9 INFINITE (OR IMPROPER) INTEGRALS AND INTEGRATION OF INFINITE SERIES

9.1. Infinite integrals.

In discussing definite integrals we have hitherto supposed that the range of integration is finite and the integrand is bounded in the range. If in an integral either the range is infinite or the integrand has an infinite discontinuity in the range (i.e., the integrand tends to infinity at some points of the range), the integral is usually called an *Infinite Integral*, and by some writers an *Improper Integral*. Simple cases of infinite integrals occur in elementary problems; for example, in the problem of finding the area between a plane curve and its asymptote. We give below the definitions of infinite integrals in different cases.

9.2. Improper Integrals.

Let us consider the integral $I = \int_a^b f(x)\,dx$.

It is said to be an improper integral of the *first kind* if the range of integration is unbounded.

Thus, $\int_0^\infty e^{-x^2}\,dx$, $\int_{-\infty}^{1} \frac{dx}{(2-x)^2}$, $\int_{-\infty}^{\infty} \tan^{-1} x\,dx$

are improper integrals of the first kind.

I is said to be an improper integral of the *second kind* if the range of integration is finite, but the integrand $f(x)$ has one or more points of infinite discontinuity within the range of integration.

Thus, $\int_0^1 \frac{dx}{x^7}$, $\int_{-1}^{2} \frac{dx}{(x-1)^2(2-x)}$

are improper integrals of the second kind.

I is said to be an improper integral of the *mixed kind* if the range of integration is unbounded; also the integrand $f(x)$ has one or more points of infinite discontinuity within the range of integration. Such an integral can be expressed as the sum of several improper integrals some of which are of the first kind and the others are of the second kind.

Thus, $\int_0^\infty \frac{dx}{x-1}$ is an improper integral of the mixed kind.

9.3. Convergence of improper integrals.

(A) *First kind.*

Consider the integral $\int_a^{\infty} f(x)\,dx$, which is an ~~inproper~~ improper integral of the first kind, if $f(x)$ is bounded in $[a, X]$ for every $X > a$. If $\lim_{X \to \infty} \int_a^X f(x)\,dx$ exists, and its value is m, say, then the above improper integral is said to be convergent and m is called the value of the integral.

Consider the integral $\int_{-\infty}^{b} f(x)\,dx$, which is an inproper integral of the first kind if $f(x)$ is bounded in $[X, b]$ for every $X < b$. If $\lim_{x \to -\infty} \int_X^b f(x)\,dx$ exists, and its value is p, say, then the above improper integral is said to be convergent and p is called the value of the integral.

If for any real number a, the improper integrals of the first kind $\int_{-\infty}^{a} f(x)\,dx$ and $\int_a^{\infty} f(x)\,dx$ both exist, we say that $\int_{-\infty}^{\infty} f(x)\,dx$ exists, and

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx.$$

(B) *Second kind.*

Consider the integral $\int_a^b f(x)\,dx$, which is an improper integral of the second kind, the end point a being the only point of infinite discontinuity of the integrand $f(x)$. If $\lim_{\varepsilon \to 0+} \int_{a+\varepsilon}^{b} f(x)\,dx$ exists, and its value is m_1, say, then the improper integral is said to be convergent and m_1 is called its value.

Again, we consider the integral $\int_a^b f(x)\,dx$, which is an improper integral of the second kind, the end point b being the only point of infinite discontinuity of the integrand $f(x)$. If $\lim_{\varepsilon \to 0+} \int_a^{b-\varepsilon} f(x)\,dx$ exists, and its value is m_2, say, then the improper integral is said to be convergent and m_2 is called its value.

Next, we consider the integral $\int_a^b f(x)\,dx$, which is an improper integral of the second kind, $c\ (a < c < b)$ being the only point of infinite discontinuity of the integrand $f(x)$, then

$$\int_a^b f(x)\,dx = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon' \to 0}} \left[\int_a^{c-\varepsilon} f(x)\,dx + \int_{c+\varepsilon'}^b f(x)\,dx \right],$$

where $a < c - \varepsilon < c < c + \varepsilon' < b$.

Note. It sometimes happens that no definite limit exists when ε and ε' tend to zero *independently*, but that a limit exists when $\varepsilon = \varepsilon'$. [See *Ex.* 11, *Article* 9.14.] When $\varepsilon = \varepsilon'$, the value of the limit on the right side, when it exists, is called the *principal value* of the improper integral and is very often denoted by

$$P \int_a^b f(x)\,dx.$$

If *a and b are both points of infinite discontinuity* and no other point of infinite discontinuity in $[a, b]$ then $\int_a^b f(x)\,dx$ is defined as $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ when these two integrals exist, as defined above, c being a point between a and b.

(C) *Mixed kind.*

Let I be an improper integral of the mixed kind. We consider the integrals as $I_1, I_2, \ldots, I_n$, where some of the improper integrals $I_1, I_2, \ldots, I_n$, are of the first kind and the others are of the second kind, where the upper limit of I_r is the lower limit of I_{r+1} $(r = 1, 2, \ldots,$

$n-1$) and the lower limit of I is that of I_1 and the upper limit of I_n is that of I. Then I is said to be convergent if each of $I_1, I_2, \ldots, I_n$ is convergent and $I = I_1 + I_2 + \ldots + I_n$.

Note 1. If an improper integral (first kind or second kind or mixed kind) be not convergent then it is called divergent.

2. A proper integral (if exists) is also said to be convergent.

9.4. Convergence of $\int_a^\infty \frac{dx}{x^n}$, $(a > 0)$.

Let $I = \int_a^\infty \frac{dx}{x^n}$, which is an improper integral of the first kind.

We have $\int \frac{dx}{x^n} = \frac{x^{1-n}}{1-n} \quad (n \neq 1)$.

Case I. When $n > 1$.

We have $$\lim_{X\to\infty} \int_a^X \frac{dx}{x^n} = \lim_{X\to\infty} \left[\frac{-1}{(n-1)x^{n-1}}\right]_a^X$$

$$= \lim_{X\to\infty} \frac{1}{n-1}\left(\frac{1}{a^{n-1}} - \frac{1}{X^{n-1}}\right)$$

$$= \frac{1}{(n-1)a^{n-1}}, \text{ which is finite.}$$

Therefore I, in this case, is convergent and its value is

$$= \frac{1}{(n-1)a^{n-1}}.$$

Case II. When $n < 1$.

We have $$\lim_{X\to\infty} \int_a^X \frac{dx}{x^n} = \lim_{X\to\infty} \left[\frac{x^{1-n}}{1-n}\right]_a^X$$

$$= \lim_{X\to\infty} \frac{X^{1-n} - a^{1-n}}{1-n} \to \infty.$$

Therefore I is divergent.

Case III. When $n = 1$.

Here $I = \int_a^{\infty} \frac{dx}{x}$.

We have $\lim\limits_{X \to \infty} = \int_a^X \frac{dx}{x} = \lim\limits_{X \to \infty} \left[\log x\right]_a^X = \lim\limits_{X \to \infty} \log\left(\frac{X}{a}\right) \to \infty.$

Therefore I is divergent.

9.5. Convergence of $\int_a^b \frac{dx}{(x-a)^n}$.

Let $I = \int_a^b \frac{dx}{(x-a)^n}$.

If $n \le 0$, I is proper; and it exists (so convergent) but if $n > 0$, I is improper of the second kind, the end point a being the only point of infinite discontinuity of the integrand $\frac{1}{(x-a)^n}$.

We have $\int \frac{dx}{(x-a)^n} = \frac{(x-a)^{1-n}}{1-n}$, $(n \neq 1)$.

Case I. When $0 < n < 1$.

We have $\lim\limits_{\varepsilon \to 0+} \int_{a+\varepsilon}^{b} \frac{dx}{(x-a)^n} = \lim\limits_{\varepsilon \to 0+} \left[\frac{(x-a)^n}{1-n}\right]_{a+\varepsilon}^{b}$

$$= \lim_{\varepsilon \to 0+} \left\{\frac{(b-a)^{1-n}}{1-n} - \frac{\varepsilon^{1-n}}{1-n}\right\} = \frac{(b-a)^{1-n}}{1-n}, \text{ which is finite.}$$

Hence I is convergent.

Case II. When $n > 1$.

We have $\lim\limits_{\varepsilon \to 0+} \int_{a+\varepsilon}^{b} \frac{dx}{(x-a)^n} = \lim\limits_{\varepsilon \to 0+} \left[\frac{1}{(n-1)(x-a)^{n-1}}\right]_{a+\varepsilon}^{b}$

$$= \lim_{\varepsilon \to 0+} \frac{1}{n-1} \left\{\frac{1}{\varepsilon^{n-1}} - \frac{1}{(b-a)^{n-1}}\right\}$$

$$\to \infty.$$

Therefore I is divergent.

Case III. When $n = 1$.

Here $I = \int_a^b \frac{dx}{x-a}$.

We have $\int \frac{dx}{x-a} = \log(x-a)$.

Therefore $\lim_{\varepsilon \to 0+} \int_{a+\varepsilon}^{b} \frac{dx}{b-x} = \lim_{\varepsilon \to 0+} \left[-\log(x-a)\right]_{a+\varepsilon}^{b}$.

$$= \lim_{\varepsilon \to 0+} \log\left(\frac{b-a}{\varepsilon}\right) \to \infty.$$

Therefore I is divergent.

Hence $\int_a^b \frac{dx}{x-a}$ is convergent if and only if $n<1$.

9.6. Convergence of $\int_a^b \frac{dx}{(b-x)^n}$.

Let $I = \int_a^b \frac{dx}{(b-x)^n}$.

If $n \le 0$, I is proper and it exists (so convergent), but if $n > 0$, I is improper of the second kind, the end point b being the only point of infinite discontinuity of the integrand $\frac{1}{(b-x)^n}$.

We have $\int \frac{dx}{(b-x)^n} = \frac{(b-x)^{1-n}}{-(1-n)}, (n \neq 1)$.

Case I. When $0 < n < 1$.

We have $\lim_{\varepsilon \to 0+} \int_a^{b-\varepsilon} \frac{dx}{(b-x)^n} = \lim_{\varepsilon \to 0+} \left[\frac{-(b-x)^{1-n}}{1-n}\right]_a^{b-\varepsilon}$

$$= \lim_{\varepsilon \to 0+} \frac{(b-a)^{1-n} - \varepsilon^{1-n}}{1-n} = \frac{(b-a)^{1-n}}{1-n}, \text{ which is finite.}$$

Hence I is convergent.

Case II. When $n > 1$.

We have $$\lim_{\varepsilon\to 0+}\int_a^{b-\varepsilon}\frac{dx}{(b-x)^n}=\lim_{\varepsilon\to 0+}\left[\frac{-(b-x)^{1-n}}{1-n}\right]_a^{b-\varepsilon}$$

$$=\lim_{\varepsilon\to 0+}\left[\frac{1}{(n-1)(b-x)^{n-1}}\right]_a^{b-\varepsilon}$$

$$=\lim_{\varepsilon\to 0+}\frac{1}{n-1}\left\{\frac{1}{\varepsilon^{n-1}}-\frac{1}{(b-a)^{n-1}}\right\}\to\infty$$

Therefore I is divergent.

Case III. When $n = 1$.

Here $$I=\int_a^b\frac{dx}{b-x}.$$

We have $$\int\frac{dx}{b-x}=-\log(b-x).$$

Therefore $$\lim_{\varepsilon\to 0+}\int_a^{b-\varepsilon}\frac{dx}{b-x}=\lim_{\varepsilon\to 0+}\left[-\log(b-x)\right]_a^{b-\varepsilon}$$

$$=\lim_{\varepsilon\to 0+}\log\left(\frac{b-a}{\varepsilon}\right)\to\infty$$

Therefore I is divergent.

Hence I is convergent if and only if $n<1$.