Beginner's Guide to Induction Theory

This guide includes both weak and strong induction examples along with providing useful tips that might come in handy while tackling such questions on assignments or during the final exams.

Disclaimer: This guide is not to be taken as a "complete and final" way of learning induction. I have tried my best to include everything that I have learnt over the semester, however they may be things that are not taught the same way I show here. There is also a possibility that the steps mentioned here do not match some of your way of writing and also possibility of minor error so sorry for that.

I highly advise you to take this as a reference and write your own notes for this topic as it will help you a long way.

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Weak Induction

This is the beginning point of induction where you learn how to proof simple statements with simple steps.

I'll mainly be highlighting the important steps to follow over here in this guide (guided by Mr. Weng Kee), for more information on the theory of how each statement is being proven please visit this amazing site for math topic guides: https://math.libretexts.org and just type in Induction. You can also use this site to cross reference my steps mentioned here and improve your own notes by making the necessary amendments.

(i) Prove by simple induction that, for each integer $n \geq 1$,

$$1 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

First and foremost, lets observe this equation carefully.

It represents the sum of squares of the first n natural numbers and is known as a finite series or a sum formula.

Understanding how the equation works is the first step to finding a way to prove its equivalency.

Let's begin with the steps:

1. Statement: Let P(n) be the statement
   P(n): (the above-mentioned equation) for $n \geq 1$, $n \in Z$

   This step sets the foundation on what symbol you are going to use to represent your to-be-proven statement throughout the process.

2. Base step:
   This step is used to prove the initial value for which both sides would be true

   RHS: right hand side, LHS: left hand side

   LHS: when n = 1, $(1)^1 = 1$

   RHS:

$$\frac{n(n+1)(2n+1)}{6}\Big|_{n=1} = 1$$

$\therefore$ INDEED, LHS = RHS and P (1) = True

3. Inductive step:

This is the most important part of induction and carries the most marks

First, write down the assumption statement:

Assume P(k) is true for an arbitrary k such that:

$$k^2 = \frac{k(k+1)(2k+1)}{6}$$

Where k ∈ Z, k ≥ 1

Then write down the step which shows the proving statement:

What to proof: Show that P (k+1) is true **based on the assumption that P (k) is true such that:**

i.e.

$$(k+1)^2 = \frac{(k+1)(k+2)(2k+2)}{6}$$

For all k values, k ≥ 1.

Now, P (k) =

$$\frac{k(k+1)(2k+1)}{6}$$

From the assumption step:

The pattern goes by $k^2 + (k+1)^2 + \cdots$

Since we have already proved that P (k) is true in our assumption step:

$$k^2 = \frac{k(k+1)(2k+1)}{6}$$

Hence,

$$\frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

Now we have obtained an equation that we can simplify,

Proceeding to simplify this,

$$\frac{k(k+1)(2k+1)}{6}+\frac{6(k+1)^2}{6}$$

$$\frac{k+1}{6}\ [k\ (2k+1)+6(k+1)]$$

$$\frac{k+1}{6}\ (2k^2+7k+6)$$

$$\frac{k+1}{6}(2k+3)(k+2)$$

$$\frac{(k+1)(2k+3)(k+2)}{6}$$

Which is the right-hand side part of the equation which we required,

Thus, proving that P (k + 1) is true.

4. Conclusion: This is one of the steps where some students may lose marks due to not providing sufficient info for justifying their proof.

   Here’s a proper way to show that:

   Since P (1) is true, (BS)
   P (2) -> P (3), (IS, k = 1)
   P (3) -> P (4), (IS, k = 2)
   ...

   ∴ By the principle of inductive step, P (n) when n ≥ 1 is true, n ∈ Z.

That’s it! We just completed our first simple induction question.

Weak Induction pointers

Now you may have some questions as to whether these steps can be applied to similar induction questions and on tackling different types of induction questions so let me add a few more pointers to try and clear your doubts.

- Yes, all of these steps can be applied to any other weak induction question

  I'll provide a template to follow below. The reason why I didn't do that initially is that having an idea about how you can use it practically first would allow you to understand the template's application on other questions quicker.

- The main thing that you would need to alter is your inductive step, your base step and statement are just replacing the lines based on the respective question, i.e.

  - Statement: Let P (n) be the statement…..for $n \geq x$…

  - Base step: If are given an equation where you have a LHS and RHS, then follow exactly how I showed above whilst replacing the equation and values according to the question using the lowest value from which your n value range starts from.

    i.e. $n \geq 1$, so check for $n = 1$ for both sides

  - There are some questions however where you may not have a RHS and LHS situation. I'll covering that type of question as the second example for the weak induction topic.

- For altering the inductive step, it all depends on how you approach the question and your understanding about it. Sadly, there's no shortcut to handling these questions and requires lots of practice, so find questions online, solve them, check answers, correct mistakes, make notes on your mistakes and repeat.

  Consider discussing whatever you understand with your friends to see what methods they would use to tackle the same question and if you could improve your own method. There aren't a huge variety of questions they could ask you for weak induction for this unit so I wouldn't worry about going too deep into the topic, however this is the best approach to follow as only reading notes does not suffice.

Back to the induction step, it's all about finding the pattern of the sequence, if the question includes a sequence, or logically proving the question using other methods like factorization or simplification if it's not a sequence, which I'll show in the next example.

Another way of representing the same equation but with summation is using the sigma notation:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

You are likely to encounter questions where the sigma notation would be used to represent a sequence, so make sure you familiarize yourself with it.

- Similarly, there is another similar notation used to represent multiplicative sequences which is the product notation.

  i.e.

  Original statement:

$$1^3 \ x \ 2^3 \ x \ 3^3 \ x \ldots \ n^3 = (n!)^3$$

  Using product notation:

$$\prod_{k=1}^{n} k^3 = (n!)^3$$

You can give this question a go and see how far you reach with it. I'll have a quick answer to this at the end of the document. Try following along the same steps as I did in the first example.

Now let's move on to the next type of weak induction question.

Weak Induction (Second type of question)

(ii) Prove that 3 divides $n^3 + 2n$ for each integer $n \geq 0$

Once again, look at this question carefully.

This one has no RHS or LHS equations but rather a simple divisibility proof.

Start off as usual with the statement and the base step.

1. Statement: Let P (n): 3 divides $n^3 + 2n$ for $n \geq 0,\ n \in Z$
2. Base step:

$$\mathrm{n}^3 + 2n|_{n=0} = 3$$

Since 3 is divisible by 0,

INDEED, P (0) is true.

3. Inductive step: Start off by assuming that P (k) is true.

Assume P (k) is true for an arbitrary k such that

$$3 \mid \mathrm{k}^3 + 2k$$

(The bar represents divisibility)

Where $k \geq 0, k \ \in Z$

What to proof: Show that P (k+1) is true, such that:

$$3|(\mathrm{k}+1)^3 + 2(k+1)$$

Based on the assumption that P (k) is true

To begin with, expand your P (k+1) equation,

$$(\mathrm{k}^2 + 2k + 1)(k+1) + 2k + 2$$

$$k^3 + 3k^2 + 3k + 3 + 2k$$

From our P(k) assumption we have,

$$k^3 + 2k$$

Since we already know it's divisible by 3, we can rewrite it as,

$$k^3 + 2k = 3a$$

Going back to our original equation, we can replace the same expression with 3a instead, giving us:

$$3a + 3k^2 + 3k + 3$$

Now we end up with an expression which is completely divisible by 3, so we can rewrite it as:

$$3a + 3k^2 + 3k + 3 \text{ can be also written as } 3(a + k^2 + k + 1) \rightarrow 3m \text{ where } m = a + k^2 + k + 1$$

Finally giving us:

$$3m$$

∴ P (k+1) is true

4. Conclusion:

   Since P (0) is true, (BS)
   P (1) -> P (2), (IS, k = 0)
   P (2) -> P (3), (IS, k= 1)
   ...

   ∴ By the principle of inductive step, P (n) is true for all n, $n \geq 0$, $n \in \mathbb{Z}$.

And that's it! You would notice that this example is also mentioned in one of the lecture slides, however the steps provided there are not sufficient enough to gain proper marks and so I redid that same example using my template over here.

These are mainly the two types of weak induction questions that you could be asked during the exam, so make sure you familiarize yourself with any complex sequences or logics by practicing plenty of such questions.

Weak Induction Template

Now for the template.

Again, keep in mind, this template is solely for guidance purposes and is not a fixed way of writing your induction working. You can alter the template based on what the lecturer tells you and if there are any mistakes, sorry for that.

1. Statement:
   Let P(n): (the equation mentioned in the question/expression) for n ≥ X, n ∈ Y

   X – range beginning mentioned in the question
   Y – What set of numbers are to be considered (real numbers, natural numbers, integers).

2. Base step:

   RHS: right hand side, LHS: left hand side

   LHS: when n = X, Substitute in LHS equation and get value

   RHS:

   $$\text{RHS equation}|_{n=\text{X}} = \text{value}$$

   Then compare both LHS and RHS values and conclude that since they are the same,

   INDEED, P (X) is true.

   For equations without LHS and RHS sides:

   Based on what the question would usually ask, you would proceed with the first step the same way,

   When n = X,

   Sub x into the single equation mentioned = value

   Then use that value to compare with what is asked to be proven, i.e. Even number, divisibility, odd number, greater than specific number, in between a specific range, natural number, rational number…etc.

3. Inductive step:
   Identify the hypothesis in the form,

   Assumption: Assume P(k) is true for an arbitrary fixed integer k,
   where $k \geq x$, $k \in y$

   What to proof: Proof that P(k+1) is true, (rewrite the expression in the question after substituting k+1) under the assumption that P(k) is true.

   Proving part: This is where you'll carry out the working for proving that P(k+1) is true

   End step: Show an indication that you have completed the proof.
   i.e. Therefore P(k+1) is true OR This completes the inductive step.

4. Conclusion:

   Since P(x) is true, (BS)
   P(x) -> P (x + 1), (IS, k=x + 1)
   P (x + 1) -> P (x + 2), (IS, k= x + 2)
   …
   $\therefore$ By the principle of inductive step, P (n) when n ≥ X is true for all n, n ∈ y.

And there, that's all you need.
With enough practice, you should be able to follow the template without needing to refer to it again and again.

Strong Induction

Now let's move on to strong induction.

There's nothing much to worry about here, the logic is quite similar to weak induction with some minor changes here and there.

In weak induction, you only need to prove the base case for the first value (usually P (1)) and then show that if the statement holds for an arbitrary integer k, it holds for k+1. This single chain of reasoning suffices to cover all integers.

However, in strong induction the inductive step involves assuming the statement is true for all values up to k, not just k. This assumption is more powerful and can be necessary for certain proofs. However, to ensure that this broader assumption is valid, you often need to prove multiple base cases.

You'll understand more once you go over the examples and start tackling these questions yourself.

(i) Given a sequence of integers a_1, a_2, a_3 ... with a_1 = 1, a_2 = 2, and

$$a_{t+1} = a_t + a_{t-1}$$

For each integer $t \geq 3$. Prove by strong induction that $a_n \leq \left(\frac{7}{4}\right)^n$ for each integer $n \geq 1$.

You'll immediately notice the difference between the weak and strong induction questions, one being that you are provided with two expressions to work with.

One of these equations represents a sequence equation and the other one is the statement that you are supposed to prove based on that sequence.

Other than that, you are also provided with two given values at the start which are obtained from the sequence equation. a_1 and a_2.

Let's start working out from the base step like usual and slowly work out everything.

1. Statement: Let P (n): $a_n \leq \left(\frac{7}{4}\right)^n$ for $n \geq 1,\ n \in$ Z.

2. Base step:

   Keep in mind we have two base steps to work out..

   When n = 1,

$$a_1 = 1\ (given)$$

   And

$$\left(\frac{7}{4}\right)^n |_{n=1} = \frac{7}{4}$$

   as $1 \leq \frac{7}{4}$, INDEED P (1) is true

Now perform the same steps for the second n value,

When n = 1,

$$a_2 = 3\ (given)$$

And

$$\left(\frac{7}{4}\right)^n |_{n=2} = \frac{49}{16}$$

As $3 \leq \frac{49}{16}$, INDEED P (2) is true

3. Inductive step:

   Assume P (k) is true for $n \leq k$, when $k \geq 3$

   What to proof: P (k+1) is true based on the assumption that

$$a_1 \leq \frac{7}{4}$$

$$a_2 \leq \frac{49}{16}$$

$$a_k \leq \left(\frac{7}{4}\right)^k$$

Now,

$$a_{k+1} = a_k + a_{k-1}$$

Since,

$$a_k \leq \left(\frac{7}{4}\right)^k \quad P(k)\ is\ True$$

$$a_{k-1} \leq \left(\frac{7}{4}\right)^{k-1} \quad P\ (k-1)\ is\ True$$

Then,

$$a_{k+1} \leq \left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^{k-1}$$

Next factorize the RHS so that you end up with "k" power alone without any "+1" or "-1".

Here's how,

$$a_{k+1} \leq \frac{\left(\frac{7}{4}\right)^k \times \left(\frac{7}{4}\right)}{\frac{7}{4}} + \frac{\left(\frac{7}{4}\right)^k \times \left(\frac{7}{4}\right)^2}{\frac{49}{16}}$$

$$a_{k+1} \leq \frac{4}{7}\left[\left(\frac{7}{4}\right)^{k+1} + \frac{4}{7}\left(\frac{7}{4}\right)^{k+1}\right]$$

$$a_{k+1} \leq \frac{4}{7}\left[\frac{11}{7}\left(\frac{7}{4}\right)^{k+1}\right]$$

$$a_{k+1} \leq \frac{44}{49}\left[\left(\frac{7}{4}\right)^{k+1}\right]$$

You'll notice that we cannot simplify anymore and are left with that expression. You'll encounter such type of situations quite often in strong induction so it's best to be prepared for it. The simplest way to relate it to what you are trying to prove is to use comparison with the value you obtain with what you have been given initially.

As,

$$\frac{44}{49} < 1 \text{ and } a_{k+1} \leq \frac{44}{49}\left[\left(\frac{7}{4}\right)^{k+1}\right]$$

Therefore,

$$a_{k+1} \leq \left(\frac{7}{4}\right)^{k+1}$$

Proving that,

P (k+1) is true.

Now this is just one example of proving such type of strong induction questions. There is another type of question that could also be asked which differs slightly but uses almost the same logic/steps.