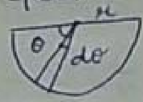
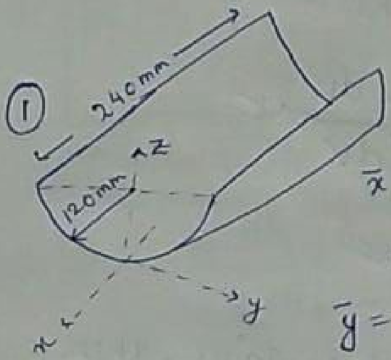
SHEET : 5.1

(COM & Centroids of Simple Geometric figures)



①

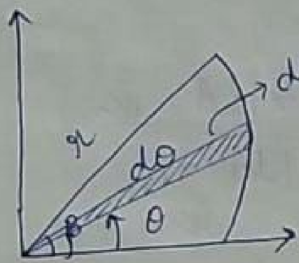


$$\bar{x} = \frac{\int_0^{240} \cancel{t}\, x\, dx}{\int_0^{240} \cancel{t}\, dx} = \frac{\left[\frac{x^2}{2}\right]_0^{240}}{[x]_0^{240}} = \frac{240}{2} = 120 \text{ mm}$$

$$\bar{y} = \frac{\int_0^{\pi} r\sin\theta \times r\, d\theta}{\int_0^{\pi} \cancel{r}\, d\theta} = \frac{r\,[-\cos\theta]_0^{\pi}}{[\theta]_0^{\pi}} = 76.4 \text{ mm}$$

$$\bar{z} = (120 - 76.4) = 43.6 \text{ mm}$$

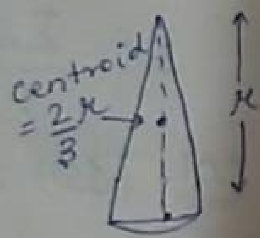
②



$dA = \frac{1}{2} \times r \times r\, d\theta$

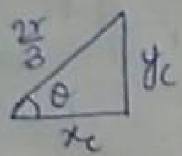
$$x_c = \frac{2}{3} r\cos\theta,$$
$$y_c = \frac{2}{3} r \sin\theta$$

$$dA = \frac{1}{2} \times r \times r\, d\theta.$$



$$\bar{x} = \frac{\int_0^{\beta} x\, dA}{\int_0^{\beta} dA} = \frac{\int_0^{\beta} \frac{2}{3} r\cos\theta \times \frac{1}{2} r^2 d\theta}{\int_0^{\beta} \frac{1}{2} r^2 d\theta}$$

$$= \frac{2}{3} r \frac{[\sin\theta]_0^{\beta}}{[\theta]_0^{\beta}} = \frac{2r}{3} \frac{\sin\beta}{\beta}.$$

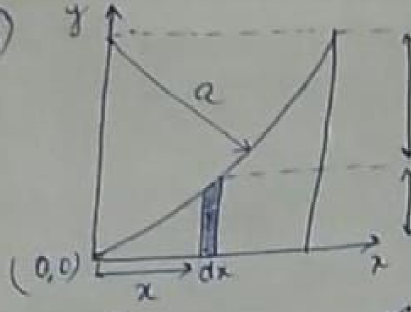


Similarly;

$$\bar{y} = \frac{\int_0^{\beta} y\, dA}{\int_0^{\beta} dA} = \frac{\int_0^{\beta} \frac{2r}{3}\sin\theta \times \frac{1}{2} r^2 d\theta}{\int_0^{\beta} \frac{1}{2} r^2 d\theta}$$

$$= \frac{2r}{3} \frac{[-\cos\theta]_0^{\beta}}{[\theta]_0^{\beta}} \Rightarrow \bar{y} = \frac{2r}{3\beta}(1 - \cos\beta)$$

(3)



from eqⁿ of circle having centre at $(0,a)$ & radius a;

$$x^2+(a-y)^2=a^2$$

$$\Rightarrow \quad x^2+y^2-2ay=0 \rightarrow (i)$$

$$\bar{x} = \frac{\int_0^a x\,dA}{\int_0^a dA} \neq \int_0^x \cancel{x}$$

$dA = y\,dx$

$y = a - \sqrt{a^2-x^2}$

$\therefore \quad dA = (a - \sqrt{a^2-x^2}\,dx)$

$$\bar{x} = \frac{\int_0^a x\,(a-\sqrt{a^2-x^2})\,dx}{\int_0^a (a-\sqrt{a^2-x^2})\,dx} \; ; \quad \bar{y} = \frac{\int_0^a y_c\,dA}{\int_0^a dA} = \frac{\int_0^a \frac{y}{2}\,dA}{\int_0^a dA}.$$

$$\bar{x} = \frac{\left[\frac{ax^2}{2} + \frac{1}{3}(a^2-x^2)^{3/2}\right]_0^a}{\left[ax - \frac{1}{2}\sqrt{a^2-x^2} - \frac{1}{2}a^2\sin^{-1}\frac{x}{a}\right]_0^a}$$

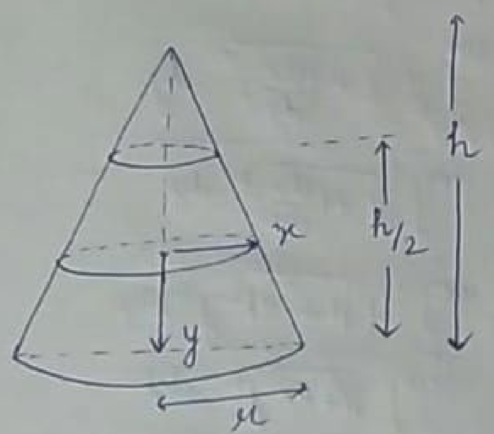
$$= \frac{\left\{\frac{a^3}{2} - \frac{a^3}{3}\right\}}{\left\{a^2 - \frac{a^2}{2}\times\frac{\pi}{2}\right\}} \Rightarrow \bar{x} = \underline{\frac{2a}{3(4-\pi)}}$$

$$\bar{y} = \frac{\int \frac{y}{2}\,y\,dx}{\int y\,dx}$$

$$= \frac{\int_0^a \frac{1}{2}\left[a-\sqrt{a^2-x^2}\right]^2 dx}{\int_0^a \left[a-\sqrt{a^2-x^2}\right] dx}$$

$$\bar{y} = \underline{\frac{10-3\pi}{3(4-\pi)}\,a}$$

(4)



$$\frac{x}{r} = \left(1-\frac{y}{h}\right)$$

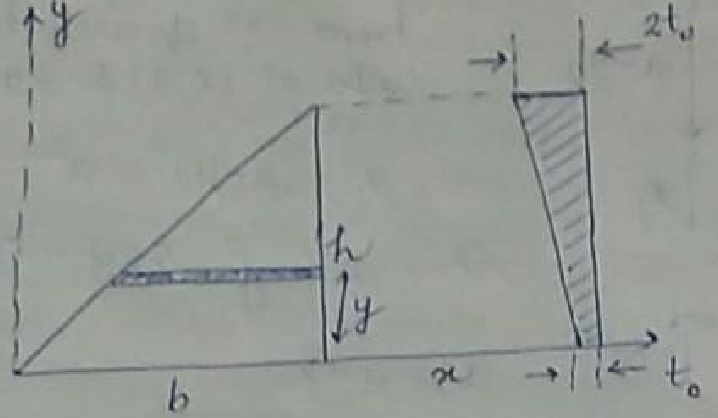
$$x = r\left(1-\frac{y}{h}\right)$$

$$\bar{h} = \frac{\int_0^{h/2} y \times \cancel{\pi} x^2\,dy}{\int_0^{h/2} \cancel{\pi} x^2\,dy}$$

$$\bar{h} = \frac{\int_0^{h/2} y \times \cancel{r^2}\left(1-\frac{y}{h}\right)^2 dy}{\int_0^{h/2} \cancel{r^2}\left(1-\frac{y}{h}\right)^2 dy} = \frac{\int_0^{h/2}\left(y + \frac{y^3}{h^2} - \frac{2y^2}{h}\right) dy}{\int_0^{h/2}\left(1 + \frac{y^2}{h^2} - \frac{2y}{h}\right) dy}$$

$$\bar{h} = \frac{11h}{56} \; //.$$

(5)

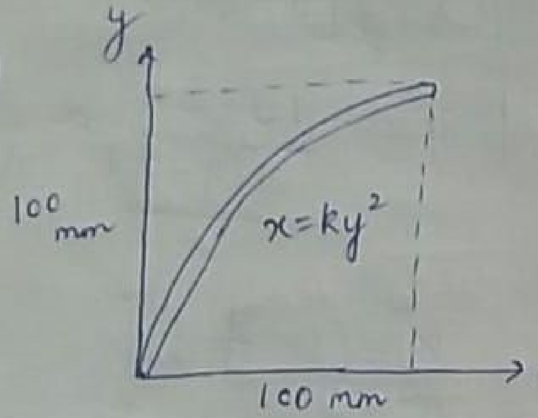


$$t = t_0\left(1+\frac{y}{h}\right) \; ; \; l = b\left(1-\frac{y}{h}\right)$$

$$\bar{y} = \frac{\int_0^h y\, \cancel{b}\left(1-\frac{y}{h}\right) \cancel{t_0}\left(1+\frac{y}{h}\right) dy}{\int_0^h \cancel{b}\left(1-\frac{y}{h}\right)\cancel{t_0}\left(1+\frac{y}{h}\right) dy}$$

$$\bar{y} = \frac{\int_0^h \left(y-\frac{y^3}{h^2}\right) dy}{\left(1-\frac{y^2}{h^2}\right) dy} = \frac{\frac{h^2}{2}-\frac{h^4}{4h^2}}{h-\frac{h^3}{3h^2}} = \frac{3h}{8}$$

(6)



$$\bar{y} = \frac{\int_0^{100} y\, dL}{\int_0^{100} dL}$$

$$= \frac{\int_0^{100} y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy}{\int_0^{100} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy}$$

$x = ky^2$

$dx = 2y\, dy\, k$

$\frac{dx}{dy} = 2yk$

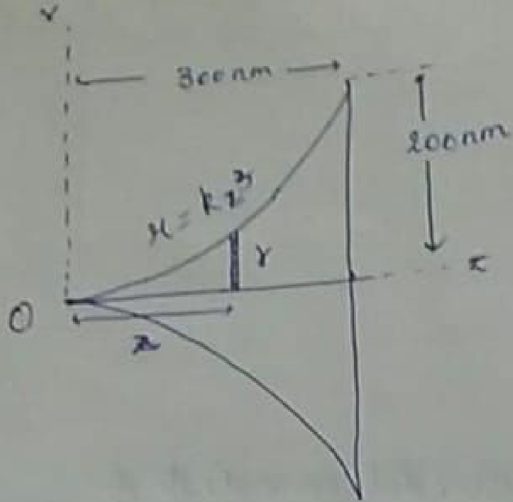
$100 = k \times 100^2$

$k = \frac{1}{100}$

$$\Rightarrow \bar{y} = \frac{\int_0^{100} y\sqrt{1+4k^2y^2}\, dy}{\int_0^{100} \sqrt{1+4k^2y^2}\, dy}$$

$\bar{y} = 57.36$

(7)



$$x = kz^3$$

$$200 = k \times 300^3,$$

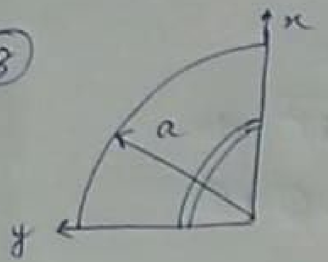
$$k = 1/135000$$

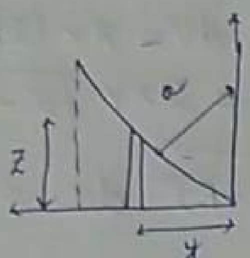
$$\bar{z} = \frac{\int_0^{300} z \times \pi \times k^2 z^6 \, dz}{\int_0^{300} \pi k^2 z^6 \, dz}$$

$$= \left[\frac{z^8}{8} \times \frac{7}{z^7} \right]_0^{300}$$

$$= \underline{262.5 \text{ mm}}$$

(8)





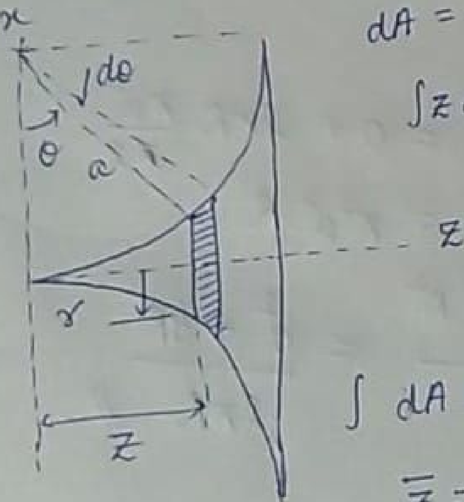
$$dV = \frac{\pi}{2}\left(ay - y\sqrt{a^2 - y^2}\right) dy$$

$$\bar{y} = \frac{\frac{\pi}{2}\int_0^a \left(ay^2 - y^2\sqrt{a^2-y^2}\right) dy}{\frac{\pi}{2}\int_0^a \left(ay - y\sqrt{a^2-y^2}\right) dy}$$

$$\bar{y} = \frac{\frac{a^4}{3} - \frac{\pi a^4}{6}}{\frac{\pi a^3}{12}} = a\left(\frac{4}{\pi} - \frac{3}{4}\right)$$

$$\bar{x} = \bar{y} = \underline{a\left(\frac{4}{\pi} - \frac{3}{4}\right)}$$ from symmetry.

(9)



$$dA = 2\pi r (a\,d\theta) = 2\pi a^2 (1 - \cos\theta)\, d\theta$$

$$\int z\, dA = \int_0^{\pi/2} (a\sin\theta)(2\pi a^2)(1-\cos\theta)\, d\theta$$

$$\Rightarrow 2\pi a^3 \int_0^{\pi/2} (\sin\theta - \sin\theta\cos\theta)\, d\theta$$

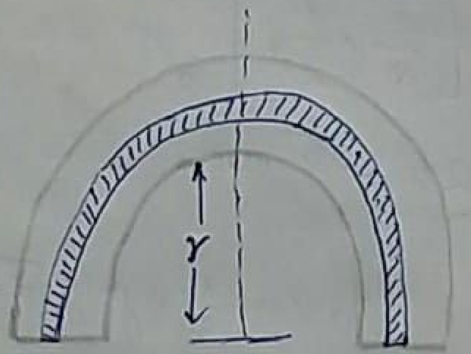
$$= 2\pi a^3\left(1 - \frac{1}{2}\right) = \pi a^3.$$

$$\int dA = 2\pi a^2 \int_0^{\pi/2} (1-\cos\theta)\, d\theta = 2\pi a^2\left(\frac{\pi}{2} - 1\right)$$

$$\bar{z} = \frac{\int z\, dA}{A} = \frac{\pi a^3}{2\pi a^2\left(\frac{\pi}{2} - 1\right)}$$

$$\Rightarrow \bar{z} = \underline{\frac{a}{(\pi - 2)}}$$

(10)



Centroidal co-ordinate of elemental ring is

$r_c = 2r/\pi$

$dV = \pi r\,(2\pi)\,dr$

$= 2\pi r \sqrt{a^2 - (r-R)^2}\;dr$

$$\int r_c\,dV = 4\int_{R-a}^{R+a} r^2\sqrt{a^2-(r-R)^2}\;dr = ① + ② + ③$$

where $\underline{u = r - R}$

$$① = 4\int_{-a}^{a} u^2\sqrt{a^2-u^2}\;du = \frac{\pi a^4}{2}$$

$$② = 4\int_{-a}^{a} 2Ru\sqrt{a^2-u^2}\;du = 0$$

$$③ = 4\int_{-a}^{a} R^2\sqrt{a^2-u^2}\;du = 2\pi a^2 R^2$$

$$\int dV = 2\pi\int_{-a}^{a}(u+R)\sqrt{a^2-u^2}\;du = 0 + 2\pi R\,\frac{\pi a^2}{2}$$

$$= \pi^2 a^2 R$$

$$\bar{r} = \frac{\int r_c\,dV}{\int dV} = \frac{\frac{\pi a^4}{2} + 2\pi a^2 R^2}{\pi^2 a^2 R} = \frac{a^2 + 4R^2}{2\pi R}\;//.$$