CHAPTER 9

Extrema for Functions of Several Variables

9.1 INTRODUCTION

To optimize something means either to maximize or minimize in some aspects of it. An important application of calculus of functions of several variables is to find the maximum and minimum values of functions and where they occur. Determination of extrema, *i.e.*, maxima or minima, is very important in the study of stability of the equilibrium states of mechanical and physical systems. Lagrange's multiplier method developed by mathematician Lagrange in 1755 is a powerful technique for evaluating extreme values of constrained functions in designing multi-stage rockets in engineering, in economics etc. In this chapter we study the extrema of functions of two and three variables.

9.2 MAXIMA AND MINIMA FOR FUNCTIONS OF TWO VARIABLES

Let $z = f(x, y)$ be a function of independent variables x and y.

Relative maximum: A point (a, b) is called a local or relative maximum point (or simply maximum point) of $f(x, y)$ if there exists a region surrounding the point (a, b) in which $f(x, y) < f(a, b)$ for all points (x, y), except (a, b), in this region.

In otherwords, $f(x, y)$ is said to have a local or relative maximum (or simply maximum) at a point (a, b) if there exist $h > 0$, $k > 0$, however small h, k may be, such that $f(a \pm h_1,\ b \pm k_1) < f(a, b)$, where $0 < h_1 \le h, 0 < k_1 \le k$.

Relative minimum: A point (a, b) is called a local or relative minimum point (or simply minimum point) of $f(x, y)$ if there exists a region surrounding the point (a, b), in which $f(x, y) > f(a, b)$ for all points (x, y), except (a, b) in this region.

In otherwords, $f(x, y)$ has a local or relative minimum (or simply minimum) at a point (a, b) if there exist $h > 0$, $k > 0$, however small h, k may be, such that

$$f(a \pm h_1,\ b \pm k_1) > f(a, b), \text{ where } 0 < h_1 \le h, 0 < k_1 \le k.$$

Note: The value of a function f at an extremum (maximum or minimum) point is known as the extremum (maximum or minimum) value of the function f.

Theorem 1: (Necessary Conditions of Extrema):

If a function $z = f(x, y)$ has a maximum or minimum point at (a, b), then

$$f_x(a,b) = f_y(a,b) = 0, \text{ provided these partial derivatives exist.}$$

Notes: *(i)* The converse of the above theorem is not true in general. Let us consider the function $f(x, y) = x^2y^3$ in support of this statement. Here $f_x = 2xy^3$, $f_y = 3x^2y^2$ and so $f_x(0,0) = f_y(0,0) = 0$. But $f(0, 0)$ is not an extreme value, since there exist no $h > 0$, $k > 0$, such that $f(0 \pm h_1, 0 \pm k_1) - f(0,0) = f(\pm h_1, \pm k_1)$ keeps the same sign for $0 < h_1 \le h$, $0 < k_1 \le k$. For example, $f(\varepsilon, \varepsilon) = \varepsilon^5 > 0$, while $f(\varepsilon, -\varepsilon) = -\varepsilon^5 < 0$ for any small $\varepsilon > 0$.

(ii) A function $f(x, y)$ may have extreme value at (a, b) though $f_x(a, b)$ and $f_y(a, b)$ do not exist. For example let us take $f(x,y) = |x| + |y|$ [*BESUS (B. Arch.) 2013*]

$$\therefore \quad \lim_{h\to 0+} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0+} \frac{h-0}{h} \qquad (\because\ h>0)$$

$$= 1 \qquad (\because\ h \neq 0)$$

and

$$\lim_{h\to 0-} \frac{f(0+h)-f(0,0)}{h} = \lim_{h\to 0-} \frac{-h-0}{h} \qquad (\because\ h<0)$$

$$= -1. \qquad (\because\ h \neq 0)$$

Hence $f_x(0, 0)$ does not exist and also, by symmetry, $f_y(0, 0)$ does not exist. But $f(0, 0) = 0$ is a minimum value of $f(x, y)$. Therefore $f(x, y)$ has minimum at $(0, 0)$ though $f_x(0, 0)$, $f_y(0, 0)$ do not exist.

(iii) For extrema $f_x = f_y = 0$ and hence $df = f_x dy + f_y dy = 0$. Therefore, $df = 0$ may be considered as the necessary condition for the existence of extrema of a function $f(x, y)$.

Theorem 2: (Conditions of Extrema):

Let $z = f(x, y)$ be a continuous function having second order partial derivatives and (a, b) be a point satisfying the equations $f_x = f_y = 0$, *i.e.*, $f_x(a, b) = f_y(a, b) = 0$. If H is defined by $H(x, y) = f_{xx}(x, y)\ f_{yy}(x, y) - \{f_{xy}(x, y)\}^2$, then

(i) $f(a, b)$ is a maximum value of $f(x, y)$ at (a, b) if $H(a, b) > 0$ and $f_{xx}(a, b) < 0$ (or $f_{yy}(a, b) < 0$),

(ii) $f(a, b)$ is a minimum value of $f(x, y)$ at (a, b) if $H(a, b) > 0$ and $f_{xx}(a, b) > 0$ (or $f_{yy}(a, b) > 0$),

(iii) $f(a, b)$ is neither a maximum nor a minimum value of $f(x, y)$ at (a, b) if $H(a, b) < 0$ and

(iv) the case is doubtful and need further investigation if $H(a, b) = 0$.

Saddle Point:

A point (a, b) is said to be a saddle point of a function $f(x, y)$ if it has neither a maximum nor a minimum at (a, b) though $f_x(a, b) = f_y(a, b) = 0$.

Note: Clearly a point (a, b) will be a saddle point of $f(x, y)$ if condition *(iii)* of Theorem 2, *i.e.*, $f_{xx}(a, b) f_{yy}(a, b) - \{f_{xy}(a, b)\}^2 < 0$ is satisfied.

Critical (or Stationary) Point

A point (a, b) is said to be a critical (or stationary) point of a function $f(x, y)$ if $f_x(a, b) = f_y(a, b) = 0$.

Working Rules

Given a function $f(x, y)$, we follow the following steps to find its extrema (*i.e.*, maxima and minima).

Step 1: Solve: $f_x = 0$, $f_y = 0$, to find critical (or stationary) points. Let (a, b) be a critical point.

Step 2: Calculate $H(a, b) = f_{xx}(a, b) f_{yy}(a, b) - \{f_{xy}(a, b)\}^2$.

Step 3: (*A*) If $H(a, b) > 0$, then $f(x, y)$ has

(*i*) maximum at (a, b) if $f_{xx}(a, b) < 0$ or $f_{yy}(a, b) < 0$,

(*ii*) minimum at (a, b) if $f_{xx}(a, b) > 0$ or $f_{yy}(a, b) > 0$.

(*B*) If $H(a, b) < 0$, then $f(x, y)$ has neither a maximum nor a minimum at (a, b). The point (a, b) is called saddle point of $f(x, y)$.

(*C*) If $H(a, b) = 0$, no information is obtained about maximum or minimum of $f(x, y)$ at (a, b) and it needs further investigation.

ILLUSTRATIVE EXAMPLES

Example 1: Find the maxima and minima of the function $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$.

[*BESUS* (*B. Arch.*) 2013]

Solution: Let $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$, so,

$$f_x = \frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 30x + 72 \text{ and } f_y = \frac{\partial f}{\partial y} = 6xy - 30y$$

The critical points are given by

$$f_x = 3x^2 + 3y^2 - 30x + 72 = 0 \quad ...(1)$$

$$f_y = 6y(x-5) = 0 \quad ...(2)$$

From (2), $y = 0$ or $x = 5$

Putting $y = 0$ in (1), we get $3x^2 - 30x + 72 = 0$

or $x^2 - 10x + 24 = 0$

or $x^2 - 6x - 4x + 24 = 0$

or $(x - 6)(x - 4)$. Therefore $x = 6, 4$.

Putting $x = 5$ in (1), we get $75 + 3y^2 - 150 + 72 = 0$

or $y^2 = 1$, therefore $y = \pm 1$.

So, the critical points are $(6, 0)$, $(4, 0)$, $(5, 1)$, $(5, -1)$.

Here $f_{xx} = 6x-30,\ f_{yy} = 6x-30,\ f_{xy} = 6y$

So, $H(x,y) = f_{xx}f_{yy}-\{f_{xy}\}^2 = 36(x-5)^2-36y^2$

$$= 36\{(x-5)^2-y^2\}.$$

(i) At the critical point (6, 0), we have $H(6, 0) = 36 > 0$, $f_{xx}(6, 0) = 6 > 0$. Therefore, (6, 0) is a minimum point of the given function and the minimum value is

$$f(6, 0) = 6^3 + 0 - 15 \times 6^2 - 0 + 72 \times 6 = 108.$$

(ii) At (4, 0), we have $H(4, 0) = 36 > 0$, $f_{xx}(4, 0) = -6 < 0$. So a maximum value of the given function occurs at (4, 0) and the maximum value is $f(4, 0) = 4^3 + 0 - 15 \times 4^2 - 0 + 72 \times 4 = 112$.

(iii) At (5, 1), we have $H(5, 1) = -36 < 0$, so (5, 1) is a saddle point, *i.e.*, the function has neither maximum nor minimum at (5, 1).

(iv) At (5, –1), we have $H(5, -1) = -36 < 0$, so the function has neither maximum nor minimum at (5, –1), *i.e.*, it is a saddle point.

Example 2: Find the maximum and minimum values of the function

$$f(x,y) = x^3+y^3-3axy.$$ (W.B.U.T. 2002, 2008)

Solution: Here $f(x,y) = x^3+y^3-3axy$, so,

$$f_x = \frac{\partial f}{\partial x} = 3x^2-3ay = 3(x^2-ay),\ f_y = \frac{\partial f}{\partial y} = 3y^2-3ax = 3(y^2-ax).$$

The critical points are given by

$$f_x = 0,\quad i.e.,\quad x^2-ay = 0 \qquad ...(1)$$
$$f_y = 0,\quad i.e.,\quad y^2-ax = 0 \qquad ...(2)$$

Subtracting (2) from (1), we get

$$x^2-y^2+a(x-y) = 0, \text{ or } (x-y)(x+y+a) = 0$$

Therefore $x = y$, or $x+y+a = 0$

Putting $y = x$ in (1), we get $x = 0, a$ and hence in this case solutions are (0, 0), (*a*, *a*).

Putting $y = -x-a$ in (1), we get $x^2+ax+a^2 = 0$, which has no real solution. Thus (0, 0), (*a*, *a*) are the only critical points of $f(x, y)$.

Also, $f_{xx} = 6x,\ f_{yy} = 6y,\ f_{xy} = -3a.$

So, $H(x, y) = f_{xx}f_{yy}-\{f_{xy}\}^2 = 36xy-9a^2$

(i) At (0, 0), we have $H(0, 0) = -9a^2 < 0$, so $f(x, y)$ has neither maximum nor minimum at (0, 0), *i.e.*, it is a saddle point.

(ii) At (*a*, *a*), we have $H(a, a) = 36a^2 - 9a^2 = 27a^2 > 0$ and $f_{xx}(a,a) = 6a \gtrless 0$ according as $a \gtrless 0$. Therefore $f(x, y)$ has a maximum when $a < 0$ and the maximum value is $f(a, a) = a^3 + a^3 - 3a^3 = -a^3$.

The function $f(x, y)$ has a minimum when $a > 0$ and the minimum value is $f(a, a) = -a^3$.