Example 6: If $y = \sin (m \sin^{-1} x)$, prove that

(*i*) $(1-x^2)y_2 - xy_1 + m^2 y = 0$

(*ii*) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 - m^2)y_n = 0.$

Solution: Here $y = \sin (m \sin^{-1} x)$

$$\therefore \quad y_1 = \frac{m}{\sqrt{1-x^2}} \cos (m \sin^{-1} x)$$

or $$y_1\sqrt{1-x^2} = m \cos (m \sin^{-1} x) \qquad \text{...(1)}$$

Differentiating both sides of (1) with respect to x, we have

$$\sqrt{1-x^2}\, y_2 - \frac{x}{\sqrt{1-x^2}} y_1 = -\frac{m^2}{\sqrt{1-x^2}} \sin (m \sin^{-1} x)$$

or $$(1-x^2)y_2 - xy_1 + m^2 y = 0 \qquad [\because y = \sin (m \sin^{-1} x)] \text{...(2)}$$

Differentiating both sides of (2) n times with respect to x by Leibnitz's theorem, we get

$$y_{n+2}(1-x^2) + {}^nC_1\, y_{n+1}(-2x) + {}^nC_2 y_n(-2) - \{y_{n+1}x + {}^nC_1 y_n\} + m^2 y_n = 0$$

or $$y_{n+2}(1-x^2) - 2nxy_{n+1} - \frac{n(n-1)}{2!} \cdot 2y_n - \{xy_{n+1} + ny_n\} + m^2 y_n = 0$$

$$\therefore \quad (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 - m^2)y_n = 0.$$

Example 7: (*i*) If $y = (x^2 - 1)^n$, prove that

$$(x^2-1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0$$

(*W.B.U.T. 2006,2009, 2010*)

(*ii*) If $y = e^{\tan^{-1} x}$, then show that

$$(1+x^2)y_{n+2} + (2nx + 2x - 1)y_{n+1} + n(n+1)y_n = 0 \qquad \text{(W.B.U.T. 2012)}$$

Solution: Here $y = (x^2 - 1)^n$

$$\therefore \quad y_1 = n(x^2-1)^{n-1} \cdot 2x$$

or $$y_1(x^2-1) = 2nx(x^2-1)^n$$

or $$y_1(x^2-1) = 2nxy \qquad [\because y = (x^2-1)^n] \text{...(1)}$$

Differentiating both sides of (1) with respect to x, we get

$$y_2(x^2-1) + y_1 2x = 2n(y + xy_1)$$

or $$(x^2-1)\, y_2 + 2(1-n)\, xy_1 - 2ny = 0 \qquad \text{...(2)}$$

Differentiating both sides of (2) n times with respect to x by Leibnitz's theorem, we get

$$y_{n+2}(x^2-1) + {}^nC_1\, y_{n+1}(2x) + {}^nC_2 y_n(2) + 2(1-n)\{y_{n+1}x + {}^nC_1 y_n\} - 2ny_n = 0$$

or $$(x^2-1)\,y_{n+2}+2nx\,y_{n+1}+\frac{n\,(n-1)}{2\,!}\cdot 2\,y_n+2\,(1-n)\{xy_{n+1}+ny_n\}-2ny_n=0$$

or $$(x^2-1)\,y_{n+2}+2xy_{n+1}-n\,(n+1)\,y_n=0.$$

(*ii*) Given $y=e^{\tan^{-1}x}$

$\therefore$ $$y_1=\frac{e^{\tan^{-1}x}}{1+x^2}$$

$\Rightarrow$ $$(1+x^2)\,y_1=y \qquad \text{...(1)}$$

Differentiating both sides of (1) with respect to x, we get

$$(1+x^2)\,y_2+2xy_1=y_1$$

or $$(1+x^2)\,y_2+(2x-1)y_1=0 \qquad \text{...(2)}$$

Differentiating both sides of (2) n times with respect to x by Leibnitz's theorem, we get

$$(1+x^2)y_{n+2}+{}^nC_1(2x)y_{n+1}+{}^nC_2(2)y_n+(2x-1)y_{n+1}+{}^nC_1(2)y_n=0$$

or $$(1+x^2)y_{n+2}+2nxy_{n+1}+n(n-1)y_n+(2x-1)y_{n+1}+2ny_n=0$$

$\therefore$ $$(1+x^2)y_{n+2}+(2nx+2x-1)y_{n+1}+n(n+1)y_n=0$$

Example 8: Show that

$$\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right)=(-1)^n\,\frac{n\,!}{x^{n+1}}\left(\log x-1-\frac{1}{2}-\frac{1}{3}-\ldots-\frac{1}{n}\right)$$ (*W.B.U.T. 2003, 2008*)

Solution: Let

$$y=\frac{\log x}{x}=uv,\text{ where } u=\frac{1}{x}\text{ and } v=\log x$$

Then $$u_n=\frac{(-1)^n\,n\,!}{x^{n+1}}\text{ and } v_n=\frac{(-1)^{n-1}(n-1)\,!}{x^n}$$

Therefore by Leibnitz's theorem

$$\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right)=(uv)_n=u_nv+{}^nC_1\,u_{n-1}v_1+{}^nC_2\,u_{n-2}v_2+{}^nC_3\,u_{n-3}v_3+\ldots+{}^nC_n\,uv_n$$

$$=\frac{(-1)^n\,n\,!}{x^{n+1}}\log x+\frac{n\,(-1)^{n-1}(n-1)\,!}{x^n}\cdot\frac{1}{x}+\frac{n\,(n-1)}{2\,!}\,\frac{(-1)^{n-2}(n-2)\,!}{x^{n-1}}\cdot\left(-\frac{1}{x^2}\right)$$

$$+\frac{n\,(n-1)\,(n-2)}{3\,!}\,\frac{(-1)^{n-3}(n-3)\,!}{x^{n-2}}\cdot\frac{(-1)\,(-2)}{x^3}+\ldots+\frac{1}{x}\cdot\frac{(-1)^{n-1}(n-1)\,!}{x^n}$$

$$=(-1)^n\,\frac{n\,!}{x^{n+1}}\left(\log x-1-\frac{1}{2}-\frac{1}{3}-\ldots-\frac{1}{n}\right).$$

Example 9: If $f(x) = \tan x$ and n is a positive integer, prove with the help of Leibnitz's theorem, that

$$f^n(0) - {}^nC_2\, f^{n-2}(0) + {}^nC_4\, f^{n-4}(0) - \ldots = \sin\left(\frac{n\pi}{2}\right). \qquad (W.B.U.T.\ 2001)$$

Solution: Here $f(x) = \tan x$, or, $f(x) \cos x = \sin x$ ...(1)

Differentiating both sides of (1) n times with respect to x by Leibnitz's theorem, we get:

$$f^n(x)\cos x + {}^nC_1\, f^{n-1}(x)\,(-\sin x) + {}^nC_2\, f^{n-2}(x)\,(-\cos x)$$

$$+ {}^nC_3\, f^{n-3}(x)\sin x + {}^nC_4\, f^{n-4}(x)\cos x + \ldots = \sin\left(\frac{n\pi}{2} + x\right)$$

Putting $x = 0$, we have

$$f^n(0) - {}^nC_2\, f^{n-2}(0) + {}^nC_4\, f^{n-4}(0) - \ldots = \sin\left(\frac{n\pi}{2}\right).$$

Example 10: If $f(x) = x^n$, prove that $f(1) + \dfrac{f'(1)}{1!} + \dfrac{f''(1)}{2!} + \dfrac{f'''(1)}{3!} + \ldots + \dfrac{f^n(1)}{n!} = 2^n$.

Solution: Here $f(x) = x^n$, therefore,

$$f'(x) = nx^{n-1},\ f''(x) = n(n-1)x^{n-2},\ f'''(x) = n(n-1)(n-2)x^{n-3}, \ldots,$$

$$f^n(x) = n(n-1)(n-2)\ldots 3.2.1 = n!$$

Putting $x = 1$, we have

$$f(1) = 1,\ \frac{f'(1)}{1!} = n,\ \frac{f''(1)}{2!} = \frac{n(n-1)}{2!},$$

$$\frac{f'''(1)}{3!} = \frac{n(n-1)(n-2)}{3!}, \ldots, \frac{f^n(1)}{n!} = 1$$

$$\therefore \quad f(1) + \frac{f'(1)}{1!} + \frac{f''(1)}{2!} + \frac{f'''(1)}{3!} + \ldots + \frac{f^n(1)}{n!}$$

$$= 1 + n + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!} + \ldots + 1$$

$$= {}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_n$$

$$= (1 + 1)^n = 2^n.$$

Example 11: If $x + y = 1$, prove that the nth derivative of $x^n y^n$ is

$$n!\{y^n - ({}^nC_1)^2 y^{n-1}x + ({}^nC_2)^2 y^{n-2}x^2 - ({}^nC_3)^2 y^{n-3}x^3 + \ldots + (-1)^n x^n\}.$$

(W.B.U.T. 2002, BESUS 2013)

Solution: Let $\quad u = x^n y^n = x^n(1-x)^n \qquad (\because x + y = 1)$

Therefore by Leibnitz's theorem, we get

$$u_n = n!(1-x)^n + {}^nC_1 \frac{n!}{1!} x \cdot n(1-x)^{n-1}(-1) + {}^nC_2 \frac{n!}{2!} x^2 \, n(n-1)(1-x)^{n-2}(-1)^2$$

$$+ {}^nC_3 \frac{n!}{3!} x^3 n(n-1)(n-2)(1-x)^{n-3}(-1)^3 + \ldots + x^n \, n!(-1)^n$$

$$\left[\because \frac{d^r}{dx^r}(x^n) = \frac{n!}{(n-r)!} x^{n-r}, r < n \atop = n!, r = n\right]$$

$$= n! \left\{ y^n - {}^nC_1 \frac{n}{1!} y^{n-1} x + {}^nC_2 \frac{n(n-1)}{2!} y^{n-2} x^2 - {}^nC_3 \frac{n(n-1)(n-2)}{3!} y^{n-3} x^3 + \ldots + (-1)^n x^n \right\}$$

$$(\because y = 1 - x)$$

$$= n!\{y^n - ({}^nC_1)^2 y^{n-1} x + ({}^nC_2)^2 y^{n-2} x^2 - ({}^nC_3)^2 y^{n-3} x^3 + \ldots + (-1)^n x^n\}.$$

Example 12: If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$, prove that

$$(x^2 - 1)\, y_{n+2} + (2n+1)\, x y_{n+1} + (n^2 - m^2)\, y_n = 0. \qquad \textit{(W.B.U.T. 2011)}$$

Solution: Here

$$y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x, \quad \text{or} \quad \left(y^{\frac{1}{m}}\right)^2 - 2xy^{\frac{1}{m}} + 1 = 0$$

$$\therefore \qquad y^{\frac{1}{m}} = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}$$

or $$y = \left(x \pm \sqrt{x^2 - 1}\right)^m$$

$$\therefore \qquad y_1 = m\left(x \pm \sqrt{x^2 - 1}\right)^{m-1} \left\{1 \pm \frac{1}{2} \cdot \frac{2x}{\sqrt{x^2 - 1}}\right\}$$

$$= \pm m \frac{(x \pm \sqrt{x^2 - 1})^m}{\sqrt{x^2 - 1}} = \pm \frac{my}{\sqrt{x^2 - 1}}$$

or $$y_1 \sqrt{x^2 - 1} = \pm my \qquad \ldots(1)$$

Differentiating both sides with respect to x, we get

$$y_1 \cdot \frac{1}{2} \frac{2x}{\sqrt{x^2 - 1}} + y_2 \sqrt{x^2 - 1} = \pm m y_1$$

or $$xy_1 + (x^2-1)y_2 = \pm m y_1\sqrt{x^2-1}$$

$$\therefore \quad (x^2-1)y_2 + xy_1 - m^2 y = 0 \qquad \text{[by (1)]}$$

Now differentiating both sides n times with respect to x by Leibnitz's theorem, we get

$$y_{n+2}(x^2-1) + {}^nC_1\, y_{n+1}(2x) + {}^nC_2\, y_n(2) + \{y_{n+1}x + {}^nC_1\, y_n\} - m^2 y_n = 0$$

or $$y_{n+2}(x^2-1) + 2n\, xy_{n+1} + \frac{n(n-1)}{2!}\cdot 2y_n + (xy_{n+1} + ny_n) - m^2 y_n = 0$$

$$\therefore \quad (x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0.$$

Example 13: If $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$, prove that

$$x^2 y_{n+2} + (2n+1)xy_{n+1} + 2n^2 y_n = 0.$$

Solution: Here

$$\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$$

$$\therefore \quad y = b\cos\left\{\log\left(\frac{x}{n}\right)^n\right\} = b\cos\left\{n\log\left(\frac{x}{n}\right)\right\}$$

$$\therefore \quad y_1 = -\frac{bn}{x}\sin\left\{n\log\left(\frac{x}{n}\right)\right\}$$

$$\therefore \quad xy_1 = -bn\sin\left\{n\log\left(\frac{x}{n}\right)\right\}$$

Differentiating both sides with respect to x, we get

$$xy_2 + y_1 = -bn\cdot\frac{n}{x}\cos\left\{n\log\left(\frac{x}{n}\right)\right\}$$

or $$x^2y_2 + xy_1 + n^2 y = 0 \qquad \left[\because y = b\cos\left\{n\log\left(\frac{x}{n}\right)\right\}\right] \ldots(1)$$

Differentiating both sides of (1) n times with respect to x by Leibnitz's theorem, we get

$$y_{n+2}\, x^2 + {}^nC_1\, y_{n+1}(2x) + {}^nC_2\, y_n(2) + \{y_{n+1}x + {}^nC_1\, y_n\} + n^2 y_n = 0$$

or $$x^2 y_{n+2} + 2n\, xy_{n+1} + n(n-1)y_n + \{xy_{n+1} + ny_n\} + n^2 y_n = 0$$

$$\therefore \quad x^2 y_{n+2} + (2n+1)\, xy_{n+1} + 2n^2 y_n = 0.$$

Example 14: If $y = 2\cos x\,(\sin x - \cos x)$ show that $(y_{10})_0 = 2^{10}$,

where $(y_{10})_0$ means the value of 10th derivative of y when $x = 0$. *(W.B.U.T. 2001)*

Solution: Here

$$y = 2\cos x\,(\sin x - \cos x) = 2\sin x \cos x - 2\cos^2 x$$
$$= \sin 2x - \cos 2x - 1$$

$$\therefore \quad y_{10} = 2^{10}\sin\left(10\cdot\frac{\pi}{2} + 2x\right) - 2^{10}\cos\left(10\cdot\frac{\pi}{2} + 2x\right)$$

$$\therefore \quad (y_{10})_0 = 2^{10}\sin 5\pi - 2^{10}\cos 5\pi = 2^{10}.$$

Example 15: If $y = \left[x + \sqrt{1+x^2}\right]^m$, find $(y_n)_0$, where $(y_n)_0$ means the value of nth derivative of y when $x = 0$.

Solution: Here

$$y = \left[x + \sqrt{1+x^2}\right]^m \qquad \text{...(1)}$$

$$\therefore \quad y_1 = m\left[x + \sqrt{1+x^2}\right]^{m-1}\left\{1 + \frac{2x}{2\sqrt{1+x^2}}\right\}$$

$$= \frac{m\left[x + \sqrt{1+x^2}\right]^m}{\sqrt{1+x^2}} \qquad \text{...(2)}$$

or $$y_1\sqrt{1+x^2} = my \qquad \text{[by (1)]} \qquad \text{...(3)}$$

Differentiating both sides with respect to x, we have

$$y_2\sqrt{1+x^2} + y_1\cdot\frac{2x}{2\sqrt{1+x^2}} = my_1$$

or $$y_2(1+x^2) + xy_1 = my_1\sqrt{1+x^2}$$

$$\therefore \quad y_2(1+x^2) + xy_1 - m^2 y = 0 \qquad \text{[by (3)]} \qquad \text{...(4)}$$

Differentiating both sides of (4) n times with respect to x by Leibnitz's theorem, we get

$$y_{n+2}(1+x^2) + {}^nC_1\, y_{n+1}(2x) + {}^nC_2\, y_n(2) + \{y_{n+1}x + {}^nC_1\, y_n\} - m^2 y_n = 0$$

or $$(1+x^2)y_{n+2} + 2nx\, y_{n+1} + n(n-1)y_n + \{xy_{n+1} + ny_n\} - m^2 y_n = 0$$

or $$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$$

$$\therefore \quad (y_{n+2})_0 = (m^2 - n^2)(y_n)_0$$

$$\therefore \quad (y_n)_0 = \{m^2 - (n-2)^2\}(y_{n-2})_0. \qquad \text{...(5)}$$

Case I: When n is even

$$(y_n)_0 = \{m^2-(n-2)^2\}(y_{n-2})_0 = \{m^2-(n-2)^2\}\{m^2-(n-4)^2\}(y_{n-4})_0$$

$$= \ldots = \{m^2-(n-2)^2\}\{m^2-(n-4)^2\}\ldots(m^2-2^2)(y_2)_0$$

$$= \{m^2-(n-2)^2\}\{m^2-(n-4)^2\}\ldots(m^2-2^2)\,m^2(y)_0 \qquad \text{[by (4)]}$$

$$= \{m^2-(n-2)^2\}\{m^2-(n-4)^2\}\ldots(m^2-2^2)m^2.$$

$[\because (y)_0 = 1 \text{ by } (1)]$

Case II: When n is odd

$$(y_n)_0 = \{m^2-(n-2)^2\}(y_{n-2})_0 = \{m^2-(n-2)^2\}\{m^2-(n-4)^2\}(y_{n-4})_0$$

$$= \ldots = \{m^2-(n-2)^2\}\{m^2-(n-4)^2\}\ldots(m^2-3^2)(y_3)_0$$

$$= \{m^2-(n-2)^2\}\{m^2-(n-4)^2\}\ldots(m^2-3^2)(m^2-1^2)(y_1)_0$$

$$= \{m^2-(n-2)^2\}\{m^2-(n-4)^2\}\ldots(m^2-3^2)(m^2-1^2)m.$$

$[\because (y_1)_0 = m \text{ by } (2)]$

Example 16: If $y = \tan^{-1}x$, then prove that

$$(1+x^2)y_{n+1} + 2nx\, y_n + n(n-1)y_{n-1} = 0.$$

Find also the value of $(y_n)_0$, where $(y_n)_0$ means the value of nth derivative of y when $x = 0$.

(*W.B.U.T. 2003*)

Solution: Given $y = \tan^{-1}x$, therefore $y_1 = \dfrac{1}{1+x^2}$.

Hence $\quad y_1(1+x^2) = 1 \qquad \ldots(1)$

Differentiating both sides of (1) n times with respect to x by Leibnitz's theorem, we get

$$y_{n+1}(1+x^2) + {}^nC_1\, y_n(2x) + {}^nC_2\, y_{n-1}(2) = 0$$

$$\therefore \quad (1+x^2)y_{n+1} + 2nx\, y_n + n(n-1)\, y_{n-1} = 0$$

$$\therefore \quad (y_{n+1})_0 = -n(n-1)(y_{n-1})_0 \qquad \ldots(2)$$

From the given expression and from (1),

$$(y)_0 = 0,\ (y_1)_0 = 1$$

From (2), putting successively $n = 1, 2, 3, 4, 5, \ldots$

$$(y_2)_0 = 0,\ (y_3)_0 = -2.1,\ (y_4)_0 = -3.2\,(y_2)_0 = 0,$$

$$(y_5)_0 = -4.3\,(y_3)_0 = (-4.3).(-2.1), \ldots$$

Hence, we conclude that

$$(y_n)_0 = \begin{cases} 0, & n \text{ even} \\ (-1)^{\frac{1}{2}(n-1)}.(n-1)!, & n \text{ odd.} \end{cases}$$

Example 17: If $y = \sin^{-1}x$, prove that

(*i*) $(1-x^2)y_2 - xy_1 = 0$,

(*ii*) $(1-x^2)\, y_{n+2} - (2n+1)x\, y_{n+1} - n^2 y_n = 0$.

Also find y_n when $x = 0$.

Solution: Here

$$y = \sin^{-1} x$$

$$\therefore \qquad y_1 = \frac{1}{\sqrt{1-x^2}}, \quad \text{or} \quad \sqrt{1-x^2}\, y_1 = 1 \qquad \ldots(1)$$

Differentiating both sides of (1) with respect to x, we get

$$y_2\sqrt{1-x^2} + y_1 \frac{1}{2\sqrt{1-x^2}}\cdot(-2x) = 0$$

$$\text{or} \qquad y_2\,(1-x^2) - xy_1 = 0 \qquad \ldots(2)$$

Differentiating both sides of (2) n times with respect to x by Leibnitz's theorem, we get

$$y_{n+2}(1-x^2) + {}^nC_1\, y_{n+1}(-2x) + {}^nC_2\, y_n(-2) - \{y_{n+1}x + {}^nC_1\, y_n\} = 0$$

$$\text{or} \qquad (1-x^2)y_{n+2} - 2nx\, y_{n+1} - n(n-1)y_n - \{xy_{n+1} + ny_n\} = 0$$

$$\text{or} \qquad (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$$

$$\therefore \qquad (y_{n+2})_0 = n^2 (y_n)_0 \qquad \ldots(3)$$

From the given expression and from (1), (2),

$$(y)_0 = 0,\ (y_1)_0 = 1,\ (y_2)_0 = 0$$

From (3), putting successively $n = 1, 2, 3, 4, 5, \ldots$

$$(y_3)_0 = 1^2\cdot(y_1)_0 = 1^2,\ (y_4)_0 = 2^2(y_2)_0 = 0,$$

$$(y_5)_0 = 3^2\cdot(y_3)_0 = 3^2\cdot 1^2,\ (y_6)_0 = 4^2(y_4)_0 = 0,$$

$$(y_7)_0 = 5^2\cdot(y_5)_0 = 5^2\cdot 3^2\cdot 1^2,$$

Therefore, we conclude that

$$(y_n)_0 = \begin{cases} 0, & \text{when } n \text{ is even} \\ 1, & \text{when } n = 1 \\ (n-2)^2(n-4)^2\ldots 3^2\cdot 1^2, & \text{when } n (\geq 3) \text{ is odd.} \end{cases}$$

Here $(y_n)_0$ means the value of nth derivative of y when $x = 0$.

Example 18: If $y = e^{m\cos^{-1}x}$ prove that

$$(1-x^2)y_{n+2} - x(2n+1)y_{n+1} - (n^2 + m^2)y_n = 0$$

and hence find the value of $(y_n)_0$, where $(y_n)_0$ means the value of nth derivative of y when $x = 0$.

Solution: Here $y = e^{m\cos^{-1}x}$

$\therefore$ $$y_1 = e^{m\cos^{-1}x}\,\frac{(-m)}{\sqrt{1-x^2}} = -\frac{my}{\sqrt{1-x^2}} \qquad \text{...(1)}$$

or $$y_1\sqrt{1-x^2} + my = 0$$

Differentiating both sides with respect to x, we get

$$y_2\sqrt{1-x^2} + y_1\frac{1}{2\sqrt{1-x^2}}\cdot(-2x) + my_1 = 0$$

or $$y_2(1-x^2) - xy_1 - m^2y = 0 \qquad \text{[by (1)]} \qquad \text{...(2)}$$

Differentiating both sides of (2) n times with respect to x by Leibnitz's theorem, we get

$$y_{n+2}(1-x^2) + {}^nC_1\,y_{n+1}(-2x) + {}^nC_2\,y_n(-2) - \{y_{n+1}x + {}^nC_1\,y_n\} - m^2y_n = 0$$

or $$(1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - \{xy_{n+1} + ny_n\} - m^2y_n = 0$$

or $$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$$

$\therefore$ $$(y_{n+2})_0 = (n^2+m^2)\,(y_n)_0$$

Hence $$(y_n)_0 = \{(n-2)^2+m^2\}\,(y_{n-2})_0 \qquad \text{...(3)}$$

Case I: When n is even.

$$(y_n)_0 = \{(n-2)^2+m^2\}\{(n-4)^2+m^2\}\ldots(4^2+m^2)\,(2^2+m^2)\,(y_2)_0$$

$$= \{(n-2)^2+m^2\}\{(n-4)^2+m^2\}\ldots(4^2+m^2)\,(2^2+m^2)\,m^2(y_0)$$

[by (2)]

$$= \{(n-2)^2+m^2\}\{(n-4)^2+m^2\}\ldots(4^2+m^2)\,(2^2+m^2)\,m^2e^{m\pi/2}$$

$[\because (y)_0 = e^{m\pi/2}]$

Case II: When n is odd.

$$(y_n)_0 = \{(n-2)^2+m^2\}\{(n-4)^2+m^2\}\ldots(3^2+m^2)\,(1^2+m^2)\,(y_1)_0$$

$$= \{(n-2)^2+m^2\}\{(n-4)^2+m^2\}\ldots(3^2+m^2)\,(1^2+m^2)\,(-me^{m\pi/2})$$

[by (1), $(y_1)_0 = -me^{m\pi/2}$]

$$= -\{(n-2)^2+m^2\}\{(n-4)^2+m^2\}\ldots(3^2+m^2)\,(1^2+m^2)\,me^{m\pi/2}.$$

Example 19: If $x = \cosh\left(\frac{1}{m}\log y\right)$, where $\cosh\theta = \frac{1}{2}(e^{\theta} + e^{-\theta})$, prove that

(*i*) $(x^2-1)y_2 + xy_1 - m^2 y = 0$,

(*ii*) $(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$.

Solution: $$\cosh^{-1}x = \frac{1}{m}\log y$$

Differentiating both sides with respect to x, we get

$$\frac{1}{\sqrt{x^2-1}} = \frac{1}{m}\cdot\frac{y_1}{y}, \quad \text{or} \quad y_1\sqrt{x^2-1} - my = 0 \qquad ...(1)$$

Again differentiating both sides of (1) with respect to x, we get

$$y_2\sqrt{x^2-1} + y_1\frac{2x}{2\sqrt{x^2-1}} - my_1 = 0$$

or $$y_2(x^2-1) + xy_1 - m^2 y = 0 \qquad \text{[using (1)]} \qquad ...(2)$$

Differentiating both sides of (2) n times w.r.t. x by Leibnitz's theorem,

$$y_{n+2}(x^2-1) + {}^nC_1\, y_{n+1}(2x) + {}^nC_2 y_n(2) + \{y_{n+1}x + {}^nC_1 y_n\} - m^2 y_n = 0$$

or $$(x^2-1)y_{n+2} + 2nxy_{n+1} + n(n-1)y_n + \{xy_{n+1} + ny_n\} - m^2 y_n = 0.$$

Hence the result.

Example 20: If $y = \cos(m\sin^{-1}x)$, then prove that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$$

Find y_n for $x = 0$. [(*W.B.U.T. 2004, 2013*); *B. Arch.* (*BESUS*), *2013*]

Solution: Here $y = \cos(m\sin^{-1}x)$

$$\therefore \quad y_1 = -\frac{m}{\sqrt{1-x^2}}\sin(m\sin^{-1}x), \quad \text{or} \quad y_1\sqrt{1-x^2} + m\sin(m\sin^{-1}x) = 0 \qquad ...(1)$$

Differentiating both sides of (1) with respect to x, we get

$$y_2\sqrt{1-x^2} + y_1\frac{1}{2\sqrt{1-x^2}}\cdot(-2x) + \frac{m^2}{\sqrt{1-x^2}}\cos(m\sin^{-1}x) = 0$$

or $$(1-x^2)y_2 - xy_1 + m^2 y = 0 \qquad [\because y = \cos(m\sin^{-1}x)] ...(2)$$

Differentiating both sides of (2) n times with respect to x by Leibnitz's theorem, we get

$$y_{n+2}(1-x^2) + {}^nC_1\, y_{n+1}(-2x) + {}^nC_2\, y_n(-2) - \{y_{n+1}x + {}^nC_1\, y_n\} + m^2 y_n = 0$$

or $$(1-x^2)y_{n+2}-2nx\,y_{n+1}-n(n-1)y_n-(xy_{n+1}+ny_n)+m^2y_n=0$$

$\therefore$ $$(1-x^2)y_{n+2}-(2n+1)\,xy_{n+1}+(m^2-n^2)y_n=0$$

$\therefore$ $$(y_{n+2})_0=(n^2-m^2)(y_n)_0 \qquad ...(3)$$

Putting $n = 1, 3, 5, \ldots$ successively in (3), we get

$(y_3)_0 = (1^2 - m^2)(y_1)_0 = 0$ [$\because (y_1)_0 = 0$, by (1)]

$(y_5)_0 = (3^2 - m^2)(y_3)_0 = 0$, $(y_7)_0 = (5^2 - m^2)(y_5)_0 = 0$

$\therefore$ $(y_n)_0 = 0$, when n is odd.

From (3), if n is even,

$$\begin{aligned}(y_n)_0 &= \{(n-2)^2-m^2\}(y_{n-2})_0\\ &= \{(n-2)^2-m^2\}\{(n-4)^2-m^2\}\ldots(4^2-m^2)(2^2-m^2)(y_2)_0\\ &= \{(n-2)^2-m^2\}\{(n-4)^2-m^2\}\ldots(4^2-m^2)(2^2-m^2)(-m^2)\end{aligned}$$

[from (2), $(y_2)_0 = -m^2$]

$\therefore$ $$(y_n)_0=-m^2(2^2-m^2)(4^2-m^2)\ldots\{(n-4)^2-m^2\}\{(n-2)^2-m^2\}$$

when n is even.

MULTIPLE CHOICE QUESTIONS

1. If $y = x^4$ then $y_4 =$
   (*a*) 4! (*b*) 5! (*c*) 0 (*d*) none of these.
2. If $y = x^n$, then $y_{n-2} =$
   (*a*) $\frac{1}{2}n!x$ (*b*) $\frac{1}{2}n!x^2$ (*c*) $n(n-1)x^2$ (*d*) $n!x^2$.
3. If $y = 10^{2x}$, then $y_n =$
   (*a*) $(10^{2x})^n$ (*b*) $(\log 10)^n 10^{2x}$ (*c*) $2^n(\log 10)^n\,10^{2x}$ (*d*) $2^n\,10^{2x}\log 10$.
4. The *n*th derivative of $(ax + b)^{10}$ when $n > 10$ is
   (*a*) a^{10} (*b*) $10!\,a^{10}$ (*c*) 0 (*d*) 10!.
   (*W.B.U.T.* 2007, 2011)
5. If $y = ax^n + b$, then $y_n =$
   (*a*) $n!$ (*b*) $n!\,a$ (*c*) 0 (*d*) none of these.
6. If $y = e^{-2x}\sin 3x$, then $y_5 =$
   (*a*) $13^{\frac{5}{3}}e^{-2x}\sin\left(3x-5\tan^{-1}\frac{3}{2}\right)$ (*b*) $13^{\frac{5}{2}}e^{-2x}\sin\left(3x+5\tan^{-1}\frac{3}{2}\right)$
   (*c*) $13^{\frac{5}{2}}e^{-2x}\sin\left(3x-5\tan^{-1}\frac{2}{3}\right)$ (*d*) $13^{\frac{5}{2}}e^{-2x}\sin\left(3x+5\tan^{-1}\frac{2}{3}\right)$.