c, $\alpha < c < \beta$, such that

$$f'(c) = \frac{f(\beta) - f(\alpha)}{\beta - \alpha} \qquad \text{...(1)}$$

Here $\qquad f(x) = x^2 + 2ax + b$, therefore $f'(x) = 2x + 2a$ and

$$f(\beta) - f(\alpha) = \beta^2 + 2a\beta + b - (\alpha^2 + 2a\alpha + b) = (\beta - \alpha)(\beta + \alpha) + 2a(\beta - \alpha)$$

$$= (\beta - \alpha)(\beta + \alpha + 2a)$$

$\therefore \qquad \dfrac{f(\beta) - f(\alpha)}{\beta - \alpha} = \beta + \alpha + 2a.$ Also $f'(c) = 2c + 2a.$

Hence, from (1), we have

$$2c + 2a = \beta + \alpha + 2a. \qquad \therefore \quad c = \frac{\alpha + \beta}{2}.$$

Therefore, the chord joining the points at $x = \alpha$ and $x = \beta$ is parallel to the tangent at the point $\dfrac{\alpha + \beta}{2}$.

Example 7: Use mean value theorem to prove the following inequalities:

(*i*) $0 < \dfrac{1}{x} \log_e \dfrac{e^x - 1}{x} < 1$ (*W.B.U.T. 2002, 2012*)

(*ii*) $\dfrac{x}{1+x} < \log_e(1+x) < x$ if $x > 0$ (*W.B.U.T. 2011*)

(*iii*) $\dfrac{x}{1+x^2} < \tan^{-1} x < x$ when $0 < x < \dfrac{\pi}{2}$ (*W.B.U.T. 2008, 2010*)

(*iv*) $x < -\ln(1-x) < \dfrac{x}{1-x}$ when $0 < x < 1$

(*v*) $\dfrac{b-a}{\sqrt{(1-a^2)}} < \sin^{-1} b - \sin^{-1} a < \dfrac{b-a}{\sqrt{(1-b^2)}}$, $0 < a < b < 1$ (*W.B.U.T. 2008*)

(*vi*) $\dfrac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \dfrac{b-a}{1+a^2}$, where $0 < a < b$ and hence deduce that

$\dfrac{\pi}{4} + \dfrac{3}{25} < \tan^{-1}\dfrac{4}{3} < \dfrac{\pi}{4} + \dfrac{1}{6}$. (*W.B.U.T. 2013*)

Solution: (*i*) Let $f(x) = e^x$, $x > 0$, it is continuous in [0, x] and derivable in (0, x).

From the mean value theorem, $f(x) = f(0) + xf'(\theta x)$, $0 < \theta < 1$.

Therefore, $\qquad e^x = 1 + xe^{\theta x} \qquad$ [since $f(0) = 1$ and $f'(x) = e^x$]

or $\qquad e^{\theta x} = \dfrac{e^x - 1}{x}$, or $\theta x = \log_e \dfrac{e^x - 1}{x}$

or
$$\theta = \frac{1}{x}\log_e \frac{e^x - 1}{x}.$$

Hence,
$$0 < \frac{1}{x}\log_e \frac{e^x - 1}{x} < 1 \qquad (\because\ 0 < \theta < 1)$$

(*ii*) Let $f(x) = \log_e(1+x)$, $x > 0$, it is continuous in $[0, x]$ and derivable in $(0, x)$.

Then from the mean value theorem, we have

$$f(x) = f(0) + xf'(\theta x),\ 0 < \theta < 1.$$

$$\therefore \quad \log(1+x) = \frac{x}{1+\theta x}, \quad \text{since } f(0) = 0 \text{ and } f'(x) = \frac{1}{1+x}. \qquad \text{...(1)}$$

Now $0 < \theta < 1$ and $x > 0$, so $0 < \theta x < x$.

$$\therefore \quad 1 < 1 + \theta x < 1 + x, \text{ or } 1 > \frac{1}{1+\theta x} > \frac{1}{1+x},$$

or
$$x > \frac{x}{1+\theta x} > \frac{x}{1+x}, \text{ or } \frac{x}{1+x} < \frac{x}{1+\theta x} < x \qquad \text{...(2)}$$

From (1) and (2), we conclude that

$$\frac{x}{1+x} < \log(1+x) < x.$$

(*iii*) Let $f(x) = \tan^{-1} x$, $0 < x < \frac{\pi}{2}$, it is continuous in $[0, x]$ and derivable in $(0, x)$.

Using mean value theorem, we have

$$f(x) = f(0) + xf'(\theta x),\ 0 < \theta < 1.$$

$$\therefore \quad \tan^{-1} x = \frac{x}{1+\theta^2 x^2}, \text{ since } f(0) = 0 \text{ and } f'(x) = \frac{1}{1+x^2}. \qquad \text{...(1)}$$

Now $0 < \theta < 1$ and $x > 0$, so $0 < \theta x < x$.

$$\therefore \quad 0 < \theta^2 x^2 < x^2, \text{ or } 1 < 1 + \theta^2 x^2 < 1 + x^2,$$

or
$$1 > \frac{1}{1+\theta^2 x^2} > \frac{1}{1+x^2}, \text{ or } x > \frac{x}{1+\theta^2 x^2} > \frac{x}{1+x^2},$$

or
$$\frac{x}{1+x^2} < \frac{x}{1+\theta^2 x^2} < x \qquad \text{...(2)}$$

From (1) and (2), we have

$$\frac{x}{1+x^2} < \tan^{-1} x < x, \text{ when } 0 < x < \frac{\pi}{2}.$$

(*iv*) Let $f(x) = \ln(1 - x)$, $0 < x < 1$, it is continuous in $[0, x]$ and derivable in $(0, x)$.

By mean value theorem, we have

$$f(x) = f(0) + xf'(\theta x),\ 0 < \theta < 1.$$

$$\therefore \qquad \ln(1-x) = -\frac{x}{1-\theta x}, \text{ since } f(0)=0 \text{ and } f'(x) = -\frac{1}{1-x}.$$

or
$$-\ln(1-x) = \frac{x}{1-\theta x} \qquad ...(1)$$

Now, $0 < \theta < 1$ and $0 < x < 1$, so $0 < \theta x < x$.

or $0 > -\theta x > -x$, or $1 > 1-\theta x > 1-x > 0$, or $1 < \frac{1}{1-\theta x} < \frac{1}{1-x}$

$$\therefore \qquad x < \frac{x}{1-\theta x} < \frac{x}{1-x} \qquad ...(2)$$

From (1) and (2), we have

$$x < -\ln(1-x) < \frac{x}{1-x}.$$

(*v*) Let $f(x) = \sin^{-1} x$, it is continuous in $[a, b]$ and derivable in (a, b), where $0 < a < b < 1$.

Therefore, by Lagrange's mean value theorem, there exists at least one value c of x, $a < c < b$, such that

$$f(b) - f(a) = (b-a)f'(c);\ a < c < b$$

or
$$\sin^{-1} b - \sin^{-1} a = \frac{(b-a)}{\sqrt{1-c^2}}, 0 < a < c < b < 1 \qquad ...(1)$$

$$\left(\text{since } f'(x) = \frac{1}{\sqrt{1-x^2}}\right)$$

Now, $0 < a < c < b < 1 \Rightarrow 1-a^2 > 1-c^2 > 1-b^2 > 0$

$$\Rightarrow \qquad \sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$$

$$\Rightarrow \qquad \frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$$

$$\Rightarrow \qquad \frac{b-a}{\sqrt{(1-a^2)}} < \frac{b-a}{\sqrt{(1-c^2)}} < \frac{b-a}{\sqrt{(1-b^2)}} \qquad (\because b > a) \qquad ...(2)$$

From (1) and (2), we have

$$\frac{b-a}{\sqrt{(1-a^2)}} < \sin^{-1} b - \sin^{-1} a < \frac{b-a}{\sqrt{(1-b^2)}}$$

(*vi*) Let $f(x) = \tan^{-1} x$, it is continuous in (a, b) and derivable in (a, b), where $0 < a < b$.

Therefore, by mean value theorem, there exists at least one value c of x, $a < c < b$, such that

$$f(b) - f(a) = (b-a)f'(c), a < c < b$$

or
$$\tan^{-1} b - \tan^{-1} a = \frac{b-a}{1+c^2}, a < c < b \qquad ...(1)$$

$$\left(\text{since } f'(x) = \frac{1}{1+x^2}\right)$$

Now, $a<c<b \Rightarrow 1+a^2<1+c^2<1+b^2$

$$\Rightarrow \quad \frac{1}{1+a^2}>\frac{1}{1+c^2}>\frac{1}{1+b^2}$$

$$\Rightarrow \quad \frac{1}{1+b^2}<\frac{1}{1+c^2}<\frac{1}{1+a^2}$$

$$\Rightarrow \quad \frac{b-a}{1+b^2}<\frac{b-a}{1+c^2}<\frac{b-a}{1+a^2} \qquad (\because b>a). \qquad ...(2)$$

From (1) and (2), we conclude that

$$\frac{b-a}{1+b^2}<\tan^{-1}b-\tan^{-1}a<\frac{b-a}{1+a^2}, \text{ where } 0<a<b. \qquad ...(3)$$

If we put $a = 1$, $b = \frac{4}{3}$ in (3), we get

$$\frac{3}{25}<\tan^{-1}\frac{4}{3}-\tan^{-1}1<\frac{1}{6}$$

or
$$\frac{\pi}{4}+\frac{3}{25}<\tan^{-1}\frac{4}{3}<\frac{\pi}{4}+\frac{1}{6} \qquad \left(\because \tan^{-1}1=\frac{\pi}{4}\right).$$

Example 8: Using mean value theorem prove the following inequalities:

(*i*) $1+\dfrac{x}{2\sqrt{1+x}}<\sqrt{1+x}<1+\dfrac{x}{2}, -1<x<0$ (*W.B.U.T. 2004*)

(*ii*) $\dfrac{\pi}{6}+\dfrac{\sqrt{3}}{15}<\sin^{-1}\left(\dfrac{3}{5}\right)<\dfrac{\pi}{6}+\dfrac{1}{8}.$ (*W.B.U.T. 2005*)

Solution: (*i*) Let $f(x) = \sqrt{1+x}, \ -1<x<0,$ it is continuous in $[x, 0]$ and derivable in $(x, 0)$. From the Lagrange's mean value theorem, we get

$$\frac{f(0)-f(x)}{-x} = f'(\theta x), \ 0<\theta<1$$

or
$$f(x) = f(0)+xf'(\theta x), \ 0<\theta<1.$$

Here $f(0) = 1$ and $f'(x)=\dfrac{1}{2\sqrt{1+x}}.$

$$\therefore \quad \sqrt{1+x} = 1+\frac{x}{2\sqrt{1+\theta x}} \qquad ...(1)$$

Now, $0<\theta<1 \Rightarrow 0>\theta x>x \quad (\because x<0)$.

$$\Rightarrow \quad 1>\sqrt{1+\theta x}>\sqrt{1+x}>0 \qquad (\because -1<x<0)$$

$$\Rightarrow \quad 1<\frac{1}{\sqrt{1+\theta x}}<\frac{1}{\sqrt{1+x}}$$

$$\Rightarrow \quad \frac{1}{2}x > \frac{1}{2}\frac{x}{\sqrt{1+\theta x}} > \frac{1}{2}\frac{x}{\sqrt{1+x}} \qquad (\because\ x<0)$$

$$\Rightarrow \quad \frac{1}{2}\frac{x}{\sqrt{1+x}} < \frac{x}{2\sqrt{1+\theta x}} < \frac{x}{2}$$

$$\Rightarrow \quad 1+\frac{1}{2}\frac{x}{\sqrt{1+x}} < 1+\frac{x}{2\sqrt{1+\theta x}} < 1+\frac{x}{2}$$

$$\Rightarrow \quad 1+\frac{x}{2\sqrt{1+x}} < \sqrt{1+x} < 1+\frac{x}{2} \qquad \text{[by (1)]}$$

where $-1 < x < 0$.

(*ii*) Let $f(x) = \sin^{-1} x$, it is continuous in $\left[\frac{1}{2}, \frac{3}{5}\right]$ and derivable in $\left(\frac{1}{2}, \frac{3}{5}\right)$ Using Lagrange's mean value theorem in $\left[\frac{1}{2}, \frac{3}{5}\right]$, we get a number c, where $\frac{1}{2} < c < \frac{3}{5}$, such that

$$\frac{f\left(\frac{3}{5}\right)-f\left(\frac{1}{2}\right)}{\frac{3}{5}-\frac{1}{2}} = f'(c)$$

or
$$10\left(\sin^{-1}\frac{3}{5}-\frac{\pi}{6}\right) = \frac{1}{\sqrt{1-c^2}} \qquad \left(\because\ f'(x)=\frac{1}{\sqrt{1-x^2}}\right)$$

$$\therefore \quad \sin^{-1}\frac{3}{5}-\frac{\pi}{6} = \frac{1}{10\sqrt{1-c^2}} \qquad \text{...(1)}$$

Now, $\frac{1}{2} < c < \frac{3}{5}$. $\quad \therefore \quad c^2 < \frac{9}{25} \Rightarrow -c^2 > -\frac{9}{25}$

$$\Rightarrow \quad 1-c^2 > 1-\frac{9}{25} = \frac{16}{25} \Rightarrow \sqrt{1-c^2} > \frac{4}{5}$$

$$\Rightarrow \quad \frac{1}{\sqrt{1-c^2}} < \frac{5}{4} \Rightarrow \frac{1}{10\sqrt{1-c^2}} < \frac{5}{4\times 10} = \frac{1}{8} \qquad \text{...(2)}$$

Again, $c > \frac{1}{2} \Rightarrow c^2 > \frac{1}{4} \Rightarrow -c^2 < -\frac{1}{4}$

$$\Rightarrow \quad 1-c^2 < 1-\frac{1}{4} = \frac{3}{4} \Rightarrow \sqrt{1-c^2} < \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \frac{1}{10\sqrt{1-c^2}} > \frac{2}{10\sqrt{3}} = \frac{\sqrt{3}}{15} \qquad \text{...(3)}$$

From (2) and (3), we have

$$\frac{\sqrt{3}}{15} < \frac{1}{10\sqrt{1-c^2}} < \frac{1}{8}, \text{ or } \frac{\sqrt{3}}{15} < \sin^{-1}\frac{3}{5} - \frac{\pi}{6} < \frac{1}{8}, \qquad \text{[by (1)]}$$

or
$$\frac{\pi}{6} + \frac{\sqrt{3}}{15} < \sin^{-1}\left(\frac{3}{5}\right) < \frac{\pi}{6} + \frac{1}{8}.$$

Example 9: If $f''(x)$ exists for all points in $[a, b]$ and

$$\frac{f(c)-f(a)}{c-a} = \frac{f(b)-f(c)}{b-c}$$

where $a < c < b$, then there is a number ξ such that $a < \xi < b$ and $f''(\xi) = 0$.

Solution: Since $f''(x)$ exists in $[a, b]$, f', f are continuous in $[a, b]$. Applying Lagrange's mean value theorem to the intervals $[a, c]$ and $[c, b]$ respectively, we get

$$\frac{f(c)-f(a)}{c-a} = f'(\xi_1), a < \text{ at least one } \xi_1 < c \qquad ...(1)$$

and
$$\frac{f(b)-f(c)}{b-a} = f'(\xi_2), c < \text{ at least one } \xi_2 < b \qquad ...(2)$$

From (1) and (2), we get on using the given relation

$$f'(\xi_1) = f'(\xi_2).$$

Now the function f' satisfies all the conditions of Rolle's theorem in $[\xi_1, \xi_2]$.

Therefore, there is a number ξ such that

$$f''(\xi) = 0 \quad \text{where} \quad \xi_1 < \xi < \xi_2 \;\; i.e., \;\; a < \xi < b.$$

Example 10: If $f(x+h) = f(x) + hf'(x+\theta h)$, $0 < \theta < 1$; find the value of θ when $f(x) = x^2$.

Solution: Here $f(x) = x^2$, therefore $f(x+h) = (x+h)^2 = x^2 + 2hx + h^2$.

Now, from $f(x+h) = f(x) + hf'(x+\theta h), 0 < \theta < 1$, we have

$$x^2 + 2hx + h^2 = x^2 + 2h(x+\theta h) \qquad (\because f'(x) = 2x)$$

or
$$h^2 = 2h^2\theta$$

$\therefore$
$$\theta = \frac{1}{2} \qquad (\because h \neq 0)$$

Example 11: Estimate $\sqrt[3]{65}$ using Lagrange's mean value theorem.

Solution: Let us consider the function $f(x) = x^{1/3}$ in [64, 65]. Evidently $f(x)$ is continuous for all values of x in [64, 65] and $f'(x) = \frac{1}{3}x^{-2/3}$ exists for all values of x in [64, 65].

By Lagrange's mean value theorem, there exists a value c, $64 < c < 65$, such that

$$f(65) - f(64) = (65-64)f'(c)$$

or
$$(65)^{1/3} - (64)^{1/3} = \frac{1}{3}(64+\theta)^{-2/3}, \quad \text{where} \quad c = 64+\theta, 0<\theta<1.$$

$$\therefore \quad \sqrt[3]{65} = 4 + \frac{1}{3} \cdot \frac{1}{(64+\theta)^{2/3}}.$$

$$\therefore \quad 4 < \sqrt[3]{65} < 4 + \frac{1}{48}, \text{ or } 4 < \sqrt[3]{65} < 4\frac{1}{48}.$$

Example 12: Prove that $\sin 46° \sim \frac{1}{2}\sqrt{2}\left(1+\frac{\pi}{180}\right)$ Is the estimate high or less?

(*W.B.U.T. 2003*)

Solution: Let $f(x) = \sin x$, which is continuous and derivable for all real values of x and $f'(x) = \cos x$.

By Lagrange's mean value theorem in $[a, a+h]$, we have

$$f(a+h) = f(a) + hf'(a+\theta h), \; 0<\theta<1.$$

Putting $a = 45°$ and $h = 1°$, we get $f(46°) = f(45°) + 1° \cos(45° + \theta \cdot 1°)$

or
$$\sin 46° = \sin 45° + \frac{\pi}{180}\cos(45°+\theta°) \quad \left(\because 1° = \frac{\pi}{180} \text{ radian}\right)$$

$$\sim \sin 45° + \frac{\pi}{180}\cos 45° \qquad (\because 0<\theta°<1°, \text{ i.e., } \theta° \text{ is very small})$$

$$\therefore \quad \sin 46° \sim \frac{1}{\sqrt{2}}\left(1+\frac{\pi}{180}\right) = \frac{1}{2}\sqrt{2}\left(1+\frac{\pi}{180}\right)$$

This estimate is high since $0 < \theta° < 1°$.

Note: Applying Lagrange's mean value theorem, approximate solution of equation $f(x) = 0$ can be obtained (Newton's method) as follows:

Let $a + h$ be the exact root of $f(x) = 0$, so

$$0 = f(a+h) = f(a) + hf'(a+\theta h), 0<\theta<1.$$

Therefore,
$$h \simeq -\frac{f(a)}{f'(a)}.$$

Hence starting at a guess value 'a', h (correction) can be calculated approximately and by iteraction a better root can be obtained.

Example 13: Calculate approximately the root of the equation $x^4 - 12x + 7 = 0$ near 2 by using Lagrange's mean value theorem.

Solution: Let $f(x) = x^4 - 12x + 7$, which is continuous and derivable for all real values of x and $f'(x) = 4x^3 - 12$.

By Lagrange's mean value theorem, $f(a+h) = f(a) + hf'(a+\theta h), \; 0<\theta<1$ so $h \simeq -\dfrac{f(a)}{f'(a)}$.

Here $\qquad f(2) = 2^4 - 12\times 2 + 7 = -1$ and $f'(2) = 4\times 2^3 - 12 = 20.$

$$\therefore \qquad h \simeq -\frac{f(2)}{f'(2)} = \frac{1}{20} = 0.05$$

Therefore, an approximate root is $x = a + h = 2 + 0.05 = 2.05$.

Observations: If we apply Lagrange's mean value theorem to two functions $f(x)$ and $g(x)$, both satisfy the conditions of the theorem in $[a, b]$, we get

$$f(b) - f(a) = (b-a)f'(c_1),\ a < \text{ at least one } c_1 < b$$

and

$$g(b) - g(a) = (b-a)g'(c_2),\ a < \text{at least one } c_2 < b.$$

Dividing we have

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c_1)}{g'(c_2)},\ c_1, c_2 \text{ are, in general different.}$$

Cauchy takes a step further to make $c_1 = c_2$ and establishes a theorem, which we are going to study in the next article.

6.4 CAUCHY'S MEAN VALUE THEOREM

If f and g be two real valued functions of a real variable x defined in the closed interval $[a, b]$ such that

(*i*) $f(x)$ and $g(x)$ both are continuous in $a \le x \le b$,

(*ii*) $f(x)$ and $g(x)$ both are derivable in $a < x < b$ and

(*iii*) $g'(x) \neq 0$ for any value of x in $a < x < b$, then there exists at least one value c of x, where $a < c < b$, such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Alternative form of Cauchy's mean value theorem

If we take $b = a + h$, $c = a + \theta h$, $0 < \theta < 1$, Cauchy's Mean Value Theorem takes the form

$$\frac{f(a+h)-f(a)}{g(a+h)-g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)}, 0 < \theta < 1,\ h > 0,$$

which is the alterntive form for Cauchy's Mean Value Theorem.

Deduction of Lagrange's Mean Value Theorem from Cauchy's Mean Value Theorem

If we take $g(x) = x$, then $g(x)$ satisfies all the stated condition in Cauchy's Mean Value Theorem and we have

(*i*) $f(x)$ is continuous for all x in $a \le x \le b$ and

(*ii*) $f(x)$ is derivable for all x in $a < x < b$, then there exists at least one value c of x, where $a < c < b$, such that

$$\frac{f(b)-f(a)}{b-a} = \frac{f'(c)}{1} \qquad (\because\ g(x) = x \text{ and } g'(x) = 1)$$

or

$$\frac{f(b)-f(a)}{b-a} = f'(c),$$

which is the Lagrange's Mean Value Theorem.

ILLUSTRATIVE EXAMPLES

Example 1: Verify Cauchy's mean value theorem for the following functions:

(*i*) $f(x) = x^4,\ g(x) = x^2$ in the interval [1, 2].

(*ii*) $f(x) = e^x,\ g(x) = e^{-x}$ in the interval [3, 7]

(*iii*) $f(x) = \cos x,\ g(x) = \sin x$ in the interval $\left[0, \frac{\pi}{2}\right]$

Solution: (*i*) Here $f(x) = x^4,\ g(x) = x^2$ both are continuous in [1, 2] and derivable in (1, 2). Now $f'(x) = 4x^3$, $g'(x) = 2x$ and $g'(x) \neq 0$ for any x in (1, 2).

Thus f and g satisfy all the conditions of Cauchy's Mean Value Theorem and therefore there should exist c, $1 < c < 2$, such that

$$\frac{f(2)-f(1)}{g(2)-g(1)} = \frac{f'(c)}{g'(c)}, \text{ or } \frac{2^4-1^4}{2^2-1^2} = \frac{4c^3}{2c},$$

or $$2c^2 = 5. \qquad \text{Hence, } c = \pm\sqrt{\frac{5}{2}}, \text{ of which } \sqrt{\frac{5}{2}} \text{ lies between 1 and 2.}$$

Therefore, Cauchy's Mean Value Theorem is verified for the given function in the interval [1, 2].

(*ii*) Here $f(x) = e^x,\ g(x) = e^{-x}$ both are continuous in [3, 7] and derivable in (3, 7).

Also $$f'(x) = e^x,\ g'(x) = -e^{-x} \text{ and } g'(x) \neq 0 \text{ for any } x \text{ in } (3, 7).$$

Thus f and g satisfy all the conditions of Cauchy's Mean Value Theorem and therefore there should exist c, $3 < c < 7$, such that

$$\frac{f(7)-f(3)}{g(7)-g(3)} = \frac{f'(c)}{g'(c)}, \text{ or } \frac{e^7-e^3}{e^{-7}-e^{-3}} = -\frac{e^c}{e^{-c}}, \text{ or } e^{2c} = e^{10}.$$

Therefore $c = 5$ which lies between 3 and 7.

Hence, Cauchy's Mean Value Theorem is verified for the given function in the interval [3, 7].

(*iii*) Since $\cos x$ and $\sin x$ are both continuous and derivable for all real x, so $f(x) = \cos x$, $g(x) = \sin x$ are continuous in $\left[0, \frac{\pi}{2}\right]$ and derivable in $\left(0, \frac{\pi}{2}\right)$.

Also $f'(x) = -\sin x,\ g'(x) = \cos x$ and $g'(x) \neq 0$ for all x in $\left(0, \frac{\pi}{2}\right)$.

Thus f and g satisfy all the conditions of Cauchy's Mean Value Theorem and therefore there should exist c, $0 < c < \frac{\pi}{2}$, such that

$$\frac{f\left(\frac{\pi}{2}\right)-f(0)}{g\left(\frac{\pi}{2}\right)-g(0)} = \frac{f'(c)}{g'(c)}, \text{ or } \frac{\cos\frac{\pi}{2}-\cos 0}{\sin\frac{\pi}{2}-\sin 0} = \frac{-\sin c}{\cos c}, \text{ or } \frac{-1}{1} = \frac{-\sin c}{\cos c},$$

or $\tan c = 1$, which gives a solution $c = \frac{\pi}{4}$ which lies between 0 and $\frac{\pi}{2}$.

Hence, Cauchy's Mean Value Theorem is verified.

Example 2: In Cauchy's Mean Value Theorem, if $f(x) = e^x$ and $g(x) = e^{-x}$, show that θ is independent of both x and h and is equal to $\frac{1}{2}$. *(W.B.U.T. 2003)*

Solution: Since $f(x) = e^x$, $g(x) = e^{-x}$ both are continuous and derivable for all real x and $g'(x) = -e^{-x} \neq 0$ for all real x, therefore by Cauchy's mean value theorem,

$$\frac{f(x+h)-f(x)}{g(x+h)-g(x)} = \frac{f'(x+\theta h)}{g'(x+\theta h)}, \quad 0<\theta<1.$$

$$\therefore \quad \frac{e^{x+h}-e^x}{e^{-(x+h)}-e^{-x}} = \frac{e^{x+\theta h}}{-e^{-(x+\theta h)}},$$

or $$\frac{e^{x+h}-e^x}{e^{-(x+h)}\cdot e^{-x}(e^x-e^{x+h})} = -e^{2(x+\theta h)}$$

or $$-e^{x+h}\cdot e^x = -e^{2(x+\theta h)}$$

or $$e^{2x+h} = e^{2x+2\theta h}$$

$$\therefore \quad 2x+h = 2x+2\theta h, \text{ or } \theta = \frac{1}{2} \qquad (\because\ h \neq 0).$$

So, θ is independent of both x and h and is equal to $\frac{1}{2}$.

Example 3: If, in the Cauchy's mean value theorem, we write

$$f(x) = \sqrt{x} \text{ and } g(x) = \frac{1}{\sqrt{x}},$$

then c is the geometric mean between a and b and if we write

$$f(x) = \frac{1}{x^2} \text{ and } g(x) = \frac{1}{x},$$

then c is the harmonic mean between a and b.

Solution: When $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{\sqrt{x}}$, we have

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}, \text{ or } \frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}} = \frac{\frac{1}{2}c^{-1/2}}{-\frac{1}{2}c^{-3/2}}$$

$$\therefore \quad -\sqrt{ab} = -c, \text{ or } c = \sqrt{ab}.$$

Therefore, c is the geometric mean between a and b.

When $$f(x) = \frac{1}{x^2} \text{ and } g(x) = \frac{1}{x}, \text{ we have}$$

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}, \text{ or } \frac{\frac{1}{b^2}-\frac{1}{a^2}}{\frac{1}{b}-\frac{1}{a}} = \frac{-2c^{-3}}{-c^{-2}}$$

or $$\left(\frac{a^2-b^2}{a-b}\right)\frac{ab}{a^2b^2} = \frac{2}{c}, \text{ or } \frac{a+b}{ab} = \frac{2}{c}$$

$\therefore$ $$c = \frac{2ab}{a+b}.$$

Therefore, c is the harmonic mean between a and b.

6.5 GENERALIZED MEAN VALUE THEOREM : TAYLOR'S THEOREM

Theorem 1: (Taylor's theorem with Lagrange's form of remainder).

Let f be a function defined on the closed interval $[a, b]$ such that

(*i*) the $(n-1)$th derivative $f^{(n-1)}$ is continuous in the closed interval $[a, b]$ and

(*ii*) the nth derivative $f^{(n)}$ exists in the open interval (a, b),

then there exists at least one value c, $a < c < b$, such that

$$f(b) = f(a)+(b-a)f'(a)+\frac{1}{2!}(b-a)^2 f''(a)+\ldots$$
$$+\frac{1}{(n-1)!}(b-a)^{n-1} f^{(n-1)}(a)+\frac{1}{n!}(b-a)^n f^{(n)}(c).$$

Alternative form of the above theorem

Let f be a function defined on the closed interval $[a, a+h]$, $h > 0$, such that

(*i*) the $(n-1)$th derivative $f^{(n-1)}$ is continuous in $[a, a+h]$ and

(*ii*) the nth derivative $f^{(n)}$ exists in $(a, a+h)$,

then there exists at least one number θ, $0 < \theta < 1$, such that

$$f(a+h) = f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)+\frac{h^n}{n!}f^n(a+\theta h).$$

Note: The last term of the above series, *i.e.*, $(n+1)$th term, is called the Lagrange's form of Remainder after n terms and is denoted by R_n.

$\therefore$ $$R_n = \frac{h^n}{n!}f^n(a+\theta h),\ 0<\theta<1.$$

Theorem 2: (Taylor's theorem with Cauchy's form of remainder)

Let f be a function defined on the closed interval $[a, b]$ such that

(*i*) the $(n-1)$th derivative $f^{(n-1)}$ is continuous in the closed interval $[a, b]$ and

(*ii*) the nth derivative $f^{(n)}$ exists in the open interval (a, b), then there exists at least one value c, $a < c < b$, such that

$$f(b) = f(a) + (b-a)f'(a) + \frac{1}{2!}(b-a)^2 f''(a) + \ldots$$

$$+ \frac{1}{(n-1)!}(b-a)^{n-1} f^{(n-1)}(a) + \frac{1}{(n-1)!}(b-a)(b-c)^{n-1} f^{(n)}(c).$$

Alternative form of the above theorem

Let f be a function defined on the closed interval $[a, a+h]$, $h > 0$, such that

(*i*) the $(n-1)$th derivative $f^{(n-1)}$ is continuous in $[a, a+h]$ and

(*ii*) the nth derivative $f^{(n)}$ exists in $(a, a+h)$,

then there exists at least one number θ, $0 < \theta < 1$, such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n(1-\theta)^{n-1}}{(n-1)!}f^n(a+\theta h).$$

Note: The last term of the above series, *i.e.*, $(n+1)$th term, is called the Cauchy's form of Remainder after n terms and is denoted by R_n.

$$\therefore \qquad R_n = \frac{h^n(1-\theta)^{n-1}}{(n-1)!}f^{(n)}(a+\theta h),\ 0 < \theta < 1.$$

Remarks: (*i*) The Taylor's theorem also holds if $h < 0$ and in this case the interval $[a, a+h]$ is to be replaced by $[a+h, a]$.

(*ii*) The result of Taylor's theorem is also known as Taylor's formula or Taylor's series for the function $f(x)$.

(*iii*) By taking $b = x$ in Taylor's theorem, we have

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_n,$$

where $$R_n = \frac{(x-a)^n}{n!}f^{(n)}\{a+\theta(x-a)\},\ 0 < \theta < 1 \qquad \text{[Lagrange's form]}$$

$$= \frac{(x-a)^n(1-\theta)^{n-1}}{(n-1)!}f^{(n)}\{a+\theta(x-a)\},\ 0 < \theta < 1 \qquad \text{[Cauchy's form]}$$

which is called the expansion of $f(x)$ about $x = a$.

(*iv*) Taylor's theorem is also known as the nth order mean value theorem or nth mean value theorem or mean value theorem of the order n. The 1st order mean value theorem is the Lagrange's mean value theorem and the 2nd order mean value theorem is

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a+\theta h),\ 0 < \theta < 1.$$