1. (a) The Laplace Transform $X(s)$ of a signal $x(t)$ is given as

$$X(s) = \frac{1}{s^2 + 5s + 4}.$$

Assume that $x(kT)$ is the sampled sequence of $x(t)$ with a sampling period T of 0.5 second. Determine the Z-transform of $x(kT)$.

(5 Marks)

(b) Solve the following difference equation:

$x(k+2) + (\alpha + 0.5)x(k+1) + 0.5\alpha\, x(k) = \delta(k)$, and $x(k) = 0$ for $k < 0$,

where α is real number and $\delta(k)$ is a unit impulse function.

(11 Marks)

(c) Discuss the convergence of $x(k)$ when $k \to \infty$ for different values of α in part 1(b).

(4 Marks)

17-S1-Q1

Q (a) $T = 0.5$ $\quad X(s) = \dfrac{1}{s^2+5s+4}$ $\quad Z(x(kT))$

Solution

$$X(s) = \frac{1}{s^2+5s+4} = \frac{1}{(s+1)(s+4)}$$

#9 $\dfrac{3}{(s+1)(s+4)}$ $\quad \dfrac{(e^{-T}-e^{-4T})z^{-1}}{(1-e^{-T}z^{-1})(1-e^{-4T}z^{-1})}$ $\quad T=0.5$

$$\text{So } Z(x(kT)) = \frac{(e^{-0.5}-e^{-2})z^{-1}}{3(1-e^{-0.5}z^{-1})(1-e^{-2}z^{-1})}$$

$$= \frac{0.4712z^{-1}}{3(1-0.6065z^{-1})(1-0.1353z^{-1})}$$

(b) Q: difference

Solution

apply z transform

$$z^2X(z) - z^2x(0) - zx(1) + (\alpha+0.5)\left[zX(z) - zx(0)\right] + 0.5\alpha X(z) = 1$$

let $k=-2$, $\quad x(0) = 0$

let $k=-1$, $\quad x(1) = 0$

$$z^2X(z) + (\alpha+0.5)zX(z) + 0.5\alpha X(z) = 1$$

$$\left[z^2+(\alpha+0.5)z+0.5\alpha\right]X(z) = 1$$

$$X(z)=\frac{1}{z^2+(\alpha+0.5)z+0.5\alpha}$$

$$=\frac{1}{(z+\alpha)(z+0.5)}$$

$$\frac{X(z)}{z}=\frac{1}{z(z+\alpha)(z+0.5)}$$

$\alpha \neq 0$ 时

① $$=\frac{A}{z}+\frac{B}{z+\alpha}+\frac{C}{z+0.5}$$ ($\alpha \neq 0.5$ 时，不涉及多极点)

$$A(z+\alpha)(z+0.5)+Bz(z+0.5)+Cz(z+\alpha)=1$$

$z=0$, $A\alpha 0.5=1 \Rightarrow A=\frac{2}{\alpha}$

$z=-\alpha$, $B(-\alpha)(-\alpha+0.5)=1 \Rightarrow B=\frac{1}{\alpha(\alpha-0.5)}$

$z=-0.5$, $C(-0.5)(-0.5+\alpha)=1 \Rightarrow C=\frac{1}{-\frac{1}{2}(-\frac{1}{2}+\alpha)}=\frac{4}{1-2\alpha}$

$$X(z)=\frac{2}{\alpha}+\frac{1}{\alpha(\alpha-0.5)}\frac{1}{1+\alpha z^{-1}}+\frac{4}{1-2\alpha}\frac{1}{1+0.5z^{-1}}$$

#18 $\quad a=-\alpha \qquad\qquad a=-0.5$

$$x(kT)=\frac{2}{\alpha}\delta_0(k)+\frac{1}{\alpha(\alpha-0.5)}(-\alpha)^k+\frac{4}{1-2\alpha}(-0.5)^k$$

② $\alpha=0.5$ 时

$$\frac{X(z)}{z}=\frac{1}{z\ (z+0.5)^2}$$

$$=\frac{A}{z}+\frac{B}{(z+0.5)^2}+\frac{C}{z+0.5}$$

$$A(z+0.5)^2 + Bz + Cz(z+0.5) = 1$$

$z=0,\ \frac{1}{4}A = 1 \quad \Rightarrow A = 4$

$z=-0.5,\ -\frac{1}{2}B = 1 \quad \Rightarrow B = -2$

考虑z^2项

$A+C=0 \quad \Rightarrow C=-4$

$$X(z) = 4 - 2\frac{z^{-1}}{(1+0.5z^{-1})^2} - 4\frac{1}{1+0.5z^{-1}}$$

#1

#20 $\quad a=-0.5 \quad \frac{z^{-1}}{(1+0.5z^{-1})^2} \quad k(-0.5)^{k-1}$

#18 $\quad a=-0.5 \quad \frac{1}{1+0.5z^{-1}} \quad (-0.5)^k$

$$X(kT) = 4\delta_0(k) - 2k(-0.5)^{k-1} - 4(-0.5)^k$$

③ $\alpha=0$时

$$\frac{X(z)}{z} = \frac{1}{z^2(z+0.5)}$$

$$= \frac{A}{z^2} + \frac{B}{z} + \frac{C}{z+0.5}$$

$$A(z+0.5) + Bz(z+0.5) + Cz^2 = 1$$

$z=0,\ \frac{1}{2}A = 1 \quad \Rightarrow A = 2$

$z=-0.5$, $\frac{1}{4}C=1 \Rightarrow C=4$

考虑 z^2 项

$B+C=0 \Rightarrow B=-4$

$X(z)=2z^{-1}-4+4\frac{1}{1+0.5z^{-1}}$

#2 $k=1$ $\delta_0(k-1)$

#1 $\delta_0(k)$

#18 $a=-0.5$ $(-0.5)^k$

$x(kT)=2\delta_0(k-1)-4\delta_0(k)+4(-0.5)^k$

(c) Q : convergence ?

Solution

$$X(z) = \frac{1}{(z+\alpha)(z+0.5)} = \frac{1}{(z-(-\alpha))(z-(-0.5))}$$

poles at $z_1 = -\alpha$ $\quad z_2 = -0.5$

$|z_1| = |\alpha|$ $\quad |z_2| = 0.5 < 1$

① all poles of $X(z)$ lie inside the unit circle with the possible exception of a simple pole at $z=1$, So we can use Final Value Theorem

$|z_1| = |\alpha| < 1 \Rightarrow \alpha \in (-1, 1)$

$z_1 = 1 \Rightarrow \alpha = -1 \Rightarrow \alpha \in [-1, 1)$

if $\alpha = -1$

$$\lim_{k \to \infty} x(k) = \lim_{z \to 1} (z-1) \frac{1}{(z-1)(z+0.5)}$$

$$= \frac{1}{1.5} = \frac{2}{3} = 0.6667$$

if $\alpha \in (-1, 1)$

$$\lim_{k \to \infty} x(k) = \lim_{z \to 1} \frac{z-1}{(z+\alpha)(z+0.5)}$$

$$= \frac{2}{3} \lim_{z \to 1} \frac{z-1}{z+\alpha}$$

$$= 0$$

② when $\alpha \in (-\infty, -1) \cup [1, +\infty)$

the prerequisites for FVT are not met

So we can't use Final Value Theorem

So, $X(kT)$ isn't converge