23 - S1 - Q4

(a) (i) K? $\rightarrow z_{12} = 0.9 \pm j0.1$

Solution: $K = [0\ 1] W_c^{-1} \alpha_c(A)$

$$W_c^{-1} = \begin{bmatrix} B & AB \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$AB = \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\alpha_c(z) = [z - (0.9 - j0.1)][z - (0.9 + j0.1)]$$

$$= z^2 - (0.9 + j0.1)z - (0.9 - j0.1)z + (0.9 - j0.1)(0.9 + j0.1)$$

$$= z^2 - [0.9 + j0.1 + 0.9 - j0.1]z + 0.82$$

$$= z^2 - 1.8z + 0.82$$

$$\alpha_c(A) = \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix} - 1.8\begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix} + 0.82\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7.62 & -2.2 \\ 4.4 & -1.18 \end{bmatrix}$$

$$K = [0\ 1]\begin{bmatrix} 1 & -2 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 7.62 & -2.2 \\ 4.4 & -1.18 \end{bmatrix}$$

$$= [0\ 0.5]\begin{bmatrix} 7.62 & -2.2 \\ 4.4 & -1.18 \end{bmatrix}$$

$$= [2.2\ -0.59]$$

(ii) Design of Deadbeat Observer Gain L_0:

A deadbeat observer requires the observer error dynamics to have eigenvalues at zero.

Step 1: Compute $A - L_0C$

Let $L_0 = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$ and $C = [2 \quad 1]$.

Compute L_0C:

$$L_0C = \begin{bmatrix} 2l_1 & l_1 \\ 2l_2 & l_2 \end{bmatrix}.$$

Then:

$$A - L_0C = \begin{bmatrix} 4 - 2l_1 & -1 - l_1 \\ 2 - 2l_2 & -l_2 \end{bmatrix}.$$

Step 2: Set Characteristic Polynomial to $z^2 = 0$

Compute the determinant of $zI - (A - L_0C)$:

$$\det(zI - (A - L_0C)) = z^2 + z(l_2 + 2l_1 - 4) + (2 + 2l_1 - 6l_2) = 0.$$

Set coefficients to zero:

1. $l_2 + 2l_1 - 4 = 0 \backslash$
2. $2 + 2l_1 - 6l_2 = 0$

Step 3: Solve for l_1 and l_2

From equation 1:

$l_2 = 4 - 2l_1$

Substitute into equation 2:

$$\begin{gathered} 2 + 2l_1 - 6(4 - 2l_1) = 0 \\ 2 + 2l_1 - 24 + 12l_1 = 0 \\ 14l_1 - 22 = 0 \\ l_1 = \frac{22}{14} = \frac{11}{7} \end{gathered}$$

Then:

$$l_2 = 4 - 2\left(\frac{11}{7}\right) = \frac{6}{7}$$

Answer:

$$L_0 = \begin{bmatrix} \frac{11}{7} \\ \frac{6}{7} \end{bmatrix}$$

(b) **Design of Control Law with Reference Input:**

We need to find k_r such that $y(k)$ reaches unity for a unit-step reference $r(k)$.

Step 1: Compute $(I - A_{\text{cl}})^{-1}B$

Using $A_{\text{cl}} = A - BK$ and K from part (a):

$$A_{\text{cl}} = \begin{bmatrix} 1.8 & -0.41 \\ 2 & 0 \end{bmatrix}$$

Compute $I - A_{\text{cl}}$:

$$I - A_{\text{cl}} = \begin{bmatrix} -0.8 & 0.41 \\ -2 & 1 \end{bmatrix}$$

Compute the inverse:

$$(I - A_{\text{cl}})^{-1} = \frac{1}{0.02} \begin{bmatrix} 1 & -0.41 \\ 2 & -0.8 \end{bmatrix} = 50 \begin{bmatrix} 1 & -0.41 \\ 2 & -0.8 \end{bmatrix}$$

Step 2: Compute the DC Gain

$$\text{DC Gain} = C(I - A_{\text{cl}})^{-1}B = [2 \quad 1] \cdot 50 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 200$$

Step 3: Compute k_r

Set the DC gain to unity:

$$1 = 200 \cdot k_r \quad \Rightarrow \quad k_r = \frac{1}{200} = 0.005$$

Answer:

$$k_r = 0.005$$

(c) **(i) Determination of Optimal Control Law:**

Given the system:

$$x(k+1) = 2x(k) + 3u(k)$$

and performance index:

$$J = \frac{1}{2} \sum_{k=0}^{\infty} \left(x^2(k) + 5u^2(k) \right)$$

Step 1: Solve the Discrete Riccati Equation

Let S satisfy:

$$S = A^2 S + Q - \frac{(ABS)^2}{r + B^2 S}$$

Substitute $A = 2$, $B = 3$, $Q = 1$, $r = 5$:

$$S = 4S + 1 - \frac{(6S)^2}{9S + 5}$$

Simplify:

$$-3S = 1 - \frac{36S^2}{9S + 5}$$

Multiply both sides by $9S + 5$:

$$-3S(9S + 5) = 1(9S + 5) - 36S^2$$

Simplify:

$$-27S^2 - 15S = 9S + 5 - 36S^2$$
$$9S^2 - 24S - 5 = 0$$

Solve the quadratic equation:

$$S = \frac{24 \pm \sqrt{24^2 + 180}}{18} = \frac{24 \pm 27.4955}{18}$$

Select the positive root:

$$S = \frac{24 + 27.4955}{18} \approx 2.8619$$

Step 2: Compute Optimal Gain K

$$K = \frac{BSA}{B^2 S + r} = \frac{3 \times 2.8619 \times 2}{9 \times 2.8619 + 5} \approx 0.5583$$

Answer:

$$u^*(k) = -0.5583\, x(k)$$

(ii) Final Value of $x(k)$ with Optimal Control:

The closed-loop system is:

$$x(k+1) = (2 - 3K)x(k) = 0.3251\, x(k)$$

Since $|0.3251| < 1$, $x(k)$ converges to zero for any non-zero initial $x(0)$. Therefore, the final value of $x(k)$ is zero.

Explanation:

The optimal control law stabilizes the system by placing the eigenvalue inside the unit circle. Consequently, the state $x(k)$ decays exponentially to zero, minimizing the performance index J.

Answer:

The final value of $x(k)$ is zero for any non-zero $x(0)$, as the optimal control law ensures the closed-loop system is stable and $x(k)$ converges to zero.