_2' + u_1' y_1 + u_2' y_2 \quad ...(2.27)$$

Let us choose u_1, u_2 in such a manner that

$$u_1' y_1 + u_2' y_2 = 0 \quad ...(2.28)$$

Then (2.27) becomes

$$y_p' = u_1 y_1' + u_2 y_2' \quad ...(2.29)$$

$$\therefore \quad y_p'' = u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2' \quad ...(2.30)$$

Putting these values of y_p, y_p', y_p'' in (2.24) and rearranging, we get

$$u_1(y_1'' + P_1 y_1' + P_2 y_1) + u_2(y_2'' + P_1 y_2' + P_2 y_2) + u_1' y_1' + u_2' y_2' = X$$

$$u_1' y_1' + u_2' y_2' = X \quad [\because y_1, y_2 \text{ are solutions of } (2.25)] \quad ...(2.31)$$

Solving (2.28) and (2.31), we get

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ X & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-y_2 X}{W} \quad \text{and} \quad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & X \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1 X}{W}$$

Here $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$ because y_1, y_2 are linearly independent functions.

Integrating we get,

$$u_1 = -\int \frac{y_2 X}{W} dx \quad \text{and} \quad u_2 = \int \frac{y_1 X}{W} dx$$

Putting these values of u_1, u_2 in (2.26), we get

$$\text{P.I.} = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx.$$

Let $y = e^{mx}$ be a trial solution, then the auxiliary equation becomes $m^2 + 4 = 0$, the roots $m = \pm 2i$.

$$\therefore \quad \text{C.F.} = c_1 \cos 2x + c_2 \sin 2x.$$

Second step: Determination of particular integral (P.I.).

Let $y_1 = \cos 2x$, $y_2 = \sin 2x$ and $X = \tan 2x$.

$$\therefore \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2 \neq 0.$$

Hence y_1, y_2 are two linearly independent solutions of the corresponding homogeneous equation of the given differential equation.

$$\therefore \quad \text{P.I.} = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$$

$$= -\cos 2x \int \frac{\sin 2x \tan 2x}{2} dx + \sin 2x \int \frac{\cos 2x \tan 2x}{2} dx$$

$$= -\frac{1}{2}\cos 2x \int \frac{1-\cos^2 2x}{\cos 2x} dx + \frac{1}{2}\sin 2x \int \sin 2x dx$$

$$= -\frac{1}{2}\cos 2x \{ \int \sec 2x\, dx - \int \cos 2x\, dx \} - \frac{1}{4}\sin 2x \cos 2x$$

$$= -\frac{1}{4}\cos 2x \{ \log(\sec 2x + \tan 2x) - \sin 2x \} - \frac{1}{4}\sin 2x \cos 2x$$

$$= -\frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x).$$

Third step: The general solution is therefore

$$y = \text{C.F.} + \text{P.I.} = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4}\cos 2x \ \log(\sec 2x + \tan 2x),$$

where c_1, c_2 are two arbitrary constants.

Example 2: Solve by the method of variation of parameters

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = \frac{e^{-x}}{x^2}.$$

Solution: Let us write the given differential equation as

$$(D^2 + 2D + 1)y = \frac{e^{-x}}{x^2}, \quad \text{or} \quad (D+1)^2 y = \frac{e^{-x}}{x^2}, \text{ where } D \equiv \frac{d}{dx}.$$

First step: Determination of complementary function (C.F.).

The complementary function is found from

$$(D+1)^2 y = 0.$$

Let $y = e^{mx}$ be a trial solution, then the auxiliary equation becomes $(m+1)^2 = 0$, the roots are $m = -1, -1$.

$$\therefore \quad \text{C.F.} = (c_1 + c_2 x)e^{-x} = c_1 e^{-x} + c_2 x e^{-x}.$$

Second step: Determination of particular integral (P.I.)

Let $y_1 = e^{-x}$, $y_2 = xe^{-x}$ and $X = \dfrac{e^{-x}}{x^2}$.

$$\therefore \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & (1-x)e^{-x} \end{vmatrix} = e^{-2x} \neq 0.$$

Hence y_1, y_2 are two linearly independent solutions of the corresponding homogeneous equation of the given differential equation.

$$\therefore \quad \text{P.I.} = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$$

$$= -e^{-x} \int \frac{xe^{-x}}{e^{-2x}} \cdot \frac{e^{-x}}{x^2} dx + xe^{-x} \int \frac{e^{-x}}{e^{-2x}} \cdot \frac{e^{-x}}{x^2} dx$$

$$= -e^{-x} \int \frac{dx}{x} + xe^{-x} \int x^{-2} dx$$

$$= -e^{-x} \log x + xe^{-x}\left(-\frac{1}{x}\right) = -(1+\log x)e^{-x}.$$

Third step: The general solution is therefore

$$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2 x)e^{-x} - (1+\log x)e^{-x},$$

where c_1, c_2 are two arbitrary constants.

Example 3: Solve by the method of variation of parameters:

$$\frac{d^2y}{dx^2} + a^2 y = \sec ax \qquad \text{(W.B.U.T. 2010)}$$

Solution: Let us write the given differential equation as

$$(D^2 + a^2)y = \sec ax, \text{ where } D \equiv \frac{d}{dx}.$$

First step: Determination of complementary function (C.F.).

The complementary function is found from

$$(D^2 + a^2)y = 0.$$

Let $y = e^{mx}$ be a trial solution, then the auxiliary equation becomes $m^2 + a^2 = 0$, the roots are $m = \pm ia$.

$$\therefore \quad \text{C.F.} = c_1 \cos ax + c_2 \sin ax.$$

Second step: Determination of particular integral (P.I.)

Let $y_1 = \cos ax$, $y_2 = \sin ax$ and $X = \sec ax$.

$$\therefore \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a \neq 0.$$

$$= -\frac{1}{a^2}\cos ax \log(\sec ax) + \frac{1}{a} x \sin ax.$$

p: The general solution is therefore

$= \text{C.F.} + \text{P.I.} = c_1 \cos ax + c_2 \sin ax - \frac{1}{a^2}\cos ax \log(\sec ax) + \frac{1}{a} x \sin ax$ (provided $a \neq 0$

re two arbitrary constants.

e 4: Apply the method of variation of parameters to solve the equation

$$\frac{d^2y}{dx^2} + y = \sec^3 x \tan x$$

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on: Let us write the given differential equation as

$$(D^2 + 1)y = \sec^3 x \tan x, \text{ where } D \equiv \frac{d}{dx}.$$

step: Determination of complementary function (C.F.).
omplementary function is found from

$$(D^2 + 1)y = 0.$$

$y = e^{mx}$ be a trial solution, then the auxiliary equation becomes $m^2 + 1 = 0$, the roots a

$$\text{C.F.} = c_1 \cos x + c_2 \sin x.$$

ond step: Determination of particular integral (P.I.)

$y_1 = \cos x$, $y_2 = \sin x$ and $X = \sec^3 x \tan x$.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1 \neq 0.$$

nce y_1, y_2 are two linearly independent solutions of the corresponding homogeneous equatio
ven differential equation.

$$\text{P.I.} = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$$

$$= -\cos x \int \sin x \sec^3 x \tan x \, dx + \sin x \int \cos x \sec^3 x \tan x \, dx$$

$$= -\cos x \int \sec^2 x \tan^2 x \, dx + \sin x \int \sec^2 x \tan x \, dx$$

$$= -\frac{1}{3}\cos x \tan^3 x + \frac{1}{2}\sin x \tan^2 x$$

$$= -\frac{1}{3}\sin x\tan^2 x + \frac{1}{2}\sin x\tan^2 x = \frac{1}{6}\sin x\tan^2 x.$$

Third step: The general solution is therefore

$$y = \text{C.F.} + \text{P.I.} = c_1\cos x + c_2\sin x + \frac{1}{6}\sin x\tan^2 x,$$

where c_1, c_2 are two arbitrary constants.

Example 5: Apply the method of variation of parameters to solve

$$\frac{d^2y}{dx^2} + 4y = 4\sec^2 2x \qquad \text{(W.B.U.T. 2006)}$$

Solution: Let us write the given differential equation as

$$(D^2+4)y = 4\sec^2 2x, \text{ where } D \equiv \frac{d}{dx}.$$

First step: Determination of complementary function (C.F.).
The complementary function is found from

$$(D^2+4)y = 0.$$

Let $y = e^{mx}$ be a trial solution, then the auxiliary equation becomes $m^2+4=0$, the roots are $m = \pm 2i$.

$$\therefore \quad \text{C.F.} = c_1\cos 2x + c_2\sin 2x.$$

Second step: Determination of particular integral (P.I.)
Let $y_1 = \cos 2x$, $y_2 = \sin 2x$ and $X \equiv 4\sec^2 2x$.

$$\therefore \quad W \equiv \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2 \neq 0.$$

Hence y_1, y_2 are two linearly independent solutions of the corresponding homogeneous equation of the given differential equation.

$$\therefore \quad \text{P.I.} = -y_1\int\frac{y_2X}{W}dx + y_2\int\frac{y_1X}{W}dx$$

$$= -\cos 2x\int\frac{1}{2}(\sin 2x)4\sec^2 2x\,dx + \sin 2x\int\frac{1}{2}(\cos 2x)4\sec^2 2x\,dx.$$

$$= -\cos 2x\int 2\sec 2x\tan 2x\,dx + \sin 2x\int 2\sec 2x\,dx$$

$$= -\cos 2x\sec 2x + \sin 2x\log(\sec 2x + \tan 2x).$$

$$= -1 + \sin 2x\log(\sec 2x + \tan 2x).$$

Third step: The general solution is therefore

$$y = \text{C.F.} + \text{P.I.} = c_1\cos 2x + c_2\sin 2x - 1 + \sin 2x\log(\sec 2x + \tan 2x),$$

where c_1, c_2 are two arbitrary constants.

Example 6: Solve by the method of variation of parameters (W.B.U.T. 200

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=9e^x$$

Solution: Let us write the given differential equation as

$$(D^2-3D+2)y=9e^x, \quad \text{where } D\equiv\frac{d}{dx}.$$

First step: Determination of complementary function (C.F.).
The complementary function is found from

$$(D^2-3D+2)y=0.$$

Let $y=e^{mx}$ be a trial solution, then the auxiliary equation becomes $m^2-3m+2=0$, $(m-2)(m-1)=0$, the roots are $m=1, 2$.

$$\therefore \quad \text{C.F.}=c_1e^x+c_2e^{2x}.$$

Second step: Determination of particular integral (P.I.)

Let $y_1=e^x$, $y_2=e^{2x}$ and $X=9e^x$.

$$\therefore \quad W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix}=e^{3x}\neq 0.$$

Hence y_1, y_2 are two linearly independent solutions of the corresponding homogeneous equation of the given differential equation.

$$\therefore \quad \text{P.I.}=-y_1\int\frac{y_2X}{W}dx+y_2\int\frac{y_1X}{W}dx$$

$$=-e^x\int\frac{e^{2x}9e^x}{e^{3x}}dx+e^{2x}\int\frac{e^x9e^x}{e^{3x}}dx$$

$$=-e^x\int 9dx+e^{2x}\int 9e^{-x}dx$$

$$=-9xe^x-9e^{2x}\cdot e^{-x}=-9(x+1)e^x.$$

Third step: The general solution is therefore

$$y=\text{C.F.}+\text{P.I.}=c_1e^x+c_2e^{2x}-9(x+1)e^x,$$

where c_1, c_2 are two arbitrary constants.

Example 7: Solve by the method of variation of parameters

$$(D^2-3D+2)y=\frac{e^x}{1+e^x}, \quad \text{where } D\equiv\frac{d}{dx}.$$

Solution:

First step: Determination of complementary function (C.F.).
The complementary function is found from

$$(D^2-3D+2)y=0.$$

Let $y=e^{mx}$ be a trial solution, then the auxiliary equation becomes $m^2-3m+2=0$, $(m-2)(m-1)=0$, the roots are $m=1, 2$.

$$\therefore \quad \text{C.F.}=c_1e^x+c_2e^{2x}.$$

Second step: Determination of particular integral (P.I.)

Let $y_1 = e^x$, $y_2 = e^{2x}$ and $X = \dfrac{e^x}{1+e^x}$.

$$\therefore \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x} \neq 0.$$

Hence y_1, y_2 are two linearly independent solutions of the corresponding homogeneous equation of the given differential equation.

$$\therefore \quad \text{P.I.} = -y_1\int \frac{y_2 X}{W}\,dx + y_2\int \frac{y_1 X}{W}\,dx$$

$$= -e^x\int \frac{e^{2x}}{e^{3x}}\cdot\frac{e^x}{(1+e^x)}\,dx + e^{2x}\int \frac{e^x}{e^{3x}}\cdot\frac{e^x}{(1+e^x)}\,dx$$

$$= -e^x\int \frac{dx}{1+e^x} + e^{2x}\int \frac{dx}{e^x(1+e^x)}$$

$$= -e^x\int \frac{dx}{1+e^x} + e^{2x}\int \frac{(1+e^x)-e^x}{e^x(1+e^x)}\,dx$$

$$= -(e^x+e^{2x})\int \frac{dx}{1+e^x} + e^{2x}\int e^{-x}\,dx$$

$$= -(e^x+e^{2x})\int \frac{e^{-x}}{e^{-x}+1}\,dx - e^{2x}\cdot e^{-x}$$

$$= (e^x+e^{2x})\log(e^{-x}+1) - e^x.$$

Third step: The general solution is therefore

$$y = \text{C.F.} + \text{P.I.} = c_1e^x + c_2e^{2x} + (e^x+e^{2x})\log(e^{-x}+1) - e^x,$$

where c_1, c_2 are two arbitrary constants.

Example 8: Solve by the method of variation of parameters

$$(D^2-2D+1)y = e^x\log x, \quad \text{where } D \equiv \frac{d}{dx}.$$

Solution:

First step: Determination of complementary function (C.F.).

The complementary function is found from

$$(D^2-2D+1)y = 0.$$

Let $y = e^{mx}$ be a trial solution, then the auxiliary equation becomes $m^2-2m+1=0$, or $(m-1)^2 = 0$ the roots are $m = 1, 1$.

$$\therefore \quad \text{C.F.} = (c_1+c_2x)e^x = c_1e^x + c_2xe^x.$$

Second step: Determination of particular integral (P.I.)

Let $y_1 = e^x$, $y_2 = xe^x$ and $X = e^x\log x$.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & (1+x)e^x \end{vmatrix} = e^{2x} \neq 0.$$

$\therefore$ Hence y_1, y_2 are two linearly independent solutions of the corresponding homogeneous equation of the given differential equation.

$$\text{P.I.} = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$$

$\therefore$

$$= -e^x \int \frac{xe^x}{e^{2x}} e^x \log x \, dx + xe^x \int \frac{e^x}{e^{2x}} e^x \log x \, dx$$

$$= -e^x \int x \log x \, dx + xe^x \int \log x \, dx$$

$$= -e^x \left(\frac{x^2}{2} \log x - \int \frac{1}{x} \cdot \frac{x^2}{2} dx \right) + xe^x \left(x \log x - \int \frac{1}{x} \cdot x \, dx \right)$$

$$= -e^x \left(\frac{x^2}{2} \log x - \frac{x^2}{4} \right) + xe^x (x \log x - x)$$

$$= \frac{1}{4} x^2 e^x (1 - 2\log x + 4\log x - 4) = \frac{1}{4} x^2 e^x (2\log x - 3).$$

Third step: The general solution is therefore

$$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2 x)e^x + \frac{1}{4} x^2 e^x (2\log x - 3),$$

where c_1, c_2 are two arbitrary constants.

Example 9: Find the general solution of

$$(1+x)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = (1+x)^2$$

by the method of variation of parameters, it is given that $y = x$ and $y = e^{-x}$ are two linearly independent solutions of the corresponding homogeneous equation.

Solution: Let us write the given equation as

$$\frac{d^2y}{dx^2} + \frac{x}{1+x}\frac{dy}{dx} - \frac{1}{1+x} y = 1 + x.$$

It is a second order linear differential equation and it is given that x, e^{-x} are two linearly independent solutions of the corresponding homogeneous equation.

$$\therefore \quad \text{C.F.} = c_1 x + c_2 e^{-x}$$

Let $y_1 = x$, $y_2 = e^{-x}$ and $X = 1 + x$.

$$\therefore \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & e^{-x} \\ 1 & -e^{-x} \end{vmatrix} = -xe^{-x} - e^{-x} = -(x+1)e^{-x} \neq 0.$$

since y_1, y_2 are linearly independent functions according to the question.

$$\therefore \quad P.I. = -y_1\int \frac{y_2 X}{W}\,dx + y_2\int \frac{y_1 X}{W}\,dx$$

$$= -x\int \frac{e^{-x}(1+x)}{-(x+1)e^{-x}}\,dx + e^{-x}\int \frac{x(1+x)}{-(x+1)e^{-x}}\,dx$$

$$= x\int dx - e^{-x}\int xe^{x}\,dx = x^2 - e^{-x}\left\{x\int e^{x}\,dx - \int\left(\frac{dx}{dx}\right)\left(\int e^{x}\,dx\right)dx\right\}$$

$$= x^2 - e^{-x}(xe^{x} - e^{x}) = x^2 - x + 1.$$

Therefore, the general solution is

$$y = C.F. + P.I. = c_1 x + c_2 e^{-x} + x^2 - x + 1,$$

where c_1, c_2 are two arbitrary constants.

Example 16: Solve by the method of variation of parameters

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \log x \ (x > 0)$$

It is given that $y = x$ and $y = \frac{1}{x}$ are two linearly independent solutions of the corresponding homogeneous differential equation.

Solution: Here $x, \frac{1}{x}$ are two linearly independent solutions of the corresponding homogeneous equation of the given differential equation.

$$\therefore \quad C.F. = c_1 x + \frac{c_2}{x}.$$

Let $y_1 = x, \quad y_2 = \frac{1}{x}$ and $X = \log x.$

$$\therefore \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix} = -\frac{2}{x} \neq 0.$$

$$\therefore \quad P.I. = -y_1\int \frac{y_2 X}{W}\,dx + y_2\int \frac{y_1 X}{W}\,dx$$

$$= -x\int \frac{1}{x}\left(-\frac{x}{2}\right)\log x\,dx + \frac{1}{x}\int x\left(-\frac{x}{2}\right)\log x\,dx$$

$$= \frac{x}{2}\int \log x\,dx - \frac{1}{2x}\int x^2\log x\,dx$$

$$= \frac{x}{2}\left(x\log x - \int \frac{1}{x}\cdot x\,dx\right) - \frac{1}{2x}\left(\frac{x^3}{3}\log x - \int \frac{1}{x}\cdot\frac{x^3}{3}\,dx\right)$$

$$= \frac{x^2}{2}(\log x - 1) - \frac{1}{2x}\left(\frac{x^3}{3}\log x - \frac{x^3}{9}\right)$$