$$y_3 = (-1)(-2)(x-1)^{-3} + (-1)(-2)(x+1)^{-3}$$
$$+ (-1)(x-1)^{-2} - (-1)(x+1)^{-2}$$
$$y_4 = (-1)(-2)(-3)(x-1)^{-4} + (-1)(-2)(-3)(x+1)^{-4}$$
$$+ (-1)(-2)(x-1)^{-3} - (-1)(-2)(x+1)^{-3}$$
$$= (-1)^3 \lfloor\underline{3}\,(x-1)^{-4} + (-1)^3 \lfloor\underline{3}\,(x+1)^{-4}$$
$$+ (-1)^2 \lfloor\underline{2}\,(x-1)^{-3} - (-1)^2 \lfloor\underline{2}\,(x+1)^{-3}$$

....

$$y_n = (-1)^{n-1} \lfloor\underline{n-1}\,(x-1)^{-n} + (-1)^{n-1} \lfloor\underline{n-1}\,(x+1)^{-n}$$
$$+ (-1)^{n-2} \lfloor\underline{n-2}\,(x-1)^{-(n-1)} - (-1)^{n-2} \lfloor\underline{n-2}\,(x+1)^{-(n-1)}$$
$$= (-1)^{n-2} \lfloor\underline{n-2}\,(x-1)^{-(n-1)} \left\{ \frac{-(n-1)}{x-1} + 1 \right\}$$
$$+ (-1)^{n-2} \lfloor\underline{n-2}\,(x+1)^{-(n-1)} \left\{ \frac{-(n-1)}{x+1} - 1 \right\}$$
$$= (-1)^{n-2} \lfloor\underline{n-2} \left\{ \frac{x-n}{(x-1)^n} - \frac{(x+n)}{(x+1)^n} \right\}$$
$$= (-1)^{n} \lfloor\underline{n-2} \left\{ \frac{x-n}{(x-1)^n} - \frac{(x+n)}{(x+1)^n} \right\}.$$

5.3 USE OF PARTIAL FRACTIONS

(*i*) In the first step we have to observe that the degree of the polynomial in the numerator must be less than the degree of the polynomial in the denominator. If this is not the case, use the process of division so as to obtain,

$$\text{Fraction (given)} = \text{Quotient} + \frac{\text{Remainder}}{\text{Divisor}},$$

in which the fractional part of right hand side meets the necessary requirements.

(*ii*) The second step is to factorize the denominator into its ultimate real factors. These factors must be of the following types:

(*a*) linear but not repeated, of the type $(ax + b)$

(*b*) linear and repeated, such as $(ax + b)^n$

(*c*) quadratic but not repeated, of the type $(ax^2 + bx + c)$

(*d*) quadratic and repeated, such as $(ax^2 + bx + c)^n$.

(*iii*) The third step is to write down the given fraction as the sum of simple fractions and this will be done according to the following rules:

(*a*) for each factor of the type $ax + b$, there should be a single fraction of the form $\dfrac{A}{ax+b}$, A is a constant,

(*b*) for each factor of the type $(ax+b)^n$, there should be fractions of the form

$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\ldots+\frac{A_n}{(ax+b)^n};\ A_1, A_2, \ldots, A_n \text{ are constants,}$$

(*c*) for ax^2+bx+c and $(ax^2+bx+c)^n$, take $\dfrac{Ax+B}{ax^2+bx+c}$ and $\dfrac{A_1x+B_1}{ax^2+bx+c}$

$$+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\ldots+\frac{A_nx+B_n}{(ax^2+bx+c)^n} \text{ respectively.}$$

(*iv*) In the next step make the numerator of the sum of component fractions identical with the numerator of the given fraction to determine A, B, A_1, B_1, ...

ILLUSTRATIVE EXAMPLES

Example 1: Given $\quad y = \dfrac{1}{x^2-9}$, find y_n.

Solution:

$$\frac{1}{x^2-9}=\frac{1}{(x-3)(x+3)}=\frac{A}{x-3}+\frac{B}{x+3}=\frac{A(x+3)+B(x-3)}{x^2-9}$$

Therefore, $A(x+3)+B(x-3)\equiv 1$

This is an identity and must be true for any value of x. Putting $x = 3, -3$, we have $A = \dfrac{1}{6}$ and $B = -\dfrac{1}{6}$.

Thus,
$$y=\frac{1}{x^2-9}=\frac{1}{6}\left\{\frac{1}{x-3}-\frac{1}{x+3}\right\}=\frac{1}{6}\left\{(x-3)^{-1}-(x+3)^{-1}\right\}$$

$\therefore$
$$y_1=\frac{1}{6}\left\{(-1)(x-3)^{-2}-(-1)(x+3)^{-2}\right\}$$

$$y_2=\frac{1}{6}\left\{(-1)(-2)(x-3)^{-3}-(-1)(-2)(x+3)^{-3}\right\}$$

$$y_3=\frac{1}{6}\left\{(-1)(-2)(-3)(x-3)^{-4}-(-1)(-2)(-3)(x+3)^{-4}\right\}$$

....

$$y_n = \left\{\frac{1}{6}(-1)(-2)(-3)\ldots(-n)(x-3)^{-(n+1)} - (-1)(-2)(-3)\ldots(-n)(x+3)^{-(n+1)}\right\}$$

$$= \frac{(-1)^n n!}{6}\left\{(x-3)^{-(n+1)} - (x+3)^{-(n+1)}\right\}.$$

Example 2: If

$$y = \frac{x^4 + 7x^3 + 21x^2 + 33x + 22}{x^3 + 6x^2 + 11x + 6}, \text{ find } y_n.$$

Solution: The degree of the numerator is greater than that of the denominator, dividing we have,

$$y = x + 1 + \frac{4x^2 + 16x + 16}{x^3 + 6x^2 + 11x + 6} = x + 1 + \frac{4x^2 + 16x + 16}{(x+1)(x+2)(x+3)}$$

Let $\dfrac{4x^2 + 16x + 16}{(x+1)(x+2)(x+3)} = \dfrac{A}{x+1} + \dfrac{B}{x+2} + \dfrac{C}{x+3}$, hence

$$4x^2 + 16x + 16 \equiv A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$$

This is an identity and must be true for any value of x.

Putting $x = -1, -2, -3$, we obtain $A = 2$, $B = 0$, $C = 2$.

Therefore, $$y = x + 1 + 2\left(\frac{1}{x+1} + \frac{1}{x+3}\right) = x + 1 + 2\left\{(x+1)^{-1} + (x+3)^{-1}\right\}$$

$\therefore$ $$y_1 = 1 + 2\left\{(-1)(x+1)^{-2} + (-1)(x+3)^{-2}\right\}$$

$$y_2 = 2\left\{(-1)(-2)(x+1)^{-3} + (-1)(-2)(x+3)^{-3}\right\}$$

....

$$y_n = 2\left\{(-1)(-2)\ldots(-n)(x+1)^{-(n+1)} + (-1)(-2)\ldots(-n)(x+3)^{-(n+1)}\right\}$$

$$= 2(-1)^n n!\left\{(x+1)^{-(n+1)} + (x+3)^{-(n+1)}\right\}.$$

5.4 USE OF DE MOIVRE'S THEOREM

De Moivre's Theorem

For all integral values of n, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ and for fractional value of n, one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$, where $i = \sqrt{-1}$.

When the denominator of a given algebraic fraction cannot be broken up into real linear factors, it is advisable to break it up into complex factors and then use De Moivre's theorem to put the final result back to the real form.

ILLUSTRATIVE EXAMPLES

Example 1: If $y = \dfrac{1}{x^2+a^2}$, find y_n.

Solution:
$$y = \frac{1}{x^2+a^2} = \frac{1}{(x+ia)(x-ia)} = \frac{1}{2ia}\left[\frac{1}{x-ia} - \frac{1}{x+ia}\right]$$
$$= \frac{1}{2ia}\left[(x-ia)^{-1} - (x+ia)^{-1}\right]$$

$\therefore$
$$y_1 = \frac{1}{2ia}\left[(-1)(x-ia)^{-2} - (-1)(x+ia)^{-2}\right]$$
$$y_2 = \frac{1}{2ia}\left[(-1)(-2)(x-ia)^{-3} - (-1)(-2)(x+ia)^{-3}\right]$$

....

$$y_n = \frac{1}{2ia}\left[(-1)(-2)\ldots(-n)(x-ia)^{-(n+1)}\right.$$
$$\left. -(-1)(-2)\ldots(-n)(x+ia)^{-(n+1)}\right]$$
$$= \frac{(-1)^n n!}{2ia}\left[(x-ia)^{-(n+1)} - (x+ia)^{-(n+1)}\right] \qquad \ldots(1)$$

Put $x = r\cos\theta$, $a = r\sin\theta$ so that $r = (x^2+a^2)^{1/2}$, $\theta = \tan^{-1}\dfrac{a}{x}$

Therefore, $(x-ia)^{-(n+1)} = r^{-(n+1)}(\cos\theta - i\sin\theta)^{-(n+1)}$

$= r^{-(n+1)}\{\cos(n+1)\theta + i\sin(n+1)\theta\}$ (by De Moivre's Theorem)

Similarly, $(x+ia)^{-(n+1)} = r^{-(n+1)}\{\cos(n+1)\theta - i\sin(n+1)\theta\}$

Therefore, from (1),
$$y_n = \frac{(-1)^n n!}{2ia} r^{-(n+1)} 2i\sin(n+1)\theta$$
$$= \frac{(-1)^n n!}{ar^{n+1}}\sin(n+1)\theta$$
$$= \frac{(-1)^n n!}{a^{n+2}}\sin^{n+1}\theta\sin(n+1)\theta \qquad \left(\because r = \frac{a}{\sin\theta}\right)$$

where $\theta = \tan^{-1}\dfrac{a}{x}$.

Example 2: Given $y = \tan^{-1}\left(\frac{x}{a}\right)$, find y_n.

Solution: Here $y = \tan^{-1}\left(\frac{x}{a}\right)$

$$\therefore \quad y_1 = \frac{1}{a}\cdot\frac{1}{1+\frac{x^2}{a^2}} = \frac{a}{(x+ia)(x-ia)} = \frac{1}{2i}\{(x-ia)^{-1}-(x+ia)^{-1}\}$$

$$y_2 = \frac{1}{2i}\{(-1)(x-ia)^{-2}-(-1)(x+ia)^{-2}\}$$

$$y_3 = \frac{1}{2i}\{(-1)(-2)(x-ia)^{-3}-(-1)(-2)(x+ia)^{-3}\}$$

....

$$y_n = \frac{1}{2i}\{(-1)(-2)\ldots(-n+1)(x-ia)^{-n}$$

$$-(-1)(-2)\ldots(-n+1)(x+ia)^{-n}\}$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i}\{(x-ia)^{-n}-(x+ia)^{-n}\} \qquad \ldots(1)$$

Put $x = r\cos\theta$, $a = r\sin\theta$ so that $r = (x^2+a^2)^{1/2}$, $\theta = \tan^{-1}\frac{a}{x}$

Therefore, $(x-ia)^{-n} = r^{-n}(\cos\theta - i\sin\theta)^{-n}$

$= r^{-n}(\cos n\theta + i\sin n\theta)$ (by De Moivre's Theorem)

Similarly, $(x+ia)^{-n} = r^{-n}(\cos n\theta - i\sin n\theta)$

Therefore, from (1), $y_n = \frac{(-1)^{n-1}(n-1)!}{2i}r^{-n}2i\sin n\theta$

$$= \frac{(-1)^{n-1}(n-1)!}{r^n}\sin n\theta$$

$$= \frac{(-1)^{n-1}(n-1)!}{a^n}\sin^n\theta\sin n\theta, \qquad \left(\because r = \frac{a}{\sin\theta}\right)$$

where $\theta = \tan^{-1}\frac{a}{x}$.

Example 3: If $y = \tan^{-1}\frac{2x}{1-x^2}$, find y_n.

Solution: Now, $y = \tan^{-1}\frac{2x}{1-x^2} = \tan^{-1}\frac{2\tan\phi}{1-\tan^2\phi}$ (putting $x = \tan\phi$)

$$= \tan^{-1} \tan 2\phi = 2\phi = 2\tan^{-1} x$$

Proceeding as in Ex. 2, we have

$$y_n = 2(-1)^{n-1}(n-1)!\ \sin^n \theta \sin n\theta, \text{ where } \theta = \tan^{-1}\frac{1}{x}.$$

Example 4: Find y_n, where

$$y = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}.$$

Solution: Here

$$y = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$$

$$= \tan^{-1}\frac{\sec\phi - 1}{\tan\phi} \qquad \text{(putting } x = \tan\phi\text{)}$$

$$= \tan^{-1}\frac{1-\cos\phi}{\sin\phi} = \tan^{-1}\tan\frac{\phi}{2} = \frac{\phi}{2} = \frac{1}{2}\tan^{-1} x$$

Proceeding as in Ex. 2, we have

$$y_n = \frac{1}{2}(-1)^{n-1}(n-1)!\ \sin^n \theta\ \sin n\theta, \text{ where } \theta = \tan^{-1}\frac{1}{x}.$$

Example 5: Given $y = x(a^2+x^2)^{-1}$, find y_n.

Solution:

$$y = \frac{x}{a^2+x^2} = \frac{x}{(x+ia)(x-ia)} = \frac{1}{2}\left\{\frac{1}{x+ia} + \frac{1}{x-ia}\right\}$$

$$= \frac{1}{2}\left\{(x+ia)^{-1} + (x-ia)^{-1}\right\}$$

$$\therefore \quad y_1 = \frac{1}{2}\left\{(-1)(x+ia)^{-2} + (-1)(x-ia)^{-2}\right\}$$

$$y_2 = \frac{1}{2}\left\{(-1)(-2)(x+ia)^{-3} + (-1)(-2)(x-ia)^{-3}\right\}$$

....

$$y_n = \frac{1}{2}\left\{(-1)(-2)\ldots(-n)(x+ia)^{-(n+1)} + (-1)(-2)\ldots(-n)(x-ia)^{-(n+1)}\right\}$$

$$= \frac{1}{2}(-1)^n n!\left\{(x+ia)^{-(n+1)} + (x-ia)^{-(n+1)}\right\} \qquad \ldots(1)$$

Put $x = r\cos\theta$, $a = r\sin\theta$ so that $r = (x^2+a^2)^{1/2}$, $\theta = \tan^{-1}\frac{a}{x}$

Therefore, $\quad (x+ia)^{-(n+1)} = r^{-(n+1)}(\cos\theta + i\sin\theta)^{-(n+1)}$

$$= r^{-(n+1)}\{\cos(n+1)\theta - i\sin(n+1)\theta\} \quad \text{(by De Moivre's Theorem)}$$

Similarly, $\quad (x-ia)^{-(n+1)} = r^{-(n+1)}\{\cos(n+1)\theta + i\sin(n+1)\theta\}$

Therefore, from (1), $\quad y_n = (-1)^n n!\, r^{-(n+1)} \cos(n+1)\theta$

$$= \frac{(-1)^n n!}{a^{n+1}} \sin^{n+1}\theta \cos(n+1)\theta \qquad \left(\because r = \frac{a}{\sin\theta}\right)$$

where $\quad \theta = \tan^{-1}\left(\frac{a}{x}\right)$.

5.5 LEIBNITZ'S THEOREM

Statement

If u and v are two functions of x, each possessing derivatives upto nth order, then $y = uv$ is derivable n times and

$$y_n = (u\,v)_n = u_n v + {}^nC_1\, u_{n-1}v_1 + {}^nC_2\, u_{n-2}v_2 + \ldots$$
$$+ {}^nC_r\, u_{n-r}v_r + \ldots + {}^nC_n\, uv_n$$

where y_r, u_r, v_r be the rth derivatives of y, u, v respectively with respect to x.

Proof: Let $y = uv$

Differentiating both sides w.r.t. x, we have

$$y_1 = u_1v + uv_1$$

Again differentiating both sides w.r.t. x, we get

$$y_2 = u_2v + u_1v_1 + u_1v_1 + uv_2$$
$$= u_2v + {}^2C_1u_1v_1 + {}^2C_2uv_2$$

Therefore, the theorem is true for $n = 1, 2$.

Let us assume that the theorem holds good for a positive integer m and so we have

$$y_m = u_mv + {}^mC_1u_{m-1}v_1 + {}^mC_2u_{m-2}v_2 + \ldots + {}^mC_{r-1}u_{m-r+1}v_{r-1} +$$
$${}^mC_ru_{m-r}v_r + \ldots + {}^mC_muv_m$$

Differentiating both sides w.r.t. x, we get

$$y_{m+1} = u_{m+1}v + u_mv_1 + {}^mC_1\{u_mv_1 + u_{m-1}v_2\} + {}^mC_2\{u_{m-1}v_2 + u_{m-2}v_3\}$$
$$+ \ldots + {}^mC_{r-1}\{u_{m-r+2}v_{r-1} + u_{m-r+1}v_r\} + {}^mC_r\{u_{m-r+1}v_r + u_{m-r}v_{r+1}\}$$
$$+ \ldots + {}^mC_m\{u_1v_m + uv_{m+1}\}$$
$$= u_{m+1}v + \{{}^mC_0 + {}^mC_1\}\, u_mv_1 + \{{}^mC_1 + {}^mC_2\}u_{m-1}v_2$$
$$+ \ldots + \{{}^mC_{r-1} + {}^mC_r\}u_{m-r+1}v_r + \ldots + {}^mC_muv_{m+1}$$

$$= u_{m+1}v + {}^{m+1}C_1 u_m v_1 + {}^{m+1}C_2 u_{m-1} v_2 + \dots$$

$$+ {}^{m+1}C_r u_{m+1-r} v_r + \dots + {}^{m+1}C_{m+1} u v_{m+1}$$

$$\left[\because {}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r \text{ and } {}^mC_m = {}^{m+1}C_{m+1} = 1\right]$$

So, the theorem is true for $m + 1$ if it is true for m. But we have proved that the theorem is true for $n = 1, 2$ and so it holds good for $2 + 1 = 3$, $3 + 1 = 4$, $4 + 1 = 5$ etc. Therefore, the theorem is true for any positive integer n.

This completes the proof.

Note: While applying Leibnitz's theorem the polynomial function is, in general, chosen as v.

ILLUSTRATIVE EXAMPLES

Example 1: Given $y = x^3 \log x$, find y_n.

Solution: Let $u = \log x$, $v = x^3$

$$\therefore \quad u_1 = \frac{1}{x} = x^{-1},\ u_2 = (-1)\,x^{-2},\ u_3 = (-1)(-2)x^{-3}, \dots,$$

$$u_n = (-1)(-2)\dots(-n+1)x^{-n} = \frac{(-1)^{n-1}(n-1)!}{x^n},$$

$$v_1 = 3x^2,\ v_2 = 6x,\ v_3 = 6,\ v_4 = 0,\ v_5 = 0,\ \dots,\ v_n = 0$$

Therefore by Leibnitz's theorem,

$$y_n = (u\,v)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2\, u_{n-2} v_2 + {}^nC_3\, u_{n-3} v_3 + \dots + {}^nC_n u\, v_n$$

$$= \frac{(-1)^{n-1}(n-1)!}{x^n} x^3 + n\frac{(-1)^{n-2}(n-2)!}{x^{n-1}} 3x^2$$

$$+ \frac{n\,(n-1)}{2!}\,\frac{(-1)^{n-3}(n-3)!}{x^{n-2}}\cdot 6x + \frac{n(n-1)\,(n-2)}{3!}\,\frac{(-1)^{n-4}(n-4)!}{x^{n-3}}\cdot 6$$

$$= \frac{(-1)^{n-4}(n-4)!}{x^{n-3}}\{-(n-1)\,(n-2)\,(n-3)$$

$$+3n\,(n-2)\,(n-3) - 3n\,(n-1)\,(n-3) + n\,(n-1)\,(n-2)\}$$

$$= \frac{(-1)^{n-4}(n-4)!}{x^{n-3}}\{\,(n-2)\,(n-3)\,(2n+1) + n\,(n-1)\,(-2n+7)\}$$

$$= \frac{(-1)^{n-4}(n-4)!}{x^{n-3}}(2n^3 - 9n^2 + 7n + 6 - 2n^3 + 9n^2 - 7n)$$

$$= \frac{(-1)^{n-4} 6 (n-4)!}{x^{n-3}}.$$

Example 2: If $y = x^3 \cos x$, find y_n.

Solution: Let $u = \cos x$, $v = x^3$, then $u_n = \cos\left(\frac{n\pi}{2} + x\right)$,

$$v_1 = 3x^2,\ v_2 = 6x,\ v_3 = 6,\ v_4 = 0,\ v_5 = 0, \ldots, v_n = 0$$

By Leibnitz's theorem,

$$y_n = (u\,v)_n = u_n v + {}^nC_1\, u_{n-1} v_1 + {}^nC_2\, u_{n-2} v_2 + {}^nC_3\, u_{n-3} v_3 + \ldots + {}^nC_n u\, v_n$$

$$= x^3 \cos\left(\frac{n\pi}{2} + x\right) + 3nx^2 \cos\left\{\frac{(n-1)\pi}{2} + x\right\}$$

$$+ \frac{n(n-1)}{2!} 6x \cos\left\{\frac{(n-2)\pi}{2} + x\right\}$$

$$+ \frac{n(n-1)(n-2)}{3!} \cdot 6 \cos\left\{\frac{(n-3)\pi}{2} + x\right\}.$$

Example 3: Find y_n when $y = e^x \log x$.

Solution: Take $u = e^x$, $v = \log x$,

then $u_n = e^x, v_n = \dfrac{(-1)^{n-1}(n-1)!}{x^n}$

By Leibnitz's theorem,

$$y_n = (u\,v)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \ldots + {}^nC_n u\ v_n$$

$$= e^x \log x + ne^x \cdot \frac{1}{x} + \frac{n(n-1)}{2!} e^x (-1) \cdot \frac{1}{x^2} + \ldots + e^x \frac{(-1)^{n-1}(n-1)!}{x^n}.$$

Example 4: If $y = \dfrac{x^n}{1+x}$, find y_n.

Solution: Take $u = x^n$ and $v = \dfrac{1}{1+x}$

Then $u_r = \dfrac{n!\,x^{n-r}}{(n-r)!}$, when $r < n$

$= n!$, when $r = n$

and $v_n = \dfrac{(-1)^n n!}{(1+x)^{n+1}}$

By Leibnitz's theorem,

$$y_n = (u\,v)_n = u_n v + {}^nC_1\, u_{n-1}\, v_1 + {}^nC_2\, u_{n-2}\, v_2 + \ldots + {}^nC_n\, u\, v_n$$

$$= n!\,\frac{1}{1+x} + {}^nC_1\,\frac{n!\,x}{1!}\,\frac{(-1)\,1!}{(1+x)^2}$$

$$+ {}^nC_2\,\frac{n!\,x^2}{2!}\,\frac{(-1)^2\,2!}{(1+x)^3} + \ldots + {}^nC_n\,\frac{x^n(-1)^n\,n!}{(1+x)^{n+1}}$$

$$= \frac{n!}{(1+x)^{n+1}}\{{}^nC_0(1+x)^n - {}^nC_1(1+x)^{n-1}x$$

$$+ {}^nC_2(1+x)^{n-2}x^2 + \ldots + (-1)^n\,{}^nC_n x^n\}$$

$$= \frac{n!}{(1+x)^{n+1}}\{(1+x)-x\}^n = \frac{n!}{(1+x)^{n+1}}.$$

Example 5: If $y = a\cos(\log x) + b\sin(\log x)$, show that

(*i*) $x^2y_2 + xy_1 + y = 0$

(*ii*) $x^2\,y_{n+2} + (2n+1)\,x\,y_{n+1} + (n^2+1)\,y_n = 0.$ (*W.B.U.T. 2007*)

Solution: Here $y = a\cos(\log x) + b\sin(\log x)$

$\therefore$ $$y_1 = -a\cdot\frac{1}{x}\sin(\log x) + b\cdot\frac{1}{x}\cos(\log x)$$

or $$xy_1 = -a\sin(\log x) + b\cos(\log x) \qquad \ldots(1)$$

Differentiating both sides of (1) with respect to x, we get

$$xy_2 + y_1 = -a\cdot\frac{1}{x}\cos(\log x) - b\cdot\frac{1}{x}\sin(\log x)$$

or $$x^2y_2 + xy_1 = -\{a\cos(\log x) + b\sin(\log x)\}$$

or $$x^2y_2 + xy_1 = -y$$

$\therefore$ $$x^2y_2 + xy_1 + y = 0 \qquad \ldots(2)$$

Differentiating both sides of (2) n times with respect to x by Leibnitz's theorem, we get

$$y_{n+2}\,x^2 + {}^nC_1\,y_{n+1}(2x) + {}^nC_2\,y_n 2 + (y_{n+1}x + {}^nC_1 y_n) + y_n = 0$$

or $$x^2y_{n+2} + 2nxy_{n+1} + \frac{n(n-1)}{2!}\cdot 2y_n + (xy_{n+1} + ny_n) + y_n = 0$$

or $$x^2y_{n+2} + (2n+1)\,xy_{n+1} + (n^2+1)\,y_n = 0.$$