知识点Z4.15

常用函数的傅里叶变换

主要内容：

常用函数的傅里叶变换

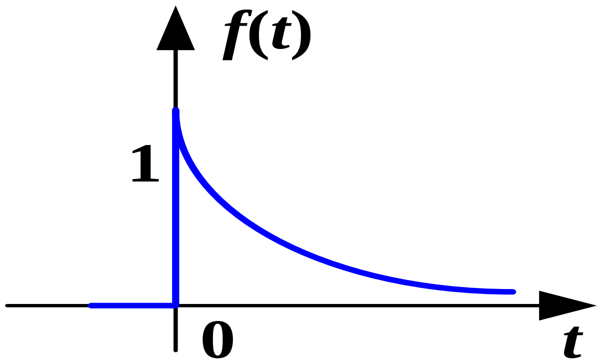
基本要求：

熟练掌握常用函数的傅里叶变换变换对

Z4.15常用函数的傅里叶变换

1. 单边指数函数

$$f(t)=\mathrm{e}^{-\alpha t}\varepsilon(t)=\begin{cases}\mathrm{e}^{-\alpha t} & t>0\\ 0 & t<0\end{cases}\qquad \alpha>0$$



$$F(j\omega)=\int_0^{\infty}\mathrm{e}^{-\alpha t}\mathrm{e}^{-j\omega t}\mathrm{d}t$$

$$=-\frac{1}{\alpha+j\omega}\mathrm{e}^{-(\alpha+j\omega)t}\Big|_0^{\infty}$$

$$=\frac{1}{\alpha+j\omega}$$

$$\boxed{\mathrm{e}^{-\alpha t}\varepsilon(t)\leftrightarrow\frac{1}{\alpha+j\omega}}$$

4.4非周期信号的频谱—傅里叶变换

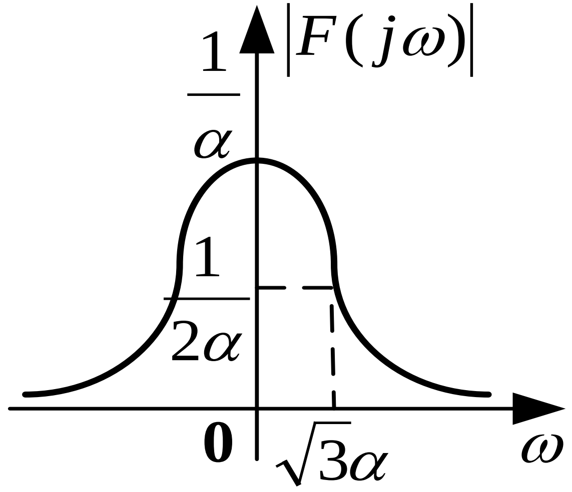
$$f(t)=\mathrm{e}^{-\alpha t}\varepsilon(t)\leftrightarrow F(j\omega)=\frac{1}{\alpha+j\omega}$$

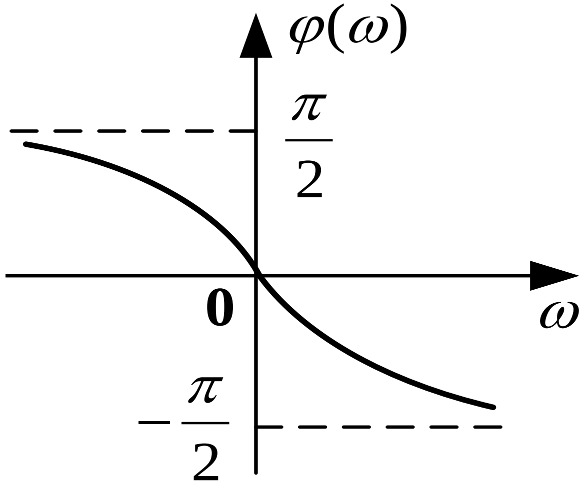
幅度频谱：

$$|F(j\omega)|=\frac{1}{\sqrt{\alpha^2+\omega^2}}$$

相位频谱：

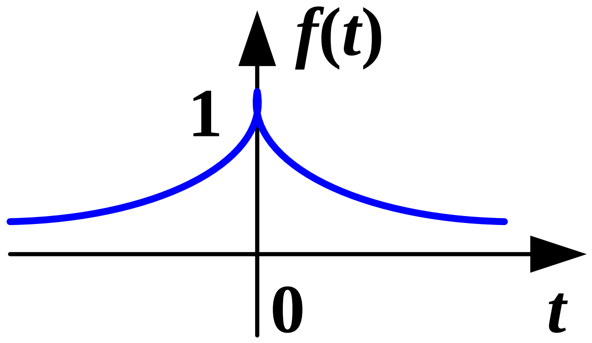
$$\varphi(\omega)=-\arctan\frac{\omega}{\alpha}$$





2. 双边指数函数

$$f(t) = e^{-\alpha|t|} = \begin{cases} e^{-\alpha t} & t > 0 \\ e^{\alpha t} & t < 0 \end{cases} \qquad \alpha > 0$$

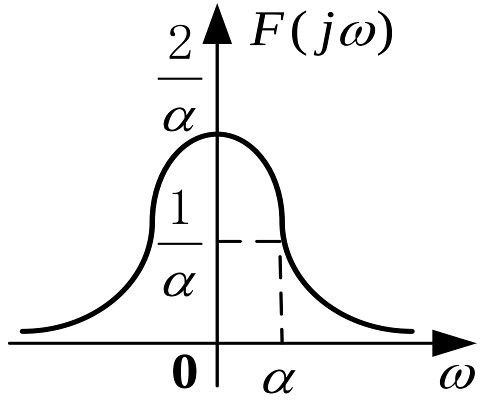


$$F(j\omega) = \int_{-\infty}^{0} e^{\alpha t} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-\alpha t} e^{-j\omega t} dt$$

$$= \frac{1}{\alpha - j\omega} + \frac{1}{\alpha + j\omega}$$

$$= \frac{2\alpha}{\alpha^2 + \omega^2}$$

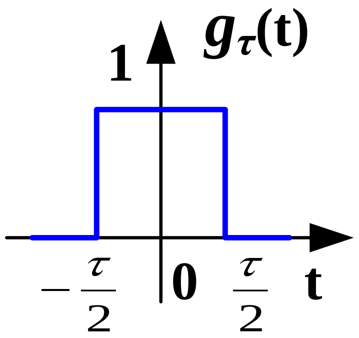
$$\boxed{e^{-\alpha|t|} \leftrightarrow \frac{2\alpha}{\alpha^2 + \omega^2}}$$



频谱图

3. 门函数(矩形脉冲)

$$g_\tau(t)=\begin{cases}1, & |t|\le\dfrac{\tau}{2}\\[2mm] 0, & |t|>\dfrac{\tau}{2}\end{cases}$$



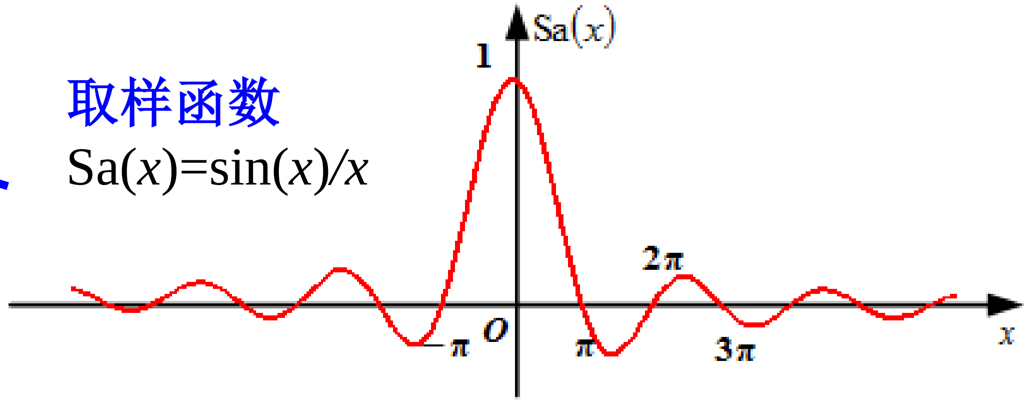
$$F(j\omega)=\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}}\mathrm{e}^{-j\omega t}\,\mathrm{d}t=\frac{\mathrm{e}^{-j\omega\frac{\tau}{2}}-\mathrm{e}^{j\omega\frac{\tau}{2}}}{-j\omega}$$

$$=\frac{2\sin(\frac{\omega\tau}{2})}{\omega}$$

$$=\tau\,\mathrm{Sa}\left(\frac{\omega\tau}{2}\right)$$

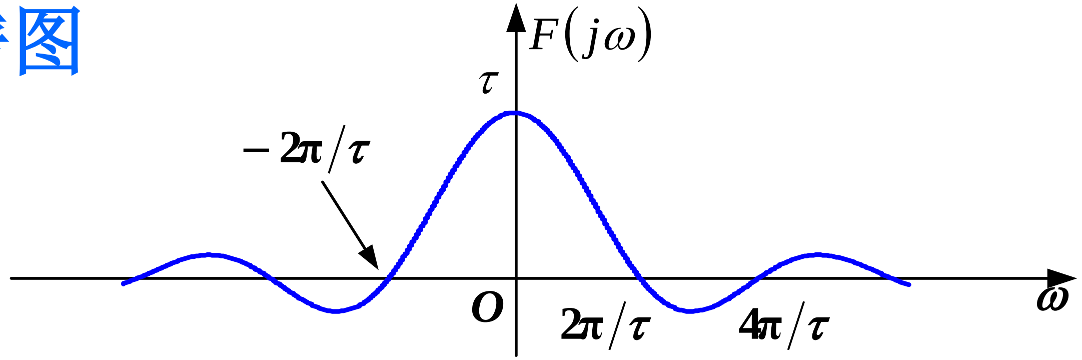
取样函数

Sa(x)=sin(x)/x



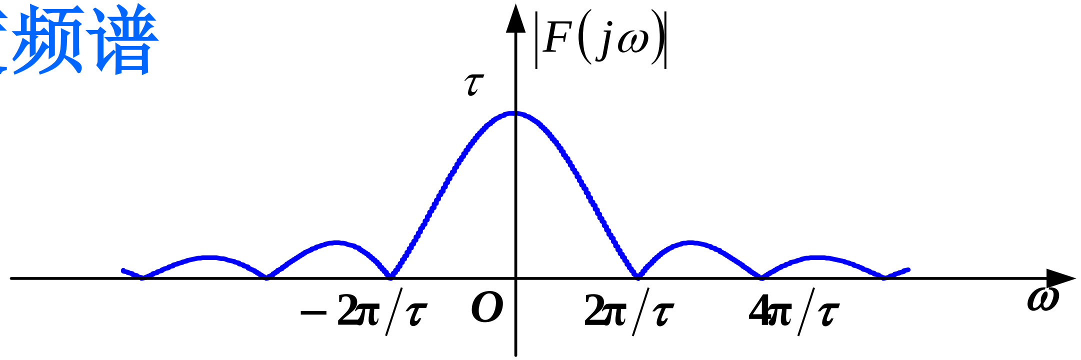
4.4非周期信号的频谱—傅里叶变换

频谱图



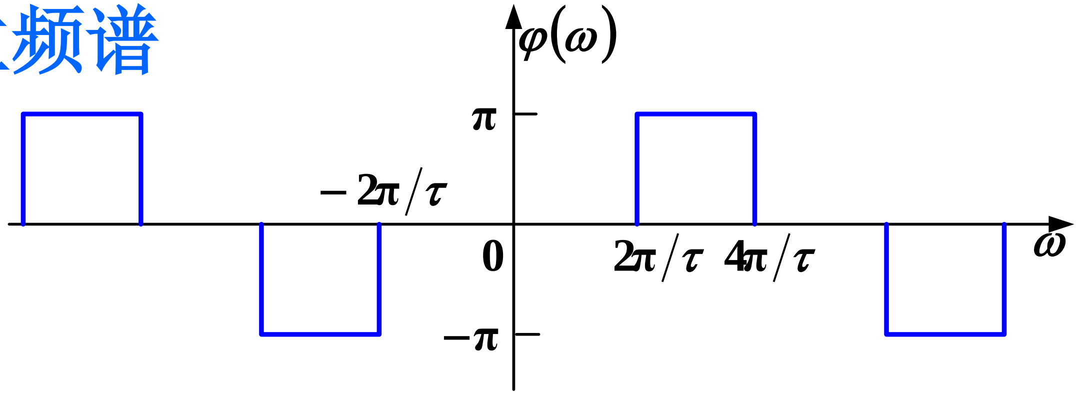
$$g_\tau(t) \leftrightarrow \tau\,\mathrm{Sa}\left(\frac{\omega\tau}{2}\right)$$

幅度频谱



$$|F(j\omega)| = \tau\left|\mathrm{Sa}\left(\frac{\omega\tau}{2}\right)\right|$$

相位频谱



频宽:

$$B_\omega \approx \frac{2\pi}{\tau} \text{或} B_f \approx \frac{1}{\tau}$$

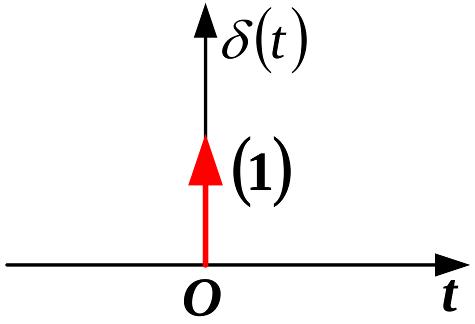
4. 冲激函数δ(t)、δ′(t)、 δ$^{(n)}$(t)

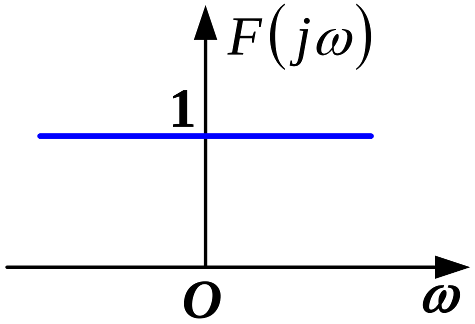
$$\delta(t) \longleftrightarrow F(j\omega)=\int_{-\infty}^{\infty}\delta(t)\mathrm{e}^{-j\omega t}\mathrm{d}t=1$$

（定义）

$$\delta'(t) \longleftrightarrow F(j\omega)=\int_{-\infty}^{\infty}\delta'(t)\mathrm{e}^{-j\omega t}\mathrm{d}t=-\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{-j\omega t}\Big|_{t=0}=j\omega$$

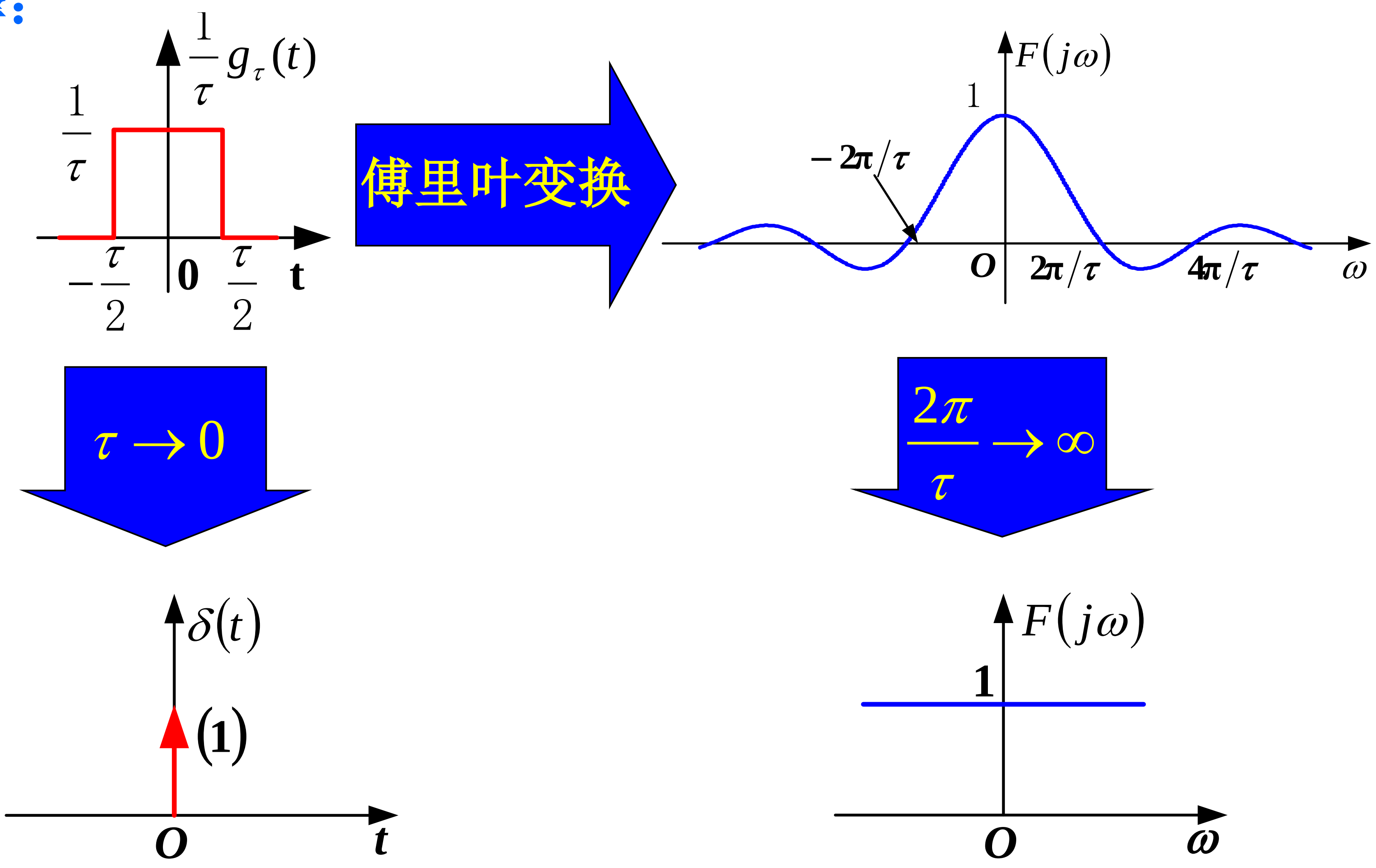
$$\delta^{(n)}(t) \longleftrightarrow (j\omega)^{n}$$





4.4非周期信号的频谱—傅里叶变换

解释：



5. 常数 1

有些函数(如1，ε(t) 等)不满足绝对可积这一充分条件，直接用定义式不易求解。可构造一函数序列{$f_n(t)$}逼近$f(t)$，即

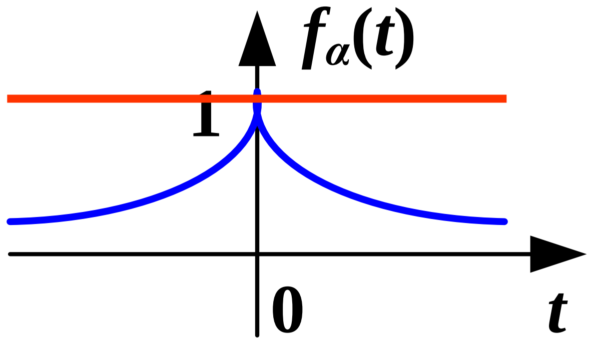
$$f(t)=\lim_{n\to\infty} f_n(t)$$

而$f_n(t)$满足绝对可积条件，并且{$f_n(t)$}的傅里叶变换所形成的序列{$F_n(j\omega)$}是极限收敛的。则$f(t)$的傅里叶变换$F(j\omega)$为

$$F(j\omega)=\lim_{n\to\infty} F_n(j\omega)$$

这样定义的傅里叶变换也称为广义傅里叶变换。

构造 $f_\alpha(t)=\mathrm{e}^{-\alpha|t|} \qquad \alpha>0$



$$F_\alpha(j\omega)=\frac{2\alpha}{\alpha^2+\omega^2}$$

$$f(t)=1=\lim_{\alpha\to 0} f_\alpha(t)$$

所以 $F(j\omega)=\lim_{\alpha\to 0} F_\alpha(j\omega)=\lim_{\alpha\to 0}\frac{2\alpha}{\alpha^2+\omega^2}=\begin{cases}0, & \omega\neq 0\\ \infty, & \omega=0\end{cases}$

又 $\lim_{\alpha\to 0}\int_{-\infty}^{\infty}\frac{2\alpha}{\alpha^2+\omega^2}d\omega=\lim_{\alpha\to 0}\int_{-\infty}^{\infty}\frac{2}{1+\left(\frac{\omega}{\alpha}\right)^2}d\frac{\omega}{\alpha}=\lim_{\alpha\to 0}2\arctan\frac{\omega}{\alpha}\Big|_{-\infty}^{\infty}=2\pi$

因此， $1\longleftrightarrow 2\pi\delta(\omega)$

另一种求法： δ(ω) 代入反变换定义式，有

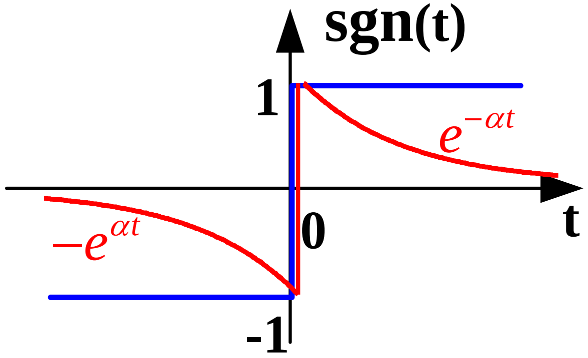
$$F^{-1}[\delta(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega)\mathrm{e}^{j\omega t}\mathrm{d}\omega = \frac{1}{2\pi}$$

$$\frac{1}{2\pi} \longleftrightarrow \delta(\omega)$$

$$1 \longleftrightarrow 2\pi\delta(\omega)$$

6. 符号函数

$$\operatorname{sgn}(t)=\begin{cases}-1, & t<0\\ 1, & t>0\end{cases}$$



构造 $$f_\alpha(t)=\mathrm{e}^{-\alpha|t|}=\begin{cases}-\mathrm{e}^{\alpha t}, & t<0\\ \mathrm{e}^{-\alpha t}, & t>0\end{cases}\qquad \alpha>0$$

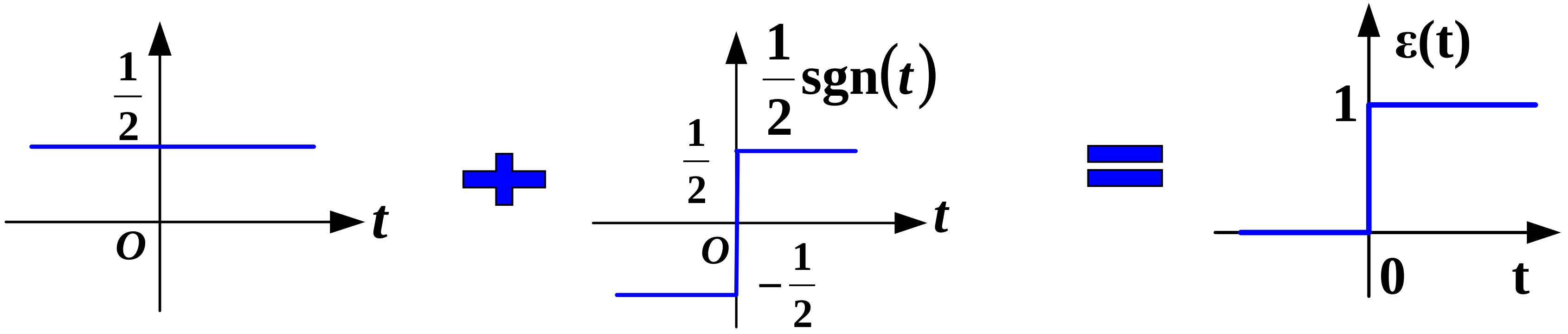
$$\operatorname{sgn}(t)=\lim_{\alpha\to 0} f_\alpha(t)$$

$$F_\alpha(j\omega)=\int_{-\infty}^{0}-e^{\alpha t}e^{-j\omega t}dt+\int_{0}^{\infty}e^{-\alpha t}e^{-j\omega t}dt=\frac{1}{\alpha+j\omega}-\frac{1}{\alpha-j\omega}=-\frac{j2\omega}{\alpha^2+\omega^2}$$

$$\boxed{\operatorname{sgn}(t)\longleftrightarrow\lim_{\alpha\to 0}F_\alpha(j\omega)=\lim_{\alpha\to 0}\left(-\frac{j2\omega}{\alpha^2+\omega^2}\right)=\frac{2}{j\omega}}$$

7. 阶跃函数 $\varepsilon(t)$

$$\varepsilon(t)=\begin{cases}0 & t<0\\ 1, & t>0\end{cases} \qquad \varepsilon(t)=\frac{1}{2}+\frac{1}{2}\mathrm{sgn}(t)$$

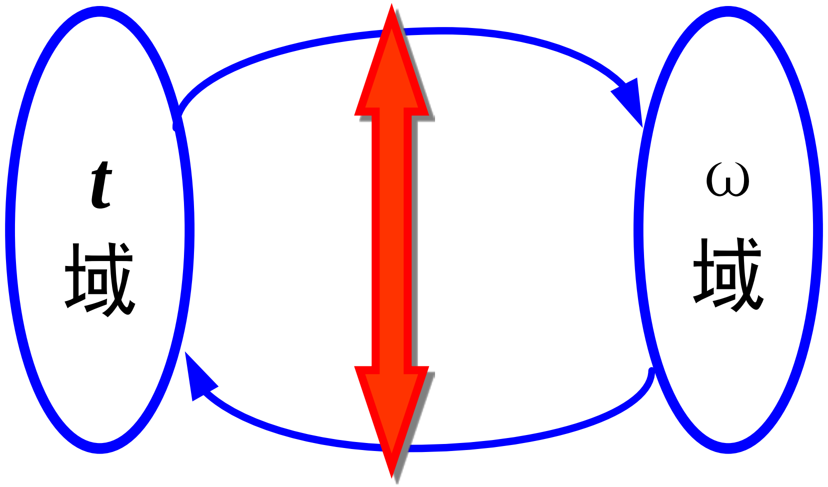


$$\frac{1}{2}\leftrightarrow\pi\delta(\omega) \qquad \frac{1}{2}\mathrm{sgn}(t)\leftrightarrow\frac{1}{j\omega} \qquad \boxed{\varepsilon(t)\leftrightarrow\pi\delta(\omega)+\frac{1}{j\omega}}$$

归纳记忆：

1. $\mathscr{F}$ 变换对

$$F(j\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,\mathrm{d}t$$



$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}\,\mathrm{d}\omega$$

2. 常用函数 $\mathscr{F}$ 变换对

$$\delta(t) \longleftrightarrow 1$$

$$1 \longleftrightarrow 2\pi\delta(\omega)$$

$$\varepsilon(t) \longleftrightarrow \pi\delta(\omega)+\frac{1}{j\omega}$$

$$e^{-\alpha t}\varepsilon(t) \longleftrightarrow \frac{1}{j\omega+\alpha}$$

$$g_\tau(t) \longleftrightarrow \tau\, Sa\left(\frac{\omega\tau}{2}\right)$$

$$\mathrm{sgn}(t) \longleftrightarrow \frac{2}{j\omega}$$

$$e^{-\alpha|t|} \longleftrightarrow \frac{2\alpha}{\alpha^2+\omega^2}$$