Sheet 5.1

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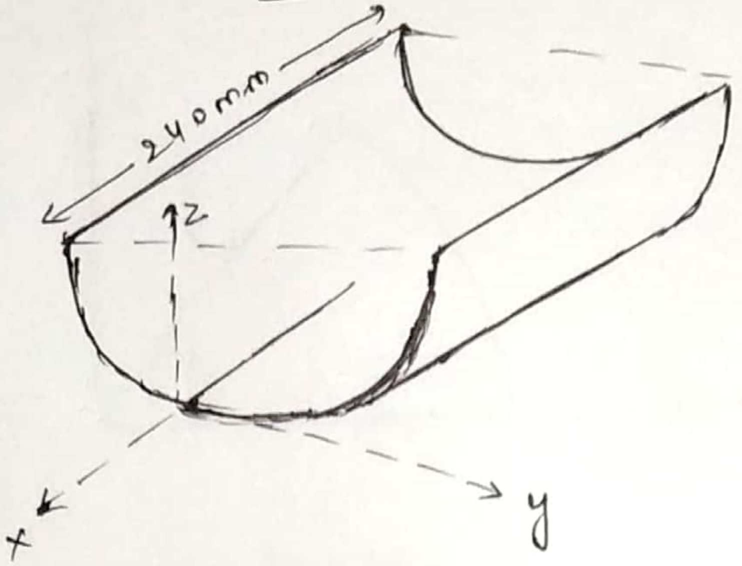
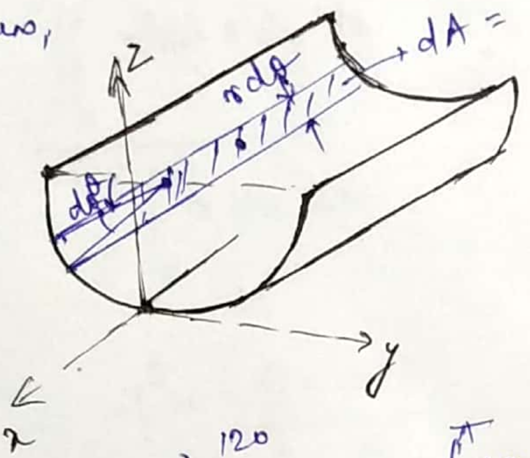
2020CEB073

(1)



From symmetry about x-z plane we can say that $\bar{y} = 0$.

Now, $dA = 240\, r\, d\theta$.



$$\bar{x} = \frac{\int x_c \, dA}{\int dA} = \frac{\int_0^{\pi} 120 \times 240 r \, d\theta}{\int_0^{\pi} 240 r \, d\theta} = 120 \text{ mm}.$$

$$\bar{z} = \frac{\int z_c \, dA}{\int dA} = \frac{\int_0^{\pi} (r - r\sin\theta) \times 240 r \, d\theta}{240 r \int_0^{\pi} d\theta}$$

$$= r \frac{[\pi - 2]}{\pi} = 43.63 \text{ mm}$$

(with $r = 120$)

So, $\bar{x} = 120$ mm, $\bar{y} = 0$ mm, $\bar{z} = 43.63$ mm Ans

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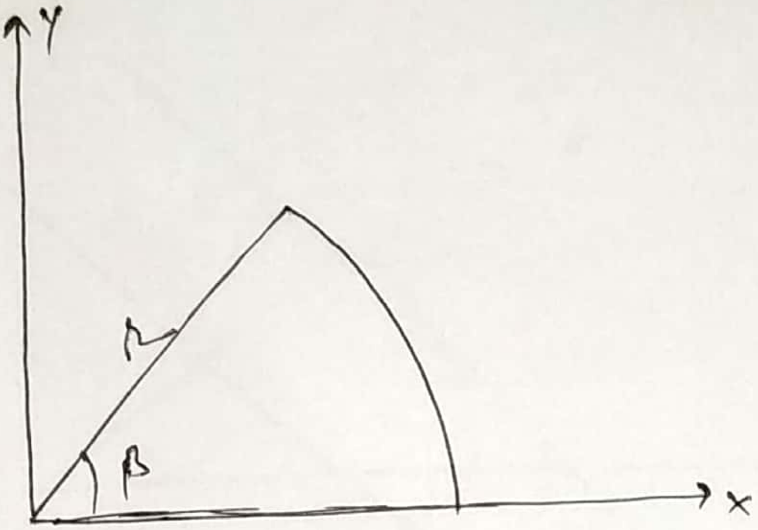
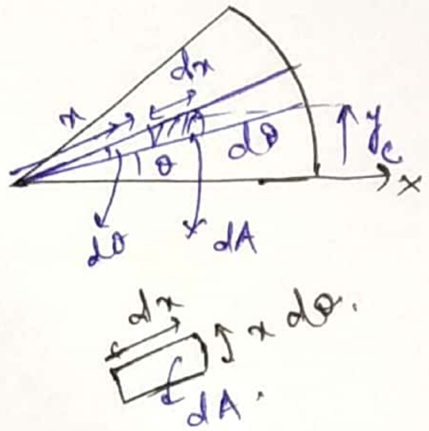


$$dA = x\,dx\,d\theta.$$



$$\bar{y} = \frac{\iint y_c\,dA}{\iint dA}$$

$$\Rightarrow \bar{y} = \frac{\int_0^r \int_0^\beta x^2 \sin\theta\,dx\,d\theta}{\int_0^r \int_0^\beta x\,dx\,d\theta}$$

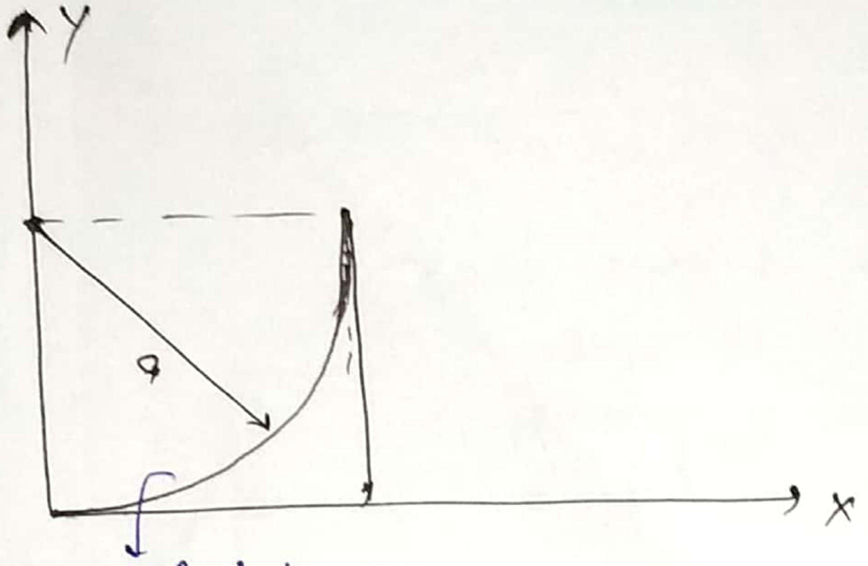
$$\Rightarrow \bar{y} = \frac{r^3}{3} \times \frac{(1-\cos\beta)}{\frac{r^2}{2}\times\beta} = \frac{2r}{3\beta}(1-\cos\beta)$$

Now; $$\bar{x} = \frac{\int_0^r \int_0^\beta \underbrace{x\cos\theta}_{x_c}\, x\,dx\,d\theta}{\int_0^r \int_0^\beta x\,dx\,d\theta} = \frac{\frac{r^3}{3}\times |\sin\theta|_0^\beta}{\frac{r^2}{2}\times\beta}$$

$$= \frac{2r}{3\beta}\sin\beta.$$

So, $$\boxed{\bar{x} = \frac{2r\sin\beta}{3\beta}\ ;\ \bar{y} = \frac{2r}{3\beta}(1-\cos\beta)}$$

Q3



circle's arc.

centre $(0, a)$

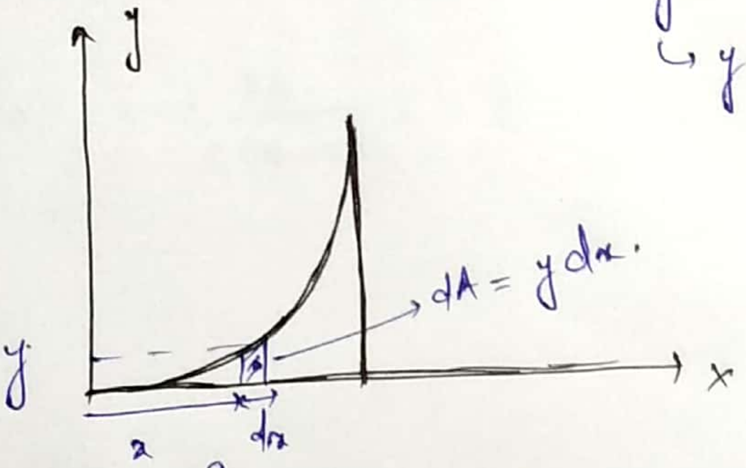
$r = a$

Eqⁿ $x^2 + y^2 - 2ay + a^2 = a^2 \Rightarrow y^2 - 2ay + x^2 = 0$

$\Rightarrow y = \dfrac{2a \pm 2\sqrt{a^2 - x^2}}{2}$

$\Rightarrow y = a - \sqrt{a^2 - x^2}$.

$\hookrightarrow y(x)$



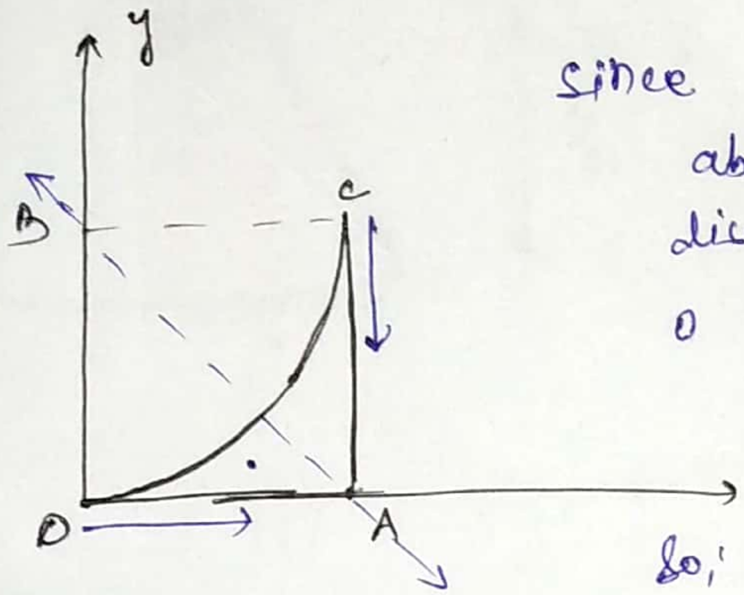
$x_c = x$

$y_c = \dfrac{y}{2} = \dfrac{a - \sqrt{a^2 - x^2}}{2}$.

$$\bar{x} = \frac{\int_0^a x\,(a - \sqrt{a^2 - x^2})\,dx}{\int_0^a (a - \sqrt{a^2 - x^2})\,dx} = \frac{\frac{a^3}{2} + \frac{1}{2} \times \int_0^a -2x\sqrt{a^2 - x^2}\,dx}{a^2 - \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a}$$

$$\Rightarrow \bar{x} = \frac{\frac{a^3}{2} + \frac{1}{2} \times \frac{2}{3}\left(a^2 - x^2\right)^{3/2}\Big|_0^a}{a^2\left(\frac{4 - \pi}{4}\right)} = \frac{2a}{3(4 - \pi)}.$$

So; $\bar{x} = \frac{2a}{3(4-\pi)}$



Since body is symmetric about AB. So; distance of ~~cm~~ $\bar{x}$ from O & $\bar{y}$ from C is same.

So; $\bar{y} = a - \frac{2a}{3(4-\pi)}$

$\Rightarrow \bar{y} = \frac{12a - 3\pi a - 2a}{3(4-\pi)}$

$\Rightarrow \bar{y} = \frac{10a - 3\pi a}{3(4-\pi)}$

So; $\bar{x} = \frac{2a}{3(4-\pi)}$; $\bar{y} = \frac{(10-3\pi)a}{3(4-\pi)}$

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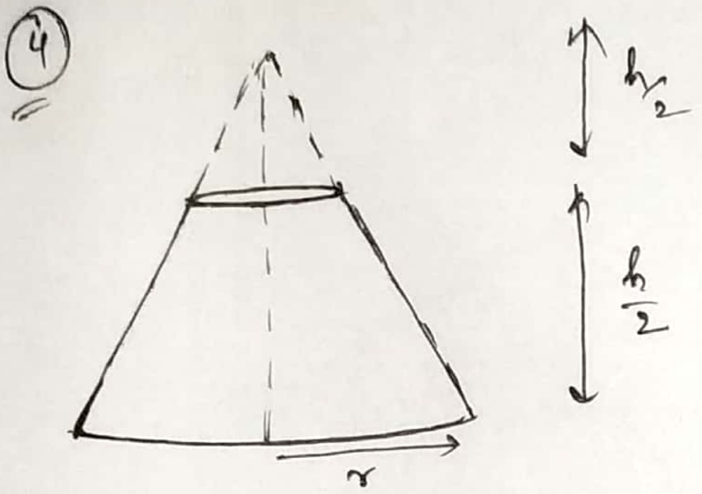

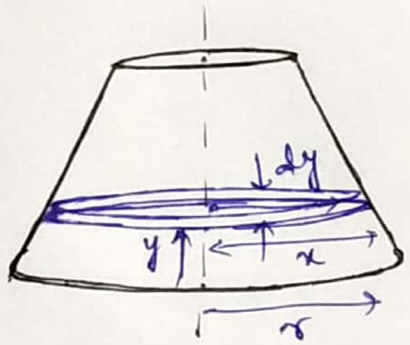

$dA = \pi x^2$.

$\Rightarrow dv = \pi x^2 \, dy$

from similarity :-

$\frac{r}{h} = \frac{x}{h-y}$

$\Rightarrow x = (h-y)\frac{r}{h}$.

Now; $dv = \frac{\pi r^2}{h^2}(h^2+y^2-2hy)\,dy$.

$$\bar{y} = \frac{\int \overset{y}{\overrightarrow{y_c}}\, dv}{\int dv} = \frac{\frac{\pi r^2}{h^2}\int_0^{h/2}(h^2y+y^3-2hy^2)\,dy}{\frac{\pi r^2}{h^2}\int_0^{h/2}(h^2+y^2-2hy)\,dy}$$

$$\Rightarrow \bar{y} = \frac{h^2\times\frac{h^2}{8}+\frac{1}{4}\times\frac{h^4}{16}-\frac{2h}{3}\times\frac{h^3}{8}}{\frac{h^3}{2}+\frac{1}{3}\cdot\frac{h^3}{8}-\frac{2h}{2}\cdot\frac{h^2}{4}} \Rightarrow \boxed{\bar{y} = \frac{11}{56}h.}$$

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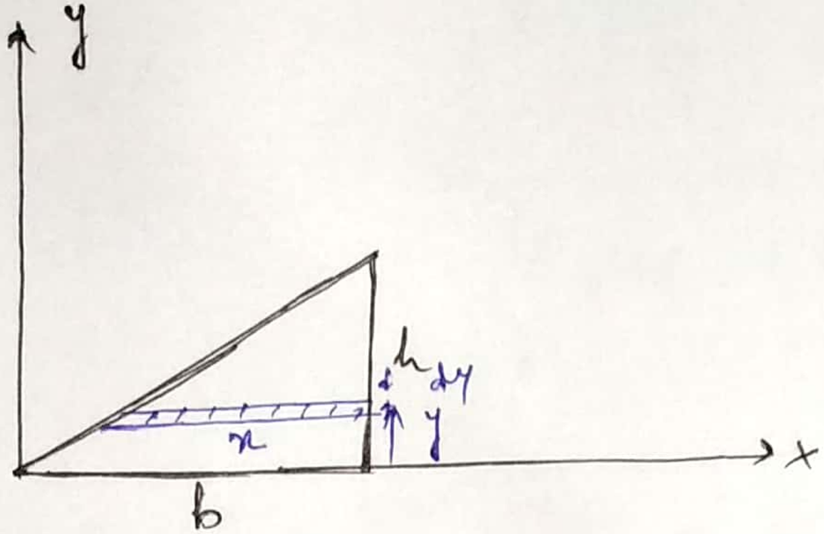

$\bar{y} = ?$

thickness is a linear fun of y.

$\Rightarrow t = ay + l.$

as t at $y = 0$ is t_0

$\Rightarrow l = t_0$

And t at $y = h$ is $2t_0$

$\Rightarrow 2t_0 = ah + t_0 \Rightarrow h = \frac{t_0}{a} \Rightarrow a = \frac{t_0}{h}$

So; $t = \left(\frac{t_0}{h}\right) y + t_0 \Rightarrow t = t_0\left(1 + \frac{y}{h}\right)$

$t(y)$

From similarity :- $\frac{b}{x} = \frac{h}{h-y} \Rightarrow x = \frac{b}{h}(h-y)$

$$dA = \frac{b}{h}(h-y)\,dy$$

$$\bar{y} = \frac{\int_0^h \cancel{\rho x_0}\left(\frac{h+y}{\cancel{h}}\right)\frac{\cancel{b}}{\cancel{h}}(h-y)\,y\,dy}{\int_0^h \cancel{\rho x_0}\left(\frac{h+y}{\cancel{h}}\right)\cdot\frac{\cancel{b}}{\cancel{h}}(h-y)\,dy}$$

$$\therefore \bar{y} = \frac{h^2\cdot\frac{h^2}{2} - \frac{h^4}{4}}{h^3 - \frac{h^3}{3}} = \frac{\frac{h^4}{4}\times 3}{2h^3} = \frac{3h}{8}$$

So; $\boxed{\bar{y} = \frac{3h}{8}}$

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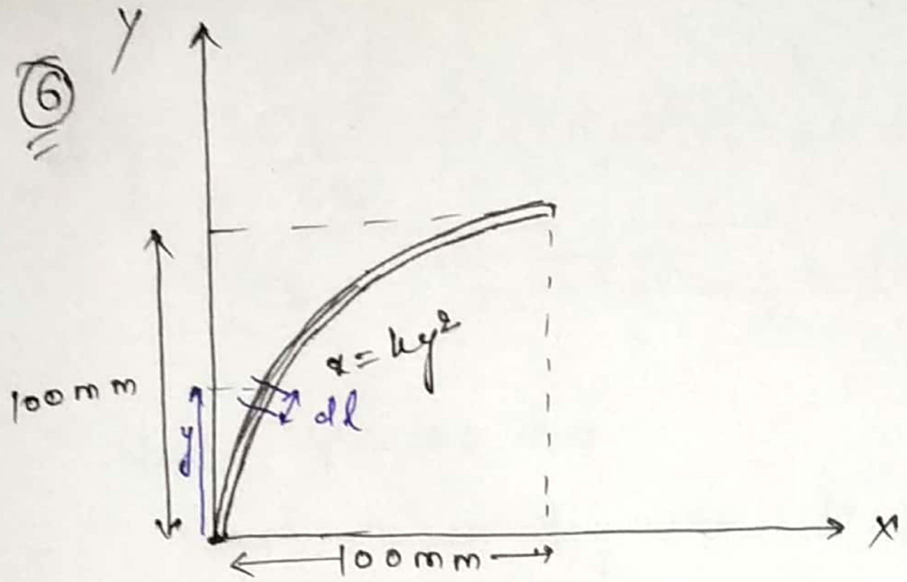
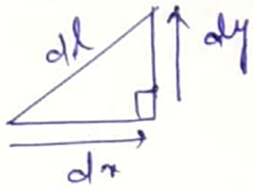


consider the differential element dl.

$$dl = \sqrt{(dx)^2 + (dy)^2}$$



Now; $$\bar{y} = \frac{\int y_c \, dl}{\int dl}$$ ($y_c \to y$)

$$\bar{y} = \frac{\int y \cdot dy \sqrt{1+\left(\frac{dx}{dy}\right)^2}}{\int \sqrt{1+\left(\frac{dx}{dy}\right)^2} \, dy} \Rightarrow \bar{y} = \frac{\int_0^Y y\sqrt{1+4k^2y^2}\,dy}{\int_0^Y \sqrt{1+4k^2y^2}\,dy}$$

($Y \to$ 100 mm)

$$\Rightarrow \bar{y} = \frac{\frac{1}{8k^2}\int_0^Y 8k^2 y \sqrt{1+4k^2y^2}\,dy}{\frac{1}{2k}\left|\frac{2ky}{2}\sqrt{1+4k^2y^2} + \frac{1}{2}\ln\left|2ky + \sqrt{1+4k^2y^2}\right|\right|_0^Y}$$

$$\Rightarrow \bar{y} = \frac{\frac{1}{4k} \times \frac{2}{3}\left(1+4k^2y^2\right)^{\frac{3}{2}}\Big|_0^Y}{kY\sqrt{1+4k^2Y^2} + \frac{1}{2}\ln\left|2kY + \sqrt{1+4k^2Y^2}\right|}$$

$$\Rightarrow \bar{y} = \frac{\frac{1}{6k}\left[(1+4k^2Y^2)^{3/2} - 1\right]}{kY\sqrt{1+k^2Y^2 \cdot 4} + \frac{1}{2}\ln\left|2kY + \sqrt{1+4k^2Y^2}\right|}$$

Now; $Y = 100$ mm.

put $x = 100$ & $y = 100$ to get k.

$\Rightarrow k = \frac{1}{100}$.

$\Rightarrow kY = 1$

So; $$\bar{y} = \frac{\frac{100}{6}(\sqrt{125} - 1)}{\sqrt{5} + \frac{1}{2}\ln(2+\sqrt{5})}$$

$$\Rightarrow \boxed{\bar{y} = 57.37 \text{ mm}}$$

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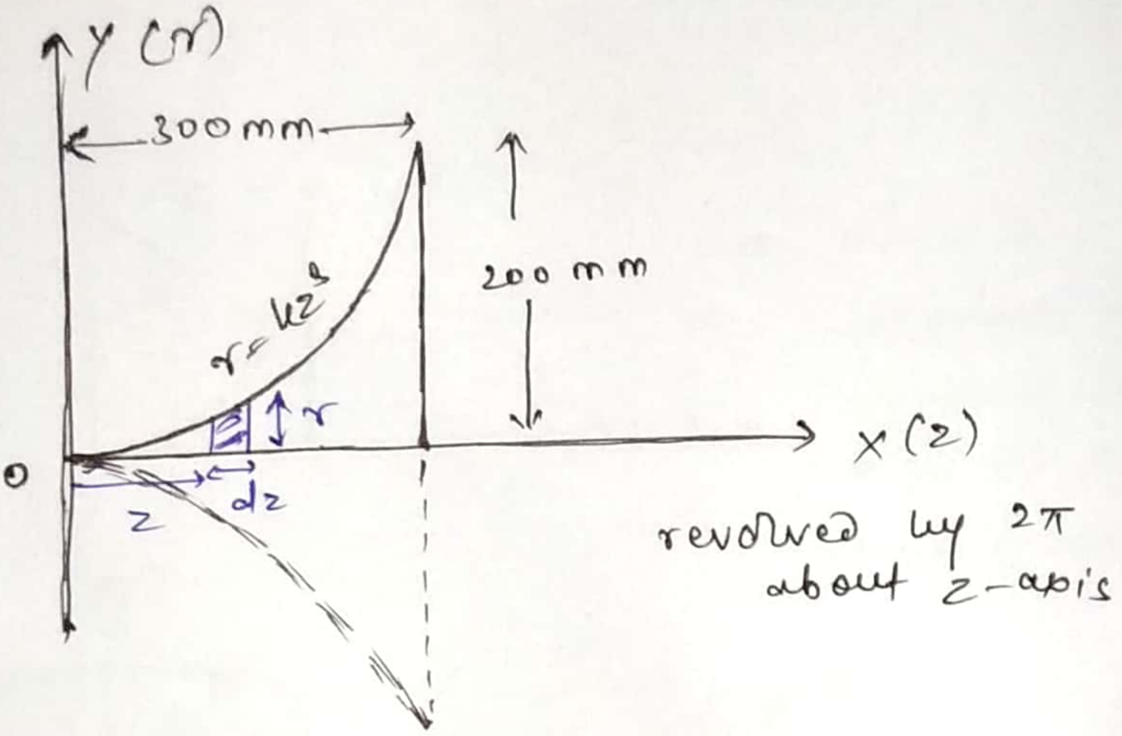

$$dv = \pi r^2 dz = \pi k^2 z^6 dz$$

$$\bar{z} = \frac{\int z_c \, dv}{\int dv} = \frac{\pi k^2 \int_0^{300} z^7 dv}{\pi k^2 \int_0^{300} z^6 dv}$$

$$\Rightarrow \bar{z} = \frac{z^8}{8} \times \frac{7}{z^7} \Big|_{z=300} \Rightarrow \bar{z} = \frac{7}{8} \times 300$$

$$\Rightarrow \boxed{\bar{z} = 262.5 \text{ mm}}$$

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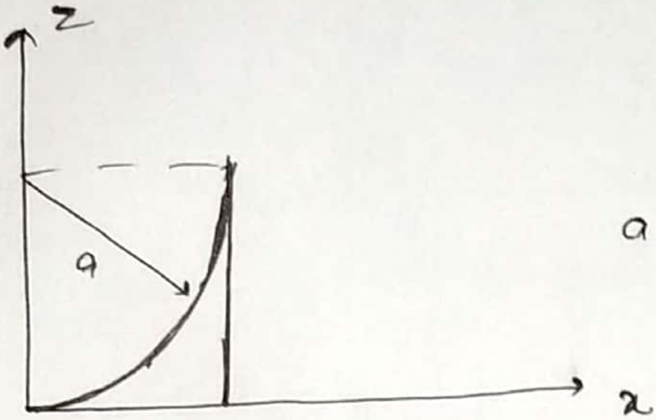
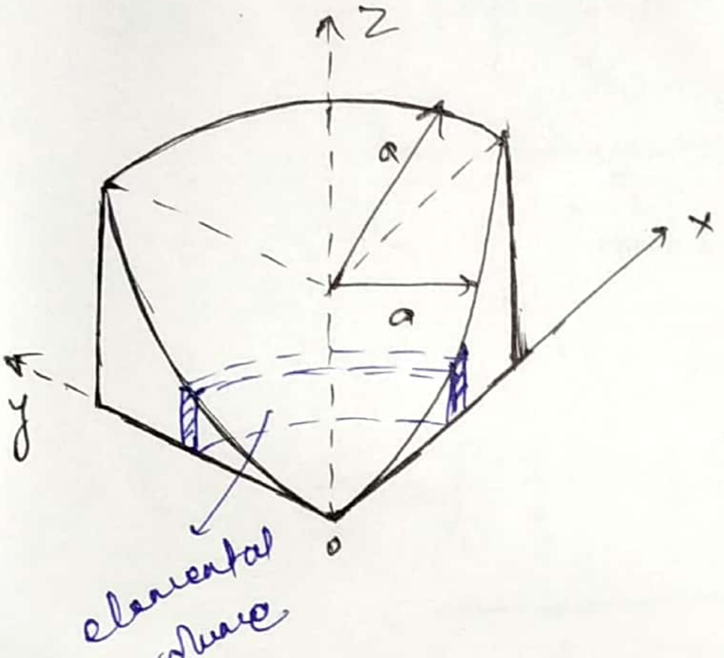
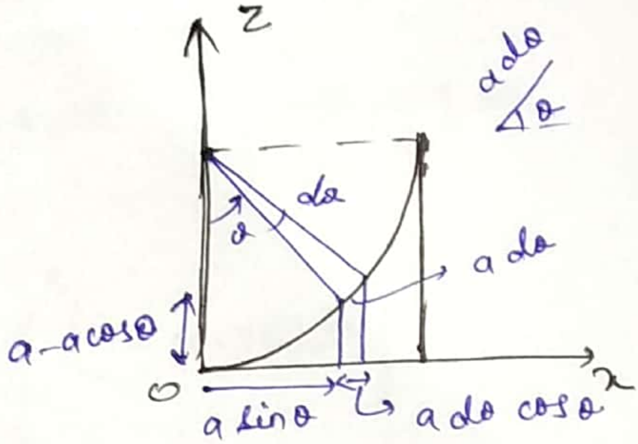
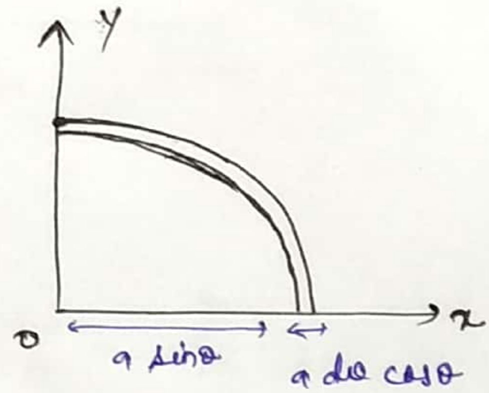


a solid is obtained by revolving this area by $\frac{\pi}{2}^c$ about z-axis.







$$dv = a(1-\cos\theta) \cdot a\,d\theta \cos\theta \cdot \frac{\pi}{2} a \sin\theta$$

(from pappus theorem)

$$\Rightarrow dv = \frac{\pi a^3}{2} \times (1-\cos\theta)\cos\theta \sin\theta\, d\theta$$

$$\bar{z} = \frac{\int z_c\, dv}{\int dv} \qquad z_c = \frac{a(1-\cos\theta)}{2}$$

$$\bar{x} = \frac{\int x_c\, dv}{\int dv} \qquad x_c = \frac{2r}{\pi} = \frac{2a\sin\theta}{\pi}$$

Also; $\vec{y_c} = \vec{x_c}$ (from symmetry)

$$So;\ \bar{x}_c = \frac{\frac{2a}{\pi}\int_0^{\frac{\pi}{2}} (1-\cos\theta)\cos\theta\sin^2\theta\, d\theta}{\int_0^{\frac{\pi}{2}} (1-\cos\theta)\cos\theta\sin\theta\, d\theta}$$

$$\Rightarrow \bar{x}_c = \frac{\frac{2a}{\pi}\left[\int_0^{\pi/2} \sin^2\theta\cos\theta\, d\theta - \frac{1}{4}\int_0^{\pi/2} \sin^2 2\theta\, d\theta\right]}{\int_0^{\pi/2} \cos\theta\sin\theta\, d\theta - \int_0^{\pi/2} \cos^2\theta\sin\theta\, d\theta}$$

$$\Rightarrow \bar{x}_c = \frac{\frac{2a}{\pi}\left[\frac{1}{3} - \frac{1}{4}\cdot\frac{1}{2}\int_0^{\pi/2} (1-\cos 4\theta)\, d\theta\right]}{\frac{1}{2} + \frac{\cos^3\theta}{3}\Big|_0^{\pi/2}}$$

$$\Rightarrow \bar{x}_c = \frac{\frac{2a}{\pi}\left[\frac{1}{3} - \frac{1}{8}\cdot\frac{\pi}{2}\right]}{\frac{1}{2} - \frac{1}{3}} = \frac{12a}{\pi} \times \frac{8}{24}\left(\frac{1}{3} - \frac{\pi}{16}\right)$$

$$\cancel{\Rightarrow \bar{x}_c = \frac{8a}{2\pi}}$$

$$\Rightarrow \boxed{\bar{x}_c = a\left(\frac{4}{\pi} - \frac{3}{4}\right) = \bar{y}_c}$$

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$$\bar{z}_c = \frac{\int z_c \, dv}{\int dv}$$

$$\Rightarrow \bar{z} = \frac{\frac{a}{2} \int_0^{\frac{\pi}{2}} (1-\cos\theta)^2 \cos\theta \sin\theta \, d\theta}{\int_0^{\frac{\pi}{2}} (1-\cos\theta) \cos\theta \sin\theta \, d\theta}$$

$\rightarrow$ ~~$\sin\theta = t$~~
$\cos\theta = t$

$$\bar{z} = \frac{a}{2} \frac{\int_1^0 t(1-t)^2 \, (-dt)}{\frac{1}{6}}$$

$$\bar{z} = 3a \cdot \int_0^1 (t^3 + t - 2t^2) \, dt$$

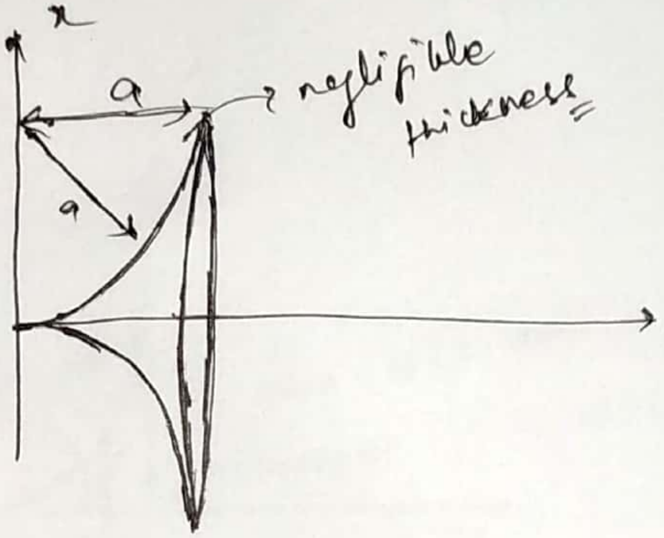
$$\Rightarrow \bar{z} = 3a\left[\frac{1}{4} + \frac{1}{2} - \frac{2}{3}\right]$$

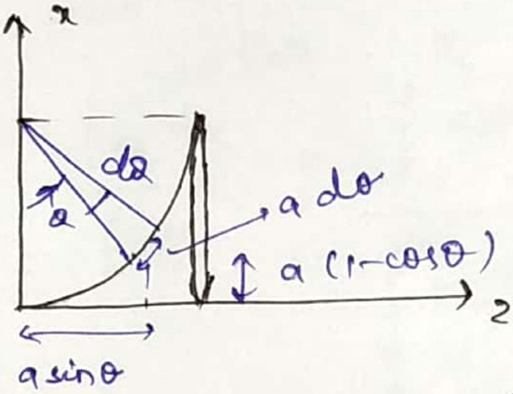
$$\bar{z} = 3a\left(\frac{3}{4} - \frac{2}{3}\right)$$

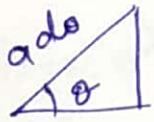
$$\Rightarrow \bar{z} = \frac{3a}{12} = \frac{a}{4}$$

so; $\boxed{\bar{z} = \frac{a}{4}}$

(9)







$$dL = 2\pi \overset{a(1-\cos\theta)}{r} \cdot a\,d\theta \Rightarrow dL = 2\pi a^2 (1-\cos\theta)\,d\theta.$$

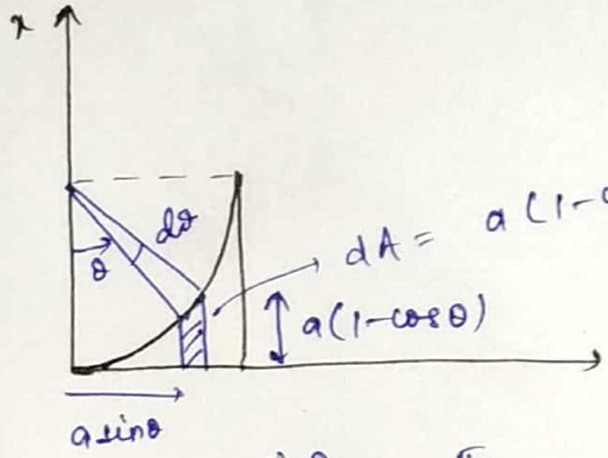
$$\Rightarrow dm = 2\pi a^2 \rho t\,(1-\cos\theta)\,d\theta$$

$$\bar{z} = \frac{\int \overset{a\sin\theta}{z_c}\,dm}{\int dm} = \frac{\int_0^{\pi/2} 2\pi a^2 \rho t\, a\sin\theta\,(1-\cos\theta)\,d\theta}{\int_0^{\pi/2} 2\pi a^2 \rho t\,(1-\cos\theta)\,d\theta}$$

$$\Rightarrow \bar{z} = \frac{a\left[(-\cos\theta)\Big|_0^{\pi/2} - \frac{\sin^2\theta}{2}\Big|_0^{\pi/2}\right]}{\frac{\pi}{2} - (\sin\theta)\Big|_0^{\pi/2}}$$

$$\Rightarrow \bar{z} = \frac{a\left(1 - \frac{1}{2}\right)}{\frac{\pi}{2} - 1} \Rightarrow \boxed{\bar{z} = \frac{a}{\pi - 2}}$$

Now;



$dA = a(1-\cos\theta)\, a\, d\theta \cos\theta$

$dv = (dA) \times \pi a (1-\cos\theta)$

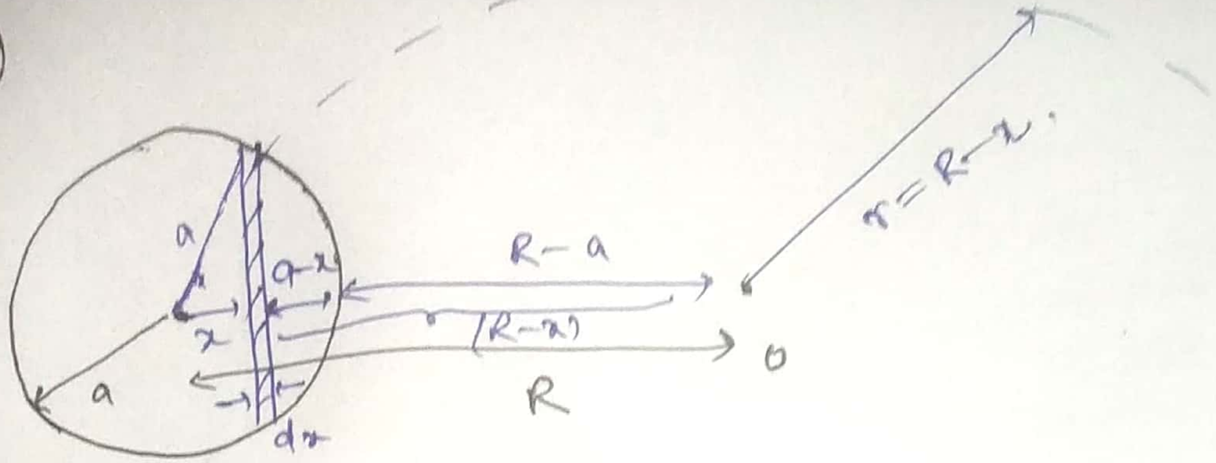
$\cos\theta = t$

$$\bar{z} = \frac{\int z_c\, dv}{\int dv} = \frac{\int_0^{\pi/2} \pi a^3 (1-\cos\theta)^2 \cos\theta \cdot a\sin\theta\, d\theta}{\int_0^{\pi/2} \pi a^3 (1-\cos\theta)^2 \cos\theta\, d\theta}$$

$$\bar{z} = \frac{a \; -\int_1^0 (1-t)^2 t\, dt}{\int_0^{\pi/2} (\cos^3\theta - 2\cos^2\theta + \cos\theta)\, d\theta}$$

$$\Rightarrow \bar{z} = \frac{a\int_0^1 t(t^2-2t+1)\, dt}{\frac{2}{3} - \frac{\pi}{2} + 1} = \frac{\frac{1}{12}\cdot a}{\frac{10-3\pi}{6}}$$

$$\Rightarrow \boxed{\bar{z} = \frac{a}{2(10-3\pi)}}$$



$$dV = \left(2\sqrt{a^2-x^2}\right) dx \times \pi (R-x)$$

$$dV = 2\pi (R-x)\left(\sqrt{a^2-x^2}\right) dx$$

$$z_c = \frac{2r}{\pi} \quad r \rightarrow (R-x)$$

$$\text{So; } \bar{z} = \frac{\int z_c \, dV}{\int dV}$$

$$\Rightarrow \bar{z} = \frac{2\pi \times \frac{2}{\pi} \int_{-a}^{a} \sqrt{a^2-x^2}\,(R^2+x^2-2Rx)\,dx}{2\pi\left(\int_{-a}^{a} R\sqrt{a^2-x^2}\,dx - \underbrace{\int_{-a}^{a} x\sqrt{a^2-x^2}\,dx}_{0 \text{ as integrand is odd fun}^n}\right)}$$

$$\Rightarrow \bar{\Sigma} = \frac{2}{\pi}\,\frac{\left(\int_{-a}^{a} (R^2+x^2)\sqrt{a^2-x^2} - \underbrace{\int_{-a}^{a} 2Rx\sqrt{a^2-x^2}\,dx}_{0 \text{ as integrand is odd function}}\right)}{\int_{-a}^{a} R\sqrt{a^2-x^2}\,dx}$$

$$\Rightarrow \bar{\Sigma} = \frac{2}{\pi}\times 2\,\frac{\left[\int_{0}^{a} R^2\sqrt{a^2-x^2}\,dx + \underbrace{\int_{0}^{a} x^2\sqrt{a^2-x^2}\,dx}_{x = a\sin\theta}\right]}{2R\int_{0}^{a}\sqrt{a^2-x^2}\,dx}$$

$$\Rightarrow \bar{\Sigma} = \frac{4\cancel{\pi}}{2R\pi}\,\frac{\left[R^2\left(\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right)\Big|_0^a + \int_0^{\pi/2} a^4 \sin^2\theta\cos^2\theta\right]}{\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\Big|_0^a}$$

$$\Rightarrow \bar{\Sigma} = \frac{2\cancel{\pi}}{\pi R}\,\frac{\left[R^2\times\frac{a^2\pi}{4} + \frac{a^4\pi}{16}\right]}{\frac{a^2}{2}\cdot\frac{\pi}{2}}$$

$$\Rightarrow \boxed{\bar{\Sigma} = \frac{a^2+4R^2}{2\pi R}}$$