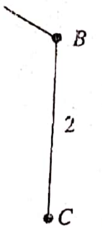


CHAPTER 5

# Improper Integrals

## 5.1 INTRODUCTION

We sometimes come across a definite integral one or both limits of which are infinite or the integrand tends to infinity at some points lying within the interval (or range) of integration. Such an integral is called **improper integral** or **infinite integral** or **generalised integral**. It will now be seen that the improper integrals are only extensions of the concept of ordinary definite integrals (also known as proper integrals) and in some cases it is possible to evaluate an improper integral.

## 5.2 TYPES OF IMPROPER INTEGRALS

Improper integrals are mainly of two types:

I. The interval (or range) increases without limit, and
II. The integrand is infinitely discontinuous at finite number of points.

**Type I:** In this type, three subcases are possible.

(*i*) The range is $[a, \infty)$.

In this case, we define

$$\int_a^\infty f(x)dx = \lim_{B\to\infty}\int_a^B f(x)dx$$

where, we assume $f(x)$ is bounded and integrable over every closed subinterval $[a, B]$, *i.e.*, $a \le x \le B$, of $[a, \infty)$.

(*ii*) The range is $(-\infty, b]$

In this case, we define

$$\int_{-\infty}^b f(x)dx = \lim_{A\to-\infty}\int_A^b f(x)dx$$

where, we assume $f(x)$ is bounded and integrable over every closed subinterval $[A, b]$ of $(-\infty, b]$.

(*iii*) The range is $(-\infty, \infty)$.

In this case, we define

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{A\to-\infty}\int_{A}^{a} f(x)dx + \lim_{B\to\infty}\int_{a}^{B} f(x)dx$$

where, we assume $f(x)$ is bounded and integrable over every closed subintervals $[A, a]$ and $[a, B]$ of $(-\infty, \infty)$.

**Notes:** 1. The existence and the value of $\int_{-\infty}^{\infty} f(x)dx$ is independent of the choice of $a$.

2. The definitions given in (*i*) – (*iii*) are in terms of limits and hence when the limits exist (*i.e.*, finite), the improper integrals are said to converge to the corresponding limiting value.

**Example :** Evaluate

(*i*) $\int_1^{\infty} \frac{1}{x^2}dx$ (*ii*) $\int_0^{\infty} \frac{1}{1+x^2}dx$ (*iii*) $\int_a^{\infty} \sin x\, dx$

(*iv*) $\int_0^{\infty} e^x dx$ (*v*) $\int_{-\infty}^{0} e^x dx$ (*vi*) $\int_{-\infty}^{\infty} xe^{-x^2} dx.$ (*M-201/11*)

(*vii*) $\int_0^{\infty} \frac{dx}{(1+x)\sqrt{x}}$ (*M-201/11*)

**Solution:** (*i*) Observe that $\frac{1}{x^2}$ is bounded and integrable in $1 \le x \le B$ for every $B > 1$. By definition

$$\int_1^{\infty} \frac{1}{x^2}dx = \lim_{B\to\infty}\int_1^{B} \frac{1}{x^2}dx = \lim_{B\to\infty}\left[-\frac{1}{x}\right]_1^B$$

$$= \lim_{B\to\infty}\left[1-\frac{1}{B}\right] = 1.$$

(*ii*) Observe that $\frac{1}{1+x^2}$ is bounded and integrable in $0 \le x \le B$ for every $B > 0$. By definition

$$\int_0^{\infty} \frac{1}{1+x^2}dx = \lim_{B\to\infty}\int_0^{B} \frac{1}{1+x^2}dx = \lim_{B\to\infty}\left[\tan^{-1} x\right]_0^B$$

$$= \lim_{B\to\infty}(\tan^{-1} B - \tan^{-1} 0)$$

$$= \lim_{B\to\infty}(\tan^{-1} B) = \frac{\pi}{2}.$$

(*iii*) Here, $\sin x$ is bounded and integrable in $a \le x \le B$ for every $B > a$. By definition

$$\int_a^{\infty} \sin x\, dx = \lim_{B\to\infty}\int_a^{B} \sin x\, dx = \lim_{B\to\infty}\left[-\cos x\right]_a^B$$

$$= \lim_{B\to\infty}(\cos a - \cos B),$$

Which oscillates finitely.

Hence, $\int_a^\infty \sin x\,dx$ does not exist.

(*iv*) Now, $\lim_{B\to\infty}\int_0^B e^x dx = \lim_{B\to\infty}\left[e^x\right]_0^B = \lim_{B\to\infty}(e^B - 1)$.

Since $(e^B - 1)$ increases beyond all bounds as $B \to \infty$, $\int_0^\infty e^x dx$ does not converge.

(*v*) Here, $e^x$ is bounded and integrable in $A \le x \le 0$ for every $A < 0$. By definition

$$\int_{-\infty}^0 e^x dx = \lim_{A\to-\infty}\int_A^0 e^x dx = \lim_{A\to-\infty}\left[e^x\right]_A^0 = \lim_{A\to-\infty}[1-e^A] = 1.$$

(*vi*) By definition, taking 0 as an intermediate point, we have

$$\int_{-\infty}^\infty xe^{-x^2} dx = \lim_{A\to-\infty}\int_A^0 xe^{-x^2} dx + \lim_{B\to\infty}\int_0^B xe^{-x^2} dx$$

$$= \lim_{A\to-\infty}\left[-\frac{1}{2}e^{-x^2}\right]_A^0 + \lim_{B\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_0^B$$

$$= \lim_{A\to-\infty}\left(\frac{1}{2}e^{-A^2} - \frac{1}{2}\right) + \lim_{B\to\infty}\left(\frac{1}{2} - \frac{1}{2}e^{-B^2}\right) = -\frac{1}{2} + \frac{1}{2} = 0.$$

(*vii*) $$\int_0^\infty \frac{dx}{(1+x)\sqrt{x}} = \lim_{\substack{\in\to0^+\\B\to\infty}}\int_\in^B \frac{dx}{(1+x)\sqrt{x}} = \lim_{\substack{\in\to0^+\\B\to\infty}}\int_{\sqrt{\in}}^{\sqrt{B}} \frac{2u}{(1+u^2)u}du \qquad \left[\begin{array}{l}\text{Put } x = u^2\\ \therefore\ dx = 2u\,du\end{array}\right]$$

$$= \lim_{\substack{\in\to0^+\\B\to\infty}} 2\int_{\sqrt{\in}}^{\sqrt{B}} \frac{du}{u^2+1} = \lim_{\substack{\in\to0^+\\B\to\infty}} 2[\tan^{-1} u]_{\sqrt{\in}}^{\sqrt{B}}$$

$$= \lim_{\substack{\in\to0^+\\B\to\infty}} 2[\tan^{-1}\sqrt{B} - \tan^{-1}\sqrt{\in}] = 2\left[\frac{\pi}{2} - 0\right] = \pi$$

**Type II:** Here also three subcases are possible.

(*i*) $f(x)$ has an infinite discontinuity (*i.e.*, unbounded) at the left hand end point $a$. In this case we define

$$\int_a^b f(x)dx = \lim_{\in\to0+}\int_{a+\in}^b f(x)dx$$

where, we assume $f(x)$ is bounded and integrable over every closed subinterval of $(a, b]$.

(*ii*) $f(x)$ has an infinite discontinuity (*i.e.*, unbounded) at the right hand end point $b$. In this case, we define

$$\int_a^b f(x)dx = \lim_{\in\to0+}\int_a^{b-\in} f(x)dx$$

where, we assume $f(x)$ is bounded and integrable over every closed subinterval of $[a, b)$.

(*iii*) $f(x)$ has an infinite discontinuity (*i.e.*, unbounded) at the point $x = c$, where $a < c < b$. In this case, we define

$$\int_a^b f(x)dx = \lim_{\epsilon \to 0+} \int_a^{c-\epsilon} f(x)dx + \lim_{\delta \to 0+} \int_{c+\delta}^b f(x)dx$$

where, we assume $f(x)$ is bounded and integrable over every closed subinterval of $[a, c)$, $(c, b]$.

**Notes:** 1. When $f(x)$ is unbounded at finitely many points, the definition of improper integral can be extended easily.

2. In some situations, where $f(x)$ is unbounded at an interior point $c$ of $[a, b]$, *i.e.*, $a < c < b$,

$$\lim_{\epsilon \to 0+} \int_a^{c-\epsilon} f(x)dx + \lim_{\delta \to 0+} \int_{c+\delta}^b f(x)dx \text{ may not exist}$$

$$\text{but } \lim_{\epsilon \to 0+} \int_a^{c-\epsilon} f(x)dx + \lim_{\epsilon \to 0+} \int_{c+\epsilon}^b f(x)dx \text{ may exist.}$$

In this case, this limiting value (when $\delta = \epsilon$) will be called the **Cauchy's principal value** of the improper integral $\int_a^b f(x)dx$. See Example 7.

3. Whenever the appropriate limits exist finitely, the corresponding improper integral is said to be convergent. When the appropriate limits fail to exist or tend to $+\infty$ (or $-\infty$), an improper integral is said to be non-convergent (*i.e.*, divergent). In case (*iii*), both limits must exist and be finite in order that the integral is to be convergent.

## ILLUSTRATIVE EXAMPLES

**Example 1:** Evaluate $\int_1^\infty \frac{dx}{x}$.

**Solution:** By definition

$$\int_1^\infty \frac{dx}{x} = \lim_{B \to \infty} \int_1^B \frac{dx}{x} = \lim_{B \to \infty} \left[\log_e x\right]_1^B$$

$$= \lim_{B \to \infty} \log_e B$$

Now, $\log_e B \to \infty$ as $B \to \infty$. Therefore, $\int_1^\infty \frac{dx}{x}$ does not exist.

**Example 2:** Evaluate $\int_0^1 \frac{dx}{x}$, if it exists.

**Solution:** Here, the integrand has an infinite discontinuity at $x = 0$. By definition

$$\int_0^1 \frac{dx}{x} = \lim_{\in \to 0+} \int_{\in}^1 \frac{dx}{x} = \lim_{\in \to 0+} \left[\log_e x\right]_{\in}^1$$

$$= \lim_{\in \to 0+} \left[-\log_e \in\right].$$

Now, $-\log_e \in \to \infty$ as $\in \to 0+$, hence, $\int_0^1 \frac{dx}{x}$ does not exist. (*i.e.*, it does not converge).

**Example 3:** Find $\int_{-\infty}^{0} \frac{dx}{(2-x)^2}$.

**Solution:** By definition

$$\int_{-\infty}^{0} \frac{dx}{(2-x)^2} = \lim_{A \to -\infty} \int_A^0 \frac{dx}{(2-x)^2} = \lim_{A \to -\infty} \left[\frac{1}{2-x}\right]_A^0$$

$$= \lim_{A \to -\infty} \left[\frac{1}{2} - \frac{1}{2-A}\right] = \frac{1}{2}.$$

**Example 4:** Evaluate $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$, if it exists.

**Solution:** By definition, taking 0 as an intermediate point, we have

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} = \lim_{A \to -\infty} \int_A^0 \frac{dx}{x^2+2x+2} + \lim_{B \to \infty} \int_0^B \frac{dx}{x^2+2x+2}$$

$$= \lim_{A \to -\infty} \int_A^0 \frac{dx}{(x+1)^2+1} + \lim_{B \to \infty} \int_0^B \frac{dx}{(x+1)^2+1}$$

$$= \lim_{A \to -\infty} \left[\tan^{-1}(x+1)\right]_A^0 + \lim_{B \to \infty} \left[\tan^{-1}(x+1)\right]_0^B$$

$$= \lim_{A \to -\infty} \left[\tan^{-1} 1 - \tan^{-1}(A+1)\right] + \lim_{B \to \infty} \left[\tan^{-1}(B+1) - \tan^{-1} 1\right]$$

$$= \left(\frac{\pi}{4} + \frac{\pi}{2}\right) + \left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \pi.$$

**Example 5:** Evaluate $\int_0^a \frac{dx}{\sqrt{a^2-x^2}}$.

**Solution:** Here, the integrand has an infinite discontinuity at $x = a$. By definition

Scanned by CamScanner

$$\int_0^a \frac{dx}{\sqrt{a^2-x^2}} = \lim_{\epsilon\to 0+}\int_0^{a-\epsilon}\frac{dx}{\sqrt{a^2-x^2}} = \lim_{\epsilon\to 0+}\left[\sin^{-1}\frac{x}{a}\right]_0^{a-\epsilon}$$

$$= \lim_{\epsilon\to 0+}\sin^{-1}\left(1-\frac{\epsilon}{a}\right) = \sin^{-1}1 = \frac{\pi}{2}.$$

**Example 6:** Find $\int_0^1 \log x\,dx$, if it exists.

**Solution:** Here, the integrand has an infinite discontinuity at $x = 0$. By definition

$$\int_0^1 \log x\,dx = \lim_{\epsilon\to 0+}\int_\epsilon^1 \log x\,dx$$

$$= \lim_{\epsilon\to 0+}\left[x\log x - x\right]_\epsilon^1 \qquad \text{(Integrating by parts)}$$

$$= -1 - \lim_{\epsilon\to 0+}\left[\epsilon\log\epsilon - \epsilon\right]$$

$$= -1 - \lim_{\epsilon\to 0+}\frac{\log\epsilon}{\frac{1}{\epsilon}} = -1 - \lim_{\epsilon\to 0+}\frac{\frac{1}{\epsilon}}{-\frac{1}{\epsilon^2}} \qquad \text{[By L'Hospital Rule]}$$

$$= -1 - \lim_{\epsilon\to 0+}(-\epsilon) \qquad [\because \epsilon \neq 0]$$

$$= -1.$$

**Example 7:** Show that $\int_{-1}^1 \frac{1}{x^3}dx$ exists in the Cauchy's principal value sense but not in the general sense.

**Solution:** Here, the integrand has an infinite discontinuity at $x = 0$.

Now,
$$\lim_{\epsilon\to 0+}\int_{-1}^{0-\epsilon}\frac{1}{x^3}dx + \lim_{\delta\to 0+}\int_{0+\delta}^{1}\frac{1}{x^3}dx.$$

$$= \lim_{\epsilon\to 0+}\left[-\frac{1}{2x^2}\right]_{-1}^{-\epsilon} + \lim_{\delta\to 0+}\left[-\frac{1}{2x^2}\right]_{\delta}^{1}$$

$$= \lim_{\epsilon\to 0+}\left(\frac{1}{2}-\frac{1}{2\epsilon^2}\right) + \lim_{\delta\to 0+}\left(-\frac{1}{2}+\frac{1}{2\delta^2}\right)$$

Here, $\lim_{\epsilon\to 0+}\frac{1}{2\epsilon^2}$ and $\lim_{\delta\to 0+}\frac{1}{2\delta^2}$ do not exist,

hence the original integral does not exist.

But $\lim_{\epsilon\to 0+}\left[\int_{-1}^{-\epsilon}\frac{1}{x^3}dx+\int_{\epsilon}^{1}\frac{1}{x^3}dx\right]$

$$= \lim_{\epsilon\to 0+}\left[\left(\frac{1}{2}-\frac{1}{2\epsilon^2}\right)+\left(-\frac{1}{2}+\frac{1}{2\epsilon^2}\right)\right]$$

= 0, because the terms involving $\epsilon$ cancel before the limit is taken.

Therefore, the Cauchy's principal value of $\int_{-1}^{1}\frac{1}{x^3}dx$ is 0, though the improper integral does not exist.

**Example 8:** Evaluate $\int_0^\infty e^{-ax}\cos bx\,dx\,(a>0)$ and hence show that $\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$.

**Solution:** By definition

$$\int_0^\infty e^{-ax}\cos bx\,dx = \lim_{B\to\infty}\int_0^B e^{-ax}\cos bx\,dx$$

$$= \lim_{B\to\infty}\left[\frac{e^{-ax}(b\sin bx - a\cos bx)}{a^2+b^2}\right]_0^B$$

$$= \lim_{B\to\infty}\left[\frac{e^{-aB}(b\sin bB - a\cos bB)}{a^2+b^2}+\frac{a}{a^2+b^2}\right]$$

$\therefore$
$$\int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}\left(\because -1\le \sin bB, \cos bB\le 1 \text{ and } \lim_{B\to\infty}e^{-aB}=0, a>0\right) \quad ...(1)$$

Let
$$I=\int_0^\infty \frac{e^{-ax}\sin bx}{x}dx,\ a>0. \quad ...(2)$$

Using the differentiation under the sign of integration, we have

$$\frac{dI}{db}=\int_0^\infty e^{-ax}\cos bx\,dx=\frac{a}{a^2+b^2} \quad \text{[By (1)]}$$

$\therefore$
$$dI=\frac{a}{a^2+b^2}db.$$

Integrating, we get
$$\int dI = a\int\frac{db}{a^2+b^2}$$

$\therefore$
$$I = a\cdot\frac{1}{a}\tan^{-1}\left(\frac{b}{a}\right)+c = \tan^{-1}\left(\frac{b}{a}\right)+c. \quad ...(3)$$

[By (2)]

when $b = 0$, $I = 0$

Therefore, from (3), we have $c = 0$.

$$\therefore \quad I = \int_0^{\infty} \frac{e^{-ax} \sin bx}{x} dx = \tan^{-1}\left(\frac{b}{a}\right).$$

when $a \to 0+$, we get

$$\int_0^{\infty} \frac{\sin bx}{x} dx = \frac{\pi}{2} \text{ or } -\frac{\pi}{2} \text{ accordingly as } b > 0 \text{ or } b < 0.$$

Putting $b = 1$, we have $\displaystyle\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$.

## 5.3 TEST FOR CONVERGENCE: TYPE I

**Definition (absolute and conditional convergence)**

The integral $\displaystyle\int_a^{\infty} f(x)dx$ is said to **converge absolutely** if

(*i*) $\displaystyle\int_a^{\infty} |f(x)|dx$ converges and

(*ii*) $f(x)$ is bounded and integrable in an arbitrary interval $a \le x \le B$ for every $B > a$.

when $\displaystyle\int_a^{\infty} f(x)dx$ converges and $\displaystyle\int_a^{\infty} |f(x)|dx$ diverges, we say that $\displaystyle\int_a^{\infty} f(x)dx$ is **conditionally convergent.**

**Theorem 1:** If $\displaystyle\int_a^{\infty} |f(x)|dx$ converges then $\displaystyle\int_a^{\infty} f(x)dx$ converges, but the converse is not necessarily true.

**Theorem 2:** **(Limit tests):** Let $f(x)$ and $g(x)$ are two integrable functions when $x \ge a$ and $g(x)$ be positive. Then, if

$$\lim_{x \to \infty} \left\{\frac{f(x)}{g(x)}\right\} = \lambda \neq 0,$$

the integrals $F = \displaystyle\int_a^{\infty} f(x)dx$ and $G = \displaystyle\int_a^{\infty} g(x)dx$, both converge absolutely or both diverge.

If $\dfrac{f}{g} \to 0$ as $x \to \infty$ and $G$ converges, then, $F$ converges absolutely.

If $\dfrac{f}{g} \to \pm\infty$ as $x \to \infty$ and $G$ diverges, then, $F$ diverges.

**A comparison integral**

Prove that the improper integral $\int_a^\infty \dfrac{dx}{x^\mu} (a > 0)$ exists if $\mu > 1$ and does not exist if $\mu \leq 1$.

**Proof:** Now by definition

$$\int_a^\infty \frac{dx}{x^\mu} = \lim_{B\to\infty} \int_a^B \frac{dx}{x^\mu} = \begin{cases} \lim\limits_{B\to\infty} \left[\dfrac{x^{-\mu+1}}{-\mu+1}\right]_a^B & \text{when } \mu \neq 1 \\ \lim\limits_{B\to\infty} (\log B - \log a) & \text{when } \mu = 1. \end{cases}$$

Here,
$$\lim_{B\to\infty} \left[\frac{x^{-\mu+1}}{-\mu+1}\right]_a^B = \lim_{B\to\infty} \frac{1}{1-\mu}\{B^{1-\mu} - a^{1-\mu}\}$$
$$= \frac{a^{1-\mu}}{\mu-1} \quad \text{if, } \mu > 1.$$

Also, $\dfrac{1}{1-\mu}\{B^{1-\mu} - a^{1-\mu}\} \to \infty$ as $B \to \infty$ if $\mu < 1$ and $\lim\limits_{B\to\infty} (\log B - \log a)$ oscillates finitely.

$$\therefore \quad \int_a^\infty \frac{dx}{x^\mu} = \begin{cases} \dfrac{a^{1-\mu}}{\mu-1} & \text{if } \mu > 1 \\ \text{does not exist for } \mu \leq 1. \end{cases}$$

**Theorem 3 (The μ-test for convergence):** Let $f(x)$ be an integrable function when $x \geq a$. Then,

(*i*) $F = \int_a^\infty f(x)dx$ converges absolutely if $\lim\limits_{x\to\infty} x^\mu f(x) = \lambda,\ \mu > 1$ and

(*ii*) $F$ diverges if $\lim\limits_{x\to\infty} x^\mu f(x) = \lambda\ (\neq 0)$ or $\pm\infty,\ \mu \leq 1$.

## ILLUSTRATIVE EXAMPLES

**Example 1:** Examine the convergence of $\int_a^\infty \dfrac{\sin^2 x}{x^2} dx$, where $a > 0$.

**Solution:** Let $f(x) = \dfrac{\sin^2 x}{x^2}$ and $g(x) = \dfrac{1}{x^{3/2}}$.

Here, $f(x)$ and $g(x)$ are both integrable and $g(x)$ is positive when $x \geq a > 0$.

Now, $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to\infty}\dfrac{\sin^2 x}{\sqrt{x}} = 0$ and also

$$\int_a^\infty g(x)\,dx = \int_a^\infty \frac{dx}{x^{3/2}} \; (a>0) \text{ converges since } \frac{3}{2}>1.$$

Therefore, by Theorem 2, $\displaystyle\int_a^\infty \frac{\sin^2 x}{x^2}\,dx$ converges.

**Example 2:** Examine the convergence of $\displaystyle\int_1^\infty \frac{x^2}{(1+x)^4}\,dx.$

**Solution:** Let $f(x) = \dfrac{x^2}{(1+x)^4}$ and $g(x) = \dfrac{1}{x^2}$.

Here, both $f(x)$ and $g(x)$ are integrable and $g(x)$ is positive when $x \geq 1$.

Now,
$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{x^4}{(1+x)^4} = \lim_{x\to\infty}\frac{1}{\left(\dfrac{1}{x}+1\right)^4}$$

$$= 1 \qquad \left(\because \frac{1}{x}\to 0 \text{ as } x\to\infty\right).$$

Also,
$$\int_1^\infty g(x)\,dx = \int_1^\infty \frac{dx}{x^2} \quad \text{converges since } 2>1.$$

Therefore, by Theorem 2, $\displaystyle\int_1^\infty \frac{x^2}{(1+x)^4}\,dx$ converges.

**Example 3:** Examine the convergence of

(*i*) $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{1+x^2}}$ (*ii*) $\displaystyle\int_0^\infty e^{-x^2}\,dx$ (*iii*) $\displaystyle\int_0^\infty \frac{x\,dx}{x^2+4}$

(*iv*) $\displaystyle\int_1^\infty e^{-x}x^n\,dx$ (*v*) $\displaystyle\int_1^\infty \frac{\log x}{x^2}\,dx$ (*vi*) $\displaystyle\int_0^\infty \frac{x^{3/2}}{3x^2+5}\,dx$

**Solution:** (*i*) Here, $\lim\limits_{x\to\infty} x^2 f(x) = \lim\limits_{x\to\infty}\dfrac{x^2}{x\sqrt{1+x^2}}$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{\frac{1}{x^2}+1}} = 1, \quad \mu = 2 > 1.$$

Therefore, the integral is convergent by μ-test.

(*ii*) Now, $$\lim_{x\to\infty} x^2 f(x) = \lim_{x\to\infty} x^2 e^{-x^2} = \lim_{x\to\infty} \frac{x^2}{e^{x^2}} \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{2x}{2xe^{x^2}} \qquad \text{(By L'Hospital rule)}$$

$$= \lim_{x\to\infty} \frac{1}{e^{x^2}} = 0, \quad \mu = 2 > 1.$$

Therefore, the integral is convergent by μ-test.

(*iii*) Here, $$\lim_{x\to\infty} xf(x) = \lim_{x\to\infty} \frac{x^2}{x^2+4} = \lim_{x\to\infty} \frac{1}{1+\frac{4}{x^2}}$$

$$= 1 \left(\because \frac{1}{x^2} \to 0 \text{ as } x \to \infty\right)$$

$$\neq 0, \quad \mu = 1.$$

Therefore, the integral is divergent by μ-test.

(*iv*) Observe that $\lim\limits_{x\to\infty} x^2 f(x) = \lim\limits_{x\to\infty} x^2 e^{-x} x^n$

$$= \lim_{x\to\infty} \frac{x^{n+2}}{e^x} = 0 \quad \text{(By L' Hospital rule)}$$

$\mu = 2 > 1.$

Hence, the integral is convergent by μ-test.

(*v*) Now, $$\lim_{x\to\infty} x^{3/2} f(x) = \lim_{x\to\infty} \frac{\log x}{x^{1/2}} \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2} x^{\frac{1}{2}-1}} \qquad \text{(By L' Hospital rule)}$$

$$= \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0, \mu = \frac{3}{2} > 1.$$

Therefore, the integral is convergent by μ-test.

## 5.4 TEST FOR CONVERGENCE: TYPE II

**Definition (absolute and conditional convergence)**

Let $a$ be the only point of infinite discontinuity of a function $f(x)$ in a finite interval $[a, b]$. Then, the integral $\int_a^b f(x)dx$ is said to be **absolutely convergent** if

(i) $\int_a^b |f(x)|dx$ converges and

(ii) $f(x)$ is bounded and integrable in an arbitrary interval $[a+\in, b]$ for every $\in$ such that $0<\in<b-a$.

When $\int_a^b f(x)dx$ converges and $\int_a^b |f(x)|dx$ diverges, we say that $\int_a^b f(x)dx$ is **conditionally convergent.**

**Theorem 4:** If $\int_a^b |f(x)|dx$ converges, where $f(x)$ has an infinite discontinuity at $x=a$ only, then $\int_a^b f(x)dx$ converges but the converse is not necessarily true.

**Theorem 5 (Limit tests):** Let $f(x)$ and $g(x)$ are two integrable functions when $a<x\le b$ and $g(x)$ be positive, where $a$ is the only point of infinite discontinuity, then if

$$\lim_{x\to a+}\left\{\frac{f(x)}{g(x)}\right\}=\lambda\neq 0, \text{ the integrals}$$

$F=\int_a^b f(x)dx$ and $G=\int_a^b g(x)dx$ both converge absolutely or both diverge.

If $\dfrac{f}{g} \to 0$ as $x \to a+$ and $G$ converges, then $F$ converges absolutely.

If $\dfrac{f}{g} \to \pm\infty$ as $x \to a+$ and $G$ diverges, then $F$ diverges.

**A Comparison Integral**

Prove that the integral $\displaystyle\int_a^b \frac{dx}{(x-a)^\mu}$ exists if $\mu < 1$ and does not exist, if $\mu \geq 1$.

**Proof:** Now $$\int_{a+\in}^{b} \frac{dx}{(x-a)^\mu} = \begin{cases} \dfrac{1}{1-\mu}\{(b-a)^{1-\mu} - \in^{1-\mu}\}, & \text{if } \mu \neq 1 \\ \log(b-a) - \log \in, & \text{if } \mu = 1 \end{cases}$$

As $\in \to 0+$, we obtain

$$\int_a^b \frac{dx}{(x-a)^\mu} = \begin{cases} \dfrac{(b-a)^{1-\mu}}{1-\mu}, & \text{when } 0 < \mu < 1 \\ \infty, & \text{when } \mu \geq 1. \end{cases}$$

**Notes:** (*i*) The integral is proper when $\mu \leq 0$.

(*ii*) Similarly $\displaystyle\int_a^b \frac{dx}{(b-x)^\mu}$ exists if $\mu < 1$ and does not exist if $\mu \geq 1$.

**Theorem 6: (The μ-test for convergence)**

(*i*) Let $f(x)$ be an integrable function in an arbitrary interval $(a + \in, b)$ for each $\in$ such that $0 < \in < b - a$, where $a$ is the only point of infinite discontinuity.

Then $$F = \int_a^b f(x)\,dx \text{ converges absolutely, if}$$

$$\lim_{x \to a+} (x-a)^\mu f(x) = \lambda \text{ for } 0 < \mu < 1 \text{ and } F \text{ diverges, if}$$

$$\lim_{x \to a+} (x-a)^\mu f(x) = \lambda\ (\neq 0) \text{ or } \pm\infty \text{ for } \mu \geq 1.$$

(*ii*) Let $f(x)$ be an integrable function in an arbitrary interval $(a, b - \in)$ for each $\in$ such that $0 < \in < b - a$, where $b$ is the only point of infinite discontinuity.

Then $$F = \int_a^b f(x)\,dx \text{ converges absolutely, if}$$

$$\lim_{x \to b-} (b-x)^\mu f(x) = \lambda \quad \text{for} \quad 0 < \mu < 1 \text{ and } F \text{ diverges, if}$$

$$\lim_{x \to b-} (b-x)^\mu f(x) = \lambda\ (\neq 0) \text{ or } \pm\infty \text{ for } \mu \geq 1.$$

## ILLUSTRATIVE EXAMPLES

**Example 1:** Examine the convergence of

(*i*) $\int_0^1 \frac{dx}{\sqrt{1-x^3}}$ (*ii*) $\int_0^{\pi/2} \frac{\sin x}{x^p}\,dx$

**Solution:** (*i*) Let $f(x) = \frac{1}{\sqrt{1-x^3}} = \frac{1}{(1-x)^{1/2}(1+x+x^2)^{1/2}}$ and $g(x) = \frac{1}{(1-x)^{1/2}}$.

Here, $f(x)$ and $g(x)$ are both integrable and $g(x)$ is positive when $0 \le x < 1$, where 1 is the only point of infinite discontinuity.

Now, $$\lim_{x\to 1-} \frac{f(x)}{g(x)} = \lim_{x\to 1-} \frac{1}{(1+x+x^2)^{1/2}} = \frac{1}{\sqrt{3}}$$ and

also, $\int_0^1 \frac{dx}{(1-x)^{1/2}}$ converges since $\mu = \frac{1}{2} < 1$. Therefore, by limit test, $\int_0^1 \frac{dx}{\sqrt{1-x^3}}$ converges.

(*ii*) Let $f(x) = \frac{\sin x}{x^p} = \frac{1}{x^{p-1}} \cdot \frac{\sin x}{x}$ and $g(x) = \frac{1}{x^{p-1}}$.

Here, $f(x)$ and $g(x)$ are both integrable and $g(x)$ is positive when $0 < x \le \frac{\pi}{2}$, where 0 is the only point of infinite discontinuity.

Now, $\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{\sin x}{x} = 1$ and $\int_0^{\pi/2} \frac{dx}{x^{p-1}}$ converges if $p < 2$ and diverges if $p \ge 2$. Therefore, by limit test $\int_0^{\pi/2} \frac{\sin x}{x^p}\,dx$ converges for $p < 2$ and diverges for $p \ge 2$.

**Example 2:** Show that the integral $\int_0^{\pi/2} \left(\frac{\sin^p x}{x^q}\right) dx$ exists if and only if $q < p + 1$.

**Solution:** Let $f(x) = \frac{\sin^p x}{x^q} = \frac{1}{x^{q-p}} \left(\frac{\sin x}{x}\right)^p$.

Now, $f(x) \to 0$ as $x \to 0+$ if $q - p < 0$ and $f(x) \to \infty$ as $x \to 0+$, if $q - p > 0$.

Therefore, it is a proper integral if $q \le p$ and improper if $q > p$, where 0 is the only point of infinite discontinuity of $f$.

When $q > p$, let $g(x) = \frac{1}{x^{q-p}}$.

Here, $f(x)$ and $g(x)$ are both integrable and $g(x)$ is positive when $0 < x \le \frac{\pi}{2}$, where 0 is the only point of infinite discontinuity.

Now, $\lim\limits_{x\to 0+}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to 0+}\left(\dfrac{\sin x}{x}\right)^p = 1$ and $\int\limits_0^{\pi/2} g(x)dx = \int\limits_0^{\pi/2}\dfrac{dx}{x^{q-p}}$ converges, if and only if $-q+p+1>0$, or $q<p+1$, therefore by limit test $\int\limits_0^{\pi/2}\dfrac{\sin^p x}{x^q}dx$ converges if and only if $q<p+1$ which includes the case $q \le p$ when the integral is proper.

**Example 3:** Examine the convergence of the integral $\int\limits_0^1 \dfrac{dx}{\sqrt{x(1-x)}}$.

**Solution:** Let $f(x) = \dfrac{1}{\sqrt{x(1-x)}}$. Here 0 and 1 are the only points of infinite discontinuity of $f$.

Now,

$$\int\limits_0^1 f(x)dx = \int\limits_0^{\frac{1}{2}} f(x)dx + \int\limits_{\frac{1}{2}}^1 f(x)dx.$$

Therefore, $\int\limits_0^1 f(x)dx$ is convergent only when $\int\limits_0^{1/2} f(x)dx$ and $\int\limits_{\frac{1}{2}}^1 f(x)dx$ both are convergent.

**Test of convergence of $\int\limits_0^{\frac{1}{2}} f(x)dx$**

Let $g(x) = \dfrac{1}{\sqrt{x}}$. Here $f(x)$ and $g(x)$ are both integrable and $g(x)$ is positive when $0 < x \le \dfrac{1}{2}$, where 0 is the only point of discontinuity.

Now, $$\lim_{x\to 0+}\frac{f(x)}{g(x)} = \lim_{x\to 0+}\frac{1}{\sqrt{1-x}} = 1 \text{ and also}$$

$$\int\limits_0^{1/2} g(x)dx = \int\limits_0^{1/2}\frac{dx}{\sqrt{x}} \quad \text{converges since } \mu = \frac{1}{2} < 1.$$

Therefore, by limit test $\int\limits_0^{1/2} f(x)dx$ is convergent.

**Test of convergence of $\int\limits_{\frac{1}{2}}^1 f(x)dx$**

Let $g(x) = \dfrac{1}{\sqrt{1-x}}$. Here $f(x)$ and $g(x)$ are both integrable and $g(x)$ is positive when $\frac{1}{2} \leq x < 1$, where 1 is the only point of discontinuity.

Now,
$$\lim_{x \to 1-} \frac{f(x)}{g(x)} = \lim_{x \to 1-} \frac{1}{\sqrt{x}} = 1 \text{ and also}$$

$$\int_{\frac{1}{2}}^{1} g(x)dx = \int_{\frac{1}{2}}^{1} \frac{dx}{\sqrt{1-x}} \text{ converges since } \mu = \frac{1}{2} < 1.$$

Therefore, by limit test $\int_{\frac{1}{2}}^{1} f(x)dx$ is convergent.

Hence
$$\int_0^1 f(x)dx = \int_0^1 \frac{dx}{\sqrt{x(1-x)}} \text{ is convergent.}$$

**Example 4:** Examine the convergence of

(i) $\displaystyle\int_0^1 \frac{dx}{(1+x)\sqrt{x}}$  (ii) $\displaystyle\int_a^b (x-a)^{-m}(b-x)^{-n}\,dx;\ 0 < m, n < 1$  (iii) $\displaystyle\int_0^1 \frac{\log x}{\sqrt{x}}dx$

(iv) $\displaystyle\int_1^2 \frac{\sqrt{x}}{\log x}dx$  (v) $\displaystyle\int_0^1 \frac{\log x}{\sqrt{1-x}}dx$  (vi) $\displaystyle\int_0^1 \frac{x^{m-1}+x^{-m}}{1+x}dx$.

**Solution:** (i) Let $f(x) = \dfrac{1}{(1+x)\sqrt{x}}$. Here 0 is the only point of infinite discontinuity of $f$.

Here,
$$\lim_{x \to 0+} (x-0)^{1/2} f(x) = \lim_{x \to 0+} \frac{\sqrt{x}}{(1+x)\sqrt{x}} = \lim_{x \to 0+} \frac{1}{1+x}$$
$$= 1, \text{ for } \mu = \frac{1}{2} < 1.$$

So, by $\mu$-test,
$$\int_0^1 f(x)\,dx = \int_0^1 \frac{dx}{(1+x)\sqrt{x}} \text{ is convergent.}$$

(ii) Let $f(x) = (x-a)^{-m}(b-x)^{-n}$. Here $a$ and $b$ are the only points of infinite discontinuity of $f$.

Now,
$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,\ a < c < b.$$

Therefore, $\int_a^b f(x)dx$ is convergent only when $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ both are convergent.

Here, $\lim_{x\to a+}(x-a)^m f(x) = (b-a)^{-n}$ and $\lim_{x\to b-}(b-x)^n f(x) = (b-a)^{-m}$.

Therefore, $$\int_a^b f(x)dx = \int_a^b (x-a)^{-m}(b-x)^{-n}\,dx$$ is convergent only when $0 < m, n < 1$.

(*iii*) Let $f(x) = \dfrac{\log x}{\sqrt{x}}$. Here, 0 is the only point of infinite discontinuity of $f$.

Now, $$\lim_{x\to 0+}(x-0)^{3/4} f(x) = \lim_{x\to 0+} x^{3/4}\frac{\log x}{\sqrt{x}}$$

$$= \lim_{x\to 0+}\frac{\log x}{x^{-1/4}}\left(\frac{\infty}{\infty}\right) = \lim_{x\to 0+}(-4x^{1/4}) \qquad \text{[By L'Hospital rule]}$$

$$= 0, \text{ for } \mu = \frac{3}{4} < 1.$$

So, by $\mu$-test, $\int_0^1 f(x)dx = \int_0^1 \frac{\log x}{\sqrt{x}}dx$ is convergent.

(*iv*) Let $f(x) = \dfrac{\sqrt{x}}{\log x}$. Here 1 is the only point of infinite discontinuity of $f$.

Now, $$\lim_{x\to 1+}(x-1)f(x) = \lim_{x\to 1+}(x-1)\frac{\sqrt{x}}{\log x}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 1+}\frac{\frac{3}{2}x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}}}{\frac{1}{x}} \qquad \text{(By L' Hospital rule)}$$

$$= 1, \text{ for } \mu = 1.$$

So, by $\mu$-test, $\int_1^2 \frac{\sqrt{x}}{\log x}dx$ is divergent.

(*v*) Let $f(x) = \dfrac{\log x}{\sqrt{1-x}}$. Here, 0 and 1 are the only points of infinite discontinuity of $f$.

Now, $$\int_0^1 f(x)dx = \int_0^{1/2} f(x)dx + \int_{\frac{1}{2}}^1 f(x)dx$$

Therefore, $\int_0^1 f(x)dx$ is convergent only when $\int_0^{1/2} f(x)dx$ and $\int_{1/2}^1 f(x)dx$ both are convergent.

**Test of convergence of** $\int_0^{1/2} f(x)dx$

Here, 0 is the only point of infinite discontinuity of $f$.

Now, $\lim_{x\to 0+} (x-0)^{1/2} f(x) = \lim_{x\to 0+} x^{\frac{1}{2}} \frac{\log x}{\sqrt{1-x}}$.

$$\lim_{x\to 0+} x^{1/2}\log x = \lim_{x\to 0+} \frac{\log x}{x^{-1/2}} \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0+} \frac{\frac{1}{x}}{-\frac{1}{2}x^{-3/2}} \quad \text{[By L'Hospital rule]}$$

$$= \lim_{x\to 0+} (-2\sqrt{x}) = 0$$

$\therefore \quad \lim_{x\to 0+} (x-0)^{1/2} f(x) = 0, \quad$ for $\quad \mu = \frac{1}{2} < 1.$

So, by $\mu$-test, $\int_0^{1/2} f(x)dx$ is convergent.

**Test of convergence of** $\int_{1/2}^1 f(x)dx$

Here, 1 is the only point of infinite discontinuity of $f$.

$$\lim_{x\to 1-} (1-x)^{1/2} f(x) = \lim_{x\to 1-} \sqrt{1-x}\,\frac{\log x}{\sqrt{1-x}}$$

$$= \lim_{x\to 1-} \log x = 0, \text{ for } \mu = \frac{1}{2} < 1.$$

So, by $\mu$-test, $\int_{1/2}^1 f(x)dx$ is convergent.

Hence $\int_0^1 f(x)dx = \int_0^1 \frac{\log x}{\sqrt{1-x}}dx$ is convergent.

(*vi*) Let $f(x) = \dfrac{x^{m-1}}{1+x}$ and $g(x) = \dfrac{x^{-m}}{1+x}$.

For $m \geq 1$, $\int_0^1 f(x)\,dx$ is a proper integral and hence it is convergent. For $m < 1$, $\int_0^1 f(x)\,dx$ is an improper integral and 0 is the only point of infinite discontinuity of $f$.

**Test of convergence of $\int_0^1 f(x)\,dx$ when $m < 1$.**

$$\lim_{x \to 0+} (x-0)^{1-m} f(x) = \lim_{x \to 0+} x^{1-m} \frac{x^{m-1}}{1+x} = \lim_{x \to 0+} \frac{1}{1+x}$$
$$= 1.$$

So, by μ-test, $\int_0^1 f(x)dx$ is convergent if $1 - m < 1$, *i.e.*, $m > 0$.

For $m \leq 0$, $\int_0^1 g(x)\,dx$ is a proper integral and hence it is convergent. For $m > 0$, $\int_0^1 g(x)\,dx$ is an improper integral and 0 is the only point of infinite discontinuity of $g$.

**Test of convergence of $\int_0^1 g(x)\,dx$ when $m > 0$**

$$\lim_{x \to 0+} (x-0)^{m} g(x) = \lim_{x \to 0+} x^{m} \frac{x^{-m}}{1+x} = \lim_{x \to 0+} \frac{1}{1+x}$$
$$= 1.$$

So, by μ-test, $\int_0^1 g(x)dx$ is convergent if $m < 1$.

Therefore, $\int_0^1 f(x)dx$ and $\int_0^1 g(x)dx$ are both convergent when $0 < m < 1$.

Hence $\displaystyle\int_0^1 \{f(x) + g(x)\}dx = \int_0^1 \frac{x^{m-1} + x^{-m}}{1+x} dx$ is convergent when $0 < m < 1$.

**Example 5:** Test for convergence and then show that

$$\int_0^{\pi/2} \log \sin x\, dx = \int_0^{\pi/2} \log \cos x\, dx = -\frac{\pi}{2} \log 2.$$

Scanned by CamScanner

**Solution:** Let $f(x) = \log \sin x$. Here 0 is the only point of infinite discontinuity of $f(x)$ when $0 \le x \le \frac{\pi}{2}$.

Now, $\lim_{x\to 0+} (x-0)^{\mu} \log \sin x = \lim_{x\to 0+} \left( x^{\mu} \log x + x^{\mu} \log \frac{\sin x}{x} \right)$

$= 0$, for $0 < \mu < 1$. $\left[ \because \lim_{x\to 0+} x^{\mu} \log x = 0 \text{ for } 0 < \mu < 1 \text{ and } \lim_{x\to 0+} \frac{\sin x}{x} = 1 \right]$.

Hence, $\int_0^{\pi/2} \log \sin x \, dx$ is convergent.

Let $g(x) = \log \cos x$. Here $\pi/2$ is the only point of infinite discontinuity of $g(x)$ when $0 \le x \le \frac{\pi}{2}$.

$$\lim_{x\to \frac{\pi}{2}-} \left( \frac{\pi}{2} - x \right)^{\mu} \log \cos x = \lim_{x\to \frac{\pi}{2}-} \left( \frac{\pi}{2} - x \right)^{\mu} \log \sin \left( \frac{\pi}{2} - x \right)$$

$$= \lim_{\theta\to 0+} \theta^{\mu} \log \sin \theta = 0, \text{ for } 0 < \mu < 1.$$

Hence, $\int_0^{\pi/2} \log \cos x \, dx$ is convergent.

We know that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ (if these integrals are convergent)

$$\therefore \quad I = \int_0^{\pi/2} \log \sin x \, dx = \int_0^{\pi/2} \log \sin \left( \frac{\pi}{2} - x \right) dx = \int_0^{\pi/2} \log \cos x \, dx.$$

$$\therefore \quad 2I = \int_0^{\pi/2} \log \sin x \, dx + \int_0^{\pi/2} \log \cos x \, dx = \int_0^{\pi/2} \log(\sin x \cos x) \, dx$$

$$= \int_0^{\pi/2} \log \left( \frac{1}{2} \sin 2x \right) dx = -\frac{\pi}{2} \log 2 + \int_0^{\pi/2} \log \sin 2x \, dx.$$

Now, $\int_{\epsilon}^{\pi/2-\delta} \log \sin 2x \, dx = \frac{1}{2} \int_{2\epsilon}^{\pi - 2\delta} \log \sin z \, dz$ (Putting $2x = z$)

For, $\epsilon, \delta \to 0+$, we have

$$\int_0^{\pi/2} \log \sin 2x \, dx = \frac{1}{2} \int_0^{\pi} \log \sin x \, dx.$$

$$\therefore \quad 2I = -\frac{\pi}{2} \log 2 + \frac{1}{2} \int_0^{\pi} \log \sin x \, dx$$

$$= -\frac{\pi}{2} \log 2 + \frac{1}{2} \cdot 2 \int_0^{\pi/2} \log \sin x \, dx$$

$$\left[ \because \int_0^a f(x)dx = \int_0^{a/2} f(x)dx + \int_0^{a/2} f(a-x)dx \right]$$

$$= -\frac{\pi}{2}\log 2 + I.$$

$$\therefore \qquad I = -\frac{\pi}{2}\log 2, \ i.e.,$$

$$\int_0^{\pi/2} \log\sin x\, dx = \int_0^{\pi/2} \log\cos x\, dx = -\frac{\pi}{2}\log 2.$$

## 5.5 BETA AND GAMMA FUNCTIONS

### Gamma Function

**Definition:** The convergent improper integral $\int_0^\infty e^{-x}x^{n-1}\,dx\,(n>0)$ is defined as **gamma function** and it is denoted by $\Gamma(n)$.

Hence, $$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx, n>0.$$

**Relation 1:** For any $a>0$,

$$\int_0^\infty e^{-ax}x^{n-1}dx = \frac{\Gamma(n)}{a^n}, n>0.$$

**Proof:** Let us put $ax = y$ whereby

$$\int_\epsilon^B e^{-ax}x^{n-1}dx = \int_{a\epsilon}^{aB} e^{-y}\frac{y^{n-1}}{a^{n-1}}\frac{dy}{a}.$$

As $B\to\infty$ and $\epsilon\to 0+$, since $a>0$,

$$\int_0^\infty e^{-ax}x^{n-1}dx = \frac{1}{a^n}\int_0^\infty e^{-y}y^{n-1}dy = \frac{\Gamma(n)}{a^n}.$$

**Relation 2:** $\Gamma(n+1) = n\Gamma(n), n>0.$ *(M–201/13)*

**Proof:** Integration by Parts gives

$$\int_\epsilon^B e^{-x}x^{n-1}dx = \left[e^{-x}\frac{x^n}{n}\right]_\epsilon^B + \frac{1}{n}\int_\epsilon^B e^{-x}x^n dx \qquad ...(1)$$

Now, $$\lim_{B\to\infty} e^{-B}B^n = \lim_{B\to\infty}\frac{B^n}{e^B}\left(\frac{\infty}{\infty}\right) = 0. \qquad \text{[By L'Hospital rule]}$$

Therefore, as $B\to\infty$ and $\epsilon\to 0+$, the integrated part of (1) vanishes at both limits and (1) becomes:

$$\int_0^\infty e^{-x}x^{n-1}dx = \frac{1}{n}\int_0^\infty e^{-x}x^n dx$$

*i.e.,* $$\Gamma(n) = \frac{1}{n}\Gamma(n+1)$$

or $$\Gamma(n+1) = n\Gamma(n), n > 0.$$

**Relation 3:** $\Gamma(1) = 1$.

**Proof:** $$\Gamma(1) = \int_0^\infty e^{-x}dx = \lim_{B\to\infty}\int_0^B e^{-x}dx$$
$$= \lim_{B\to\infty}[-e^{-x}]_0^B = \lim_{B\to\infty}(1-e^{-B}) = 1.$$

**Relation 4:** $\Gamma(n+1) = n!$, if $n$ is a positive integer.

**Proof:** Combining Relation 2 and Relation 3, when $n$ is a positive integer,
$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = n(n-1)(n-2)\Gamma(n-2)$$
$$= n(n-1)(n-2)\ldots 3.2.1\,\Gamma(1) = n!$$

## Beta Function

**Definition:** The convergent improper integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx\ (m>0, n>0)$ is defined as Beta function and it is denoted by $B(m, n)$ or $\beta(m, n)$. So,

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx, m>0, n>0.$$

**Relation 5:** $B(m, n) = B(n, m), m > 0, n > 0.$

**Proof:** Put $x = 1 - y$, then

$$\int_\epsilon^{1-\delta} x^{m-1}(1-x)^{n-1}dx = \int_{1-\epsilon}^{\delta}(1-y)^{m-1}y^{n-1}(-dy)$$
$$= \int_\delta^{1-\epsilon}(1-y)^{m-1}y^{n-1}dy$$

As $\epsilon \to 0+, \delta \to 0+$, we have

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx = B(m,n) = \int_0^1 y^{n-1}(1-y)^{m-1}dy = B(n, m).$$

**Relation 6:** $$B(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$= \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}}\,dx = B(n,m),\ m>0,\ n>0.$$

**Proof:** Put $x = \frac{1}{1+y}$, *i.e.*, $y = \frac{1}{x} - 1$, then

$$\int_{\epsilon}^{1-\delta} x^{m-1}(1-x)^{n-1}\,dx = \int_{\frac{1}{\epsilon}-1}^{\frac{\delta}{1-\delta}} \frac{1}{(1+y)^{m-1}} \frac{y^{n-1}}{(1+y)^{n-1}}(-1)\frac{1}{(1+y)^2}\,dy$$

$$= \int_{\frac{\delta}{1-\delta}}^{\frac{1}{\epsilon}-1} \frac{y^{n-1}}{(1+y)^{m+n}}\,dy.$$

As $\epsilon \to 0+$, $\delta \to 0+$, we have

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = B(m,n) = \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$$

Also since $B(m,n) = B(n,m)$, we have

$$B(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}}\,dx = B(n,m),\ m>0,\ n>0.$$

**Relation 7:** $$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta,\ m>0,\ n>0.$$

**Proof:** Put $x = \sin^2\theta$, then

$$\int_{\epsilon}^{1-\delta} x^{m-1}(1-x)^{n-1}\,dx = \int_{\alpha}^{\beta} \sin^{2m-2}\theta \cos^{2n-2}\theta (2\sin\theta\cos\theta)\,d\theta$$

where $\alpha = \sin^{-1}\sqrt{\epsilon}$ and $\beta = \sin^{-1}\sqrt{1-\delta}$. As $\epsilon \to 0+$, $\delta \to 0+$, we have $\alpha \to 0$, $\beta \to \frac{\pi}{2}$ and

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta,\ m>0,\ n>0.$$

**Relation 8:** $B\left(\frac{1}{2},\frac{1}{2}\right) = \pi.$

**Proof:** Put $m = n = \frac{1}{2}$ in Relation 7, we have

$$B\left(\frac{1}{2},\frac{1}{2}\right) = 2\int_0^{\pi/2} d\theta = 2\cdot\frac{\pi}{2} = \pi.$$

We state below (with proof) the **relation between Beta and Gamma functions.**

**Relation 9:** $B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, m > 0, n > 0.$ *(M–201/11)*

**Proof:** We know that

$$\Gamma(m) = \int_0^\infty e^{-t}\, t^{m-1}\, dt = 2\int_0^\infty e^{-x^2}\, x^{2m-1}\, dx$$

[Put $t = x^2 \Rightarrow dt = 2x\, dx$] ...(1)

Similarly, $$\Gamma(n) = 2\int_0^\infty e^{-y^2}\, y^{2n-1}\, dy$$

$\therefore$ $$\Gamma(m)\Gamma(n) = 4\int_0^\infty e^{-x^2}\, x^{2m-1}\, dx \int_0^\infty e^{-y^2}\, y^{2n-1}\, dy$$

$$= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^{2m-1}\, y^{2n-1}\, dx\, dy$$

Let $x = r\cos\theta$, $y = r\sin\theta$ and so $dx\, dy = r\, d\theta\, dr.$

$\therefore$ $$\Gamma(m)\Gamma(n) = 4\int_{r=0}^\infty \int_{\theta=0}^{\pi/2} e^{-r^2}\, r^{2(m+n)-1} \cos^{2m-1}\theta \sin^{2n-1}\theta\, dr\, d\theta$$

$$= 2\int_0^{\pi/2}\left[2\int_0^\infty e^{-r^2}\, r^{2(m+n)-1}\, dr\right]\cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \Gamma(m+n) \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta$$ [Using (1)]

$$= \Gamma(m+n)\left[2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta\right]$$

$$= \Gamma(m+n)B(m, n)$$

[Using relation 7 and since $B(m, n) = B(n, m)$]

Hence, $$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

**Relation 10:** $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

**Proof:** From Relation 9, putting $m = n = \frac{1}{2}$, we have

$$B\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)}$$

Now from Relation 8, we have $B\left(\frac{1}{2},\frac{1}{2}\right) = \pi$ and from Relation 3, $\Gamma(1) = 1$, therefore

$$\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = \pi, \quad \text{or} \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

**Relation 11:** $\displaystyle\int_0^{\pi/2} \sin^p\theta\cos^q\theta\, d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)};\ p, q > -1.$

**Proof:** From Relation 7, we have

$$B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta;\ m, n > 0.$$

Putting $2m-1 = p, 2n-1 = q$, we have $m = \frac{1}{2}(p+1), n = \frac{1}{2}(q+1)$.

$$\therefore \quad \int_0^{\pi/2} \sin^p\theta\cos^q\theta\, d\theta = \frac{1}{2}B\left(\frac{p+1}{2},\frac{q+1}{2}\right)$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)};\ p, q > -1. \qquad \text{[Using Relation 9 and since}$$

$m, n > 0$ implies $m = \frac{1}{2}(p+1) > 0, n = \frac{1}{2}(q+1) > 0$, *i.e.*, $p, q > -1$].

**Relation 12:** $\displaystyle\Gamma(n) = 2\int_0^{\infty} e^{-x^2}x^{2n-1}dx, n > 0;$

$$\int_0^{\infty} e^{-x^2}dx = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2} \text{ and } \int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}.$$

**Proof:** Now, $\displaystyle\int_{\epsilon}^{B} e^{-x}x^{n-1}dx = 2\int_{\sqrt{\epsilon}}^{\sqrt{B}} e^{-y^2}y^{2n-1}dy$ [Putting $x = y^2$]

As $\epsilon \to 0+$ and $B \to \infty$, we have

$$\Gamma(n) = \int_0^{\infty} e^{-x}x^{n-1}dx = 2\int_0^{\infty} e^{-y^2}y^{2n-1}dy = 2\int_0^{\infty} e^{-x^2}x^{2n-1}dx, n > 0.$$

Putting $n = \frac{1}{2}$, we get

$$\int_0^\infty e^{-x^2} dx = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$ [Using Relation 10]

Also $$\int_{-\infty}^\infty e^{-x^2} dx = \lim_{A\to\infty} \int_{-A}^{A} e^{-x^2} dx = \lim_{A\to\infty} 2\int_0^A e^{-x^2} dx$$ [$\because e^{-x^2}$ is an even function]

$$= 2\int_0^\infty e^{-x^2} dx = 2\cdot\frac{\sqrt{\pi}}{2} = \sqrt{\pi}.$$

We state below two important relations without proofs.

**Relation 13:** $2^{2m-1}\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2m), m > 0.$

**Relation 14:** $\Gamma(m)\Gamma(1-m) = \pi \operatorname{cosec} m\pi, 0 < m < 1.$

**Note:** Relation 13 is called **Duplication Formula.**

## ILLUSTRATIVE EXAMPLES

**Example 1:** Evaluate:

(*i*) $\int_0^{\pi/2} \sin^4 x \cos^5 x\, dx$ (*ii*) $\int_0^1 (x \log x)^3 dx$

(*iii*) $\int_0^{\pi/2} \sin^9 x\, dx$ (*iv*) $\int_0^1 x^4(1-x)^3 dx$

(*v*) $\int_0^1 x^3(1-x^2)^{5/2} dx$ (*vi*) $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}} dx$

(*vii*) $\int_0^\infty x^{1/4} e^{-\sqrt{x}} dx$ (*viii*) $\int_0^1 x^4(1-\sqrt{x})^5 dx$

**Solution:** (*i*) $$\int_0^{\pi/2} \sin^4 x \cos^5 x\, dx = \frac{\Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{5+1}{2}\right)}{2\Gamma\left(\frac{4+5+2}{2}\right)}$$ [Using Relation 11]

$$= \frac{\Gamma\left(\frac{5}{2}\right)\Gamma(3)}{2\Gamma\left(\frac{11}{2}\right)} = \frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)2!}{2\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}$$ [Using Relations 2 and 4]

$$= \frac{8}{315}.$$

*(ii)* Put $x = e^{-y}$, hence $\log x = -y$. $\therefore\ dx = -e^{-y}\,dy$

$$\therefore \quad \int_0^1 (x\log x)^3\,dx = \lim_{\epsilon\to 0+}\int_\epsilon^1 (x\log x)^3\,dx = \lim_{\epsilon\to 0+}\int_{-\log\epsilon}^{0} (-e^{-y}y)^3(-e^{-y}dy)$$

$$= \lim_{\epsilon\to 0+} -\int_0^{-\log\epsilon} e^{-4y}y^3\,dy = -\int_0^\infty e^{-4y}y^{4-1}\,dy$$

$$= -\frac{\Gamma(4)}{4^4} \qquad \text{[Using Relation 1]}$$

$$= -\frac{3!}{4^4} = -\frac{3}{128}.$$

*(iii)* $$\int_0^{\pi/2}\sin^9 x\,dx = \int_0^{\pi/2}\sin^9 x\cos^0 x\,dx$$

$$= \frac{\Gamma\left(\frac{9+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{9+0+2}{2}\right)} \qquad \text{[Using Relation 11]}$$

$$= \frac{\Gamma(5)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{11}{2}\right)} = \frac{4!\,\Gamma\left(\frac{1}{2}\right)}{2\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} \qquad \text{[Using Relations 2 and 4]}$$

$$= \frac{128}{315}.$$

*(iv)* $$\int_0^1 x^4(1-x)^3\,dx = \int_0^1 x^{5-1}(1-x)^{4-1}\,dx = B(5,4)$$

$$= \frac{\Gamma(5)\Gamma(4)}{\Gamma(5+4)} \qquad \text{[Using Relation 9]}$$

$$= \frac{3!\,\Gamma(5)}{8.7.6.5\,\Gamma(5)} = \frac{1}{280}.$$

*(v)* Put $x = \sin\theta$, then $dx = \cos\theta\,d\theta$.

$$\therefore \quad \int_0^1 x^3(1-x^2)^{5/2}\,dx = \int_0^{\pi/2}\sin^3\theta\cos^5\theta\cos\theta\,d\theta$$

$$= \int_0^{\pi/2}\sin^3\theta\cos^6\theta\,d\theta = \frac{\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{6+1}{2}\right)}{2\Gamma\left(\frac{3+6+2}{2}\right)} \qquad \text{[Using Relation 11]}$$

$$= \frac{\Gamma(2)\Gamma\left(\frac{7}{2}\right)}{2\Gamma\left(\frac{11}{2}\right)} = \frac{\Gamma\left(\frac{7}{2}\right)}{2\cdot\frac{9}{2}\cdot\frac{7}{2}\Gamma\left(\frac{7}{2}\right)} \quad \text{[Using Relations 2 and 4]}$$

$$= \frac{2}{63}.$$

(*vi*)
$$\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx = \lim_{B\to\infty}\int_0^B \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx$$

$$= \lim_{B\to\infty}\int_0^B \frac{x^8}{(1+x)^{24}}\,dx - \lim_{B\to\infty}\int_0^B \frac{x^{14}}{(1+x)^{24}}\,dx$$

$$= \int_0^\infty \frac{x^8}{(1+x)^{24}}\,dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}\,dx$$

$$= \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}}\,dx - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}}\,dx$$

$$= B(9, 15) - B(15, 9)$$

$$= 0 \quad [\because B(m, n) = B(n, m)]. \quad \text{[Using Relation 6]}$$

(*vii*) Put $\sqrt{x} = t$, or $x = t^2$ then $dx = 2t\,dt$.

$$\therefore \quad \int_0^\infty x^{1/4}e^{-\sqrt{x}}\,dx = \lim_{B\to\infty}\int_0^B x^{1/4}e^{-\sqrt{x}}\,dx = \lim_{B\to\infty}\int_0^{\sqrt{B}} t^{1/2}e^{-t}\,2t\,dt$$

$$= 2\int_0^\infty t^{3/2}e^{-t}\,dt = 2\int_0^\infty e^{-t}t^{\frac{5}{2}-1}\,dt = 2\Gamma\left(\frac{5}{2}\right)$$

$$= 2\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) \quad \text{[Using Relation 2]}$$

$$= \frac{3}{2}\sqrt{\pi} \quad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right].$$

(*viii*) Put $\sqrt{x} = t$, or $x = t^2$ then $dx = 2t\,dt$.

$$\therefore \quad \int_0^1 x^4(1-\sqrt{x})^5\,dx = \int_0^1 (t^2)^4(1-t)^5\,2t\,dt$$

$$= 2\int_0^1 t^9(1-t)^5\,dt = 2\int_0^1 t^{10-1}(1-t)^{6-1}\,dt$$

$$= 2B(10, 6) = 2\frac{\Gamma(10)\Gamma(6)}{\Gamma(10+6)} \quad \text{[Using Relation 9]}$$

$$= 2\frac{9!5!}{(15)!} \qquad \text{[Using Relation 4]}$$

$$= 2\cdot\frac{5!}{15\cdot14\cdot13\cdot12\cdot11\cdot10}$$

$$= \frac{2\cdot5\cdot4\cdot3\cdot2\cdot1}{15\cdot14\cdot13\cdot12\cdot11\cdot10} = \frac{1}{15015}.$$

**Example 2:** Assuming the convergence of the integral, prove that $\int_0^\infty \sqrt{x}e^{-x^3}\,dx = \frac{1}{3}\sqrt{\pi}$.

**Solution:** Let us put $x = y^3$, whereby

$$\int_\in^B e^{-x}n^{n-1}dx = \int_{\in^{1/3}}^{B^{1/3}} e^{-y^3}y^{3n-3}3y^2\,dy \qquad [\because dx = 3y^2dy]$$

$$= 3\int_{\in^{1/3}}^{B^{1/3}} e^{-y^3}y^{3n-1}dy.$$

As $B \to \infty$ and $\in \to 0+$, we have

$$\int_0^\infty e^{-x}x^{n-1}dx = 3\int_0^\infty e^{-y^3}y^{3n-1}dy$$

$\therefore$
$$\int_0^\infty e^{-y^3}y^{3n-1}dy = \frac{1}{3}\int_0^\infty e^{-x}x^{n-1}dx = \frac{1}{3}\Gamma(n).$$

Putting $n = \frac{1}{2}$, we get

$$\int_0^\infty e^{-y^3}y^{1/2}dy = \frac{1}{3}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{3} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$\therefore$
$$\int_0^\infty \sqrt{x}e^{-x^3}dx = \frac{1}{3}\sqrt{\pi}.$$

**Example 3:** Assuming convergence of the integrals and using the relation $\Gamma(m)\Gamma(1-m) = \pi\,\text{cosec}\,(m\pi)$, $0 < m < 1$, show that

$$\int_0^\infty e^{-x^4}dx \times \int_0^\infty x^2e^{-x^4}dx = \frac{\pi}{8\sqrt{2}}.$$

**Solution:**

Now,
$$\int_\in^B e^{-x}x^{n-1}dx = \int_{\in^{1/4}}^{B^{1/4}} e^{-y^4}y^{4n-4}(4y^3dy) \qquad [\text{Putting } x = y^4]$$

$$= 4\int_{\epsilon^{1/4}}^{B^{1/4}} e^{-y^4} y^{4n-1} dy$$

As $B \to \infty, \epsilon \to 0+$, we have

$$\int_0^\infty e^{-x} x^{n-1} dx = 4\int_0^\infty e^{-y^4} y^{4n-1} dy$$

$\therefore$
$$\int_0^\infty e^{-y^4} y^{4n-1} dy = \frac{1}{4}\int_0^\infty e^{-x} x^{n-1} dx = \frac{1}{4}\Gamma(n). \quad \text{...(1)}$$

Putting $n = \frac{1}{4}$ in (1), we get

$$\int_0^\infty e^{-y^4} dy = \frac{1}{4}\Gamma\left(\frac{1}{4}\right)$$

Putting $n = \frac{3}{4}$ in (1), we get

$$\int_0^\infty e^{-y^4} y^2 dy = \frac{1}{4}\Gamma\left(\frac{3}{4}\right)$$

$\therefore$
$$\int_0^\infty e^{-x^4} dx \times \int_0^\infty x^2 e^{-x^4} dx = \frac{1}{16}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$

$$= \frac{1}{16}\pi \operatorname{cosec}\left(\frac{\pi}{4}\right) \quad [\because \Gamma(m)\Gamma(1-m) = \pi \operatorname{cosec}(m\pi), 0 < m < 1]$$

$$= \frac{\sqrt{2}\pi}{16} = \frac{\pi}{8\sqrt{2}}.$$

**Example 4:** Show that $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \frac{\pi}{\sqrt{2}}$. *(W.B.T.U. 2012)*

**Solution:** We know that

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}; p, q > -1.$$

$\therefore$
$$\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2} \sin^{\frac{1}{2}}\theta \cos^{-\frac{1}{2}}\theta\, d\theta$$

$$= \frac{\Gamma\left(\dfrac{\frac{1}{2}+1}{2}\right)\Gamma\left(\dfrac{-\frac{1}{2}+1}{2}\right)}{2\Gamma\left(\dfrac{\frac{1}{2}-\frac{1}{2}+2}{2}\right)} = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2\Gamma(1)}$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) \qquad [\because \Gamma(1) = 1]$$

$$= \frac{1}{2}\pi \operatorname{cosec} \frac{\pi}{4} \qquad [\because \Gamma(m)\Gamma(1-m) = \pi \operatorname{cosec} m\pi,\ 0 < m < 1]$$

$$= \frac{1}{2}\pi\sqrt{2} = \frac{\pi}{\sqrt{2}}.$$

**Example 5:** Show that $\displaystyle\int_0^{\pi/2} \sin^{p-1} x\, dx = \frac{\sqrt{\pi}\,\Gamma\left(\frac{p}{2}\right)}{2\Gamma\left(\frac{p+1}{2}\right)}, p > 0.$

**Solution:** We know that

$$B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1} x \cos^{2n-1} x\, dx,\ m > 0, n > 0.$$

$$\therefore \quad \int_0^{\pi/2} \sin^{p-1} x\, dx = \int_0^{\pi/2} \sin^{2\cdot\frac{p}{2}-1} x \cos^{2\cdot\frac{1}{2}-1} x\, dx$$

$$= \frac{1}{2}B\left(\frac{p}{2}, \frac{1}{2}\right) = \frac{1}{2}\frac{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{p}{2}+\frac{1}{2}\right)} \qquad \left[\because B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right]$$

$$= \frac{\sqrt{\pi}\,\Gamma\left(\frac{p}{2}\right)}{2\Gamma\left(\frac{p+1}{2}\right)}.$$

**Example 6:** Show that $\displaystyle\int_0^{\infty} x^m e^{-ax^n} dx = \frac{1}{na^{\frac{m+1}{n}}}\Gamma\left(\frac{m+1}{n}\right)$, where $m > 0, n > 0, a > 0.$

**Solution:** We know that

$$\Gamma\left(\frac{m+1}{n}\right) = \int_0^{\infty} e^{-x} x^{\frac{m+1}{n}-1} dx$$

Scanned by CamScanner

$$= \int_0^\infty e^{-x} x^{\frac{m-n+1}{n}} dx.$$

$$\therefore \quad \int_\epsilon^B e^{-x} x^{\frac{m-n+1}{n}} dx = \int_{\left(\frac{\epsilon}{a}\right)^{1/n}}^{\left(\frac{B}{a}\right)^{1/n}} e^{-ay^n} (ay^n)^{\frac{m-n+1}{n}} nay^{n-1} dy \qquad [\text{Putting } x = ay^n]$$

As $B \to \infty$ and $\epsilon \to 0+$, we have

$$\int_0^\infty e^{-x} x^{\frac{m-n+1}{n}} dx = na^{\frac{m+1}{n}} \int_0^\infty e^{-ay^n} y^m dy.$$

$$\therefore \quad \Gamma\left(\frac{m+1}{n}\right) = na^{\frac{m+1}{n}} \int_0^\infty x^m e^{-ax^n} dx.$$

or

$$\int_0^\infty x^m e^{-ax^n} dx = \frac{1}{na^{\frac{m+1}{n}}} \Gamma\left(\frac{m+1}{n}\right).$$

**Example 7:** For $m > -1, n > -1$, prove that $\int_a^b (x-a)^m (b-x)^n dx = (b-a)^{m+n+1} \dfrac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}$.

**Solution:** We know that

$$B(m+1, n+1) = \int_0^1 x^m (1-x)^n dx, m > -1, n > -1.$$

Put $x = \dfrac{y-a}{b-a}$, then $dx = \dfrac{dy}{b-a}$, $y = (b-a)x + a$,

whereby

$$\int_\epsilon^{1-\delta} x^m (1-x)^n dx = \int_{(b-a)\epsilon+a}^{(b-a)(1-\delta)+a} \left(\frac{y-a}{b-a}\right)^m \left\{1 - \left(\frac{y-a}{b-a}\right)\right\}^n \frac{dy}{b-a}.$$

As $\epsilon \to 0+, \delta \to 0+$, we have

$$\int_0^1 x^m (1-x)^n dx = \frac{1}{(b-a)^{m+n+1}} \int_a^b (y-a)^m (b-y)^n dy.$$

$$= \frac{1}{(b-a)^{m+n+1}} \int_a^b (x-a)^m (b-x)^n dx.$$

$$\therefore \quad \int_a^b (x-a)^m (b-x)^n dx = (b-a)^{m+n+1} B(m+1, n+1)$$

Scanned by CamScanner

$$= (b-a)^{m+n+1}\frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)} \qquad \text{[Using Relation 9]}$$

**Example 8:** Prove that

(*i*) $B(m,n) = B(m+1,n)+B(m,n+1),\ m,n>0$

(*ii*) $\dfrac{B(m+1,n)}{m} = \dfrac{B(m,n+1)}{n} = \dfrac{B(m,n)}{m+n},\ m,n>0.$

**Solution:** (*i*) R.H.S. $= B(m+1,n)+B(m,n+1)$

$$= \frac{\Gamma(m+1)\Gamma(n)}{\Gamma(m+n+1)}+\frac{\Gamma(m)\Gamma(n+1)}{\Gamma(m+n+1)} \qquad \text{[Using Relation 9]}$$

$$= \frac{m\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)}+\frac{n\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)} \qquad \text{[Using Relation 2]}$$

$$= \frac{(m+n)\Gamma(m)\Gamma(n)}{(m+n)\Gamma(m+n)} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$= B(m,n) =$ L.H.S.

(*ii*) Now, $B(m+1,n) = \dfrac{\Gamma(m+1)\Gamma(n)}{\Gamma(m+n+1)}$ [By Relation 9]

$$= \frac{m\Gamma(m)\Gamma(n)}{(m+n)\Gamma(m+n)} \qquad \text{[By Relation 2]}$$

$$= \frac{mB(m,n)}{m+n}.$$

$\therefore \quad \dfrac{B(m+1,n)}{m} = \dfrac{B(m,n)}{m+n}$

Similarly, $\dfrac{B(m,n+1)}{n} = \dfrac{B(m,n)}{m+n}.$

Hence, the result follows.

**Example 9:** Show that $\displaystyle\int_0^1 \frac{x^2dx}{\sqrt{1-x^4}}\times\int_0^1\frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4\sqrt{2}}$

**Solution:** Now, $\displaystyle\int_0^1 \frac{x^2dx}{\sqrt{1-x^4}} = \int_0^{\pi/2}\frac{\sin\theta}{\cos\theta}\frac{\cos\theta}{2\sqrt{\sin\theta}}d\theta$ [Putting $x^2 = \sin\theta$]

$$= \frac{1}{2}\int_0^{\pi/2}\sqrt{\sin\theta}\,d\theta$$

$$= \frac{1}{4}\cdot 2\int_0^{\pi/2}\sin^{2\cdot\frac{3}{4}-1}\theta\cos^{2\cdot\frac{1}{2}-1}\theta\,d\theta$$

Scanned by CamScanner

$$= \frac{1}{4} B\left(\frac{3}{4}, \frac{1}{2}\right) \qquad \text{[By Relation 7]}$$

$$= \frac{1}{4} \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{2}\right)} = \frac{1}{4} \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)}$$

$$= \frac{1}{4} \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)} = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}\right)} \qquad \text{...(1)}$$

$$\int_0^1 \frac{dx}{\sqrt{1+x^4}} = \int_0^{\pi/4} \frac{1}{\sec\theta} \frac{\sec^2\theta \, d\theta}{2\sqrt{\tan\theta}} \qquad \left[\text{Putting } x^2 = \tan\theta, \text{ then } dx = \frac{\sec^2\theta \, d\theta}{2\sqrt{\tan\theta}}\right]$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/4} \frac{d\theta}{\sqrt{\sin 2\theta}} = \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \sin^{-1/2}\varphi \, d\varphi \qquad [\text{Putting } 2\theta = \varphi]$$

$$= \frac{1}{4\sqrt{2}} \cdot 2 \int_0^{\pi/2} \sin^{2\cdot\frac{1}{4}-1}\varphi \cos^{2\cdot\frac{1}{2}-1}\varphi \, d\varphi$$

$$= \frac{1}{4\sqrt{2}} B\left(\frac{1}{4}, \frac{1}{2}\right) \qquad \text{[By Relation 7]}$$

$$= \frac{1}{4\sqrt{2}} \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)} = \frac{1}{4\sqrt{2}} \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} \qquad \text{...(2)}$$

From (1) and (2), we have

$$\int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{1}{4\sqrt{2}} \left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = \frac{\pi}{4\sqrt{2}}.$$

**Example 10:** Prove that (*i*) $B\left(m, \frac{1}{2}\right) = 2^{2m-1} B(m, m)$

(*ii*) $\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma(2m)$.

**Solution:** (*i*) We know that

$$B(m, n) = 2 \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta \qquad \text{...(1)}$$

Putting $n = \frac{1}{2}$ in (1), we get

$$B\left(m, \frac{1}{2}\right) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta \qquad \text{...(2)}$$

Again putting $n = m$ in (1), we get

$$B(m, m) = 2\int_0^{\pi/2} (\sin\theta\cos\theta)^{2m-1} d\theta$$

$$= \frac{1}{2^{2m-2}}\int_0^{\pi/2} \sin^{2m-1} 2\theta \, d\theta$$

$$= \frac{1}{2^{2m-1}}\int_0^{\pi} \sin^{2m-1} \varphi \, d\varphi \qquad \text{[Putting } 2\theta = \varphi]$$

$$= \frac{1}{2^{2m-1}} 2\int_0^{\pi/2} \sin^{2m-1} \varphi \, d\varphi$$

$$\left[\because \int_0^a f(x)dx = \int_0^{a/2} f(x)dx + \int_0^{a/2} f(a-x)dx\right]$$

$$\therefore \quad 2^{2m-1} B(m, m) = 2\int_0^{\pi/2} \sin^{2m-1} \varphi \, d\varphi = B\left(m, \frac{1}{2}\right) \quad \text{[By (2)].} \qquad \text{...(3)}$$

(*ii*) From (3), using the relation $B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, we get

$$2^{2m-1}\frac{\Gamma(m)\Gamma(m)}{\Gamma(m+m)} = \frac{\Gamma(m)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)}$$

$$\therefore \quad \Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m) \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

**Example 11:** Evaluate $\int_0^1 x^4 \left\{\log\left(\frac{1}{x}\right)\right\}^3 dx$.

**Solution:** $$\int_\epsilon^1 x^4 \left\{\log\left(\frac{1}{x}\right)\right\}^3 dx = -\frac{1}{625}\int_{5\log\left(\frac{1}{\epsilon}\right)}^{0} e^{-t} t^3 dt \qquad \left[\begin{array}{l}\text{Put } x = e^{-\frac{t}{5}}, \text{ then} \\ \log\left(\frac{1}{x}\right) = \frac{t}{5}, dx = -\frac{1}{5}e^{-\frac{t}{5}}dt\end{array}\right]$$

## MULTIPLE CHOICE QUESTIONS

1. The value of $\int_{2}^{\infty} \frac{1}{x^2} dx$ is

(a) 0 (b) 1 (c) 2 (d) $\frac{1}{2}$.

2. The value of $\int_{1}^{\infty} \frac{1}{1+x^2} dx$ is

(a) 1 (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) 0.

3. The value of $\int_{-\infty}^{1} e^x dx$ is

(a) 1 (b) $e$ (c) $e^2$ (d) 0.

4. The value of $\int_{-\infty}^{0} \frac{dx}{(1-x)^2}$ is

(a) 1 (b) 2 (c) $\frac{1}{2}$ (d) none of these.

5. The value of $\int_{-\infty}^{\infty} \frac{dx}{a^2+x^2} (a>0)$ is

(a) $\pi$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{a}$ (d) $-\frac{\pi}{a}$.

6. The value of $\int_{0}^{\pi/2} \tan x \, dx$ is

(a) 0 (b) $\frac{\pi}{2}$ (c) $\pi$ (d) does not exist.

7. The value of $\int_{-a}^{a} \frac{dx}{\sqrt{a^2-x^2}}$ is

(a) $\frac{\pi}{2}$ (b) $\pi$ (c) $2\pi$ (d) none of these.

8. The value of $\int_{-1}^{1} \frac{dx}{x^{2/3}}$ is

(a) 0 (b) 3 (c) 6 (d) does not exist.

9. The integral $\int_{a}^{\infty} \frac{dx}{x^{\mu}} (a > 0)$ exists if

(a) $\mu > 1$ (b) $\mu = 1$ (c) $\mu < 1$ (d) none of these.

10. If $\lim_{x \to \infty} x^{\mu} f(x)$ exists, then $\int_{a}^{\infty} f(x)dx$ converges absolutely, when

(a) $\mu < 1$ (b) $\mu > 1$ (c) $\mu = 1$ (d) none of these.

11. If $\lim_{x \to \infty} x^{\mu} f(x) = \lambda (\neq 0)$ or $\pm \infty$, then $\int_{a}^{\infty} f(x)dx$ diverges when

(a) $\mu > 1$ (b) $\mu \to \infty$ (c) $\mu \leq 1$ (d) none of these.

12. The improper integral $\int_{1}^{\infty} \frac{xdx}{(1+x)^3}$

(a) converges (b) diverges to $+\infty$
(c) oscillatory (d) none of these.

13. The integral $\int_{0}^{\pi/2} \frac{\sqrt{x}}{\sin x} dx$

(a) diverges to $+\infty$ (b) converges (c) oscillatory (d) none of these.

14. The integral $\int_{a}^{b} \frac{dx}{(x-a)^{\mu}}$ exists if

(a) $\mu > 1$ (b) $\mu = 1$ (c) $\mu < 1$ (d) none of these.
(M-201/12)

15. If $a$ is the only point of infinite discontinuity of $f(x)$, $a \leq x \leq b$, and $\lim_{x \to a+} (x-a)^{\mu} f(x)$ exists then $\int_{a}^{b} f(x)dx$ converges absolutely when

(a) $\mu \leq 0$ (b) $\mu > 1$ (c) $\mu = 1$ (d) $0 < \mu < 1$.

16. If $a$ is the only point of infinite discontinuity of $f(x)$, $a \leq x \leq b$, and $\lim_{x \to a+} (x-a)^{\mu} f(x)$ $= \lambda (\neq 0)$ or $\pm \infty$ then $\int_{a}^{b} f(x)dx$ diverges when

(a) $\mu \geq 1$ (b) $0 < \mu < 1$ (c) $\mu \leq 0$ (d) none of these.

17. The integral $\int_0^1 \frac{e^{-x}}{\sqrt{x}}\,dx$

(a) diverges to $+\infty$ (b) converges
(c) oscillatory (d) none of these.

18. The integral $\int_0^2 \frac{dx}{2-x}$

(a) does not exist (b) exists (c) none of these.

19. $\int_0^\infty e^{-x^2}\,dx =$

(a) $\pi$ (b) $\frac{\pi}{2}$ (c) $\sqrt{\pi}$ (d) $\frac{\sqrt{\pi}}{2}$.

20. $B\left(\frac{1}{2}, \frac{1}{2}\right) =$

(a) $\sqrt{\pi}$ (b) $\frac{\sqrt{\pi}}{2}$ (c) $\pi$ (d) $\frac{\pi}{2}$. *(M-201/11)*

21. $\Gamma(3.5) =$

(a) $\sqrt{\pi}$ (b) $\frac{15}{8}\sqrt{\pi}$ (c) $\frac{15}{16}\sqrt{\pi}$ (d) none of these.

22. $\frac{\Gamma(6)}{\Gamma(3)} =$

(a) 60 (b) 120 (c) 240 (d) none of these.

23. $\frac{\Gamma(7/2)}{\Gamma(1/2)} =$

(a) $\frac{3}{4}$ (b) $\frac{3}{8}$ (c) $\frac{15}{8}$ (d) none of these.

24. $\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{7}{2}\right) =$

(a) $\frac{45}{32}\pi^2$ (b) $\frac{45}{32}\pi$ (c) $\frac{45}{64}\pi$ (d) $\frac{45}{64}\pi^2$

25. $B\left(\frac{3}{2}, 2\right) =$

(a) $\frac{4}{15}\pi$ (b) $\frac{4}{15}$ (c) $\frac{4}{15}\sqrt{\pi}$ (d) none of these.

26. The value of $\int_0^\infty \frac{\sin t}{t}\,dt$ is equal to

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$ *(W.B.U.T. 2009)*

**27.** The singularities of the integral $\int_{-1}^{2} \frac{dx}{x(x-1)}$ are

(*a*) 0, 1 (*b*) 1, 2
(*c*) −1, 2 (*d*) 0, 2 *(M–201/11)*

**28.** The value of $\Gamma\left(\frac{1}{2}\right)$ is

(*a*) $2\pi$ (*b*) $\sqrt{\pi}$
(*c*) $\frac{\pi}{2}$ (*d*) none of these *(M–201/11)*

**29.** The value of $\Gamma(m)\Gamma(1-m)$ is.

(*a*) $\frac{2\pi}{\sin \pi}$ (*b*) $\frac{3\pi}{\sin m\pi}$
(*c*) $\frac{\pi}{\sin m\pi}$ (*d*) none of these *(M–201/11)*

**30.** The value of $\int_0^\infty e^{-x}\, x^{3/2}\, dx$ is

(*a*) $\frac{3}{4}\sqrt{\pi}$ (*b*) $\frac{5}{4}\sqrt{5}$
(*c*) $\frac{3}{5}\sqrt{\pi}$ (*d*) $\frac{1}{4}\sqrt{\pi}$ *(M–201/12)*

**31.** The value of $\Gamma(6)$ is

(*a*) 720 (*b*) 5
(*c*) 6 (*d*) 120 *(M–201/12)*

**32.** $\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)$ equals to

(*a*) $\frac{2\pi}{\sqrt{3}}$ (*b*) $\frac{3\pi}{\sqrt{2}}$
(*c*) $\frac{\pi}{\sqrt{3}}$ (*d*) $\frac{\pi}{\sqrt{2}}$ *(M–201/13)*

[**Hint:** $\Gamma(m)\,\Gamma(1-m) = \pi \operatorname{cosec} m\pi,\ 0 < m < 1$]

**33.** $\int_{-\infty}^{\infty} xe^{-x^2}\, dx =$

(*a*) −1 (*b*) 0
(*c*) 1 (*d*) none of these *(M–201/13)*

[See page 256, Ex. (*vi*)]

| | | | | |
|---|---|---|---|---|
| 1. (*d*) | 2. (*c*) | 3. (*b*) | 4. (*a*) | 5. (*c*) |
| 6. (*d*) | 7. (*b*) | 8. (*c*) | 9. (*a*) | 10. (*b*) |
| 11. (*c*) | 12. (*a*) | 13. (*b*) | 14. (*c*) | 15. (*d*) |
| 16. (*a*) | 17. (*b*) | 18. (*a*) | 19. (*d*) | 20. (*c*) |
| 21. (*b*) | 22. (*a*) | 23. (*c*) | 24. (*d*) | 25. (*b*) |
| 26. (*d*) | 27. (*d*) | 28. (*b*) | 29. (*c*) | 30. (*a*) |
| 31. (*d*) | 32. (*a*) | 33. (*b*) | | |

## PROBLEMS

1. Show that

(*i*) $\int_0^\infty e^{-x}dx = 1$

(*ii*) $\int_2^\infty \frac{dx}{x^2-1} = \frac{1}{2}\log 3$

(*iii*) $\int_1^\infty \frac{dx}{x^{3/2}} = 2$

(*iv*) $\int_0^\infty \frac{dx}{4+9x^2} = \frac{\pi}{12}$

(*v*) $\int_0^4 \frac{dx}{\sqrt{16-x^2}} = \frac{\pi}{2}$

(*vi*) $\int_{-1}^1 \sqrt{\frac{1+x}{1-x}}dx = \pi$

(*vii*) $\int_0^\infty \frac{dx}{(1+x^2)^4} = \frac{5}{32}\pi$

(*viii*) $\int_{-\infty}^\infty \frac{x\,dx}{x^4+1} = 0.$

2. Show that the Cauchy's principal value of

(*i*) $\int_{-\infty}^\infty \frac{dx}{x^3}$ is 0.

(*ii*) $\int_1^2 \frac{dx}{x}$ is log 2.

(*iii*) $\int_{-1}^1 \frac{dx}{|x|}$ does not exist.

3. Show that

(*i*) $\int_0^\infty \frac{dx}{x}$ and $\int_a^b \frac{dx}{x-a}$ $(0<a<b)$ diverge to $+\infty$.

(*ii*) $\int_a^\infty \sin x\,dx$ and $\int_{-\infty}^\infty \sin x\,dx$ are oscillatory.

(*iii*) $\int_0^a \frac{dx}{x\log x}$ and $\int_a^1 \frac{dx}{x\log x}$ diverge to $-\infty$, where $0<a<1$.

(*iv*) $\int_a^\infty \frac{dx}{x\log x}$ $(a>1)$ diverges to $\infty$.

(*v*) $\int_a^\infty e^{-pt}dt$ and $\int_{-\infty}^b e^{pt}dt$ converge for any constant $p>0$ and diverge for $p<0$.

4. Prove that the following integrals converge:

(i) $\int_0^\infty \frac{\sin x}{1+x^2}dx$ (ii) $\int_0^\infty \frac{\sin x\, dx}{1+\cos x+e^x}$

(iii) $\int_0^\infty \frac{x^2 dx}{(a^2+x^2)^2}, a>0$ (iv) $\int_0^\infty \frac{(x-1)\sqrt{x}}{1+x+x^2+\cos x}dx$

(v) $\int_0^1 \frac{dx}{x+\sqrt{x}}$ (vi) $\int_0^1 \frac{x^{m-1}}{1+x}dx, m>0$

(vii) $\int_0^1 \frac{\log x}{1+x}dx$ (viii) $\int_0^1 \frac{\log x}{1-x^2}dx.$

5. Prove that the following integrals are divergent:

(i) $\int_1^\infty \frac{dx}{(x^2+x)^{1/3}}$ (ii) $\int_0^\infty \frac{dx}{\sqrt{x^2+1}}$

(iii) $\int_0^\infty \frac{x^2}{x^3+1}dx$ (iv) $\int_0^1 \frac{dx}{\sqrt{x(1+x)}}$

(v) $\int_0^a \frac{dx}{x^2(1+x)^2}, a>0$ (vi) $\int_0^\pi \frac{\sqrt{x}dx}{\sin x}.$

6. Prove that

(i) $\Gamma(n) = \int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx, n>0.$

(ii) $B(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx;\ m,n>0.$

7. Show that

(i) $\int_0^1 \frac{dx}{\sqrt{(1-x^4)}} = \frac{\sqrt{\pi}}{4}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$ (ii) $\int_0^\infty \frac{x^c}{c^x}dx = \frac{\Gamma(c+1)}{(\log c)^{c+1}}.$

8. Prove that

(i) $\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \frac{\pi}{\sqrt{2}}.$ (ii) $\int_0^{\pi/2} \sqrt{\sin\theta}d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \pi.$

(iii) $\int_0^1 \frac{x\,dx}{\sqrt{1-x^5}} = \frac{1}{5}B\left(\frac{2}{5},\frac{1}{2}\right).$ (iv) $\int_0^1 x^3(1-\sqrt{x})^5 dx = 2B(8,6).$

(v) $\int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{1}{4\sqrt{2}}B\left(\frac{1}{4},\frac{1}{2}\right)$

4. Prove that the following integrals converge:

(i) $\int_0^\infty \frac{\sin x}{1+x^2}dx$ (ii) $\int_0^\infty \frac{\sin x\, dx}{1+\cos x+e^x}$

(iii) $\int_0^\infty \frac{x^2 dx}{(a^2+x^2)^2}, a>0$ (iv) $\int_0^\infty \frac{(x-1)\sqrt{x}}{1+x+x^2+\cos x}dx$

(v) $\int_0^1 \frac{dx}{x+\sqrt{x}}$ (vi) $\int_0^1 \frac{x^{m-1}}{1+x}dx, m>0$

(vii) $\int_0^1 \frac{\log x}{1+x}dx$ (viii) $\int_0^1 \frac{\log x}{1-x^2}dx.$

5. Prove that the following integrals are divergent:

(i) $\int_1^\infty \frac{dx}{(x^2+x)^{1/3}}$ (ii) $\int_0^\infty \frac{dx}{\sqrt{x^2+1}}$

(iii) $\int_0^\infty \frac{x^2}{x^3+1}dx$ (iv) $\int_0^1 \frac{dx}{\sqrt{x(1+x)}}$

(v) $\int_0^a \frac{dx}{x^2(1+x)^2}, a>0$ (vi) $\int_0^\pi \frac{\sqrt{x}dx}{\sin x}.$

6. Prove that

(i) $\Gamma(n) = \int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx, n>0.$

(ii) $B(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx;\ m,n>0.$

7. Show that

(i) $\int_0^1 \frac{dx}{\sqrt{(1-x^4)}} = \frac{\sqrt{\pi}}{4}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$ (ii) $\int_0^\infty \frac{x^c}{c^x}dx = \frac{\Gamma(c+1)}{(\log c)^{c+1}}.$

8. Prove that

(i) $\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \frac{\pi}{\sqrt{2}}.$ (ii) $\int_0^{\pi/2} \sqrt{\sin\theta}\,d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \pi.$

(iii) $\int_0^1 \frac{x\,dx}{\sqrt{1-x^5}} = \frac{1}{5}B\left(\frac{2}{5},\frac{1}{2}\right).$ (iv) $\int_0^1 x^3(1-\sqrt{x})^5 dx = 2B(8,6).$

(v) $\int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{1}{4\sqrt{2}}B\left(\frac{1}{4},\frac{1}{2}\right)$