CHAPTER 6

Mean Value Theorems and Expansions of Functions

6.1 INTRODUCTION

Calculus is one of the most beautiful intellectual achievements of Mathematicians and it deals with the mathematical study of change, motion, growth or decay etc. One of the most important ideas of differential calculus is derivative which measures the rate of change of a given function and concept of derivative is very useful in engineering, science, economics, medicine and computer science. There are many real valued functions of a real variable which are continuous in a finite and closed interval $[a, b]$ and also derivable on the open interval (a, b). Such functions possess some interesting and very useful properties and these properties are formulated in the form of theorems known as mean value theorems. In this chapter we shall deal with Rolle's theorem, Lagrange's mean value theorem which connects the average rate of change of a function over an interval with the instantaneous rate of change of the function at a point within that interval, Taylor's theorem (generalized mean value theorem) which enables us to express any differentiable function in power series, namely Taylor's and Maclaurin's series.

6.2 ROLLE'S THEOREM

Let f be a function defined on a finite closed interval $[a, b]$ satisfying the following conditions:

(*i*) $f(x)$ is continuous for all x in $a \le x \le b$,

(*ii*) $f(x)$ is derivable for all x in $a < x < b$ and

(*iii*) $f(a) = f(b)$.

Then there exists at least one value of x, say c, $a < c < b$, such that $f'(c) = 0$.

Notes: 1. Here we have assumed continuity in a closed interval, we have assumed derivability only in the open interval, *i.e.*, to say for the conclusion to be valid, we do not need derivability at the end points a, b.

2. If $f(x)$ be constant in $a \le x \le b$, then $f(a) = f(b)$ and $f'(x) = 0$ at every point in $a < x < b$.

3. If a, b are two roots of the equation $f(x) = 0$, *i.e.*, $f(a) = f(b) = 0$, then the equation $f'(x) = 0$ will have at least one root between a and b, provided

   (*i*) $f(x)$ is continuous in $a \le x \le b$ and

   (*ii*) $f'(x)$ exists in $a < x < b$.

If $f(x)$ be a polynomial in x, the conditions (*i*) and (*ii*) are obviously satisfied.

Geometrical Interpretation of Rolle's Theorem

If the graph of $y = f(x)$ has the ordinates at two points $A \equiv (a, f(a)), B \equiv (b, f(b))$ equal, *i.e.*, $f(a) = f(b)$, and if the graph be continuous (*i.e.*, without break) throughout the interval from A to B and if the curve has a unique tangent at each point on it from A to B except possibly at the two end points A and B, then there must exist at least one point on the arc AB, where the tangent is parallel to the x-axis.

Compare the figures 6.1 and 6.2

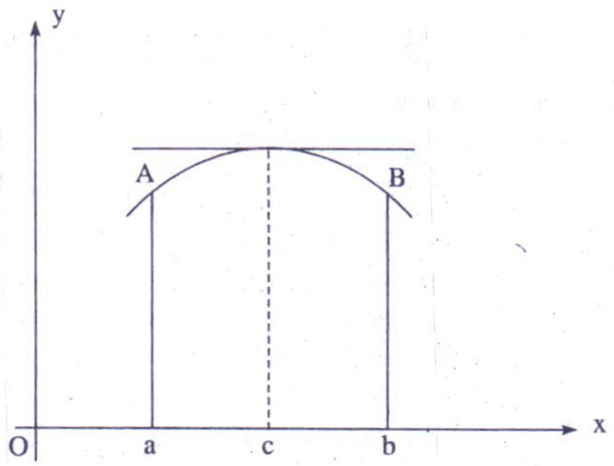


Fig. 6.1

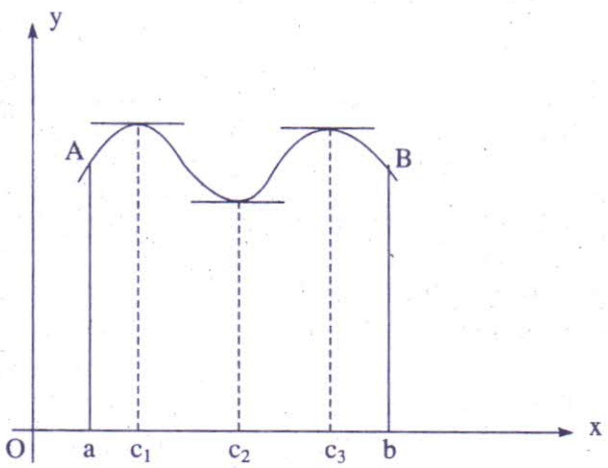


Fig. 6.2

ILLUSTRATIVE EXAMPLES

Example 1: Verify Rolle's theorem in each of the following functions:

(*i*) $f(x) = |x|$ in $-1 \le x \le 1$ *(BESUS 2013, W.B.U.T. 2003, 2012)*

(*ii*) $f(x) = \cos^2 x$ in $\left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$

(*iii*) $f(x) = 4x^3 + x^2 - 4x - 1$ in $[-1, 1]$

(*iv*) $f(x) = \sin x$ in $[0, \pi]$

(*v*) $f(x) = \dfrac{1}{x} + \dfrac{1}{1-x}$ in $[0, 1]$.

Solution: (*i*) Here $f(x) = |x|, -1 \le x \le 1$

$$f(x) = -x, \text{ for } -1 \le x \le 0$$
$$= x, \text{ for } 0 < x \le 1.$$

Obviously $f(x)$ is continuous for all x in $-1 \le x \le 1$ except possibly at $x = 0$.

Now, $$f(0+0) = \lim_{x\to 0+} f(x) = \lim_{x \to 0+} x \quad [\because\ x \to 0+ \Rightarrow x > 0]$$
$$= 0$$

$$f(0-0) = \lim_{x\to 0-} f(x) = \lim_{x\to 0-} (-x) \qquad [\because\ x\to 0- \Rightarrow x<0]$$
$$= 0$$

Also, $f(0) = 0$. Therefore, $f(0+0) = f(0-0) = f(0)$ and hence $f(x)$ is continuous at $x = 0$.

Thus, $f(x)$ is continuous for all x in $-1 \le x \le 1$.

Here $f(x)$ is derivable for all x in $-1 < x < 1$ except possibly at $x = 0$.

Now, $$\mathrm{R}f'(0) = \lim_{h\to 0+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0+} \frac{h-0}{h} \qquad [\because\ h\to 0+ \Rightarrow h>0]$$
$$= 1$$

$$\mathrm{L}f'(0) = \lim_{h\to 0-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0-} \frac{-h-0}{h} \qquad [\because\ h\to 0- \Rightarrow h<0]$$
$$= -1.$$

Therefore, $\mathrm{R}f'(0) \neq \mathrm{L}f'(0)$ and hence $f(x)$ is not derivable in $-1 < x < 1$.

Therefore, we conclude that Rolle's theorem is not applicable to the function $f(x) = |x|$ in $-1 \le x \le 1$.

(*ii*) Here (*a*) $f(x)$ is continuous in $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$.

(*b*) $f'(x) = -\sin 2x$ exists in $-\frac{\pi}{4} < x < \frac{\pi}{4}$.

(*c*) $f\left(-\frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right) = \frac{1}{2}$.

All the conditions of Rolle's theorem are satisfied and as such there exists a c, where $f'(c) = -\sin 2c = 0$, namely $c = 0$, $-\frac{\pi}{4} < 0 < \frac{\pi}{4}$.

Hence Rolle's theorem is verified.

(*iii*) Since every polynomial in x is continuous and derivable for all real values of x, so

(*a*) $f(x)$ is continuous in $-1 \le x \le 1$.

(*b*) $f'(x) = 12x^2 + 2x - 4$ in $-1 < x < 1$.

Also (*c*) $f(-1) = f(1) = 0$.

Thus $f(x)$ satisfies all the conditions of Rolle's theorem. By Rolle's theorem, we have

$$f'(c) = 0, \text{ where } -1 < c < 1.$$

Here $f'(c) = 0$ gives $12c^2 + 2c - 4 = 0$, or, $(3c + 2)(2c - 1) = 0$ whose two solutions are $c = -\frac{2}{3}, \frac{1}{2}$ and $-1 < -\frac{2}{3} < 1$ as well as $-1 < \frac{1}{2} < 1$.

Hence Rolle's theorem is verified for the given function.

(*iv*) Here (*a*) $f(x)$ is continuous in $0 \le x \le \pi$.

(*b*) $f'(x) = \cos x$ exists in $0 < x < \pi$.

(*c*) $f(0) = f(\pi) = 0$.

All the conditions of Rolle's theorem are satisfied and as such there exists a c, where $f'(c) = \cos c = 0$, namely $c = \frac{\pi}{2}$, $0 < \frac{\pi}{2} < \pi$.

Hence Rolle's theorem is verified.

(*v*) Here (*a*) $f(x)$ is continuous in $0 < x < 1$ (not in $0 \le x \le 1$)

(*b*) $f'(x) = \frac{1}{(1-x)^2} - \frac{1}{x^2}$ exists in $0 < x < 1$

(*c*) $f(0) \ne f(1)$, both are undefined

Thus the conditions of Rolle's theorem do not hold good. But get there exists a c, where $f'(c) = 0$, namely $c = \frac{1}{2}$, where $0 < \frac{1}{2} < 1$.

Note: The above examples lead to the following conclusion:

If $f(x)$ satisfies all the conditions of Rolle's theorem in $[a, b]$ then the result $f'(c) = 0$, where $a < c < b$ is assured, but if any of the conditions is not satisfied then Rolle's theorem will not be necessarily true, it may still be true but the truth is not ensured.

Example 2: Show that Rolle's theorem is not applicable to $f(x) = \tan x$ in $[0, \pi]$, although $f(0) = f(\pi)$. (*W.B.U.T. 2004, 2006, 2011*)

Solution: The function $f(x) = \tan x$ is not continuous everywhere in $[0, \pi]$ since $\tan x \to \infty$ as $x \to \frac{\pi}{2}-$ and $\tan \frac{\pi}{2}$ is undefined, where $0 < \frac{\pi}{2} < \pi$.

Thus the conditions of Rolle's theorem do not hold and hence Rolle's theorem is not applicable to the function $f(x) = \tan x$ in $[0, \pi]$, although $f(0) = f(\pi) = 0$.

Example 3: If $f(x) = (x-a)^m (x-b)^n$ where m, n are positive integers, show that c in Rolle's theorem divides the segment $a \le x \le b$ in the ratio $m : n$.

Solution: Since every polynomial in x is continuous and derivable for all real values of x, so

(*i*) $f(x) = (x-a)^m (x-b)^n$ is continuous in $a \le x \le b$,

(*ii*) $f'(x) = (x-a)^{m-1}(x-b)^{n-1}\{m(x-b)+n(x-a)\}$ exists in $a < x < b$.

Also (*iii*) $f(a) = f(b) = 0$.

Therefore by Rolle's theorem, there exists c, $a < c < b$, such that $f'(c) = (c-a)^{m-1}(c-b)^{n-1}$ $\{m(c-b)+n(c-a)\} = 0$.

Hence $m(c-b)+n(c-a) = 0$ or $c = \frac{mb+na}{m+n}$, which divides the segment $a \le x \le b$ in the ratio $m : n$.

Example 4. (*a*) Verify Rolle's theorem in the following cases:

(*i*) $f(x) = |\cos(x)|$ in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

[*BESUS (B.Arch.) 2013, W.B.U.T. (B.Arch.) 2013*]

(*ii*) $f(x) = |\sin(x)|$ in the interval $[-\pi, \pi]$ [*BESUS (B.Arch.) 2013*]

(*b*) Let $f(x) = \begin{vmatrix} \sin(x) & \sin(\alpha) & \sin(\beta) \\ \cos(x) & \cos(\alpha) & \cos(\beta) \\ \tan(x) & \tan(\alpha) & \tan(\beta) \end{vmatrix}$, $0 < \alpha < \beta < \frac{\pi}{2}$,

then show that there exists $c(\alpha < c < \beta)$ such that $f'(c) = 0$. [*BESUS (B. Arch.)2013*]

Solution: (*a*) (*i*) $f(x) = |\cos(x)| = \cos(x)$ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Obviously $f(x) = |\cos(x)|$ is continuous and derivable for all x in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so

(*I*) $f(x)$ is continuous in $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$,

(*II*) $f'(x) = -\sin x$ in $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Also (*III*) $f\left(-\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right) = 0$.

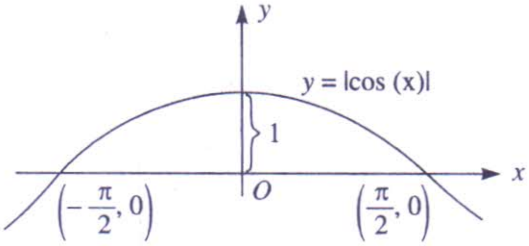


Fig. 6.3

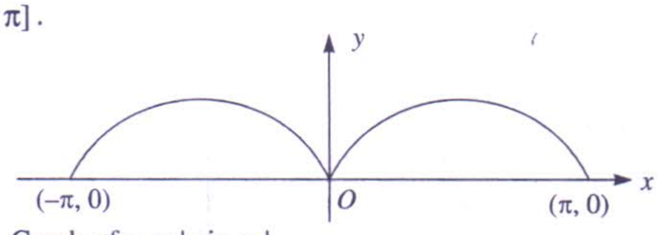
Thus $f(x)$ satisfies all the conditions of Rolle's theorem and so there exists $c = 0\left(-\frac{\pi}{2} < 0 < \frac{\pi}{2}\right)$ such that $f'(0) = -\sin 0 = 0$.

Hence Rolle's theorem is verified for the given function.

(*ii*) Here $f(x) = |\sin(x)|$ in $[-\pi, \pi]$

$$\therefore \quad f(x) = \begin{cases} -\sin x, \text{ for } -\pi \le x \le 0 \\ \sin x, \text{ for } 0 < x \le \pi \end{cases}$$

Obviously $f(x)$ is continuous for all x in $[-\pi, \pi]$.



Graph of $y = |\sin x|$

Fig. 6.4

Now, $$Rf'(0) = \lim_{h\to 0+} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0+} \frac{\sin h - 0}{h} = 1$$

and $$Lf'(0) = \lim_{h\to 0-} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0-} \frac{-\sin h - 0}{h} = -1$$

Therefore, $Rf'(0) \neq Lf'(0)$ and hence $f(x)$ is not derivable at $x = 0$. Thus $f(x)$ is not derivable for all x in $-\pi < x < \pi$.

So, we conclude that Rolle's theorem is not applicable to the function $f(x) = |\sin(x)|$ in $[-\pi, \pi]$

(*b*) Obiously $f(x)$ is continuous and derivable for all x in $0 < \alpha < \beta < \frac{\pi}{2}$.

Also, $$f(\alpha) = \begin{vmatrix} \sin(\alpha) & \sin(\alpha) & \sin(\beta) \\ \cos(\alpha) & \cos(\alpha) & \cos(\beta) \\ \tan(\alpha) & \tan(\alpha) & \tan(\beta) \end{vmatrix}$$

$$= \begin{vmatrix} \sin(\beta) & \sin(\alpha) & \sin(\beta) \\ \cos(\beta) & \cos(\alpha) & \cos(\beta) \\ \tan(\beta) & \tan(\alpha) & \tan(\beta) \end{vmatrix}$$

$$= f(\beta) = 0.$$

Thus $f(x)$ satisfies all the conditions of Rolle's theorem in $0 < \alpha < \beta < \frac{\pi}{2}$ and so there exists $c(\alpha < c < \beta)$ such that $f'(c) = 0$.

6.3 LAGRANGE'S MEAN VALUE THEOREM

Let f be a function defined on a finite closed interval $[a, b]$ be such that it is

(*i*) continuous for all values of x in $a \leq x \leq b$ and

(*ii*) derivable for all values of x in $a < x < b$ then there exists at least one value of x, say c, $a < c < b$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

Note:

(*i*) This theorem is also known as 'Formula of finite increment of Lagrange' or 'Average Value Theorem' or 'The Law of Mean'.

(*ii*) If $f(a) = f(b)$ then $f'(c) = 0$, $a < c < b$. Thus Lagrange's Mean Value Theorem becomes Rolle's theorem when $f(a) = f(b)$.

Alternative form of Lagrange's Mean Value Theorem

Let f be a function defined on a finite closed interval $[a, a + h]$ be such that it is

(*i*) continuous for all values of x in $a \leq x \leq a + h$ and

(*ii*) derivable for all values of x in $a < x < a + h$ then $f(a+h) = f(a) + h\,f'(a+\theta h)$, where $0 < \theta < 1$.

Notes:

(*i*) For the interval $[0, h]$, the above form reduces to the Maclaurin's Formula,

$$f(h) = f(0) + h\,f'(\theta h),\ 0 < \theta < 1.$$

(*ii*) Here $\dfrac{f(b)-f(a)}{b-a}$ measures the mean (or average) rate of increase of the function f in the interval $[a, b]$. Therefore the theorem expresses the fact that, under the stated conditions, the mean rate of increase in any interval is equal to the actual rate of increase at some point within the interval. For example, the mean velocity of a moving point in any interval of time is equal to the actual velocity at some instant within the interval. This is the justification of the name Mean Value Theorem.

Geometrical Interpretation: Consider the following Fig. 6.5

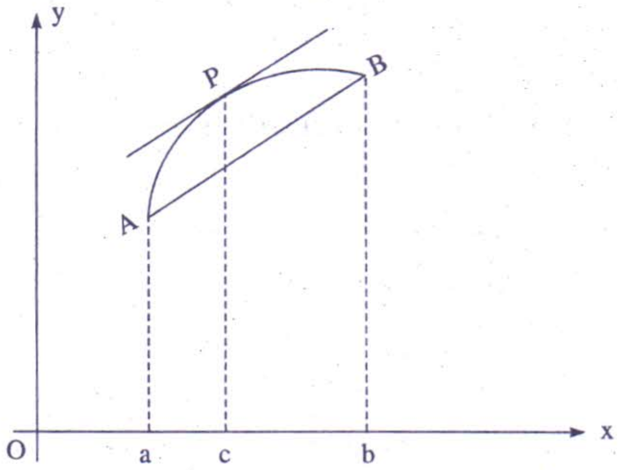


Fig. 6.5

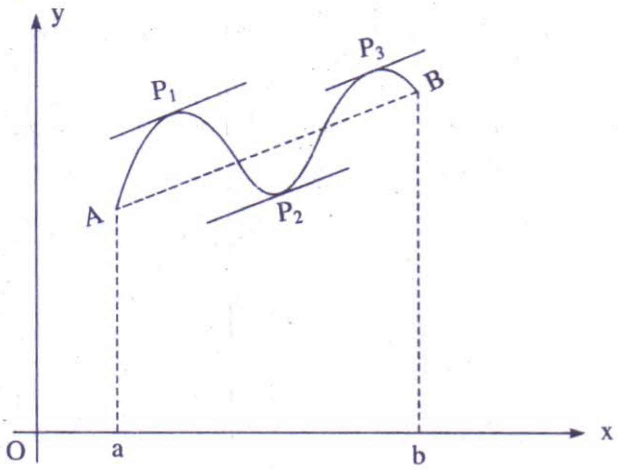


Fig. 6.6

Let $y = f(x)$ be represented by the curve AB. This curve has a tangent at every point in the interval $[a, b]$.

Slope of the chord $AB = \dfrac{f(b)-f(a)}{b-a}$.

Here $P(x = c)$ be a point on the curve such that

$$\frac{f(b)-f(a)}{b-a} = f'(c) = \text{slope of the tangent at } P.$$

$\therefore$ Slope of the tangent at P = slope of the chord AB.

Therefore the tangent at P is parallel to the chord AB.

The same is true for Fig. 6.6

Hence we have the following geometrical interpretation of Lagrange's Mean Value Theorem:

"If the graph of $y = f(x)$ be represented by an arc AB without any break on $[a, b]$ having tangent at every point on AB, then there must exist at least one point C between A and B on arc AB at which the tangent is parallel to the chord AB joining the two points $A \equiv (a, f(a))$ and $B \equiv (b, f(b))$."

ILLUSTRATIVE EXAMPLES

Example 1: Verify Lagrange's mean value theorem for the following functions in the specified intervals:

(*i*) $f(x) = x^3 - x^2 - 5x + 3,\ 0 \le x \le 4$

(*ii*) $\left.\begin{aligned} f(x) &= x\sin\frac{1}{x}, && \text{for } x \neq 0 \\ &= 0, && \text{for } x = 0 \end{aligned}\right\}$ in $[-1, 1]$

(*iii*) $f(x) = \log_e x,\ 1 \le x \le e$

(*iv*) $f(x) = x^{1/3},\ -1 \le x \le 1$

Solution: (*i*) We know that every polynomial in x is continuous and derivable for all real values of x, therefore

$$f(x) = x^3 - x^2 - 5x + 3$$

is continuous on [0, 4] and derivable on (0, 4).

Also $$f'(x) = 3x^2 - 2x - 5, 0 < x < 4.$$

Therefore $f(x)$ satisfies all conditions of Lagrange's mean value theorem and hence there exists c, $0 < c < 4$, such that

$$f'(c) = \frac{f(4) - f(0)}{4 - 0}, \text{ or } 3c^2 - 2c - 5 = \frac{31 - 3}{4},$$

or $$3c^2 - 2c - 12 = 0.$$

It has two roots, namely $c = \dfrac{1 \pm \sqrt{37}}{3}$, of which $\dfrac{1 + \sqrt{37}}{3}$ lies between 0 and 4.

Thus, Lagrange's mean value theorem is verified for the given function in the interval [0, 4].

(*ii*) Since x and $\sin\dfrac{1}{x}$ are derivable for all $x \neq 0$, so $f(x)$ is derivable for all $x \neq 0$.

$$\therefore \quad f'(x) = \frac{d}{dx}\left(x\sin\frac{1}{x}\right) = \sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x}, \text{ when } x \neq 0.$$

But $$f'(0) = \lim_{h \to 0}\frac{f(0+h) - f(0)}{h} = \lim_{h \to 0}\frac{h\sin\frac{1}{h} - 0}{h} = \lim_{h \to 0}\sin\frac{1}{h},$$

which does not exist.

Hence $f(x)$ is not derivable in $(-1, 1)$.

Therefore Lagrange's mean value theorem is not applicable for the given function in $[-1, 1]$.

(*iii*) Since $\log_e x$ is continuous and derivable for all x in $(0, \infty)$, so $f(x) = \log_e x$ is continuous on $1 \le x \le e$ and derivable on $1 < x < e$.

Also $$f'(x) = \frac{1}{x}, 1 < x < e.$$

Therefore $f(x)$ satisfies all conditions of Lagrange's mean value theorem and hence there exists c, $1 < c < e$, such that

$$f'(c) = \frac{f(e) - f(1)}{e-1}, \text{ or } \frac{1}{c} = \frac{1-0}{e-1}, \text{ or } c = e-1$$

which lies in $[1, e]$, since $2 < e < 3$.

Thus, Lagrange's mean value theorem is verified for the given function in $[1, e]$.

(*iv*) Here $f(x) = x^{1/3}$. Therefore, $f'(x) = \frac{1}{3} \cdot x^{-2/3} = \frac{1}{3x^{2/3}}$.

Thus $f'(x)$ does not exist at $x = 0$ and hence $f(x)$ is not derivable in $(-1, 1)$.

Therefore the Lagrange's mean value theorem is not applicable here.

Example 2: If $f'(x) = 0$ in $[a, b]$, then by using Lagrange's Mean Value Theorem prove that $f(x)$ is constant in $[a, b]$.

Solution: Suppose x_1, x_2 are two arbitrary points in $[a, b]$ such that $a \le x_1 < x_2 \le b$. By Lagrange's mean value theorem, we get

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c), \; x_1 < c < x_2$$
$$= 0 \text{ (it is given)}$$

Therefore, $f(x_2) - f(x_1) = 0$, or $f(x_2) = f(x_1)$.

Since x_1, x_2 are any two arbitrary points in $[a, b]$ therefore it follows that $f(x)$ is constant in $[a, b]$.

Example 3: If $f(x)$ is continuous in $a \le x \le b$ and $f'(x) > 0$ in $a < x < b$, then by using mean value theorem show that $f(x)$ is a strictly increasing function in $[a, b]$.

Solution: Suppose x_1, x_2 are two arbitrary points in (a, b) such that $a < x_1 < x_2 < b$. By Lagrange's mean value theorem, we get

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c), \; x_1 < c < x_2,$$

or $$\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0, \quad \text{since } f'(c) > 0 \text{ by the given condition.}$$

But $x_2 - x_1 > 0$, therefore $f(x_2) - f(x_1) > 0$, *i.e.*, $f(x_2) > f(x_1)$ whenever $x_2 > x_1$.

Hence $f(x)$ is a strictly increasing function in (a, b).

Example 4: Deduce Lagrange's Mean Value Theorem from Rolle's Theorem.

Solution: Let us choose $g(x) = f(x) - f(a) - A(x - a)$, where A is a constant, f is continuous in $[a, b]$ and derivable in (a, b). Therefore $g(x)$ is continuous in $[a, b]$ and derivable in (a, b). Also

$g(a) = 0$ and determine A such that

$$g(b) = f(b) - f(a) - A(b-a) = 0$$

$$\Rightarrow \qquad A = \frac{f(b)-f(a)}{b-a}.$$

Thus g satisfies all the conditions of Rolle's theorem and therefore there exists a number c, $a < c < b$, such that

$$g'(c) = 0$$

or $$f'(c) - A = 0, \quad \text{since } g(x) = f(x) - f(a) - A(x-a)$$

Therefore, $$f'(c) = A = \frac{f(b)-f(a)}{b-a}, \; a < c < b,$$

which is the result of Lagrange's Mean Value Theorem.

Example 5: In the mean value theorem $f(h) = f(0) + hf'(\theta h),\ 0 < \theta < 1$, prove that the limiting value of θ as $h \to 0$ is $\frac{1}{2}$ if $f(x) = \cos x$.

Solution: Here $f(x) = \cos x$, therefore $f(0) = 1$ and $f'(x) = -\sin x$.

From the given relation $f(h) = f(0) + hf'(\theta h)$, we have

$$\cos h = 1 + h(-\sin\theta h), \text{ or } \sin\theta h = \frac{1-\cos h}{h}$$

or $$\theta \cdot \frac{\sin\theta h}{\theta h} = \frac{1}{2}\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2$$

$\therefore$ $$\lim_{h\to 0}\left(\theta \cdot \frac{\sin\theta h}{\theta h}\right) = \frac{1}{2}\lim_{h\to 0}\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2$$

or $$\lim_{h\to 0}\theta \cdot \lim_{\theta h\to 0}\frac{\sin\theta h}{\theta h} = \frac{1}{2}\left(\lim_{\frac{h}{2}\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2$$

$\therefore$ $$\lim_{h\to 0}\theta = \frac{1}{2}.$$

Example 6: Apply Lagrange's mean value theorem to prove that the chord on the parabola $y = x^2 + 2ax + b$ joining the points at $x = \alpha$ and $x = \beta$ is parallel to its tangent at the point $x = \frac{1}{2}(\alpha + \beta)$.

Solution: Let $f(x) = x^2 + 2ax + b$. Since $f(x)$ is a polynomial in x, therefore $f(x)$ is continuous in $[\alpha, \beta]$ and derivable in (α, β). Hence by Lagrange's mean value theorem, there exists a real number