* Taylor's Theorem in generalised form of remainder (Generalised M.V.T) :-

Statement : If a fnc. $f(x)$ defined in $a \le x \le a+h$ be st.

i) $f^{(n-1)}(x)$ be continuous in $a \le x \le a+h$.

ii) $f^{(n)}(x)$ exists in $a < x < a+h$.

& iii) p is any given positive integer then $\exists$ atleast one θ b/w 0 to 1 such that

· Taylor's Expansion

$$f(a+h) = f(a) + h \cdot f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n (1-\theta)^{n-p}}{(n-1)! \cdot p} f^{(n)}(a+\theta h)$$

$$0 < \theta < 1$$

Proof :- Continuity of $f^{(n-1)}(x)$ in $[a, a+h]$ implies continuity of $f(x), f''(x), f'''(x) \dots$ upto $f^{(n-2)}(x)$ in $[a, a+h]$

Construct a fnc. $\phi(x)$ s.t :-

$$\phi(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + \cdots + \frac{(a+h-x)^{n-1}}{(n-1)!} f^{(n-1)}(x) + (a+h-x)^p A \rightarrow ①$$

Choose the const. A s.t : ~~$\phi(x)$ satis~~

$\phi(a) = \phi(a+h)$ [i.e. $\phi(x)$ satisfies the 3rd cond. of Rolle's Theorem]

$$\Rightarrow f(a) + h \cdot f'(a) + \frac{h^2}{2!} f''(a) \cdots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + h^p A = f(a+h) \longrightarrow ②$$

Diff. ① w.r.t. x, we get.

$$\phi'(x) = \cancel{f'(x)} + \left[-\cancel{f'(x)} + (a+h-x) \cancel{f''(x)}\right] + \left[-\cancel{(a+h-x) f''(x)} + \cancel{\frac{(a+h-x)^2}{2!}} f'''(x)\right] + \cdots + \cancel{\left[-\frac{(a+h-x)^{n-2}}{(n-2)!} f^{(n-1)}(x)\right.} + \left.\frac{(a+h-x)^{n-1}}{(n-1)!} f^{(n)}(x)\right] - p(a+h-x)^{p-1} A \rightarrow ③$$

exists in $(a, a+h)$

$\Rightarrow \phi'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!} f^n(x) - p(a+h-x)^{p-1} A$

Now, we observe that :-

i) $\phi(x)$ is cont. in $[a, a+h]$ because $f(x)$ is cont. in $[a+ a+h]$ & the rem. polyn fnc are also cont.

ii) $\phi'(x)$ exists in $(a, a+h)$

iii) $\phi(a) = \phi(a+h)$ from the construction

$\therefore \phi(x)$ satisfies all the three conditions of Rolle's Theorem in $[a, a+h]$

$\therefore$ According to Rolle's Theorem:-

$$\phi'(\xi) = 0$$

$$\Rightarrow \phi'(a+\theta h) = 0$$

$$\Rightarrow \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h) - p\, h^{p-1}(1-\theta)^{p-1} A = 0$$

$$\Rightarrow A = \frac{h^{n-p}(1-\theta)^{n-p}}{(n-1)!\cdot p} f^n(a+\theta h)$$

Substituting the value of A from ④ in ②, we get

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \dots$$
$$+ \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + h^p \cdot \frac{h^{n-p}(1-\theta)^{n-p}}{(n-1)! \cdot p} f^n(a+\theta h),$$
$$0 < \theta < 1$$

Types of Remainder:-

The last term in Taylor's expansion is called the remainder & is denoted by R_n i.e.

i) $R_n = \dfrac{h^n (1-\theta)^{n-p}}{(n-1)! \cdot p} f^n(a+\theta h)$.

(Generalized form or Schlömilch Roche form) $\boxed{\dfrac{h^n (1-\theta)^{n-p}}{(n-1)! \cdot p} f^n(a+\theta h)}$

ii) If $p = n$

$R_n = \dfrac{h^n}{n!} f^n(a+\theta h)$

(Lagrange's Form of Remainder)

iii) If $p = 1$

$$R_n = \frac{h^n (1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h)$$

(Cauchy's Remainder)

~~Maclau~~

*** Taylor's Theorem in different intervals:**
(in Lagrange's Form of Remainder)

$$f(a+h) = f(a) + h \cdot f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^n(a+\theta h)$$

where, $0 < \theta < 1$, $a \le x \le a+h$

1. Interval $[x, x+h]$

$$f(x+h) = f(x) + h \cdot f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(x) + \frac{h^n}{n!} f^n(x+\theta h)$$

$0 < \theta < 1$

2. Interval $[a, b]$ (Here $a = x$, $b = x+h$ $\therefore h = b-a$)

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(a) + \frac{(b-a)^3}{3!} f'''(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(b-a)^n}{n!} f^n(\xi)$$

3. Interval [

$f(x) = f(a) +$

$+ \cdots$

Now,

i) Put

from

$\boxed{f(a+h)}$

ii) Put

$\boxed{f(}$

and

3. Interval $[a, x]$ (Here, $a = x$, $h = x-a$)

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots + \cdots + \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(x-a)^n}{n!} f^n(a + \theta(x-a))$$

where $0 < \theta < 1$

Now,

i) Put $n=1$ in Taylor's Theorem (in Lagrange's Form of Remainder)

$$\boxed{f(a+h) = f(a) + h f'(a+\theta h)}, \quad 0 < \theta < 1$$

which is Lagrange's MVT.

ii) Put $n=2$, we get

$$\boxed{f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a+\theta h)}$$

$0 < \theta < 1$,

→ MVT of 2nd order.

and so on.

* Maclaurin's theorem in generalised form of remainder :-

Putting $\boxed{a = 0 \ \& \ h = x}$ in Taylor's Theorem, we get,

i.e. interval is $[0, x]$..

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + \frac{x^n (1-\theta)^{n-p}}{(n-1)! \cdot p} f^n(\theta x),$$

$$0 < \theta < 1.$$

This theorem holds if

i) $f^{n-1}(x)$ is continuous in $[0, x]$

ii) $f^n(x)$ exists in $(0, x)$.

& iii) p is any given positive integer.

$\therefore$ Maclaurin's Remainder is:

$$R_n = \frac{x^n (1-\theta)^{n-p}}{(n-1)! \cdot p} f^n(\theta x), \quad 0 < \theta < 1$$

(Generalised form of Remainder)

(Schlömilch Roche Form)

i) $\boxed{p=n}$

$$R_n = \frac{x^n}{n!} f^n(\theta x), \quad 0 < \theta < 1$$

(Lagrange's form of Remainder)

ii) $\boxed{p = 1}$

$$R_n = \frac{x^n (1-\theta)^{n-1}}{(n-1)!} f^n(\theta x), \quad 0 < \theta < 1$$

(Cauchy's Form of Remainder)

* Taylor's Infinite Series