21 – S1 – Q1

Q:

$$x(kT) = \begin{cases} \cos(\frac{\pi}{2}kT), & k \geqslant 0 \\ 0, & k < 0 \end{cases}$$

已知 $T = 0.5$ $\quad k \in N$

一定代

$$y(kT) = \sum_{n \leq 0}^{k} x(hT)$$

Z - transform of $y(kT)$?

$\lim_{k \to \infty} x(kT) \leftarrow$ Final Value Theorem ?

Solution

$$y(kT) = y[(k-1)T] + x(kT)$$

$$Y(z) = z^{-1} Y(z) + X(z)$$

$$Y(z) = \frac{X(z)}{1 - z^{-1}}$$

$$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$Y(z) = \frac{1 - z^{-1} \cos\frac{\pi}{2} T}{(1 - 2z^{-1}\cos\frac{\pi}{2}T + z^{-2})(1 - z^{-1})}$$

$T = \frac{1}{2}$

$$= \frac{1 - \frac{\sqrt{2}}{2} z^{-1}}{(1 - \sqrt{2} z^{-1} + z^{-2})(1 - z^{-1})} = \frac{1 - \frac{\sqrt{2}}{2} z^{-1}}{1 - z^{-1} - \sqrt{2} z^{-1} + \sqrt{2} z^{-2} + z^{-2} - z^{-3}}$$

$$= \frac{1-\frac{\sqrt{2}}{2}z^{-1}}{1-(1+\sqrt{2})z^{-1}+(\sqrt{2}+1)z^{-2}-z^{-3}} = \frac{z^3-\frac{\sqrt{2}}{2}z^2}{z^3-(1+\sqrt{2})z^2+(\sqrt{2}+1)z-1}$$

let $1-2z^{-1}\cos\frac{\pi}{2}T+z^{-2}=0$

$$z^2-2z\cos\frac{\pi}{2}T+1=0$$

$$z_1,z_2 = -\frac{-2\cos\frac{\pi}{2}T \pm\sqrt{4\cos^2\frac{\pi}{2}T-4}}{2} \sim \frac{b\pm\sqrt{\Delta}}{2a} \quad \Delta=b^2-4ac$$

$$=\cos\frac{\pi}{2}T \mp \sqrt{\cos^2\frac{\pi}{2}T-1}$$

let $z_1=\cos\frac{\pi}{2}T-\sqrt{\cos^2\frac{\pi}{2}T-1}$

$$z_2=\cos\frac{\pi}{2}T+\sqrt{\cos^2\frac{\pi}{2}T-1}$$

由于要与1比较，所以不能出现虚数

$\because \cos^2\frac{\pi}{2}T \in [0,1]$

$\because \cos^2\frac{\pi}{2}T-1 \geq 0$

$\therefore \cos^2\frac{\pi}{2}T=1$

$z_1=\pm1$, $T=2k\pi$, $k\in N^+$ So, $z_1\leq 1$

$z_2=\pm1$, $T=2k\pi$, $k\in N^+$ So $z_2\leq 1$

Obviously $z_3=1$, can use Final value

Final Value Theorem

if $\lim\limits_{k\to\infty} x(kT)$ exists

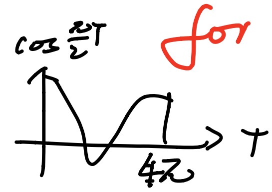
$$\lim_{k\to\infty} x(kT)=\lim_{z\to1}(z-1)X(z)$$

$$\lim_{k\to\infty} x(kT)=\cos\frac{\pi}{4}k.$$

oscillates indefinitely

not exist

conditions for applying FVT are not meet



(b) $x(k+2) - x(k+1) + x(k) = 1$ Theorem

$u(t)$

$x(0) = 0 \qquad x(1) = 0$

$$z^2 X(z) - z^2 x(0) - z\,x(1) - [z X(z) - z\,x(0)] + X(z) = 1 \quad \frac{1}{1-z^{-1}}$$

$$z^2 X(z) - z X(z) + X(z) = 1 \quad \frac{1}{1-z^{-1}}$$

$$X(z) = \frac{1}{z^2 - z + 1} \cdot \frac{1}{1-z^{-1}}$$

$$= \cancel{\frac{z^{-2}}{1 - z^{-1} + z^{-2}}}$$

$z^2 - z + 1$

$1 \quad 1$

$1 \quad 1 \qquad -\frac{-1}{2\times 1} = \frac{1}{2}$

$\frac{1}{4} - \frac{1}{2} + 1$

$\Delta = -1 - 4 = -5 < 0$

无根，无法 PFE

long division $\quad \frac{X(z)}{z} = \frac{1}{(z^2 - z + 1)(z-1)}$

$$
\begin{array}{r|l}
 & z^{-2} + z^{-3} + z^{-5} + z^{-6} \\
\hline
1 - z^{-1} + z^{-2} & z^{-2} \\
 & z^{-2} - z^{-3} + z^{-4} \\
\hline
 & z^{-3} - z^{-4} \\
 & z^{-3} - z^{-4} + z^{-5} \\
\hline
 & z^{-5} \\
 & z^{-5} - z^{-6} + z^{-7} \\
\hline
 & z^{-6} - z^{-7} \\
 & z^{-6} - z^{-7} + z^{-8}
\end{array}
$$

k	0	1	2	3	4 …
$x(k)$	0	0	1	1	1 …

PFE

$$X(z) = \frac{A}{z-1} + \frac{Bz + C}{z^2 - z + 1}$$

$$Az^2 - Az + A + \underline{(Bz + C)(z-1)}$$

$$= Az^2 - Az + A + Bz^2 - Bz + Cz - C$$

$$= (A+B)z^2 + (-A-B+C)\,z + (A-C)$$

$$\begin{cases} A+B=0 \\ -A-B+C=0 \\ A-C=1 \end{cases} \Rightarrow \begin{cases} B=-A \\ -A+A+A-1=0 \\ C=A-1 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=-1 \\ C=0 \end{cases}$$

$$\frac{X(z)}{z} = \frac{1}{z-1} + \frac{-z}{z^2-z+1}$$

$$X(z) = \frac{1}{1-z^{-1}} - \boxed{\frac{1}{1-z^{-1}+z^{-2}}}$$

#3 $u(k)$

PFE ~~X~~

let $A(x) = \dfrac{1}{1-z^{-1}+z^{-2}} = \dfrac{z^2}{z^2-z+1}$

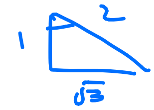
$$\frac{A(x)}{z} = \frac{z}{z^2-z+1} = \frac{z}{\left(z-\frac{1+\sqrt{3}i}{2}\right)\left(z-\frac{1-\sqrt{3}i}{2}\right)}$$

$$\frac{1}{1-z^{-1}+z^{-2}}$$

#14 $\dfrac{z^{-1}\sin\omega T}{1-2z^{-1}\cos\omega T+z^{-2}}$

#15 $\dfrac{1 - z^{-1}\cos wT}{1 - 2z^{-1}\cos wT + z^{-2}}$

let $-2\cos wT = -1$ $\quad \cos wT = \frac{1}{2}$ $\quad \sin wT = \frac{\sqrt{3}}{2}$



$\frac{\pi}{3}$

$$1 - \frac{1}{2}z^{-1} + \frac{\sqrt{3}}{2}z^{-1} \times \frac{1}{\sqrt{3}} = 1$$

$$Z^{-1}\left(\frac{1}{1 - z^{-1} + z^{-2}}\right) = \cos wkT + \frac{1}{\sqrt{3}}\sin wkT$$

$$= \cos\frac{\pi}{3}k + \frac{1}{\sqrt{3}}\sin\frac{\cancel{\sqrt{3}}\,\pi}{3}k$$

$$X(z) = 1 - \cos\frac{\pi}{3}k - \frac{\sqrt{3}}{3}\sin\frac{\cancel{\sqrt{3}}\,\pi}{3}k \quad , k = 0, 1, 2 \ldots$$