$= f(x)$ be a derivable function of x in a given interval of x. Then its derivative is also a function. It may happen that the derivative of $y = f(x)$ denoted by

$$y_1,\ y',\ Dy,\ \frac{dy}{dx},\ f'(x),\ \frac{d}{dx}\{f(x)\},\ Df(x)$$

again a derivable function of x in a certain interval. This derivative is called the second order derivative of $y = f(x)$ and it is denoted by

$$y_2,\ y'',\ D^2y,\ \frac{d^2y}{dx^2},\ f''(x),\ \frac{d^2}{dx^2}\{f(x)\},\ D^2f(x).$$

If possible, this process of differentiation can be continued and is known successive differentiation of the function $y = f(x)$. The nth order derivative of $y = f(x)$, if it exists, is denoted by

$$y_n,\ y^{(n)},\ D^ny,\ \frac{d^ny}{dx^n},\ f^n(x),\ \frac{d^n}{dx^n}\{f(x)\},\ D^nf(x).$$

.2 THE NTH ORDER DERIVATIVES OF SOME FUNCTIONS

I. $y = x^m$ $(x > 0)$, m is any real number.

Differentiating successively with respect to x, we get

$$\frac{dy}{dx} = y_1 = mx^{m-1}$$

$$\frac{d^2y}{dx^2} = y_2 = m(m-1)x^{m-2}$$

$$\frac{d^3y}{dx^3} = y_3 = m(m-1)(m-2)x^{m-3}$$

....

$$\frac{d^n y}{dx^n} = y_n = m(m-1)(m-2)\ldots(m-n+1)\,x^{m-n}$$

$$= \frac{m!}{(m-n)!}\,x^{m-n}, \text{ when } m > n \text{ and } m \text{ is a positive integer}$$

$$= n!, \text{ when } m = n$$

$$= 0, \text{ when } m < n \text{ and } m \text{ is a positive integer.}$$

Note: Verification by Mathematical Induction.

We observe that $y_1 = mx^{m-1}$, $y_2 = m(m-1)\,x^{m-2}$. Therefore the result is true for $n = 1, 2$.

Let us assume that the result holds good for a particular value of n, say r, so that we have

$$y_r = m(m-1)(m-2)\ldots(m-r+1)\,x^{m-r}.$$

Differentiating once again, we have

$$y_{r+1} = m(m-1)(m-2)\ldots(m-r+1)(m-r)\,x^{m-r-1}$$

from which we see that if the result is true for any positive integral value r of n, then it is also true for $r + 1$.

But the result holds good for $n = 1, 2$ and hence it must be true for $n = 2 + 1$, *i.e.*, 3, $n = 3 + 1$, *i.e.*, 4, $n = 4 + 1$, *i.e.*, 5 and similarly for every positive integral value of n.

II. $y = (ax + b)^m$, m is any real number. (W.B.U.T. 20[illegible])

Proceeding as in I, we get

$$y_n = m(m-1)(m-2)\ldots(m-n+1)(ax+b)^{m-n}\cdot a^n$$

$$= \frac{m!}{(m-n)!}\,a^n (ax+b)^{m-n}, \text{ when } m > n \text{ and } m \text{ is a positive integer}$$

$$= n!\,a^n, \text{ when } m = n$$

$$= 0, \text{ when } m < n \text{ and } m \text{ is a positive integer.}$$

Note: This result can be verified with the help of Mathematical Induction.

III. $$y = \frac{1}{ax+b} = (ax+b)^{-1}.$$

Differentiating successively with respect to x, we get

$$y_1 = (-1)(ax+b)^{-2}\,a$$

$$y_2 = (-1)(-2)(ax+b)^{-3}\,a^2$$

$$y_3 = (-1)(-2)(-3)(ax+b)^{-4}\,a^3$$

....

$$y_n = (-1)(-2)(-3)\ldots(-n)(ax+b)^{-(n+1)}\,a^n = \frac{(-1)^n n!\,a^n}{(ax+b)^{n+1}}$$

Note: Verify with the help of Mathematical Induction.

IV. $$y = e^{ax+b}$$

Differentiating successively with respect to x, we get

$$y_1 = a\, e^{ax+b}$$

$$y_2 = a^2\, e^{ax+b}$$

....

$$y_n = a^n\, e^{ax+b}$$

Note: $$y = a^{kx} = e^{kx \log a}$$

$\therefore$ $$y_n = (k \log a)^n\, e^{kx \log a}$$

$$= (k \log a)^n\, a^{kx}$$

V. $$y = \log (ax + b)$$

Differentiating successively with respect to x, we get

$$y_1 = \frac{1}{ax+b} \cdot a = (ax+b)^{-1} a$$

$$y_2 = (-1)\,(ax + b)^{-2}\, a^2$$

$$y_3 = (-1)\,(-2)\,(ax + b)^{-3}\, a^3$$

....

$$y_n = (-1)\,(-2) \ldots (-n + 1)\,(ax + b)^{-n}\, a^n$$

$$= \frac{(-)^{n-1}(n-1)!\,a^n}{(ax+b)^n}$$

VI. $$y = \sin (ax + b)$$ (*W.B.U.T. 2003*)

Differentiating successively with respect to x, we get

$$y_1 = a \cos (ax + b) = a \sin \left(\frac{\pi}{2} + ax + b\right)$$

$$y_2 = a^2 \cos\left(\frac{\pi}{2} + ax + b\right) = a^2 \sin\left(2 \cdot \frac{\pi}{2} + ax + b\right)$$

$$y_3 = a^3 \cos\left(2 \cdot \frac{\pi}{2} + ax + b\right) = a^3 \sin\left(3 \cdot \frac{\pi}{2} + ax + b\right)$$

....

$$y_n = a^n \sin\left(\frac{n\pi}{2} + ax + b\right)$$

VII. $$y = \cos (ax + b)$$

Differentiating successively with respect to x, we get

$$y_1 = -a \sin (ax + b) = a \cos\left(\frac{\pi}{2} + ax + b\right)$$

$$y_2 = -a^2 \sin\left(\frac{\pi}{2} + ax + b\right) = a^2 \cos\left(2 \cdot \frac{\pi}{2} + ax + b\right)$$

$$y_3 = -a^3 \sin\left(2 \cdot \frac{\pi}{2} + ax + b\right) = a^3 \cos\left(3 \cdot \frac{\pi}{2} + ax + b\right)$$

....

$$y_n = a^n \cos\left(\frac{n\pi}{2} + ax + b\right)$$

Note: The forms $\sin mx \cos nx$, $\sin^2 mx$, etc. may conveniently be differentiated with the help of the following trigonometrical formulae.

$$\sin mx \cos nx = \frac{1}{2}\{\sin(m+n)x + \sin(m-n)x\},$$

$$\sin mx \sin nx = \frac{1}{2}\{\cos(m-n)x - \cos(m+n)x\},$$

$$\cos mx \cos nx = \frac{1}{2}\{\cos(m+n)x + \cos(m-n)x\},$$

$$\sin mx \cos mx = \frac{1}{2}\sin 2mx,$$

$$\sin^2 mx = \frac{1}{2}(1 - \cos 2mx),$$

$$\cos^2 mx = \frac{1}{2}(1 + \cos 2mx),$$

$$\sin^3 mx = \frac{1}{4}(3\sin mx - \sin 3mx),$$

$$\cos^3 mx = \frac{1}{4}(3\cos mx + \cos 3mx).$$

Example 1: Find y_n if $y = \sin 7x \cos 3x$.

Solution:

$$y = \frac{1}{2}(\sin 10x + \sin 4x)$$

$\therefore$

$$y_1 = \frac{1}{2}(10\cos 10x + 4\cos 4x)$$

$$= \frac{1}{2}\left\{10\sin\left(\frac{\pi}{2} + 10x\right) + 4\sin\left(\frac{\pi}{2} + 4x\right)\right\}$$

$$y_2 = \frac{1}{2}\left\{10^2\cos\left(\frac{\pi}{2} + 10x\right) + 4^2\cos\left(\frac{\pi}{2} + 4x\right)\right\}$$

$$= \frac{1}{2}\left\{10^2 \sin\left(2\cdot\frac{\pi}{2}+10x\right)+4^2 \sin\left(2\cdot\frac{\pi}{2}+4x\right)\right\}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$y_n = \frac{1}{2}\left\{10^n \sin\left(n\cdot\frac{\pi}{2}+10x\right)+4^n \sin\left(n\cdot\frac{\pi}{2}+4x\right)\right\}.$$

Example 2: Find y_n if $y = \sin^3 x \cos^3 x$.

Solution:

$$y = (\sin x \cos x)^3 = \frac{1}{8}\sin^3 2x = \frac{1}{32}(3\sin 2x - \sin 6x)$$

$\therefore$

$$y_1 = \frac{1}{32}(3\cdot 2\cos 2x - 6\cos 6x)$$

$$= \frac{1}{32}\left\{3\cdot 2\sin\left(\frac{\pi}{2}+2x\right)-6\sin\left(\frac{\pi}{2}+6x\right)\right\}$$

$$y_2 = \frac{1}{32}\left\{3\cdot 2^2 \cos\left(\frac{\pi}{2}+2x\right)-6^2\cos\left(\frac{\pi}{2}+6x\right)\right\}$$

$$= \frac{1}{32}\left\{3\cdot 2^2 \sin\left(2\cdot\frac{\pi}{2}+2x\right)-6^2\sin\left(2\cdot\frac{\pi}{2}+6x\right)\right\}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$y_n = \frac{1}{32}\left\{3\cdot 2^n \sin\left(n\cdot\frac{\pi}{2}+2x\right)-6^n\sin\left(n\cdot\frac{\pi}{2}+6x\right)\right\}.$$

VIII. $y = e^{ax}\sin bx$

Differentiating successively with respect to x, we get

$$y_1 = ae^{ax}\sin bx + be^{ax}\cos bx$$

$$= e^{ax}(a\sin bx + b\cos bx)$$

Put $a = r\cos\theta$, $b = r\sin\theta$ so that $r = +\sqrt{a^2+b^2}$ and θ is found from $\cos\theta = \frac{a}{r}$, $\sin\theta = \frac{b}{r}$, $-\pi < \theta \le +\pi$.

Also $\tan\theta = \frac{b}{a}$

$\therefore$

$$y_1 = re^{ax}(\sin bx\cos\theta + \cos bx\sin\theta)$$

$$= re^{ax}\sin(bx+\theta)$$

Similarly, $y_2 = r^2e^{ax}\sin(bx+2\theta)$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$a r e^{ax} \sin(bx+\theta) + r e^{ax} \cos(bx+\theta)\cdot b$

$r e^{ax}$ ($\cos\theta$ ...)

$$y_n = r^n e^{ax} \sin (bx + n\theta)$$

$$= r^n e^{ax} \sin \left(bx + n \tan^{-1} \frac{b}{a} \right).$$

IX. $$y = e^{ax} \cos bx$$

Differentiating successively with respect to x, we get

$$y_1 = ae^{ax} \cos bx - be^{ax} \sin bx$$

$$= e^{ax} (a \cos bx - b \sin bx)$$

Put $a = r \cos \theta$, $b = r \sin \theta$ so that $r = +\sqrt{a^2 + b^2}$ and θ is found from $\cos \theta = \frac{a}{r}$, $\sin \theta =$ $-\pi < \theta \leq +\pi$

Also $$\tan \theta = \frac{b}{a}$$

$\therefore$ $$y_1 = re^{ax} (\cos bx \cos\theta - \sin bx \sin\theta)$$

$$= re^{ax} \cos (bx + \theta)$$

Similarly, $$y_2 = r^2 e^{ax} \cos (bx + 2\theta)$$

....

$$y_n = r^n e^{ax} \cos (bx + n\theta) = r^n e^{ax} \cos \left(bx + n \tan^{-1} \frac{b}{a} \right).$$

Example 3: Find y_n if $y = e^x \sin^2 2x$

Solution: $$y = \frac{1}{2} e^x (1 - \cos 4x) = \frac{1}{2} e^x - \frac{1}{2} e^x \cos 4x$$

$\therefore$ $$y_n = \frac{1}{2} e^x - \frac{1}{2} (1^2 + 4^2)^{n/2} e^x \cos \left(4x + n \tan^{-1} \frac{4}{1} \right)$$

$$= \frac{1}{2} e^x - \frac{1}{2} (17)^{n/2} e^x \cos (4x + n \tan^{-1} 4).$$

on:

e $$y=\frac{a-x}{a+x}=\frac{2a-(a+x)}{a+x}=\frac{2a}{a+x}-1=2a\,(a+x)^{-1}-1.$$

ntiating successively with respect to x, we get

$$y_1=(-1)\,2a\,(a+x)^{-2}$$
$$y_2=(-1)\,(-2)\,2a\,(a+x)^{-3}$$
$$y_3=(-1)\,(-2)\,(-3)\,2a\,(a+x)^{-4}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$y_n=(-1)\,(-2)\,(-3)\ldots(-n)\,2a\,(a+x)^{-(n+1)}$$
$$=\frac{2a(-1)^n n!}{(a+x)^{n+1}}.$$

e $$y=\log\left(\frac{a+x}{a-x}\right)=\log(a+x)-\log(a-x),\ x<a.$$

ntiating successively with respect to x, we get

$$y_1=\frac{1}{a+x}+\frac{1}{a-x}=(a+x)^{-1}+(a-x)^{-1}$$
$$y_2=(-1)\,(a+x)^{-2}+(a-x)^{-2}$$
$$y_3=(-1)(-2)(a+x)^{-3}+1\cdot 2\,(a-x)^{-3}$$
$$y_4=(-1)(-2)(-3)(a+x)^{-4}+1\cdot 2\cdot 3\,(a-x)^{-4}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$y_n=(-1)(-2)(-3)\ldots(-n+1)\,(a+x)^{-n}$$
$$+1\cdot 2\cdot 3.\ \ldots\ (n-1)\,(a-x)^{-n}$$
$$=\frac{(-1)^{n-1}(n-1)!}{(a+x)^n}+\frac{(n-1)!}{(a-x)^n}.$$

re $$y=\frac{x^n}{x-1}=\frac{(x^n-1)+1}{x-1}=\frac{x^n-1}{x-1}+\frac{1}{x-1}$$
$$=\frac{(x-1)\,(x^{n-1}+x^{n-2}+\ldots+x+1)}{x-1}+(x-1)^{-1}$$
$$=x^{n-1}+x^{n-2}+\ldots+x+1+(x-1)^{-1}$$

ie nth order derivatives of x^{n-1}, x^{n-2}, …, x are zero and

$$\frac{d^n}{dx^n}(x-1)^{-1}=(-1)(-2)(-3)\ldots(-n)\,(x-1)^{-(n+1)},$$

$$y_n=\frac{(-1)^n n!}{(x-1)^{n+1}}.$$

(*iv*) Here
$$y = \sin^4 x = (\sin^2 x)^2 = \frac{1}{4}(1-\cos 2x)^2$$
$$= \frac{1}{4}(1-2\cos 2x+\cos^2 2x)$$
$$= \frac{1}{4}-\frac{1}{2}\cos 2x+\frac{1}{8}(1+\cos 4x) = \frac{3}{8}-\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x$$

Differentiating successively with respect to x, we get
$$y_n = -\frac{1}{2}2^n\cos\left(\frac{n\pi}{2}+2x\right)+\frac{1}{8}\cdot 4^n\cos\left(\frac{n\pi}{2}+4x\right)$$
$$= -2^{n-1}\cos\left(\frac{n\pi}{2}+2x\right)+2^{2n-3}\cos\left(\frac{n\pi}{2}+4x\right)$$

(*v*) Here
$$y = \cos x\cos 2x\sin 3x = \frac{1}{2}(\cos 3x+\cos x)\sin 3x$$
$$= \frac{1}{2}(\sin 3x\cos 3x+\sin 3x\cos x)$$
$$= \frac{1}{4}(\sin 6x+\sin 4x+\sin 2x)$$
$$\therefore \quad y_n = \frac{1}{4}\left\{6^n\sin\left(\frac{n\pi}{2}+6x\right)+4^n\sin\left(\frac{n\pi}{2}+4x\right)+2^n\sin\left(\frac{n\pi}{2}+2x\right)\right\}$$

(*vi*) Here
$$y = 10^{a+bx} = 10^a\cdot 10^{bx} = 10^a e^{bx\log_e 10}$$

Differentiating successively with respect to x, we get
$$y_1 = 10^a b\log_e 10\ e^{bx\log_e 10}$$
$$y_2 = 10^a(b\log_e 10)^2\ e^{bx\log_e 10}$$
....
$$y_n = 10^a(b\log_e 10)^n\ e^{bx\log_e 10}.$$

Example 2: If $u = \sin ax+\cos ax$, show that
$$u_n = a^n[1+(-1)^n\sin 2ax]^{1/2}.$$
(W.B.U.T. 200[illegible])

Solution: Here $u = \sin ax+\cos ax$
$$\therefore \quad u_n = a^n\sin\left(\frac{n\pi}{2}+ax\right)+a^n\cos\left(\frac{n\pi}{2}+ax\right)$$
$$= a^n\left[\left\{\sin\left(\frac{n\pi}{2}+ax\right)+\cos\left(\frac{n\pi}{2}+ax\right)\right\}^2\right]^{1/2}$$
$$= a^n\left[1+2\sin\left(\frac{n\pi}{2}+ax\right)\cos\left(\frac{n\pi}{2}+ax\right)\right]^{1/2}$$

$$= a^n[1+\sin(n\pi+2ax)]^{1/2}$$

$$= a^n[1+(-1)^n \sin 2ax]^{1/2}.$$

Example 3: If $y = x^{2n}$, where n is a positive integer, prove that

$$y_n = 2^n\{1.3.5\ldots(2n-1)\}x^n.$$

Solution: Here $y = x^{2n}$

Differentiating successively with respect to x, we get

$$y_1 = 2n\, x^{2an-1}$$

$$y_2 = 2n\,(2n-1)\, x^{2n-2}$$

$$y_3 = 2n\,(2n-1)\,(2n-2)\, x^{2n-3}$$

$$\ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots$$

$$y_n = 2n\,(2n-1)\,(2n-2)\ldots\{2n-(n-1)\}\, x^{2n-n}$$

$$= \frac{2n\,(2n-1)\,(2n-2)\ldots(n+1)\,n\,(n-1)\ldots3.2.1}{n!}x^n$$

$$= \frac{\{2.4.6\ldots(2n-2).2n\}\{1.3.5\ldots(2n-1)\}}{n!}x^n$$

$$= \frac{2^n(1.2.3\ldots n)\{1.3.5\ldots(2n-1)\}}{n!}x^n$$

$$= \frac{2^n n!}{n!}\{1.3.5\ldots(2n-1)\}\,x^n = 2^n\{1.3.5\ldots(2n-1)\}x^n$$

Example 4: If $u_n = D^n(x^n \log x)$, show that $u_n = nu_{n-1} + \lfloor n-1$ and hence show that

$$u_n = \lfloor n\left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)$$

where $D \equiv \dfrac{d}{dx}$. *(W.B.U.T. 2007)*

Solution: First part

$$u_n = D^n(x^n \log x) = D^{n-1}\left\{\frac{d}{dx}(x^n \log x)\right\}$$

$$= D^{n-1}\left\{x^n\cdot\frac{1}{x} + n\,x^{n-1}\log x\right\}$$

$$= D^{n-1}(x^{n-1}) + nD^{n-1}(x^{n-1}\log x)$$

$$= \lfloor n-1 + n\,u_{n-1} \qquad (\because D^n x^n = \lfloor n)$$

Second part

Now $u_n = nu_{n-1} + \lfloor n-1$

$$\therefore \qquad \frac{u_n}{\lfloor n} = \frac{n\,u_{n-1}}{\lfloor n} + \frac{\lfloor n-1}{\lfloor n} = \frac{u_{n-1}}{\lfloor n-1} + \frac{1}{n}$$

or
$$\frac{u_n}{\lfloor n} - \frac{u_{n-1}}{\lfloor n-1} = \frac{1}{n}$$

Putting $n = 1, 2, 3, \ldots$, we get

$$\frac{u_1}{\lfloor 1} - \frac{u_0}{\lfloor 0} = \frac{1}{1}$$

$$\frac{u_2}{\lfloor 2} - \frac{u_1}{\lfloor 1} = \frac{1}{2}$$

$$\frac{u_3}{\lfloor 3} - \frac{u_2}{\lfloor 2} = \frac{1}{3}$$

....

$$\frac{u_n}{\lfloor n} - \frac{u_{n-1}}{\lfloor n-1} = \frac{1}{n}$$

Adding, we get
$$\frac{u_n}{\lfloor n} - \frac{u_0}{\lfloor 0} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$$

or
$$u_n = \lfloor n\left[u_0 + 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right]$$

$$= \lfloor n\left[\log x + 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right] \qquad (\because u_0 = \log x)$$

10 **Example 5:** If $y = x\log\dfrac{x-1}{x+1}$, show that $y_n = (-1)^n \lfloor n-2 \left\{\dfrac{x-n}{(x-1)^n} - \dfrac{(x+n)}{(x+1)^n}\right\}$.

Solution: Here

$$y = x\log\frac{x-1}{x+1} = x\log(x-1) - x\log(x+1)$$

$$\therefore \qquad y_1 = \frac{x}{x-1} + \log(x-1) - \frac{x}{x+1} - \log(x+1)$$

$$= \frac{x-1+1}{x-1} - \frac{(x+1-1)}{x+1} + \log(x-1) - \log(x+1)$$

$$= 1 + (x-1)^{-1} - 1 + (x+1)^{-1} + \log(x-1) - \log(x+1)$$

$$= (x-1)^{-1} + (x+1)^{-1} + \log(x-1) - \log(x+1)$$

$$y_2 = (-1)(x-1)^{-2} + (-1)(x+1)^{-2} + (x-1)^{-1} - (x+1)^{-1}$$

$$\begin{aligned}
y_3 &= (-1)(-2)(x-1)^{-3} + (-1)(-2)(x+1)^{-3} \\
&\quad + (-1)(x-1)^{-2} - (-1)(x+1)^{-2} \\
y_4 &= (-1)(-2)(-3)(x-1)^{-4} + (-1)(-2)(-3)(x+1)^{-4} \\
&\quad + (-1)(-2)(x-1)^{-3} - (-1)(-2)(x+1)^{-3} \\
&= (-1)^3 \lfloor\!\underline{3}\,(x-1)^{-4} + (-1)^3 \lfloor\!\underline{3}\,(x+1)^{-4} \\
&\quad + (-1)^2 \lfloor\!\underline{2}\,(x-1)^{-3} - (-1)^2 \lfloor\!\underline{2}\,(x+1)^{-3}
\end{aligned}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\begin{aligned}
y_n &= (-1)^{n-1} \lfloor\!\underline{n-1}\,(x-1)^{-n} + (-1)^{n-1} \lfloor\!\underline{n-1}\,(x+1)^{-n} \\
&\quad + (-1)^{n-2} \lfloor\!\underline{n-2}\,(x-1)^{-(n-1)} - (-1)^{n-2} \lfloor\!\underline{n-2}\,(x+1)^{-(n-1)} \\
&= (-1)^{n-2} \lfloor\!\underline{n-2}\,(x-1)^{-(n-1)} \left\{\frac{-(n-1)}{x-1} + 1\right\} \\
&\quad + (-1)^{n-2} \lfloor\!\underline{n-2}\,(x+1)^{-(n-1)} \left\{\frac{-(n-1)}{x+1} - 1\right\} \\
&= (-1)^{n-2} \lfloor\!\underline{n-2} \left\{\frac{x-n}{(x-1)^n} - \frac{(x+n)}{(x+1)^n}\right\} \\
&= (-1)^{n} \lfloor\!\underline{n-2} \left\{\frac{x-n}{(x-1)^n} - \frac{(x+n)}{(x+1)^n}\right\}.
\end{aligned}$$

5.3 USE OF PARTIAL FRACTIONS

(*i*) In the first step we have to observe that the degree of the polynomial in the numerator must be less than the degree of the polynomial in the denominator. If this is not the case, use the process of division so as to obtain,

$$\text{Fraction (given)} = \text{Quotient} + \frac{\text{Remainder}}{\text{Divisor}},$$

in which the fractional part of right hand side meets the necessary requirements.

(*ii*) The second step is to factorize the denominator into its ultimate real factors. These factors must be of the following types:

(*a*) linear but not repeated, of the type $(ax + b)$

(*b*) linear and repeated, such as $(ax + b)^n$

(*c*) quadratic but not repeated, of the type $(ax^2 + bx + c)$

(*d*) quadratic and repeated, such as $(ax^2 + bx + c)^n$.

(*iii*) The third step is to write down the given fraction as the sum of simple fractions and this be done according to the following rules:

(*a*) for each factor of the type $ax + b$, there should be a single fraction of the form $\frac{A}{ax+b}$, A is a constant,

(*b*) for each factor of the type $(ax+b)^n$, there should be fractions of the form

$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\ldots+\frac{A_n}{(ax+b)^n};\ A_1, A_2, \ldots, A_n \text{ are constants,}$$

(*c*) for ax^2+bx+c and $(ax^2+bx+c)^n$, take $\dfrac{Ax+B}{ax^2+bx+c}$ and $\dfrac{A_1x+B_1}{ax^2+bx+c}$

$$+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\ldots+\frac{A_nx+B_n}{(ax^2+bx+c)^n} \text{ respectively.}$$

(*iv*) In the next step make the numerator of the sum of component fractions identical with the numerator of the given fraction to determine A, B, A_1, B_1, ...

....

$$y_n = \left\{\frac{1}{6}(-1)(-2)(-3)\ldots(-n)(x-3)^{-(n+1)} - (-1)(-2)(-3)\ldots(-n)(x+3)^{-(n+1)}\right\}$$

$$= \frac{(-1)^n n!}{6}\left\{(x-3)^{-(n+1)} - (x+3)^{-(n+1)}\right\}.$$

Example 2: If

$$y = \frac{x^4+7x^3+21x^2+33x+22}{x^3+6x^2+11x+6}, \text{ find } y_n.$$

Solution: The degree of the numerator is greater than that of the denominator, dividing we have,

$$y = x+1+\frac{4x^2+16x+16}{x^3+6x^2+11x+6} = x+1+\frac{4x^2+16x+16}{(x+1)(x+2)(x+3)}$$

Let $\dfrac{4x^2+16x+16}{(x+1)(x+2)(x+3)} = \dfrac{A}{x+1}+\dfrac{B}{x+2}+\dfrac{C}{x+3}$, hence

$$4x^2+16x+16 \equiv A(x+2)(x+3)+B(x+1)(x+3)+C(x+1)(x+2)$$

This is an identity and must be true for any value of x.

Putting $x = -1, -2, -3$, we obtain $A = 2$, $B = 0$, $C = 2$.

Therefore, $$y = x+1+2\left(\frac{1}{x+1}+\frac{1}{x+3}\right) = x+1+2\left\{(x+1)^{-1}+(x+3)^{-1}\right\}$$

$\therefore$ $$y_1 = 1+2\left\{(-1)(x+1)^{-2}+(-1)(x+3)^{-2}\right\}$$

$$y_2 = 2\left\{(-1)(-2)(x+1)^{-3}+(-1)(-2)(x+3)^{-3}\right\}$$

....

$$y_n = 2\left\{(-1)(-2)\ldots(-n)(x+1)^{-(n+1)}+(-1)(-2)\ldots(-n)(x+3)^{-(n+1)}\right\}$$

$$= 2(-1)^n n!\left\{(x+1)^{-(n+1)}+(x+3)^{-(n+1)}\right\}.$$

Example 1: If $y = \dfrac{1}{x^2 + a^2}$, find y_n.

Solution:

$$y = \frac{1}{x^2 + a^2} = \frac{1}{(x+ia)(x-ia)} = \frac{1}{2ia}\left[\frac{1}{x-ia} - \frac{1}{x+ia}\right]$$

$$= \frac{1}{2ia}\left[(x-ia)^{-1} - (x+ia)^{-1}\right]$$

$\therefore$

$$y_1 = \frac{1}{2ia}\left[(-1)(x-ia)^{-2} - (-1)(x+ia)^{-2}\right]$$

$$y_2 = \frac{1}{2ia}\left[(-1)(-2)(x-ia)^{-3} - (-1)(-2)(x+ia)^{-3}\right]$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$y_n = \frac{1}{2ia}\left[(-1)(-2)\ldots(-n)(x-ia)^{-(n+1)}\right.$$

$$\left. - (-1)(-2)\ldots(-n)(x+ia)^{-(n+1)}\right]$$

$$= \frac{(-1)^n n!}{2ia}\left[(x-ia)^{-(n+1)} - (x+ia)^{-(n+1)}\right] \quad \ldots(1)$$

Put $x = r\cos\theta$, $a = r\sin\theta$ so that $r = (x^2 + a^2)^{1/2}$, $\theta = \tan^{-1}\dfrac{a}{x}$

Therefore, $(x-ia)^{-(n+1)} = r^{-(n+1)}(\cos\theta - i\sin\theta)^{-(n+1)}$

$$= r^{-(n+1)}\{\cos(n+1)\theta + i\sin(n+1)\theta\} \quad \text{(by De Moivre's Theorem)}$$

Similarly, $(x+ia)^{-(n+1)} = r^{-(n+1)}\{\cos(n+1)\theta - i\sin(n+1)\theta\}$

Therefore, from (1), $y_n = \dfrac{(-1)^n n!}{2ia} r^{-(n+1)} 2i\sin(n+1)\theta$

$$= \frac{(-1)^n n!}{ar^{n+1}}\sin(n+1)\theta$$

$$= \frac{(-1)^n n!}{a^{n+2}}\sin^{n+1}\theta\sin(n+1)\theta \quad \left(\because r = \frac{a}{\sin\theta}\right)$$

where

$$\theta = \tan^{-1}\frac{a}{x}.$$

Example 2: Given $y = \tan^{-1}\left(\dfrac{x}{a}\right)$ find y_n.

Solution: Here $y = \tan^{-1}\left(\dfrac{x}{a}\right)$

$$y_1 = \frac{1}{a}\cdot\frac{1}{1+\dfrac{x^2}{a^2}} = \frac{a}{(x+ia)(x-ia)} = \frac{1}{2i}\{(x-ia)^{-1}-(x+ia)^{-1}\}$$

$$y_2 = \frac{1}{2i}\{(-1)(x-ia)^{-2}-(-1)(x+ia)^{-2}\}$$

$$y_3 = \frac{1}{2i}\{(-1)(-2)(x-ia)^{-3}-(-1)(-2)(x+ia)^{-3}\}$$

....

$$y_n = \frac{1}{2i}\{(-1)(-2)\ldots(-n+1)(x-ia)^{-n}$$

$$-(-1)(-2)\ldots(-n+1)(x+ia)^{-n}\}$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i}\{(x-ia)^{-n}-(x+ia)^{-n}\} \qquad \ldots(1)$$

Put $x = r\cos\theta$, $a = r\sin\theta$ so that $r = (x^2+a^2)^{1/2}$, $\theta = \tan^{-1}\dfrac{a}{x}$

Therefore, $(x-ia)^{-n} = r^{-n}(\cos\theta - i\sin\theta)^{-n}$

$= r^{-n}(\cos n\theta + i\sin n\theta)$ (by De Moivre's Theorem)

Similarly, $(x+ia)^{-n} = r^{-n}(\cos n\theta - i\sin n\theta)$

Therefore, from (1), $y_n = \dfrac{(-1)^{n-1}(n-1)!}{2i}r^{-n}2i\sin n\theta$

$$= \frac{(-1)^{n-1}(n-1)!}{r^n}\sin n\theta$$

$$= \frac{(-1)^{n-1}(n-1)!}{a^n}\sin^n\theta\sin n\theta, \qquad \left(\because r = \frac{a}{\sin\theta}\right)$$

$$\theta = \tan^{-1}\frac{a}{x}.$$

Example 3: If $y = \tan^{-1}\dfrac{2x}{1-x^2}$, find y_n.

Solution: Now, $y = \tan^{-1}\dfrac{2x}{1-x^2} = \tan^{-1}\dfrac{2\tan\phi}{1-\tan^2\phi}$ (putting $x = \tan\phi$)

Scanned by CamScanner

$$= \tan^{-1}\tan 2\phi = 2\phi = 2\tan^{-1} x$$

Proceeding as in Ex. 2, we have

$$y_n = 2(-1)^{n-1}(n-1)!\ \sin^n\theta \sin n\theta, \text{ where } \theta = \tan^{-1}\frac{1}{x}.$$

Example 4: Find y_n, where

$$y = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}.$$

Solution: Here
$$y = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$$

$$= \tan^{-1}\frac{\sec\phi-1}{\tan\phi} \qquad \text{(putting } x = \tan\phi\text{)}$$

$$= \tan^{-1}\frac{1-\cos\phi}{\sin\phi} = \tan^{-1}\tan\frac{\phi}{2} = \frac{\phi}{2} = \frac{1}{2}\tan^{-1} x$$

Proceeding as in Ex. 2, we have

$$y_n = \frac{1}{2}(-1)^{n-1}(n-1)!\ \sin^n\theta\ \sin n\theta, \text{ where } \theta = \tan^{-1}\frac{1}{x}.$$

Example 5: Given $y = x(a^2+x^2)^{-1}$, find y_n.

Solution:
$$y = \frac{x}{a^2+x^2} = \frac{x}{(x+ia)(x-ia)} = \frac{1}{2}\left\{\frac{1}{x+ia}+\frac{1}{x-ia}\right\}$$

$$= \frac{1}{2}\left\{(x+ia)^{-1}+(x-ia)^{-1}\right\}$$

$\therefore$
$$y_1 = \frac{1}{2}\left\{(-1)(x+ia)^{-2}+(-1)(x-ia)^{-2}\right\}$$

$$y_2 = \frac{1}{2}\left\{(-1)(-2)(x+ia)^{-3}+(-1)(-2)(x-ia)^{-3}\right\}$$

....

$$y_n = \frac{1}{2}\{(-1)(-2)\ldots(-n)(x+ia)^{-(n+1)}$$
$$+(-1)(-2)\ldots(-n)(x-ia)^{-(n+1)}\}$$
$$= \frac{1}{2}(-1)^n n!\ \left\{(x+ia)^{-(n+1)}+(x-ia)^{-(n+1)}\right\}$$

Put $x = r\cos\theta$, $a = r\sin\theta$ so that $r = (x^2+a^2)^{1/2}$, $\theta = \tan^{-1}\frac{a}{x}$

Therefore, $\quad (x+ia)^{-(n+1)} = r^{-(n+1)}(\cos\theta+i\sin\theta)^{-(n+1)}$

$$= r^{-(n+1)}\{\cos(n+1)\theta - i\sin(n+1)\theta\} \quad \text{(by De Moivre's Theorem)}$$

Similarly, $\quad (x-ia)^{-(n+1)} = r^{-(n+1)}\{\cos(n+1)\theta + i\sin(n+1)\theta\}$

Therefore, from (1), $\quad y_n = (-1)^n n!\, r^{-(n+1)} \cos(n+1)\theta$

$$= \frac{(-1)^n n!}{a^{n+1}} \sin^{n+1}\theta \cos(n+1)\theta \qquad \left(\because r = \frac{a}{\sin\theta}\right)$$

where $\quad \theta = \tan^{-1}\left(\frac{a}{x}\right).$

5.5 LEIBNITZ'S THEOREM

Statement

If u and v are two functions of x, each possessing derivatives upto nth order, then $y = uv$ is derivable n times and

$$y_n = (u\,v)_n = u_n v + {}^nC_1\, u_{n-1}v_1 + {}^nC_2\, u_{n-2}v_2 + \dots$$
$$+ {}^nC_r\, u_{n-r}v_r + \dots + {}^nC_n\, uv_n$$

where y_r, u_r, v_r be the rth derivatives of y, u, v respectively with respect to x.

Proof: Let $y = uv$

Differentiating both sides w.r.t. x, we have

$$y_1 = u_1 v + uv_1$$

Again differentiating both sides w.r.t. x, we get

$$y_2 = u_2 v + u_1 v_1 + u_1 v_1 + uv_2$$
$$= u_2 v + {}^2C_1 u_1 v_1 + {}^2C_2 uv_2$$

Therefore, the theorem is true for $n = 1, 2$.

Let us assume that the theorem holds good for a positive integer m and so we have

$$y_m = u_m v + {}^mC_1 u_{m-1}v_1 + {}^mC_2 u_{m-2}v_2 + \dots + {}^mC_{r-1}u_{m-r+1}v_{r-1} +$$
$${}^mC_r u_{m-r}v_r + \dots + {}^mC_m uv_m$$

Differentiating both sides w.r.t. x, we get

$$y_{m+1} = u_{m+1}v + u_m v_1 + {}^mC_1\{u_m v_1 + u_{m-1}v_2\} + {}^mC_2\{u_{m-1}v_2 + u_{m-2}v_3\}$$
$$+ \dots + {}^mC_{r-1}\{u_{m-r+2}v_{r-1} + u_{m-r+1}v_r\} + {}^mC_r\{u_{m-r+1}v_r + u_{m-r}v_{r+1}\}$$
$$+ \dots + {}^mC_m\{u_1 v_m + uv_{m+1}\}$$
$$= u_{m+1}v + \{{}^mC_0 + {}^mC_1\}\, u_m v_1 + \{{}^mC_1 + {}^mC_2\} u_{m-1}v_2$$
$$+ \dots + \{{}^mC_{r-1} + {}^mC_r\} u_{m-r+1}v_r + \dots + {}^mC_m uv_{m+1}$$

$$= u_{m+1}v + {}^{m+1}C_1 u_m v_1 + {}^{m+1}C_2 u_{m-1} v_2 + \ldots$$

$$+ {}^{m+1}C_r u_{m+1-r} v_r + \ldots + {}^{m+1}C_{m+1} u v_{m+1}$$

$$\left[\because {}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r \text{ and } {}^mC_m = {}^{m+1}C_{m+1}\right]$$

So, the theorem is true for $m + 1$ if it is true for m. But we have proved that the theorem is true for $n = 1, 2$ and so it holds good for $2 + 1 = 3$, $3 + 1 = 4$, $4 + 1 = 5$ etc. Therefore, the theorem is true for any positive integer n.

This completes the proof.

Note: While applying Leibnitz's theorem the polynomial function is, in general, chosen as v.

ILLUSTRATIVE EXAMPLES

Example 1: Given $y = x^3 \log x$, find y_n.

Solution: Let $u = \log x,\ v = x^3$

$\therefore$

$$u_1 = \frac{1}{x} = x^{-1},\ u_2 = (-1)\,x^{-2},\ u_3 = (-1)(-2)x^{-3}, \ldots,$$

$$u_n = (-1)(-2)\ldots(-n+1)x^{-n} = \frac{(-1)^{n-1}(n-1)!}{x^n},$$

$$v_1 = 3x^2,\ v_2 = 6x,\ v_3 = 6,\ v_4 = 0,\ v_5 = 0,\ \ldots,\ v_n = 0$$

Therefore by Leibnitz's theorem,

$$y_n = (u\,v)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2\, u_{n-2} v_2 + {}^nC_3\, u_{n-3} v_3 + \ldots + {}^nC_n u\, v_n$$

$$= \frac{(-1)^{n-1}(n-1)!}{x^n} x^3 + n\frac{(-1)^{n-2}(n-2)!}{x^{n-1}} 3x^2$$

$$+ \frac{n(n-1)}{2!}\,\frac{(-1)^{n-3}(n-3)!}{x^{n-2}} \cdot 6x + \frac{n(n-1)(n-2)}{3!}\,\frac{(-1)^{n-4}(n-4)!}{x^{n-3}} \cdot 6$$

$$= \frac{(-1)^{n-4}(n-4)!}{x^{n-3}} \{-(n-1)(n-2)(n-3)$$

$$+3n(n-2)(n-3) - 3n(n-1)(n-3) + n(n-1)(n-2)\}$$

$$= \frac{(-1)^{n-4}(n-4)!}{x^{n-3}} \{(n-2)(n-3)(2n+1) + n(n-1)(-2n+7)\}$$

$$= \frac{(-1)^{n-4}(n-4)!}{x^{n-3}}(2n^3 - 9n^2 + 7n + 6 - 2n^3 + 9n^2 - 7n)$$

$$= \frac{(-1)^{n-4}\, 6\,(n-4)!}{x^{n-3}}.$$

Example 2: If $y = x^3 \cos x$, find y_n.

Solution: Let $u = \cos x,\ v = x^3$, then $u_n = \cos\left(\frac{n\pi}{2} + x\right)$

$$v_1 = 3x^2,\ v_2 = 6x,\ v_3 = 6,\ v_4 = 0,\ v_5 = 0, \ldots, v_n = 0$$

By Leibnitz's theorem,

$$y_n = (u\,v)_n = u_n v + {}^nC_1\, u_{n-1} v_1 + {}^nC_2\, u_{n-2} v_2 + {}^nC_3\, u_{n-3} v_3 + \ldots + {}^nC_n u\, v_n$$

$$= x^3 \cos\left(\frac{n\pi}{2} + x\right) + 3nx^2 \cos\left\{\frac{(n-1)\pi}{2} + x\right\}$$

$$+ \frac{n(n-1)}{2!}\, 6x \cos\left\{\frac{(n-2)\pi}{2} + x\right\}$$

$$+ \frac{n(n-1)(n-2)}{3!} \cdot 6 \cos\left\{\frac{(n-3)\pi}{2} + x\right\}.$$

Example 3: Find y_n when $y = e^x \log x$.

Solution: Take $u = e^x,\ v = \log x$,

then $u_n = e^x,\ v_n = \dfrac{(-1)^{n-1}(n-1)!}{x^n}$

By Leibnitz's theorem,

$$y_n = (u\,v)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \ldots + {}^nC_n u\, v_n$$

$$= e^x \log x + ne^x \cdot \frac{1}{x} + \frac{n(n-1)}{2!} e^x (-1) \cdot \frac{1}{x^2} + \ldots + e^x \frac{(-1)^{n-1}(n-1)!}{x^n}.$$

Example 4: If $y = \dfrac{x^n}{1+x}$, find y_n.

Solution: Take $u = x^n$ and $v = \dfrac{1}{1+x}$

Then $u_r = \dfrac{n!\, x^{n-r}}{(n-r)!}$, when $r < n$

$= n!$, when $r = n$

$$v_n = \frac{(-1)^n\, n!}{(1+x)^{n+1}}$$

By Leibnitz's theorem,

$$y_n = (u\,v)_n = u_n v + {}^nC_1\, u_{n-1}\, v_1 + {}^nC_2\, u_{n-2}\, v_2 + \ldots + {}^nC_n\, u\, v_n$$

$$= n!\,\frac{1}{1+x} + {}^nC_1\,\frac{n!\,x}{1!}\,\frac{(-1)\,1!}{(1+x)^2}$$

$$+ {}^nC_2\,\frac{n!\,x^2}{2!}\,\frac{(-1)^2\,2!}{(1+x)^3} + \ldots + {}^nC_n\,\frac{x^n\,(-1)^n\,n!}{(1+x)^{n+1}}$$

$$= \frac{n!}{(1+x)^{n+1}}\{{}^nC_0(1+x)^n - {}^nC_1(1+x)^{n-1}x$$

$$+ {}^nC_2(1+x)^{n-2}x^2 + \ldots + (-1)^n\,{}^nC_n x^n\}$$

$$= \frac{n!}{(1+x)^{n+1}}\{(1+x) - x\}^n = \frac{n!}{(1+x)^{n+1}}.$$

Example 5: If $y = a\cos(\log x) + b\sin(\log x)$, show that

(i) $x^2 y_2 + x y_1 + y = 0$

(ii) $x^2 y_{n+2} + (2n+1)\,x\,y_{n+1} + (n^2+1)\,y_n = 0.$ (W.B.U.T. 2007)

Solution: Here $y = a\cos(\log x) + b\sin(\log x)$

$\therefore$ $$y_1 = -a\cdot\frac{1}{x}\sin(\log x) + b\cdot\frac{1}{x}\cos(\log x)$$

or $$xy_1 = -a\sin(\log x) + b\cos(\log x) \quad \ldots(1)$$

Differentiating both sides of (1) with respect to x, we get

$$xy_2 + y_1 = -a\cdot\frac{1}{x}\cos(\log x) - b\cdot\frac{1}{x}\sin(\log x)$$

or $$x^2y_2 + xy_1 = -\{a\cos(\log x) + b\sin(\log x)\}$$

or $$x^2y_2 + xy_1 = -y$$

$\therefore$ $$x^2y_2 + xy_1 + y = 0$$

Differentiating both sides of (2) n times with respect to x by Leibnitz's theorem, we get

$$y_{n+2}\,x^2 + {}^nC_1\,y_{n+1}(2x) + {}^nC_2\,y_n 2 + (y_{n+1}x + {}^nC_1 y_n) + y_n = 0$$

or $$x^2 y_{n+2} + 2nxy_{n+1} + \frac{n(n-1)}{2!}\cdot 2y_n + (xy_{n+1} + ny_n) + y_n = 0$$

or $$x^2 y_{n+2} + (2n+1)\,xy_{n+1} + (n^2+1)\,y_n = 0.$$

Example 6: If $y = \sin(m \sin^{-1} x)$, prove that

(*i*) $(1-x^2)y_2 - xy_1 + m^2 y = 0$

(*ii*) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 - m^2)y_n = 0.$

Solution: Here $\quad y = \sin(m \sin^{-1} x)$

$\therefore$ $$y_1 = \frac{m}{\sqrt{1-x^2}} \cos(m \sin^{-1} x)$$

or $$y_1\sqrt{1-x^2} = m\cos(m\sin^{-1} x) \qquad \ldots(1)$$

Differentiating both sides of (1) with respect to x, we have

$$\sqrt{1-x^2}\, y_2 - \frac{x}{\sqrt{1-x^2}} y_1 = -\frac{m^2}{\sqrt{1-x^2}} \sin(m \sin^{-1} x)$$

or $$(1-x^2)y_2 - xy_1 + m^2 y = 0 \qquad [\because y = \sin(m\sin^{-1} x)] \ldots(2)$$

Differentiating both sides of (2) n times with respect to x by Leibnitz's theorem, we get

$$y_{n+2}(1-x^2) + {}^nC_1\, y_{n+1}(-2x) + {}^nC_2 y_n(-2) - \{y_{n+1}x + {}^nC_1 y_n\} + m^2 y_n = 0$$

or $$y_{n+2}(1-x^2) - 2nxy_{n+1} - \frac{n(n-1)}{2!} \cdot 2y_n - \{xy_{n+1} + ny_n\} + m^2 y_n = 0$$

$\therefore$ $$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 - m^2)y_n = 0.$$

Example 7: (*i*) If $y = (x^2 - 1)^n$, prove that

$$(x^2-1)\,y_{n+2} + 2xy_{n+1} - n(n+1)\,y_n = 0$$

(*W.B.U.T. 2006, 2009, 2010*)

(*ii*) If $y = e^{\tan^{-1} x}$, then show that

$$(1+x^2)y_{n+2} + (2nx + 2x - 1)y_{n+1} + n(n+1)y_n = 0 \qquad \textit{(W.B.U.T. 2012)}$$

Solution: Here $\quad y = (x^2-1)^n$

$\therefore$ $$y_1 = n(x^2-1)^{n-1} \cdot 2x$$

or $$y_1(x^2-1) = 2nx(x^2-1)^n$$

or $$y_1(x^2-1) = 2nxy \qquad [\because y = (x^2-1)^n] \ldots(1)$$

Differentiating both sides of (1) with respect to x, we get

$$y_2(x^2-1) + y_1 2x = 2n(y + xy_1)$$

or $$(x^2-1)\,y_2 + 2(1-n)\,xy_1 - 2ny = 0 \qquad \ldots(2)$$

Differentiating both sides of (2) n times with respect to x by Leibnitz's theorem, we get

$$y_{n+2}(x^2-1) + {}^nC_1\, y_{n+1}(2x) + {}^nC_2 y_n(2) + 2(1-n)\{y_{n+1}x + {}^nC_1 y_n\} - 2ny_n = 0$$

or $$(x^2-1)\,y_{n+2}+2nx\,y_{n+1}+\frac{n\,(n-1)}{2\,!}\cdot 2\,y_n+2\,(1-n)\{xy_{n+1}+ny_n\}-2ny_n=0$$

or $$(x^2-1)\,y_{n+2}+2xy_{n+1}-n\,(n+1)\,y_n=0.$$

(*ii*) Given $y=e^{\tan^{-1}x}$

$\therefore$ $$y_1=\frac{e^{\tan^{-1}x}}{1+x^2}$$

$\Rightarrow$ $$(1+x^2)\,y_1=y \qquad ...(1)$$

Differentiating both sides of (1) with respect to x, we get

$$(1+x^2)\,y_2+2xy_1=y_1$$

or $$(1+x^2)\,y_2+(2x-1)y_1=0 \qquad ...(2)$$

Differentiating both sides of (2) n times with respect to x by Leibnitz's theorem, we get

$$(1+x^2)y_{n+2}+{}^nC_1(2x)y_{n+1}+{}^nC_2(2)y_n+(2x-1)y_{n+1}+{}^nC_1(2)y_n=0$$

or $$(1+x^2)y_{n+2}+2nxy_{n+1}+n(n-1)y_n+(2x-1)y_{n+1}+2ny_n=0$$

$\therefore$ $$(1+x^2)y_{n+2}+(2nx+2x-1)y_{n+1}+n(n+1)y_n=0$$

Example 8: Show that

$$\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right)=(-1)^n\frac{n\,!}{x^{n+1}}\left(\log x-1-\frac{1}{2}-\frac{1}{3}-\ldots-\frac{1}{n}\right)$$ (*W.B.U.T. 2003, 2008*)

Solution: Let

$$y=\frac{\log x}{x}=uv,\ \text{where } u=\frac{1}{x}\ \text{and } v=\log x$$

Then $$u_n=\frac{(-1)^n n\,!}{x^{n+1}}\ \text{and } v_n=\frac{(-1)^{n-1}(n-1)\,!}{x^n}$$

Therefore by Leibnitz's theorem

$$\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right)=(uv)_n=u_nv+{}^nC_1\,u_{n-1}v_1+{}^nC_2\,u_{n-2}v_2+{}^nC_3\,u_{n-3}v_3+\ldots+{}^nC_n\,uv_n$$

$$=\frac{(-1)^n n\,!}{x^{n+1}}\log x+\frac{n\,(-1)^{n-1}(n-1)\,!}{x^n}\cdot\frac{1}{x}+\frac{n\,(n-1)}{2\,!}\,\frac{(-1)^{n-2}(n-2)\,!}{x^{n-1}}\cdot\left(-\frac{1}{x^2}\right)$$

$$+\frac{n\,(n-1)\,(n-2)}{3!}\,\frac{(-1)^{n-3}(n-3)\,!}{x^{n-2}}\cdot\frac{(-1)\,(-2)}{x^3}+\ldots+\frac{1}{x}\cdot\frac{(-1)^{n-1}(n-1)!}{x^n}$$

$$=(-1)^n\frac{n\,!}{x^{n+1}}\left(\log x-1-\frac{1}{2}-\frac{1}{3}-\ldots-\frac{1}{n}\right).$$

Example 9: If $f(x) = \tan x$ and n is a positive integer, prove with the help of Leibnitz's theorem, that

$$f^n(0) - {}^nC_2\, f^{n-2}(0) + {}^nC_4\, f^{n-4}(0) - \ldots = \sin\left(\frac{n\pi}{2}\right).$$ *(W.B.U.T. 2001)*

Solution: Here $f(x) = \tan x$, or, $f(x)\cos x = \sin x$ ...(1)

Differentiating both sides of (1) n times with respect to x by Leibnitz's theorem, we get:

$$f^n(x)\cos x + {}^nC_1\, f^{n-1}(x)\,(-\sin x) + {}^nC_2\, f^{n-2}(x)\,(-\cos x)$$

$$+ {}^nC_3\, f^{n-3}(x)\sin x + {}^nC_4\, f^{n-4}(x)\cos x + \ldots = \sin\left(\frac{n\pi}{2} + x\right)$$

Putting $x = 0$, we have

$$f^n(0) - {}^nC_2\, f^{n-2}(0) + {}^nC_4\, f^{n-4}(0) - \ldots = \sin\left(\frac{n\pi}{2}\right).$$

Example 10: If $f(x) = x^n$, prove that $f(1) + \frac{f'(1)}{1!} + \frac{f''(1)}{2!} + \frac{f'''(1)}{3!} + \ldots + \frac{f^n(1)}{n!} = 2^n$.

Solution: Here $f(x) = x^n$, therefore,

$$f'(x) = nx^{n-1},\ f''(x) = n(n-1)x^{n-2},\ f'''(x) = n(n-1)(n-2)x^{n-3}, \ldots,$$

$$f^n(x) = n(n-1)(n-2)\ldots 3.2.1 = n!$$

Putting $x = 1$, we have

$$f(1) = 1,\ \frac{f'(1)}{1!} = n,\ \frac{f''(1)}{2!} = \frac{n(n-1)}{2!},$$

$$\frac{f'''(1)}{3!} = \frac{n(n-1)(n-2)}{3!}, \ldots, \frac{f^n(1)}{n!} = 1$$

$$\therefore \quad f(1) + \frac{f'(1)}{1!} + \frac{f''(1)}{2!} + \frac{f'''(1)}{3!} + \ldots + \frac{f^n(1)}{n!}$$

$$= 1 + n + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!} + \ldots + 1$$

$$= {}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_n$$

$$= (1+1)^n = 2^n.$$

Example 11: If $x + y = 1$, prove that the nth derivative of $x^n y^n$ is

$$n!\{y^n - ({}^nC_1)^2 y^{n-1}x + ({}^nC_2)^2 y^{n-2}x^2 - ({}^nC_3)^2 y^{n-3}x^3 + \ldots + (-1)^n x^n\}.$$

(W.B.U.T. 2002, BESUS 2013)

Solution: Let $u = x^n y^n = x^n(1-x)^n$ $(\because x + y = 1)$

Therefore by Leibnitz's theorem, we get

$$u_n = n!(1-x)^n + {}^nC_1 \frac{n!}{1!} x \cdot n(1-x)^{n-1}(-1) + {}^nC_2 \frac{n!}{2!} x^2\, n(n-1)(1-x)^{n-2}(-1)^2$$

$$+ {}^nC_3 \frac{n!}{3!} x^3 n(n-1)(n-2)(1-x)^{n-3}(-1)^3 + \ldots + x^n\, n!(-1)^n$$

$$\left[\because \frac{d^r}{dx^r}(x^n) = \frac{n!}{(n-r)!}x^{n-r}, r<n \atop = n!, r = n\right]$$

$$= n!\left\{y^n - {}^nC_1 \frac{n}{1!} y^{n-1}x + {}^nC_2 \frac{n(n-1)}{2!} y^{n-2}x^2 - {}^nC_3 \frac{n(n-1)(n-2)}{3!} y^{n-3}x^3 + \ldots + (-1)^n x^n\right\}$$

$$(\because y = 1-x)$$

$$= n!\{y^n - ({}^nC_1)^2 y^{n-1}x + ({}^nC_2)^2 y^{n-2}x^2 - ({}^nC_3)^2 y^{n-3}x^3 + \ldots + (-1)^n x^n\}.$$

Example 12: If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$, prove that

$$(x^2-1)\,y_{n+2} + (2n+1)\,xy_{n+1} + (n^2-m^2)\,y_n = 0. \qquad \text{(W.B.U.T. 2011)}$$

Solution: Here

$$y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x, \quad \text{or} \quad \left(y^{\frac{1}{m}}\right)^2 - 2xy^{\frac{1}{m}} + 1 = 0$$

$$\therefore \quad y^{\frac{1}{m}} = \frac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1}$$

or

$$y = \left(x \pm \sqrt{x^2-1}\right)^m$$

$$\therefore \quad y_1 = m\left(x \pm \sqrt{x^2-1}\right)^{m-1}\left\{1 \pm \frac{1}{2}\cdot\frac{2x}{\sqrt{x^2-1}}\right\}$$

$$= \pm m\frac{(x \pm \sqrt{x^2-1})^m}{\sqrt{x^2-1}} = \pm\frac{my}{\sqrt{x^2-1}}$$

or

$$y_1\sqrt{x^2-1} = \pm my \qquad \ldots(1)$$

Differentiating both sides with respect to x, we get

$$y_1 \cdot \frac{1}{2}\frac{2x}{\sqrt{x^2-1}} + y_2\sqrt{x^2-1} = \pm my_1$$

$$xy_1 + (x^2-1)y_2 = \pm my_1\sqrt{x^2-1}$$

or

$$\therefore \quad (x^2-1)y_2 + xy_1 - m^2y = 0 \qquad \text{[by (1)]}$$

Now differentiating both sides n times with respect to x by Leibnitz's theorem, we get

$$y_{n+2}(x^2-1) + {}^nC_1\, y_{n+1}(2x) + {}^nC_2\, y_n(2) + \{y_{n+1}x + {}^nC_1\, y_n\} - m^2y_n = 0$$

or

$$y_{n+2}(x^2-1) + 2n\, xy_{n+1} + \frac{n(n-1)}{2!}\cdot 2y_n + (xy_{n+1} + ny_n) - m^2y_n = 0$$

$$\therefore \quad (x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0.$$

Example 13: If $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$, prove that

$$x^2y_{n+2} + (2n+1)xy_{n+1} + 2n^2y_n = 0.$$

Solution: Here

$$\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$$

$$\therefore \quad y = b\cos\left\{\log\left(\frac{x}{n}\right)^n\right\} = b\cos\left\{n\log\left(\frac{x}{n}\right)\right\}$$

$$\therefore \quad y_1 = -\frac{bn}{x}\sin\left\{n\log\left(\frac{x}{n}\right)\right\}$$

$$\therefore \quad xy_1 = -bn\sin\left\{n\log\left(\frac{x}{n}\right)\right\}$$

Differentiating both sides with respect to x, we get

$$xy_2 + y_1 = -bn\cdot\frac{n}{x}\cos\left\{n\log\left(\frac{x}{n}\right)\right\}$$

or

$$x^2y_2 + xy_1 + n^2y = 0 \qquad \left[\because y = b\cos\left\{n\log\left(\frac{x}{n}\right)\right\}\right] \quad ...(1)$$

Differentiating both sides of (1) n times with respect to x by Leibnitz's theorem, we get

$$y_{n+2}\, x^2 + {}^nC_1\, y_{n+1}(2x) + {}^nC_2\, y_n(2) + \{y_{n+1}x + {}^nC_1\, y_n\} + n^2y_n = 0$$

or

$$x^2y_{n+2} + 2n\, xy_{n+1} + n(n-1)y_n + \{xy_{n+1} + ny_n\} + n^2y_n = 0$$

$$\therefore \quad x^2y_{n+2} + (2n+1)\, xy_{n+1} + 2n^2y_n = 0.$$

Example 14: If $y = 2\cos x\,(\sin x - \cos x)$ show that $(y_{10})_0 = 2^{10}$,

where $(y_{10})_0$ means the value of 10th derivative of y when $x = 0$. (W.B.U.T.

Solution: Here

$$y = 2\cos x\,(\sin x - \cos x) = 2\sin x\cos x - 2\cos^2 x$$
$$= \sin 2x - \cos 2x - 1$$

$$\therefore \quad y_{10} = 2^{10}\sin\left(10\cdot\frac{\pi}{2} + 2x\right) - 2^{10}\cos\left(10\cdot\frac{\pi}{2} + 2x\right)$$

$$\therefore \quad (y_{10})_0 = 2^{10}\sin 5\pi - 2^{10}\cos 5\pi = 2^{10}.$$

Example 15: If $y = \left[x + \sqrt{1+x^2}\right]^m$, find $(y_n)_0$, where $(y_n)_0$ means the value of nth derivative of y when $x = 0$.

Solution: Here

$$y = \left[x + \sqrt{1+x^2}\right]^m \qquad \text{...(1)}$$

$$\therefore \quad y_1 = m\left[x + \sqrt{1+x^2}\right]^{m-1}\left\{1 + \frac{2x}{2\sqrt{1+x^2}}\right\}$$

$$= \frac{m\left[x + \sqrt{1+x^2}\right]^m}{\sqrt{1+x^2}} \qquad \text{...(2)}$$

or $$y_1\sqrt{1+x^2} = my \qquad \text{[by (1)]} \qquad \text{...(3)}$$

Differentiating both sides with respect to x, we have

$$y_2\sqrt{1+x^2} + y_1\cdot\frac{2x}{2\sqrt{1+x^2}} = my_1$$

or $$y_2(1+x^2) + xy_1 = my_1\sqrt{1+x^2}$$

$$\therefore \quad y_2(1+x^2) + xy_1 - m^2y = 0 \qquad \text{[by (3)]} \qquad \text{...(4)}$$

Differentiating both sides of (4) n times with respect to x by Leibnitz's theorem, we get

$$y_{n+2}(1+x^2) + {}^nC_1\, y_{n+1}(2x) + {}^nC_2\, y_n(2) + \{y_{n+1}x + {}^nC_1\, y_n\} - m^2y_n = 0$$

or $$(1+x^2)y_{n+2} + 2nx\, y_{n+1} + n(n-1)y_n + \{xy_{n+1} + ny_n\} - m^2y_n = 0$$

or $$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$$

$$\therefore \quad (y_{n+2})_0 = (m^2 - n^2)(y_n)_0$$

$$\therefore \quad (y_n)_0 = \{m^2 - (n-2)^2\}(y_{n-2})_0.$$

Case I: When n is even

$$(y_n)_0 = \{m^2-(n-2)^2\}(y_{n-2})_0 = \{m^2-(n-2)^2\}\{m^2-(n-4)^2\}(y_{n-4})_0$$
$$= \dots = \{m^2-(n-2)^2\}\{m^2-(n-4)^2\}\dots(m^2-2^2)(y_2)_0$$
$$= \{m^2-(n-2)^2\}\{m^2-(n-4)^2\}\dots(m^2-2^2)\,m^2(y)_0 \qquad \text{[by (4)]}$$
$$= \{m^2-(n-2)^2\}\{m^2-(n-4)^2\}\dots(m^2-2^2)m^2.$$
$$[\because (y)_0 = 1 \text{ by (1)}]$$

Case II: When n is odd

$$(y_n)_0 = \{m^2-(n-2)^2\}(y_{n-2})_0 = \{m^2-(n-2)^2\}\{m^2-(n-4)^2\}(y_{n-4})_0$$
$$= \dots = \{m^2-(n-2)^2\}\{m^2-(n-4)^2\}\dots(m^2-3^2)(y_3)_0$$
$$= \{m^2-(n-2)^2\}\{m^2-(n-4)^2\}\dots(m^2-3^2)(m^2-1^2)(y_1)_0$$
$$= \{m^2-(n-2)^2\}\{m^2-(n-4)^2\}\dots(m^2-3^2)(m^2-1^2)m.$$
$$[\because (y_1)_0 = m \text{ by (2)}]$$

Example 16: If $y = \tan^{-1}x$, then prove that

$$(1+x^2)y_{n+1} + 2nx\,y_n + n(n-1)y_{n-1} = 0.$$

Find also the value of $(y_n)_0$, where $(y_n)_0$ means the value of nth derivative of y when $x = 0$.

(*W.B.U.T. 2003*)

Solution: Given $y = \tan^{-1}x$, therefore $y_1 = \dfrac{1}{1+x^2}$.

Hence $\qquad y_1(1+x^2) = 1 \qquad \dots(1)$

Differentiating both sides of (1) n times with respect to x by Leibnitz's theorem, we get

$$y_{n+1}(1+x^2) + {}^nC_1\, y_n(2x) + {}^nC_2\, y_{n-1}(2) = 0$$

$$\therefore \quad (1+x^2)y_{n+1} + 2nx\,y_n + n(n-1)\,y_{n-1} = 0$$

$$\therefore \quad (y_{n+1})_0 = -n(n-1)(y_{n-1})_0 \qquad \dots(2)$$

From the given expression and from (1),

$$(y)_0 = 0,\ (y_1)_0 = 1$$

From (2), putting successively $n = 1, 2, 3, 4, 5, \dots$

$$(y_2)_0 = 0,\ (y_3)_0 = -2.1,\ (y_4)_0 = -3.2\,(y_2)_0 = 0,$$
$$(y_5)_0 = -4.3\,(y_3)_0 = (-4.3).(-2.1),\dots$$

Hence, we conclude that

$$(y_n)_0 = \begin{cases} 0, & n \text{ even} \\ (-1)^{\frac{1}{2}(n-1)}.(n-1)!, & n \text{ odd.} \end{cases}$$

Example 17: If $y = \sin^{-1}x$, prove that

(*i*) $(1-x^2)y_2 - xy_1 = 0$,

(*ii*) $(1-x^2)\,y_{n+2} - (2n+1)x\,y_{n+1} - n^2 y_n = 0$.

Also find y_n when $x = 0$.

Solution: Here

$$y = \sin^{-1} x$$

$$\therefore \qquad y_1 = \frac{1}{\sqrt{1-x^2}}, \quad \text{or} \quad \sqrt{1-x^2}\,y_1 = 1$$

Differentiating both sides of (1) with respect to x, we get

$$y_2\sqrt{1-x^2} + y_1\frac{1}{2\sqrt{1-x^2}}\cdot(-2x) = 0$$

or
$$y_2\,(1-x^2) - xy_1 = 0$$

Differentiating both sides of (2) n times with respect to x by Leibnitz's theorem, we get

$$y_{n+2}(1-x^2) + {}^nC_1\, y_{n+1}(-2x) + {}^nC_2\, y_n(-2) - \{y_{n+1}x + {}^nC_1\, y_n\} = 0$$

or
$$(1-x^2)y_{n+2} - 2nx\,y_{n+1} - n(n-1)y_n - \{xy_{n+1} + ny_n\} = 0$$

or
$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$$

$$\therefore \qquad (y_{n+2})_0 = n^2(y_n)_0$$

From the given expression and from (1), (2),

$$(y)_0 = 0,\ (y_1)_0 = 1,\ (y_2)_0 = 0$$

From (3), putting successively $n = 1, 2, 3, 4, 5, \ldots$

$$(y_3)_0 = 1^2\cdot(y_1)_0 = 1^2,\ (y_4)_0 = 2^2(y_2)_0 = 0,$$

$$(y_5)_0 = 3^2\cdot(y_3)_0 = 3^2\cdot 1^2,\ (y_6)_0 = 4^2(y_4)_0 = 0,$$

$$(y_7)_0 = 5^2\cdot(y_5)_0 = 5^2\cdot 3^2\cdot 1^2,$$

Therefore, we conclude that

$$(y_n)_0 = \begin{cases} 0, & \text{when } n \text{ is even} \\ 1, & \text{when } n = 1 \\ (n-2)^2(n-4)^2\ldots 3^2\cdot 1^2, & \text{when } n(\geq 3) \text{ is odd.} \end{cases}$$

Here $(y_n)_0$ means the value of nth derivative of y when $x = 0$.

Example 18: If $y = e^{m\cos^{-1}x}$ prove that

$$(1-x^2)y_{n+2} - x(2n+1)y_{n+1} - (n^2+m^2)y_n = 0$$

and hence find the value of $(y_n)_0$, where $(y_n)_0$ means the value of nth derivative of y when $x = 0$.

Solution: Here $y = e^{m\cos^{-1}x}$

$$\therefore \quad y_1 = e^{m\cos^{-1}x}\,\frac{(-m)}{\sqrt{1-x^2}} = -\frac{my}{\sqrt{1-x^2}} \qquad \ldots(1)$$

$$y_1\sqrt{1-x^2} + my = 0$$

Differentiating both sides with respect to x, we get

$$y_2\sqrt{1-x^2} + y_1\frac{1}{2\sqrt{1-x^2}}\cdot(-2x) + my_1 = 0$$

$$y_2(1-x^2) - xy_1 - m^2y = 0 \qquad \text{[by (1)]} \qquad \ldots(2)$$

Differentiating both sides of (2) n times with respect to x by Leibnitz's theorem, we get

$$y_{n+2}(1-x^2) + {}^nC_1\,y_{n+1}(-2x) + {}^nC_2\,y_n(-2) - \{y_{n+1}x + {}^nC_1\,y_n\} - m^2y_n = 0$$

$$(1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - \{xy_{n+1} + ny_n\} - m^2y_n = 0$$

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$$

$$\therefore \quad (y_{n+2})_0 = (n^2+m^2)\,(y_n)_0$$

Hence $$(y_n)_0 = \{(n-2)^2+m^2\}\,(y_{n-2})_0 \qquad \ldots(3)$$

Case I: When n is even.

$$(y_n)_0 = \{(n-2)^2+m^2\}\{(n-4)^2+m^2\}\ldots(4^2+m^2)\,(2^2+m^2)\,(y_2)_0$$

$$= \{(n-2)^2+m^2\}\{(n-4)^2+m^2\}\ldots(4^2+m^2)\,(2^2+m^2)\,m^2(y_0) \qquad \text{[by (2)]}$$

$$= \{(n-2)^2+m^2\}\{(n-4)^2+m^2\}\ldots(4^2+m^2)\,(2^2+m^2)\,m^2e^{m\pi/2} \qquad [\because (y)_0 = e^{m\pi/2}]$$

Case II: When n is odd.

$$(y_n)_0 = \{(n-2)^2+m^2\}\{(n-4)^2+m^2\}\ldots(3^2+m^2)\,(1^2+m^2)\,(y_1)_0$$

$$= \{(n-2)^2+m^2\}\{(n-4)^2+m^2\}\ldots(3^2+m^2)\,(1^2+m^2)\,(-me^{m\pi/2})$$

$$\text{[by (1), } (y_1)_0 = -me^{m\pi/2}]$$

$$= -\{(n-2)^2+m^2\}\{(n-4)^2+m^2\}\ldots(3^2+m^2)\,(1^2+m^2)\,me^{m\pi/2}.$$

Example 19: If $x = \cosh\left(\frac{1}{m}\log y\right)$, where $\cosh\theta = \frac{1}{2}(e^{\theta} + e^{-\theta})$, prove that

(*i*) $(x^2-1)y_2 + xy_1 - m^2 y = 0$,

(*ii*) $(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$.

Solution: $\cosh^{-1}x = \frac{1}{m}\log y$

Differentiating both sides with respect to x, we get

$$\frac{1}{\sqrt{x^2-1}} = \frac{1}{m}\cdot\frac{y_1}{y}, \quad \text{or} \quad y_1\sqrt{x^2-1} - my = 0 \qquad \text{...(1)}$$

Again differentiating both sides of (1) with respect to x, we get

$$y_2\sqrt{x^2-1} + y_1\frac{2x}{2\sqrt{x^2-1}} - my_1 = 0$$

or $$y_2(x^2-1) + xy_1 - m^2 y = 0 \qquad \text{[using (1)]} \qquad \text{...(2)}$$

Differentiating both sides of (2) n times w.r.t. x by Leibnitz's theorem,

$$y_{n+2}(x^2-1) + {}^nC_1\, y_{n+1}(2x) + {}^nC_2 y_n(2) + \{y_{n+1}x + {}^nC_1 y_n\} - m^2 y_n = 0$$

or $$(x^2-1)y_{n+2} + 2nxy_{n+1} + n(n-1)y_n + \{xy_{n+1} + ny_n\} - m^2 y_n = 0.$$

Hence the result.

Example 20: If $y = \cos(m\sin^{-1}x)$, then prove that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2)y_n = 0$$

Find y_n for $x = 0$. [(*W.B.U.T. 2004, 2013*); *B. Arch.* (*BESUS*), *2013*]

Solution: Here $y = \cos(m\sin^{-1}x)$

$\therefore$ $$y_1 = -\frac{m}{\sqrt{1-x^2}}\sin(m\sin^{-1}x), \quad \text{or} \quad y_1\sqrt{1-x^2} + m\sin(m\sin^{-1}x) = 0 \qquad \text{...(1)}$$

Differentiating both sides of (1) with respect to x, we get

$$y_2\sqrt{1-x^2} + y_1\frac{1}{2\sqrt{1-x^2}}\cdot(-2x) + \frac{m^2}{\sqrt{1-x^2}}\cos(m\sin^{-1}x) = 0$$

or $$(1-x^2)y_2 - xy_1 + m^2 y = 0 \qquad [\because y = \cos(m\sin^{-1}x)] \qquad \text{...(2)}$$

Differentiating both sides of (2) n times with respect to x by Leibnitz's theorem, we get

$$y_{n+2}(1-x^2) + {}^nC_1\, y_{n+1}(-2x) + {}^nC_2\, y_n(-2) - \{y_{n+1}x + {}^nC_1\, y_n\} + m^2 y_n = 0$$

$$(1-x^2)y_{n+2}-2nx\,y_{n+1}-n(n-1)y_n-(xy_{n+1}+ny_n)+m^2y_n=0$$

$$(1-x^2)y_{n+2}-(2n+1)\,xy_{n+1}+(m^2-n^2)y_n=0$$

$$(y_{n+2})_0=(n^2-m^2)(y_n)_0 \quad \ldots(3)$$

Putting $n=1,3,5,\ldots$ successively in (3), we get

$$(y_3)_0=(1^2-m^2)(y_1)_0=0 \quad [\because (y_1)_0=0, \text{ by } (1)]$$

$$(y_5)_0=(3^2-m^2)(y_3)_0=0,\ (y_7)_0=(5^2-m^2)(y_5)_0=0$$

$(y_n)_0=0$, when n is odd.

From (3), if n is even,

$$(y_n)_0=\{(n-2)^2-m^2\}(y_{n-2})_0$$

$$=\{(n-2)^2-m^2\}\{(n-4)^2-m^2\}\ldots(4^2-m^2)(2^2-m^2)(y_2)_0$$

$$=\{(n-2)^2-m^2\}\{(n-4)^2-m^2\}\ldots(4^2-m^2)(2^2-m^2)(-m^2)$$

[from (2), $(y_2)_0=-m^2$]

$$\therefore \quad (y_n)_0=-m^2(2^2-m^2)(4^2-m^2)\ldots\{(n-4)^2-m^2\}\{(n-2)^2-m^2\}$$

when n is even.

MULTIPLE CHOICE QUESTIONS

1. If $y=x^4$ then $y_4=$
   (a) 4! (b) 5! (c) 0 (d) none of these.
2. If $y=x^n$, then $y_{n-2}=$
   (a) $\frac{1}{2}n!x$ (b) $\frac{1}{2}n!x^2$ (c) $n(n-1)x^2$ (d) $n!x^2$.
3. If $y=10^{2x}$, then $y_n=$
   (a) $(10^{2x})^n$ (b) $(\log 10)^n 10^{2x}$ (c) $2^n(\log 10)^n\, 10^{2x}$ (d) $2^n\, 10^{2x}\log 10$.
4. The nth derivative of $(ax+b)^{10}$ when $n>10$ is
   (a) a^{10} (b) $10!\,a^{10}$ (c) 0 (d) 10!.
   (*W.B.U.T.* 2007, 2011)
5. If $y=ax^n+b$, then $y_n=$
   (a) $n!$ (b) $n!\,a$ (c) 0 (d) none of these.
6. If $y=e^{-2x}\sin 3x$, then $y_5=$
   (a) $13^{\frac{5}{3}}e^{-2x}\sin\left(3x-5\tan^{-1}\frac{3}{2}\right)$ (b) $13^{\frac{5}{2}}e^{-2x}\sin\left(3x+5\tan^{-1}\frac{3}{2}\right)$
   (c) $13^{\frac{5}{2}}e^{-2x}\sin\left(3x-5\tan^{-1}\frac{2}{3}\right)$ (d) $13^{\frac{5}{2}}e^{-2x}\sin\left(3x+5\tan^{-1}\frac{2}{3}\right)$.