22-S1-Q4

Q (a) (i) K ? → deadbeat control

$$x(k+1) = [A - BK]\,x(k)$$

$$[A - BK] = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} K$$

$$K = [0 \; 1]\, W_c^{-1} \alpha_c(A)$$

$$W_c^{-1} = [B \;\; AB]^{-1}$$

$$AB = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1}$$

$$= -\begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\alpha_c(z) = z^2 = A^2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$K = [0 \; 1] \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = [1 \; -1] \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = [3 \; -1]$$

$$[A - BK] = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} [3 \; -1]$$

$$= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix}$$

$\exists q > 1$, $[A - BK]^q = 0$, $x(k) = [A - BK]^q x(0) = 0$

$$[A - BK]^2 = \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad q = 2.$$

(ii) $\hat{x}_1(k)$ 是什么？ 还没学到 尝试写

$$\begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

$$u(k) = -\hat{K} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

状态转移矩阵

定义 $x(k) = P w(k)$ ... lecture 2-5-P4

$$x(k) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \hat{x}(k)$$

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$A_w = P^{-1} A P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$$

$$B_w = P^{-1}B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$C_w = CP = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\hat{x}(k+1) = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \hat{x}(k) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u(k)$$

$$\hat{y}(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{x}(k)$$

$$\hat{K} = \begin{bmatrix} 0 & 1 \end{bmatrix} \hat{W}_c^{-1} \alpha_c(A_w)$$

$$\hat{W}_c^{-1} = \begin{bmatrix} B_w & A_w B_w \end{bmatrix}^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = - \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

$$A_w B_w = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\alpha_c(z) = z^2$$

$$\alpha_c(A_w) = A_w^2 = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$$

$$\hat{K} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -4 \end{bmatrix}$$

(a) (ii) Determine $\hat{K}$ for deadbeat control with new state variables.

Given the new state variables:

$$\hat{x}_1(k) = x_1(k) + x_2(k), \quad \hat{x}_2(k) = x_2(k)$$

Define the transformation matrix:

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Compute the transformed matrices:

$$\hat{A} = TAT^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}, \quad \hat{B} = TB = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The closed-loop system:

$$\hat{\mathbf{x}}(k+1) = (\hat{A} - \hat{B}\hat{K})\hat{\mathbf{x}}(k)$$

Compute $\hat{A}_{\text{cl}} = \hat{A} - \hat{B}\hat{K}$:

$$\hat{B}\hat{K} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} \hat{k}_1 & \hat{k}_2 \end{bmatrix} = \begin{bmatrix} 2\hat{k}_1 & 2\hat{k}_2 \\ \hat{k}_1 & \hat{k}_2 \end{bmatrix}$$

$$\hat{A}_{\text{cl}} = \begin{bmatrix} -2\hat{k}_1 & 1 - 2\hat{k}_2 \\ -1 - \hat{k}_1 & 2 - \hat{k}_2 \end{bmatrix}$$

Set the determinant and trace to zero:

1. Trace:

$$\text{Trace}(\hat{A}_{\text{cl}}) = -2\hat{k}_1 + 2 - \hat{k}_2 = 0$$

Simplify:

$$-2\hat{k}_1 - \hat{k}_2 + 2 = 0 \quad \Rightarrow \quad 2 - 2\hat{k}_1 - \hat{k}_2 = 0 \quad (1)$$

2. Determinant:

$$\det(\hat{A}_{\text{cl}}) = (-2\hat{k}_1)(2 - \hat{k}_2) - (1 - 2\hat{k}_2)(-1 - \hat{k}_1) = 0$$

Simplify:

$$-4\hat{k}_1 + 2\hat{k}_1\hat{k}_2 + (1 - 2\hat{k}_2)(1 + \hat{k}_1) = 0$$

$$-4\hat{k}_1 + 2\hat{k}_1\hat{k}_2 + 1 + \hat{k}_1 - 2\hat{k}_2 - 2\hat{k}_1\hat{k}_2 = 0$$

Combine like terms:

$$-3\hat{k}_1 - 2\hat{k}_2 + 1 = 0 \quad (2)$$

From equation (1):

$$2 - 2\hat{k}_1 - \hat{k}_2 = 0 \quad \Rightarrow \quad \hat{k}_2 = -2\hat{k}_1 + 2$$

Substitute $\hat{k}_2$ into equation (2):

$$-3\hat{k}_1 - 2(-2\hat{k}_1 + 2) + 1 = 0$$

Simplify:

$$\begin{gathered} -3\hat{k}_1 + 4\hat{k}_1 - 4 + 1 = 0 \\ \hat{k}_1 - 3 = 0 \\ \hat{k}_1 = 3 \end{gathered}$$

Then:

$$\hat{k}_2 = -2(3) + 2 = -6 + 2 = -4$$

Therefore, $\hat{K} = \begin{bmatrix} 3 & -4 \end{bmatrix}$**.**

(b) Obtain an expression for the transfer function $\frac{\hat{X}(z)}{U(z)}$.

Starting from the estimator equation:

$$\hat{\mathbf{x}}(k+1) = A\hat{\mathbf{x}}(k) + Bu(k) + L_0\left(y(k) - C\hat{\mathbf{x}}(k)\right)$$

Taking the Z-transform:

$$z\hat{X}(z) = A\hat{X}(z) + BU(z) + L_0\left(Y(z) - C\hat{X}(z)\right)$$

Rewriting:

$$\left[zI - (A - L_0C)\right]\hat{X}(z) = BU(z) + L_0Y(z)$$

Since $Y(z)$ depends on $U(z)$ through the plant dynamics (not specified here), the transfer function can be expressed as:

$$\frac{\hat{X}(z)}{U(z)} = \left[zI - (A - L_0C)\right]^{-1}\left(B + L_0\frac{Y(z)}{U(z)}\right)$$

Without further information, this is the simplest form.

(c) Determine the optimal feedback gains and $u^*(k)$ for $k = 0, 1, 2$.

Given:

$$x(k+1) = 5x(k) + u(k)$$

$$J = \frac{1}{2} \sum_{k=0}^{2} \left(8x^2(k) + u^2(k)\right)$$

Initialize $S(3) = 0$.

Backward Recursion:

1. **At $k = 2$:**

$$K(2) = \left(B^T S(3) B + r\right)^{-1} B^T S(3) A = 0$$

$$S(2) = A^T S(3) A + Q = (5)^2 \cdot 0 + 8 = 8$$

2. **At $k = 1$:**

$$K(1) = \frac{B^T S(2) A}{B^T S(2) B + r} = \frac{(1)(8)(5)}{(1)(8)(1) + 1} = \frac{40}{9}$$

$$S(1) = (A - BK(1))^2 S(2) + K(1)^2 r + Q$$

Calculate $A - BK(1) = 5 - 1 \cdot \frac{40}{9} = \frac{5}{9}$.

$$S(1) = \left(\frac{5}{9}\right)^2 \cdot 8 + \left(\frac{40}{9}\right)^2 \cdot 1 + 8 = \frac{200}{81} + \frac{1600}{81} + 8 = \frac{1800}{81} + 8 = \frac{200}{9} + 8 = \frac{272}{9}$$

3. **At $k = 0$:**

$$K(0) = \frac{(1)S(1)(5)}{(1)S(1)(1) + 1} = \frac{\frac{272}{9} \cdot 5}{\frac{272}{9} + 1} = \frac{1360}{281}$$

Compute Optimal Control Inputs:

1. **At $k = 0$:**

$$u^*(0) = -K(0)x(0) = -\frac{1360}{281}x(0)$$

2. **At $k = 1$:**

   First, find $x(1)$:

$$x(1) = 5x(0) + u^*(0) = 5x(0) - \frac{1360}{281}x(0) = \left(5 - \frac{1360}{281}\right)x(0)$$

$$u^*(1) = -K(1)x(1) = -\frac{40}{9}x(1)$$

3. **At $k = 2$:**

   Since $K(2) = 0$:

$$u^*(2) = -K(2)x(2) = 0$$

Therefore, the optimal control inputs are:

$$u^*(0) = -\frac{1360}{281}x(0)$$

$$u^*(1) = -\frac{40}{9}x(1)$$

$$u^*(2) = 0$$

Expressing $u^*(1)$ in terms of $x(0)$:

$$x(1) = \left(5 - \frac{1360}{281}\right)x(0)$$

$$u^*(1) = -\frac{40}{9}\left(\left(5 - \frac{1360}{281}\right)x(0)\right)$$