## Cauchy's Root Test.

Given a series $\sum_{n=1}^{\infty} a_n$, put $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$

Then

(1) if $\alpha < 1$, $\sum a_n$ converges.

(2) if $\alpha > 1$, $\sum a_n$ diverges.

(3) if $\alpha = 1$, the test gives no information.

**N. B.** A common, but less general, theorem may be obtained by $\lim_{n\to\infty} \sup \sqrt[n]{|a_n|}$ by $\lim_{n\to\infty} \sqrt[n]{|a_n|}$. Recall that if $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists, then $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{|a_n|}$.

Proof (1) Here $\alpha < 1$. Choose $\varepsilon > 0$ such that $\beta = \alpha + \varepsilon < 1$.

Since $\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|} = \alpha$, $\sqrt[n]{|a_n|} > \alpha + \varepsilon = \beta$ for a <u>finite number</u> of values of $n$.

In other words, $\sqrt[n]{|a_n|} \leq \alpha + \varepsilon$ for all but a finite number of values of $n$.

$\therefore$ $\exists$ an integer $N$ such that

$$\sqrt[n]{|a_n|} \leq \beta \quad \forall \; n \geq N$$

This means $|a_n| \leq \beta^n \;\; \forall \; n \geq N$. Now the G. P. series $\sum \beta^n$ converges since $\beta < 1$. Therefore $\sum a_n$ converges by comparison test.

(2) If $\alpha > 1$, choose $\varepsilon > 0$ such that $\alpha - \varepsilon = \gamma > 1$.

We have $\sqrt[n]{|a_n|} > \alpha - \varepsilon > 1$ for infinitely many values of $n$.

$\therefore$ $|a_n| > 1$ for infinitely many values of $n$.

This means the sequence $\{a_n\}$ cannot tend to $0$, and hence $\sum a_n$ diverges.

(3) To prove this part, consider the series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$.

For both the series $\alpha = 1$ (since $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$), but the first series diverges and the second series converges.

<u>Another simple version of the theorem</u> :

If for the series $\sum a_n$, $a_n > 0$, $\sqrt[n]{a_n} \leq r < 1$

for $n \geq N$, then $\sum a_n$ converges. If $\sqrt[n]{a_n} > 1$ for infinitely many values of $n$, then $\sum a_n$ diverges.

e.g. $$\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$$

$$\sqrt[n]{a_n} = \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(1+\frac{1}{n}\right)^n} \to \frac{1}{e} < 1.$$

$\therefore$ The series converges.

$$\sum_{n=2}^{\infty} (\log_e n)^{-n}.$$

$$\sqrt[n]{a_n} = \frac{1}{\log_e n} \to 0 < 1 \quad \text{as } n \to \infty.$$

$\therefore$ The series converges by Root Test.

## D'Alembert's Ratio Test.

Suppose for a series $\sum a_n$, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$

Then

(1) if $L < 1$, $\sum a_n$ converges.

(2) if $L > 1$, $\sum a_n$ diverges.

(3) if $L = 1$, the test fails.

<u>Proof</u>: If $L < 1$, choose $\varepsilon > 0$ s.t. $M = L + \varepsilon < 1$. Then there is an integer $N$ such that

$$\left|\frac{a_{n+1}}{a_n}\right| < L + \varepsilon = M \quad \text{if } n \geq N.$$

$$\therefore \quad |a_{n+1}| < |a_n| M \quad \text{if } n \geq N.$$

In particular,

$$|a_{N+1}| < |a_N| \cdot M$$

$$|a_{N+2}| < |a_{N+1}| M < |a_N| M^2$$

$$\cdots\cdots$$

$$|a_{N+k}| < |a_N| \cdot M^k, \quad k \geq 1.$$

or, $$|a_n| < |a_N| M^{n-N} \quad \text{if } n \geq N$$

$$= \frac{|a_N|}{M^N} M^n.$$

The G. P. series $\sum_{n=1}^{\infty} \frac{|a_N|}{M^N} M^n = \frac{|a_N|}{M^N} \sum_n M^n$ converges since $M < 1$.

$\therefore$ $\Sigma a_n$ converges by comparison test.

(2) If $L > 1$, choose $\varepsilon > 0$ s.t. $R = L - \varepsilon > 1$.

$\therefore$ $\exists$ an integer $N$ s.t.

$$\left|\frac{a_{n+1}}{a_n}\right| > L - \varepsilon = R \text{ if } n \geq N.$$

As before, we have $|a_n| > |a_N| R^{n-N}$

$$= \frac{|a_N|}{R^N} R^n \text{ if } n \geq N.$$

Since the G. P. series $\sum_n \frac{|a_N|}{R^N} R^n$ diverges ($R$ being greater than $1$), $\Sigma a_n$ diverges by comparison test.

(3) Consider $\Sigma \frac{1}{n}$, $\Sigma \frac{1}{n^2}$.

For the first series, $\frac{a_{n+1}}{a_n} = \frac{n}{n+1} \to 1$ as $n \to \infty$.

and the series is divergent.

For the second series $\dfrac{a_{n+1}}{a_n} = \dfrac{n^2}{(n+1)^2} \to 1$ as $n \to \infty$, and the series converges.

e.g. $\displaystyle\sum \frac{n!}{n^n}$. Here $a_n = \dfrac{n!}{n^n}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \left(\frac{n}{n+1}\right)^n$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^n} \to \frac{1}{e} < 1.$$

$\therefore$ The series converges by ratio test.

Example. For the series $\sum_n 2^{-n-(-1)^n}$, the root test works, but the ratio test is not applicable.

$$a_n^{1/n} = 2^{-1-\frac{(-1)^n}{n}} \to \frac{1}{2} < 1 \quad \text{as } n \to \infty.$$

So the series is convergent by root test.

$$\frac{a_{n+1}}{a_n} = \frac{2^{-n-1-(-1)^{n+1}}}{2^{-n-(-1)^n}} = 2^{-1-(-1)^{n+1}+(-1)^n}$$

$$= \begin{cases} 2 & \text{if } n \text{ is EVEN,} \\ \frac{1}{8} & \text{if } n \text{ is ODD.} \end{cases}$$

$$\therefore \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = 2.$$

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{8}.$$

So the ratio test is not applicable.

Exc. 1. Show that the following series is convergent :

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \cdots\cdots$$

Exc. 2. Prove that the series

$$\frac{1}{3} + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{7}\right)^3 + \cdots\cdots$$

is convergent.

$$a_n = \left(\frac{n}{2n+1}\right)^n$$

$$\sqrt[n]{a_n} = \frac{n}{2n+1} = \frac{1}{\frac{2n+1}{n}} = \frac{1}{2+\frac{1}{n}} \to \frac{1}{2} \text{ as } n \to \infty.$$

$\therefore$ By root test, the series is convergent, as $\frac{1}{2} < 1$.

Exc. 3. For a series of positive terms $\sum a_n$, show that

(1) if $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$, then $\lim\limits_{n\to\infty} a_n^{\frac{1}{n}} = \infty$.

(2) if $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = L$, then $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = L$ also.

Proof: (1) Take any positive number $M$. Then $\exists$ an integer $N$ such that

$$\frac{a_{n+1}}{a_n} > 2M \quad \text{if} \quad n \geq N.$$

Then $\underbrace{\frac{a_{N+1}}{a_N}, \frac{a_{N+2}}{a_{N+1}}, \cdots\cdots, \frac{a_n}{a_{n-1}}}_{n-N} > 2M$

On multiplication, $\dfrac{a_n}{a_N} > (2M)^{n-N}$,

or, $a_n > a_N (2M)^{n-N}$

or, $a_n^{1/n} > \left[\dfrac{a_N}{(2M)^N}\right]^{\frac{1}{n}} 2M$

Now, $\left[\dfrac{a_N}{(2M)^N}\right]^{\frac{1}{n}} \to 1$ as $n \to \infty$.

$\therefore$ For $\varepsilon = \frac{1}{2}$, $\exists$ an integer $N_0$ (and we say suppose $N_0 \geq N$) such that

$$\left(\frac{a_N}{(2M)^N}\right)^{1/n} > \frac{1}{2} \text{ if } n \geq N_0.$$

$\therefore$ $a_n^{1/n} > \left[\dfrac{a_N}{(2M)^N}\right]^{1/n} \cdot 2M > \dfrac{1}{2} \cdot 2M = M$ if $n \geq N_0$.

This means $a_n^{1/n} \to \infty$ as $n \to \infty$.

(2) Let $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = L$, $L \neq 0$.

Then for any $\varepsilon > 0$, $\exists$ an integer $N$ s.t.

$$L - \varepsilon < \frac{a_{n+1}}{a_n} < L + \varepsilon \quad \text{if } n \geq N.$$

$\therefore$ $L - \varepsilon < \dfrac{a_{N+1}}{a_N}, \dfrac{a_{N+2}}{a_{N+1}}, \ldots, \dfrac{a_n}{a_{n-1}} < L + \varepsilon$

On multiplication,

$$(L-\varepsilon)^{n-N} < \frac{a_n}{a_N} < (L+\varepsilon)^{n-N}$$

The first inequality yields

$$(L-\varepsilon)^n < \frac{a_n}{a_N} (L-\varepsilon)^N < \frac{a_n}{a_N} L^N$$

and the second inequality gives

$$(L+\varepsilon)^n > \frac{a_n}{a_N}(L+\varepsilon)^N > \frac{a_n}{a_N} L^N$$

$\therefore$
$$(L-\varepsilon)^n < \frac{a_n}{a_N} L^N < (L+\varepsilon)^n$$

or,
$$L-\varepsilon < \left(\frac{L^N}{a_N}\right)^{\frac{1}{n}} \cdot a_n^{1/n} < L+\varepsilon$$

or,
$$(L-\varepsilon)\left(\frac{a_N}{L^N}\right)^{\frac{1}{n}} < a_n^{1/n} < (L+\varepsilon)\left(\frac{a_N}{L^N}\right)^{\frac{1}{n}}$$

Take limits as $n \to \infty$, and note that $\left(\frac{a_N}{L^N}\right)^{\frac{1}{n}} \to 1$. Then

$$L-\varepsilon \le \lim_{n\to\infty} a_n^{\frac{1}{n}} \le L+\varepsilon.$$

Now take limits as $\varepsilon \to 0$.

$$\therefore \lim_{n\to\infty} a_n^{1/n} = L.$$

If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$, then by writing $a_n = \frac{1}{b_n}$, this case can be reduced to (1) above.

__Exc. 4.__ Show that if $a_n > 0$,

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \le \liminf_{n\to\infty} \sqrt[n]{a_n} \le \limsup_{n\to\infty} \sqrt[n]{a_n} \le \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

__Cauchy's Condensation Test.__

If $a_1 \ge a_2 \ge a_3 \ge \cdots \ge 0$, then the series $\sum_{n=1}^{\infty} a_n$ converges iff the series $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges.

(The second series is called the <u>condensed series</u>.)

<u>Proof</u>: It is sufficient to consider the boundedness of the partial sums. Let

$$S_n = a_1 + a_2 + \cdots + a_n \quad , \quad n \geq 1$$

$$T_k = a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k} \quad , \quad k \geq 0 .$$

Then if $n \leq 2^k$,

$$S_n \leq a_1 + (a_2 + a_3) + \cdots + \underbrace{(a_{2^k} + \cdots + a_{2^{k+1}-1})}_{2^k \text{ terms}}$$

$$< a_1 + 2a_2 + \cdots + 2^k a_{2^k} = T_k .$$

$\therefore$ $T_k$ bounded $\Rightarrow$ $S_n$ bounded.

If $n \geq 2^k$, then

$$S_n \geq a_1 + a_2 + (a_3 + a_4) + \cdots + \underbrace{(a_{2^{k-1}+1} + \cdots + a_{2^k})}_{2^{k-1} \text{ terms}}$$

$$> \frac{1}{2} a_1 + a_2 + 2a_4 + \cdots + 2^{k-1} a_{2^k}$$

$$= \frac{1}{2}\left(a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k}\right) = \frac{1}{2} T_k .$$

$\therefore$ $S_n$ bounded $\Rightarrow$ $T_k$ bounded. $\square$

<u>Ex. 1.</u> $\sum_{n=1}^{\infty} \frac{1}{n^p}$ <u>converges if $p > 1$</u> ; $\sum_{n=1}^{\infty} \frac{1}{n^p}$ <u>diverges if $p \leq 1$.</u>

$$a_n = \frac{1}{n^p} .$$

The condensed series is $\sum_{k=0}^{\infty} 2^k a_{2^k} = \sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^p}$

$$= \sum_{k=0}^{\infty} 2^{(1-p)k}$$ , which is a G.P. series with $r = 2^{1-p}$.

Now, $2^{(1-p)} < 1$ if $1-p<0$, i.e. if $p>1$. Therefore the G.P. series converges if $p>1$, and diverges if $p \leq 1$.

Ex. 2. $\sum_{n=2}^{\infty} \frac{1}{n(\log_e n)^p}$ converges if $p>1$ and diverges if $p \leq 1$.

In particular, the series $\sum_{n=2}^{\infty} \frac{1}{n \log_e n}$, which is known as Abel's series, diverges.

The condensed series is $\sum_{k=1}^{\infty} 2^k \frac{1}{2^k (\log_e 2^k)^p}$

$$= (\log_e 2)^{-p} \sum_{k=1}^{\infty} \frac{1}{k^p}$$

The p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p>1$, and diverges if $p \leq 1$. This establishes the assertion.

Ex. 3. The series $\sum_{n=2}^{\infty} \frac{1}{n \log_e n \log_e(\log_e n)}$ diverges.

The condensed series is

$$\sum_k 2^k \frac{1}{2^k \log_e 2^k \log_e(\log_e 2^k)}$$

$$= \frac{1}{\log_e 2} \sum \frac{1}{k \log_e(\log_e 2^k)}$$

Now, $2 < e$

$\log_e 2 < \log_e e = 1$

$$\log_e 2^k = k \log_e 2 < k$$

$$\log_e(\log_e 2^k) < \log_e k$$

$$k \log_e(\log_e 2^k) < k \log_e k$$

$$\frac{1}{k \log_e k} < \frac{1}{k \log_e(\log_e 2^k)}.$$

Therefore, the condensed series diverges by comparison with Abel's series.

__Remark__. The ratio test will follow from this by taking $b_n = 1 \ \forall \ n$.

$$\therefore \quad c_n = 1 - \frac{a_{n+1}}{a_n}.$$

$\therefore$ if $\lim_{n \to \infty} c_n > 0$, $\Sigma a_n$ converges (by Kummer's test (limit form))

i.e. if $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} < 1$, $\Sigma a_n$ converges.

Again, if $\lim_{n \to \infty} c_n < 0$ and $\Sigma \frac{1}{b_n}$ diverges, $\Sigma a_n$ diverges.

i.e. if $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} > 1$, $\Sigma a_n$ diverges as $\sum_{n=1}^{\infty} \frac{1}{b_n} = \sum_{n=1}^{\infty} 1$ diverges.

## __Raabe's Test.__

Let $\Sigma a_n$ be a series of __positive__ terms.

(1) If $\exists$ a __positive__ number $r$ such that

$$\frac{a_{n+1}}{a_n} \leq 1 - \frac{1}{n} - \frac{r}{n} \quad \text{for } n \geq N,$$

then $\sum a_n$ converges.

(2) if $\frac{a_{n+1}}{a_n} \geq 1 - \frac{1}{n}$ for $n \geq N$, $\sum a_n$ diverges.

Proof: (1) This follows from Kummer's test by taking $b_{n+1} = n$.

( Note that $\sum \frac{1}{b_{n+1}} = \sum \frac{1}{n}$ diverges. This is used for (2).) We need to show that $\exists$ $r > 0 \ni$

$$r \leq c_n = n - 1 - \frac{n a_{n+1}}{a_n} \quad \forall \ n \geq N.$$

Suppose $\exists$ $r > 0 \ni$

$$\frac{a_{n+1}}{a_n} \leq 1 - \frac{1}{n} - \frac{r}{n} \quad \forall \ n \geq N.$$

$$\therefore \quad \frac{r}{n} \leq 1 - \frac{1}{n} - \frac{a_{n+1}}{a_n} \quad \forall \ n \geq N.$$

$$\therefore \quad r \leq n - 1 - \frac{n a_{n+1}}{a_n} \quad \forall \ n \geq N.$$

Thus (1) follows from Kummer's test.

(2) $$\frac{a_{n+1}}{a_n} \geq 1 - \frac{1}{n} \quad \forall \ n \geq N.$$

$$\Rightarrow \quad \frac{n a_{n+1}}{a_n} \geq n - 1 \quad \forall \ n \geq N.$$

$$\Rightarrow \quad n - 1 - \frac{n a_{n+1}}{a_n} \leq 0 \quad \forall \ n \geq N.$$

i.e. $c_n \leq 0 \ \forall \ n \geq N$, taking $b_{n+1} = n$ in Kummer's test.

Also, $\sum \frac{1}{b_{n+1}} = \sum \frac{1}{n}$ diverges.

$\therefore$ $\sum a_n$ diverges by (2) in Kummer's test.

## Another form of Raabe's Test.

Let $\Sigma a_n$ be a series of positive terms.

(1) If there is a number $\rho > 1$ and an integer $N \geq 1$ s.t.

$n\left(1 - \frac{a_{n+1}}{a_n}\right) \geq \rho \quad \forall\ n \geq N$, then $\Sigma a_n$ converges.

(2) If $n\left(1 - \frac{a_{n+1}}{a_n}\right) \leq 1 \quad \forall\ n \geq N$, then $\Sigma a_n$ diverges.

Proof: (1) Take $\rho = 1 + r$, $r > 0$ and apply the previous version of Raabe's test to get

$$n\left(1 - \frac{a_{n+1}}{a_n}\right) \geq 1 + r \quad \forall\ n \geq N.$$

or $$1 - \frac{a_{n+1}}{a_n} \geq \frac{1}{n} + \frac{r}{n} \quad \forall\ n \geq N.$$

or $$\frac{a_{n+1}}{a_n} \leq 1 - \frac{1}{n} - \frac{r}{n} \quad \forall\ n \geq N.$$

$\therefore$ $\Sigma a_n$ is convergent.

(2) If $n\left(1 - \frac{a_{n+1}}{a_n}\right) \leq 1 \ \forall\ n \geq N$, then

$\frac{a_{n+1}}{a_n} \geq 1 - \frac{1}{n} \quad \forall\ n \geq N$, so $\Sigma a_n$ diverges.

## Raabe's Test (Limit Form).

Let $\Sigma a_n$ be a series of positive terms and

$$\lim_{n \to \infty} n\left(1 - \frac{a_{n+1}}{a_n}\right) = L.$$

Then if $L > 1$, the series converges.

If $L < 1$, the series diverges.

Proof: Apply Kummer's test (limit form) taking $b_{n+1} = n$.

Then $\lim_{n\to\infty} n\left(1 - \frac{a_{n+1}}{a_n}\right) \gtrless 1$ according as

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \left(n - 1 - \frac{n\, a_{n+1}}{a_n}\right) \gtrless 0.$$

Note that for the divergence, we also require $\Sigma \frac{1}{n}$ to diverge, which it does.

## Gauss's Test.

Let $\Sigma a_n$ be a series of positive terms. Let there exist an integer $N \geq 1$, a number $s > 1$, and $M > 0 \ni$

$$\frac{a_{n+1}}{a_n} = 1 - \frac{A}{n} + \frac{f(n)}{n^s} \quad \text{for } n \geq N, \quad \text{——— (1)}$$

where $A$ is a constant and $f(n)$ is a function of $n$ such that $|f(n)| < M \;\forall\, n$.

Then if $A > 1$, $\Sigma a_n$ converges
and if $A \leq 1$, $\Sigma a_n$ diverges.

Remark. (1) can be written as

$$\frac{a_{n+1}}{a_n} = 1 - \frac{A}{n} + O\left(\frac{1}{n^s}\right).$$

[ The notation 'Capital Oh' will be explained later]

Proof: Apply Raabe's test.

Observe that