9.8. Theorem 9.2. *Let* $f(x)$ and $g(x)$ *be two functions such that, in* $a < x \le b$, *they are both positive;* $f(x) \le g(x)$ *and have point of infinite discontinuity at* $x = a$, *then*

(i) $\int_a^b f(x)\,dx$ *converges if* $\int_a^b g(x)\,dx$ *converges ;*

(ii) $\int_a^b g(x)\,dx$ *does not converge if* $\int_a^b f(x)\,dx$ *does not.*

a

9.9. Theorem 9.3. (*Comparison Test*).

Let $f(x)$ *and* $g(x)$ *be two functions such that in* $a < x \leq b$, *they are both positive and have point of infinite discontinuity at* $x = a$. *If*

$\lim_{x \to a+} \frac{f(x)}{g(x)} = l$, *where* l *is a non-zero finite number, then the two integrals*

$\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ *either both converge or both do not* converge.

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9.10. μ-test for Improper Integrals of the Second Kind.

If $f(x)$ be an integrable function in $a+\varepsilon \leq x \leq b$, where

$a < \varepsilon < b - a$, and a be the only point of infinite disconfinuity of $f(x)$ in $a \le x \le b$, then the integral $\int_a^b f(x)\,dx$ converges if

$$\lim_{x \to a+0} (x-a)^{\mu} f(x) = l,$$

a non zero finite number and $0 < \mu < 1$, and it diverges if $\mu \ge 1$.

a

9.12. Theorem 9.5. (Comparison Test)

(i) *If the functions* $f(x)$ *and* $g(x)$ *are both positive and* $g(x) \le f(x)$ *in* $a \le x \le X$, *then* $\int_a^\infty g(x)\,dx$ *converges if* $\int_a^\infty f(x)\,dx$ *be convergent.*

(ii) *If* $f(x)$ *and* $g(x)$ *are positive functions for* $x \ge a$ *and* $\lim_{x\to\infty} \dfrac{f(x)}{g(x)}$ *exists and is equal to a non-zero finite number, then* $\int_a^\infty f(x)\,dx$ *and* $\int_a^\infty g(x)\,dx$ *have the same nature.*

9.13. μ-test for Improper Integrals of the First Kind.

Let $f(x)$ be bounded and integrable in $a \le x \le X$, where $a > 0$. If there exists a number $\mu > 1$ such that $\lim_{x\to\infty} x^{\mu} f(x)$ exists and be equal to a non-zero finite number, then $\int_a^\infty f(x)\,dx$ is convergent.

If there exists a number $\mu \le 1$ such that $\lim\limits_{x \to \infty} x^{\mu} f(a)$ exists and is a non-zero finite number, then $\int\limits_a^{\infty} f(x)\, dx$ is divergent. The same is also true if $\lim\limits_{x \to \infty} x^{\mu} f(x) \to \infty$ or $-\infty$.

9.14. Illustrative Examples.

Ex. 1. *Examine the convergence of* $\int\limits_0^{\infty} e^{-x} dx$.

$I = \int\limits_0^{\infty} e^{-x}\, dx$ is an improper integral of the first kind.

$$\text{We have } \lim_{X \to \infty} \int\limits_0^{X} e^{-x} = \lim_{X \to \infty} \left[-e^{-x}\right]_0^X$$

$$= \lim_{X \to \infty} (1 - e^{-X}) = 1, \text{ a finite number.}$$

Hence I is convergent and its value is 1.

Ex. 2. *Discuss the convergence of* $\int\limits_0^{\infty} \cos tx\, dx$.

$I = \int\limits_0^{\infty} \cos tx\, dx$ is an improper integral of the first kind.

$$\text{We have } \lim_{X \to \infty} \int\limits_0^{X} \cos tx\, dx = \lim_{X \to \infty} \left[\frac{\sin tx}{t}\right]_0^X$$

$$= \lim_{X \to \infty} \frac{\sin tX}{t};$$

but this limit does not exist.

Hence I does not exist.

Ex. 3. *Evaluate, if possible,* $\int\limits_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

$I = \int\limits_{-\infty}^{\infty} \frac{dx}{1+x^2}$ is an improper integral of the first kind.

We consider the integrals I_1, I_2

where $I_1 = \int_{-\infty}^{a} \frac{dx}{1+x^2}, I_2 = \int_{a}^{\infty} \frac{dx}{1+x^2}$.

We have $\lim_{X\to\infty} \int_{-X}^{a} \frac{dx}{1+x^2} = \lim_{X\to\infty} \left[\tan^{-1} x\right]_{-X}^{a}$

$$= \lim_{X\to\infty} (\tan^{-1} a + \tan^{-1} X)$$

$= \tan^{-1} a + \frac{\pi}{2}$, a finite number.

Therefore I_1 is convergent and its value is $\tan^{-1} a + \frac{\pi}{2}$.

Again, $\lim_{X\to\infty} \int_{a}^{X} \frac{dx}{1+x^2} = \lim_{X\to\infty} [\tan^{-1} x]_{a}^{X}$.

$\lim_{X\to\infty} [\tan^{-1} X - \tan^{-1} a] = \frac{\pi}{2} - \tan^{-1} a$, a finite number.

Therefore I_2 is convergent and its value is $\frac{\pi}{2} - \tan^{-1} a$.

Since I_1 and I_2 are both convergent, I is convergent and its value is

$$\tan^{-1} a + \frac{\pi}{2} + \frac{\pi}{2} - \tan^{-1} a = \pi.$$

Ex. 4. *Test for convergence of the integral* $\int_0^1 \frac{dx}{x^{2/3}}$.

$\int_0^1 \frac{dx}{x^{2/3}}$ is an improper integral of the second kind, 0 being the only point infinite discontinuity of the integrand in [0, 1].

We have $\lim_{\varepsilon\to 0+} \int_{\varepsilon}^{1} \frac{dx}{x^{\frac{2}{3}}} = \lim_{\varepsilon\to 0+} \left[3x^{\frac{1}{3}}\right]_{\varepsilon}^{1}$

$= \lim_{\varepsilon\to 0+} (3 - 3\varepsilon^{1/3}) = 3$, a finite number.

Hence I is convergent and its value is 3.

Ex 5. *Examine the convergence of* $\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$

$\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$ is an improper integral of the mixed kind.

We consider the integrals I_1 and I_2

where $I_1 = \int_0^a \frac{dx}{(1+x)\sqrt{x}}$ and $I_2 = \int_a^\infty \frac{dx}{(1+x)\sqrt{x}}$, $(a>0)$.

We have $\int \frac{dx}{(1+x)\sqrt{x}} = \int \frac{2z\,dz}{z(1+z^2)}$

$$= 2\tan^{-1} z = 2\tan^{-1}\sqrt{x}.$$

$I_1 = \int_0^a \frac{dx}{(1+x)\sqrt{x}}$ is an improper integral of the second kind, 0 being the only point of infinite discontinuity of the integrand in $0 \le x \le a$.

We have $\lim\limits_{\varepsilon \to 0+} \int_\varepsilon^a \frac{dx}{(1+x)\sqrt{x}} = \lim\limits_{\varepsilon \to 0+} [2\tan^{-1}\sqrt{x}]_\varepsilon^a$

$$= \lim_{\varepsilon \to 0+} (2\tan^{-1}\sqrt{a} - 2\tan^{-1}\sqrt{\varepsilon})$$

$= 2\tan^{-1}\sqrt{a}$, a finite number.

Therefore I_1 is convergent whose value is $2\tan^{-1}\sqrt{a}$.

$I_2 = \int_a^\infty \frac{dx}{(1+x)\sqrt{x}}$ is an improper integral of the first kind.

We have $\lim\limits_{X \to \infty+} \int_a^X \frac{dx}{(1+x)\sqrt{x}} = \lim\limits_{\varepsilon \to 0+} [2\tan^{-1}\sqrt{x}]_a^X$

$$= \lim_{X \to \infty} (2\tan^{-1}\sqrt{X} - 2\tan^{-1}\sqrt{a})$$

$$= 2.\frac{\pi}{2} - 2\tan^{-1}\sqrt{a} = \pi - 2\tan^{-1}\sqrt{a}$$

which is a finite number.

Therefore I_2 is convergent whose value is $\pi - 2\tan^{-1}\sqrt{a}$.

Therefore I is convergent whose value is

$$2\tan^{-1}\sqrt{a} + \pi - 2\tan^{-1}\sqrt{a} = \pi.$$

Ex. 6. *Test for convergence of the integral* $\int_1^{\infty} \frac{\sin^2 x}{x^2} dx$.

$I = \int_1^{\infty} \frac{\sin^2 x}{x^2} dx$ is an improper integral of the first kind.

We have $\frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$ for all $x \in [1, \infty)$

It is known that $\int_1^{\infty} \frac{dx}{x^{\mu}}$ is convergent if $\mu > 1$.

Also $\frac{\sin x}{x^2} \geq 0$, $\frac{1}{x^2} > 0$, for all $x \in [1, \infty)$.

Therefore $\int_1^{\infty} \frac{dx}{x^2}$ is convergent by comparison test.

Hence I is convergent.

Ex. 7. *Examine the convergence of* $\int_0^{\frac{\pi}{2}} x^m \operatorname{cosec}^n x \, dx$.

$$I = \int_0^{\frac{\pi}{2}} x^m \operatorname{cosec}^n x \, dx = \int_0^{\frac{\pi}{2}} \frac{x^m}{\sin^n x} dx = \int_0^{\frac{\pi}{2}} \left(\frac{x}{\sin x}\right)^n \frac{1}{x^{n-m}} dx$$

Since $\lim_{x \to 0} \left(\frac{x}{\sin x}\right)^n = 1$, I is an improper integral of the second kind if $n > m$, 0 being the only point of infinite discontinuity of the integrand $f(x) = x^m \operatorname{cosec}^n x$ in $0 \leq x \leq \frac{\pi}{2}$.

Take $g(x) = \frac{1}{x^{n-m}}$.

Therefore $\lim\limits_{x\to0}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to0}\left(\dfrac{x}{\sin x}\right)^n=1$, a non-zero finite number.

By Comparison test, I will have the same nature as $\int\limits_0^{\frac{\pi}{2}}\dfrac{dx}{x^{n-m}}$, which converges if $n-m<1$.

Ex. 8. *Test the convergence of* $\int\limits_1^{\infty}\dfrac{x^2dx}{(1+x^2)^2}$.

$\int\limits_1^{\infty}\dfrac{x^2dx}{(1+x^2)^2}$ is an improper integral of the first kind.

Let $f(x)=\dfrac{x^2}{(1+x^2)^2}$.

Take $g(x)=\dfrac{x^2}{x^4}=\dfrac{1}{x^2}$.

Therefore $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to\infty}\dfrac{x^4}{(1+x^2)^2}$

$$=\lim_{x\to\infty}\frac{1}{\left(1+\frac{1}{x^2}\right)^2}=1\text{, a non-zero } \textit{finite number.}$$

Therefore $\int\limits_1^{\infty}f(x)\,dx$ and $\int\limits_1^{\infty}g(x)\,dx$ have the same nature.

It is known that $\int\limits_1^{\infty}\dfrac{dx}{x^{\mu}}$ is convergent if $\mu>1$. Therefore $\int\limits_1^{\infty}g(x)\,dx$ is convergent.

Hence $\int\limits_1^{\infty}f(x)\,dx=\int\limits_1^{\infty}\dfrac{x^2\,dx}{(1+x^2)^2}$ is convergent.

To evaluate this integral we use the transformation $x=\tan\theta$.

Then $I=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\tan^2\theta\sec^2\theta\,d\theta}{\sec^4\theta}=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\sin^2\theta\,d\theta$

$$=\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(1-\cos 2\theta)\,d\theta=\frac{1}{2}\left[\theta-\frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$=\frac{1}{2}\left(\frac{\pi}{2}-\frac{\pi}{4}+\frac{1}{2}\right)=\frac{1}{2}\left(\frac{\pi}{4}+\frac{1}{2}\right)=\frac{\pi+2}{8}.$$

Ex. 9 *Apply* μ *-test to examine the convergence of* $\int_1^{\infty}\frac{x\,dx}{(1+x)^3}$.

$I=\int_1^{\infty}\frac{x\,dx}{(1+x)^3}$ is an improper integral of the first kind.

Take $\mu=2$.

We have $\lim_{x\to\infty}x^{\mu}.\frac{x}{(1+x)^3}=\lim_{x\to\infty}\frac{x^3}{(1+x)^3}=1$, a non-zero finite number.

Since $\mu>1$, I is convergent.

Ex. 10 *Apply* μ *-test to test the convergence of* $\int_0^{\frac{\pi}{2}}\frac{\sqrt{x}}{\sin x}dx$.

$I=\int_0^{\frac{\pi}{2}}\frac{\sqrt{x}}{\sin x}dx$ is an improper integral of the second kind, 0 being the only point of infinite discontinuity of the integrand $f(x)=\frac{\sqrt{x}}{\sin x}$.

Take $\mu=\frac{1}{2}$.

Then $\lim_{x\to 0}x^{\mu}f(x)=\lim_{x\to 0}\frac{x}{\sin x}=1$, a non-zero finite number.

Since $0<\mu<1$, I is convergent.

Ex. 11) *Test for convergence of the integral* $\int_{-1}^{+1} \frac{dx}{x}$.

$\int_{-1}^{+1} \frac{dx}{x}$ is an improper integral of the second kind, 0 being the only point of infinite discontinuity of the integrand $f(x) = \frac{1}{x}$ in $[-1, +1]$.

$f(x) = \frac{1}{x}$ in $[-1, 1]$

Here we see that

$$\lim_{\substack{\epsilon \to 0 \\ \epsilon' \to 0}} \left[\int_{-1}^{0-\epsilon} \frac{1}{x} dx + \int_{0+\epsilon'}^{1} \frac{1}{x} dx \right], \quad 0 < \epsilon < 1, \quad 0 < \epsilon' < 1$$

$$= \lim_{\substack{\epsilon \to 0 \\ \epsilon' \to 0}} \left[\log|x| \Big|_{-1}^{-\epsilon} + \log|x| \Big|_{\epsilon'}^{1} \right]$$

$$= \lim_{\substack{\epsilon \to 0 \\ \epsilon' \to 0}} \left[\log|-\epsilon| - \log|\epsilon'| \right]$$

$$= \lim_{\substack{\epsilon \to 0 \\ \epsilon' \to 0}} \left[\log\left(\frac{\epsilon}{\epsilon'}\right) \right], \text{ (Here } |\epsilon| = \epsilon, |\epsilon'| = \epsilon' \text{ since } \epsilon > 0, \epsilon' > 0)$$

Now we see that for the choice

$$\epsilon = 3\epsilon', \quad \lim_{\substack{\epsilon \to 0 \\ \epsilon' \to 0}} \log\left(\frac{\epsilon}{\epsilon'}\right) = \log 3$$

where as for the choice $\varepsilon = 2\varepsilon'$, $\lim_{\substack{\epsilon \to 0 \\ \epsilon' \to 0}} \log\left(\frac{\epsilon}{\epsilon'}\right) = \log 2 \neq \log 3$.

So $\lim_{\substack{\epsilon \to 0 \\ \epsilon' \to 0}} \log \frac{\epsilon}{\epsilon'}$ does not exist and consequently the given integral is not convergent and so it has no value.

But if we put $\varepsilon = \varepsilon'$, we get $\int_{-1}^{+1} \frac{dx}{x} = \lim_{\varepsilon \to 0} \log 1 = 0$.

Thus although the general value of the integral does not exist, its principal value exists.