Infinite Series

An expression of the form

$$u_1 + u_2 + \cdots + u_n + \cdots \quad \text{——— (1)}$$

or $\sum_{n=1}^{\infty} u_n$ is called an INFINITE SERIES.

To understand/study infinite series, we must first understand the concept of a (real) sequence.

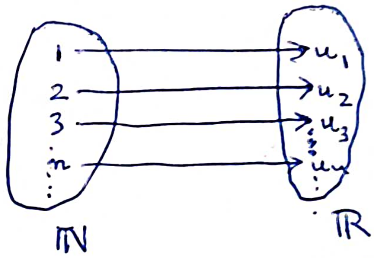
Sequence of real numbers $\{u_n\}_{n=1}^{\infty}$ or $\{u_n\}$ is a function from $\mathbb{N}$ to $\mathbb{R}$.

$\mathbb{N}$: set of natural numbers $= \{1, 2, 3, \cdots\}$

$\mathbb{R}$: set of real numbers $= \{x : x > 0\} \cup \{x : x < 0\} \cup \{0\}$.

$\{u_n\}$

$u_1, u_2, u_3, \cdots\cdots$



Defn. A sequence $\{u_n\}$ is said to converge to a real number L if for every $\varepsilon > 0$, $\exists$ an integer $N \geq 1 \ni$

$$|u_n - L| < \varepsilon \quad \forall \; n \geq N.$$

(Here $\exists$: there exists, $\ni$: such that)

For infinite series (1), define

$$S_n = u_1 + u_2 + \cdots + u_n, \quad n = 1, 2, 3, \cdots$$

The sequence $\{S_n\}$ is called the sequence of nth partial sums of the series above.

The series $\sum u_n$ is said to be convergent if the sequence $\{S_n\}$ converges/is convergent, and $\lim_{n \to \infty} S_n$ is called the SUM OF THE SERIES.

The series $\sum u_n$ is said to be divergent if the sequence $\{S_n\}$ is divergent,

e.g., the geometric series

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots, \quad a \neq 0$$

converges if $|r| < 1$, and the sum of the series is $\dfrac{a}{1-r}$.

The series diverges if $|r| \geq 1$.

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$
$$= \frac{a}{1-r} - \frac{ar^n}{1-r} \text{ ——— } (2)$$

If $|r| < 1$, let $|r| = \dfrac{1}{1+p}$, $p > 0$. Then

$(1+p)^n > \binom{n}{k} p^k$ for all integers $k = 1, 2, \cdots, n$.

$$(1+p)^n > \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} p^k \quad \forall\ k = 1, 2, \cdots, n.$$

Take $k = 1$, $(1+p)^n > np$ so that

$$0 < |r|^n = \frac{1}{(1+p)^n} < \frac{1}{np} \longrightarrow 0 \text{ as } n \to \infty.$$

$\therefore$ If $|r| < 1$, $\lim\limits_{n\to\infty} r^n = 0$.

Now, the desired result follows from (2).

If $r > 1$, then $r^n > 1\ \forall\ n$, and $r^{n+1} = r \cdot r^n > r^n$.

So the sequence $\{r^n\}$ is monotone increasing.

Let $\lim\limits_{n\to\infty} r^n = L$.

Then $L > 1$ as $r^n > 1\ \forall\ n$. So L is either a positive no. > 1 or $L = +\infty$.

Suppose L is <u>finite</u>. Then $\lim\limits_{n\to\infty} r^{n+1} = r \lim\limits_{n\to\infty} r^n = rL$.

$\therefore$ $L = rL$ or $L(r-1) = 0$. $\therefore$ $L = 0$ as $r \neq 1$. Contradiction!

So if $r > 1$, $\lim\limits_{n\to\infty} r^n = +\infty$.

If $r = 1$, $S_n = an \longrightarrow +\infty$ or $-\infty$ acc. as $a > 0$ or $a < 0$.

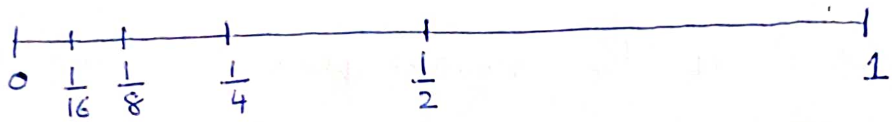
If $r = -1$, $S_n = a - a + a - a + \cdots$

$$= \begin{cases} 0 & \text{if } n \text{ is EVEN} \\ a & \text{if } n \text{ is ODD.} \end{cases}$$

So, in this case $\lim\limits_{n\to\infty} S_n$ <u>does not exist</u>. Hence the series diverges.

Zeno's Paradox or Racecourse Paradox:

A runner cannot reach the end of a racecourse because he must cover the half of any distance before he covers the whole.



Let T be the time taken to cover the distance between the point 1 to the point $\frac{1}{2}$.

Then the total time taken

$$= T + \frac{T}{2} + \frac{T}{4} + \cdots\cdots + \frac{T}{2^n} + \cdots = 2T;$$ observe that

$\frac{T}{2^n}$ = time taken to cover the distance from $\frac{1}{2^n}$ to $\frac{1}{2^{n+1}}$.

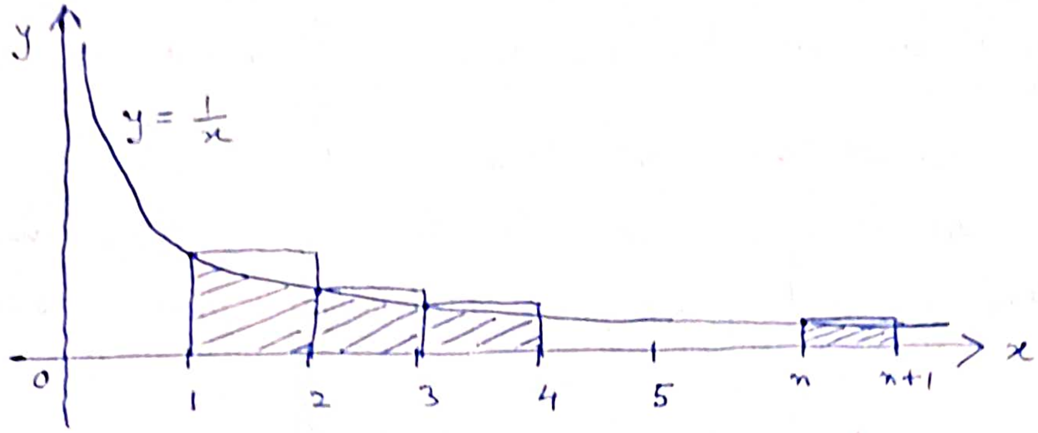
Now, let us make a small but important change in the foregoing analysis of the racecourse paradox. Instead of assuming that the speed of the runner is constant, suppose that his speed gradually decreases in such a way that

he requires time T to go from 1 to $\frac{1}{2}$,
time $\frac{T}{2}$ to go from $\frac{1}{2}$ to $\frac{1}{4}$,
time $\frac{T}{3}$ to go from $\frac{1}{4}$ to $\frac{1}{8}$,

in general, time $\frac{T}{n}$ to go from $\frac{1}{2^{n-1}}$ to $\frac{1}{2^n}$.

Then the time taken to reach the end of the course $= T + \frac{T}{2} + \frac{T}{3} + \cdots\cdots + \frac{T}{n} + \cdots\cdots$, which is infinity.

The harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots\cdots + \frac{1}{n} + \cdots\cdots$ diverges to $+\infty$.



Sum of the areas of the rectangles

$$= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = S_n$$

Area of the shaded region

$$= \int_1^{n+1} \frac{dx}{x} = \log_e (n+1) .$$

$\therefore \quad S_n \geq \log_e (n+1)$ for all $n \geq 1$.

This means/shows that $\lim_{n \to \infty} S_n = +\infty$, since $\log_e (n+1)$ increases without bound as $n \to \infty$.

<u>Definition</u>. Suppose f is <u>continuous</u> in the interval $[a, \infty)$.

Then $\int_a^\infty f(x)\,dx$ is defined to be $\lim_{R \to \infty} \int_a^R f(x)\,dx$ provided the limit exists.

The integral $\int_a^\infty f(x)\,dx$ is called an <u>improper integral</u>.

We say that $\int_a^\infty f(x)\,dx$ is <u>convergent</u> if $\lim_{R \to \infty} \int_a^R f(x)\,dx$ exists.

In this case, we write

$$\int_a^\infty f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx .$$

The integral $\int_a^\infty f(x)\,dx$ is said to be divergent if $\lim_{R\to\infty} \int_a^R f(x)\,dx$ does not exist.

Theorem. A series $\sum_{n=1}^\infty u_n$ of non-negative terms converges iff its partial sums form a bounded sequence.

Proof: Since $u_n \geq 0 \ \forall\ n$, the sequence $\{S_n\}$ of partial sums is a monotone increasing sequence, and hence $\{S_n\}$ is convergent iff it is bounded above.

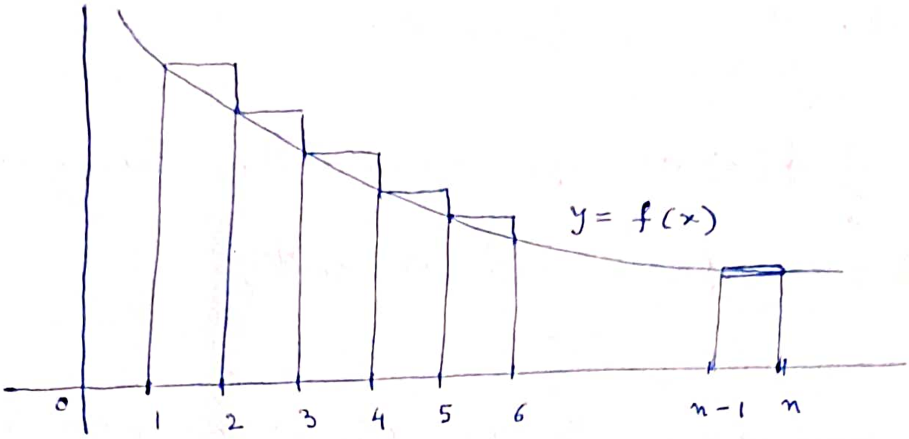
Cauchy's Integral Test.

If f is continuous, non-negative and non-increasing on $[1, \infty)$, and if $\sum_{n=1}^\infty u_n$ is a series with $u_n = f(n)$, then

(a) $\sum_{n=1}^\infty u_n$ converges if the improper integral $\int_1^\infty f(x)\,dx$ converges.

(b) $\sum_{n=1}^\infty u_n$ diverges if $\int_1^\infty f(x)\,dx$ diverges.

Proof: See diagram overleaf.



Since $f(x)$ is <u>non-negative</u> and <u>non-increasing</u>, we have for $n \geq 2$,

$$\sum_{k=2}^{n} u_k \leq \int_1^n f(x)\,dx \leq \sum_{k=1}^{n-1} u_k \leq \sum_{k=1}^{n} u_k$$

or,
$$S_n - u_1 \leq \int_1^n f(x)\,dx \leq S_n .$$

Define $F(X) = \int_1^X f(x)\,dx$. Then $F(X)$ is non-decreasing, as $f(x)$ is non-negative.

If $\int_1^\infty f(x)\,dx$ converges, then $F(X)$ tends to a limit as $X \to \infty$, and hence $F(X)$ is bounded above. Let $F(X) \leq K$ for all X

Therefore $S_n - u_1 \leq K$, or $S_n \leq u_1 + K$ for all n. Thus the sequence $\{S_n\}$ is bounded above, and therefore $\sum_{n=1}^{\infty} u_n$ is convergent.

If $\int_1^\infty f(x)\,dx$ diverges, then $F(X) \to \infty$ as $X \to \infty$. In particular $F(n) \to \infty$ as $n \to \infty$. Since $F(n) \leq S_n$, the sequence $\{S_n\}$ diverges to $+\infty$, and therefore $\sum_{n=1}^{\infty} u_n$ is divergent.

e.g. The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p>1$, and diverges if $p \leq 1$.

$$\int_1^{\infty} \frac{dx}{x^p} = \lim_{R\to\infty} \int_1^R \frac{dx}{x^p} = \lim_{R\to\infty} \left[\frac{R^{1-p}}{1-p} - \frac{1}{1-p} \right]$$

$$= \begin{cases} \frac{1}{p-1} & \text{if } p>1 \\ \infty & \text{if } p<1. \end{cases}$$

If $p = 1$,

$$\int_1^{\infty} \frac{dx}{x^p} = \lim_{R\to\infty} \int_1^R \frac{dx}{x} = \lim_{R\to\infty} \log_e R = \infty.$$

Thus the improper integral $\int^{\infty} \frac{dx}{x^p}$ converges to $\frac{1}{p-1}$ if $p>1$, and it diverges if $p \leq 1$.

<u>N. B.</u> The above is called the p-series

$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots\cdots + \frac{1}{\sqrt{n}} + \cdots$ is divergent as here $p = \frac{1}{2} < 1$.

Cauchy Criterion.

A series $\sum_{n=1}^{\infty} u_n$ converges iff for every $\varepsilon > 0$, $\exists$ an integer $N > 0$ such that

$$|S_n - S_m| = |u_{m+1} + \cdots\cdots + u_n| = \left| \sum_{k=m+1}^{n} u_k \right| < \varepsilon$$

if $n \geq m \geq N$.

In particular, taking $n = m+1$, we get

$$|u_n| < \varepsilon \text{ if } n \geq N.$$

This means —

<u>Theorem.</u> If the series $\sum_{n=1}^{\infty} u_n$ converges, then $\lim_{n\to\infty} u_n = 0$.

The converse is not true. For example, the harmonic series $\sum \frac{1}{n}$ diverges, but $\frac{1}{n} \to 0$ as $n \to \infty$.

Example. Consider the series

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{n}{n+1} + \cdots$$

$$u_n = \frac{n}{n+1} \to 1 \text{ as } n \to \infty.$$

So the series is divergent.

Comparison Test.

(a) If $|u_n| \le a_n$ for all $n \ge N_0$ where N_0 is some fixed positive integer, and if $\sum a_n$ converges, then $\sum u_n$ also converges.

(b) If $u_n \ge b_n \ge 0$ for all $n \ge N_0$, and if $\sum b_n$ diverges, then $\sum u_n$ diverges.

Proof: (a) Since $\sum a_n$ is convergent, for every $\varepsilon > 0$, $\exists$ an integer $N \ge N_0$ such that

$$|a_{m+1} + \cdots + a_n| < \varepsilon \text{ if } n \ge m \ge N.$$

(by Cauchy Criterion). Therefore

$$|u_{m+1} + \cdots + u_n| \le a_{m+1} + \cdots + a_n < \varepsilon \text{ if } n \ge m \ge N$$

So $\sum u_n$ converges by Cauchy Criterion.

The part (b) follows from (a), because if $\sum u_n$ is convergent, then $\sum b_n$ must converge, leading to a contradiction.

Show that the series

$$\frac{8}{1}\cdot 1 + \frac{11}{2}\cdot\frac{1}{2} + \frac{14}{3}\cdot\frac{1}{2^2} + \cdots\cdots + \frac{3n+5}{n}\left(\frac{1}{2}\right)^{n-1} + \cdots\cdots$$

converges.

$$u_n = \frac{3n+5}{n}\left(\frac{1}{2}\right)^{n-1}$$

If $n \geq 5$, $\frac{3n+5}{n} \leq 4$, and therefore

$$u_n = \frac{3n+5}{n}\cdot\frac{1}{2^{n-1}} \leq 4\cdot\frac{1}{2^{n-1}} = a_n \text{ if } n \geq 5.$$

The G. P. series $\sum a_n$ converges. So $\sum u_n$ converges by comparison test.

Ex. 1. Show that a series of positive terms is either convergent, or it diverges to $+\infty$.

Ex. 2. Let $\sum a_n$ be a convergent series. Prove that there is a number M such that $|a_n| \leq M \ \forall\ n$.

Ex. 3. If $\sum a_n$ is convergent, then show (without using the Cauchy criterion) that $\lim_{n\to\infty} a_n = 0$.

[Hints : If $\lim_{n\to\infty} S_n = S$, then $\lim_{n\to\infty} S_{n-1} = S$ also.

$\therefore\ a_n = S_n - S_{n-1} \to S - S = 0$ as $n \to \infty$.]

Ex. 4. If $\sum a_n$ converges to A and $\sum b_n$ converges to B, then show that $\sum(a_n + b_n)$ converges to $A+B$.