20 — S1 — Q3

Q(a) $\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$

$y(t) = [1 \quad 1]\, x(t)$

$T = 1$ discretised

Solution ① state tranfer matrix in s domain

$$[sI - A]^{-1} = \begin{bmatrix} s & -1 \\ 3 & s+4 \end{bmatrix}^{-1} = \frac{1}{s(s+4)+3} \begin{bmatrix} s+4 & 1 \\ -3 & s \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s+4}{s^2+4s+3} & \frac{1}{s^2+4s+3} \\ -\frac{3}{s^2+4s+3} & \frac{s}{s^2+4s+3} \end{bmatrix}$$

② t domain

$$\Phi(t) = L^{-1}\{[sI - A]^{-1}\}$$

where $s^2+4s+3 = (s+1)(s+3)$

#9 $\frac{b-a}{(s+a)(s+b)} = \frac{2}{(s+1)(s+3)}$ $\quad L^{-1}\left\{\frac{2}{(s+1)(s+3)}\right\} = e^{-t} - e^{-3t}$

#17

$\frac{s+a}{(s+a)^2+w^2} = \frac{s+2}{(s+2)^2 - 1}$ × 无法化为 w^2 PFE

$$\frac{s}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}$$

$$A(S+3)+B(S+1) = AS+3A+BS+B$$

$$= (A+B)S+3A+B$$

$$\begin{cases} A+B=1 \\ 3A+B=0 \end{cases} \Rightarrow \begin{cases} A=-\frac{1}{2} \\ B=1-A=1+\frac{1}{2}=\frac{3}{2} \end{cases}$$

$$\frac{S}{(S+1)(S+3)} = \frac{-\frac{1}{2}}{S+1} + \frac{\frac{3}{2}}{S+3}$$

$$L^{-1}\left\{\frac{S}{(S+1)(S+3)}\right\} = -\frac{1}{2}e^{-t} + \frac{3}{2}e^{-3t}$$

$$L^{-1}\left\{\begin{bmatrix} \frac{S+4}{S^2+4S+3} & \frac{1}{S^2+4S+3} \\ -\frac{3}{S^2+4S+3} & \frac{S}{S^2+4S+3} \end{bmatrix}\right\} = \begin{bmatrix} \frac{3}{2}e^{-t}-\frac{1}{2}e^{-3t} & \frac{1}{2}e^{-t}-\frac{1}{2}e^{-3t} \\ -\frac{3}{2}e^{-t}+\frac{3}{2}e^{-3t} & -\frac{1}{2}e^{-t}+\frac{3}{2}e^{-3t} \end{bmatrix}$$

$$L^{-1}\left\{\frac{S+4}{S^2+4S+3}\right\} = -\frac{1}{2}e^{-t}+\frac{3}{2}e^{-3t}+2\left(e^{-t}-e^{-3t}\right)$$

$$= \frac{3}{2}e^{-t}-\frac{1}{2}e^{-3t}$$

③ T domain

$$\Phi(T) = L^{-1}\left\{[SI-A]^{-1}\right\}\Big|_{t=T}$$

$$= \begin{bmatrix} \frac{3}{2}e^{-t}-\frac{1}{2}e^{-3t} & \frac{1}{2}e^{-t}-\frac{1}{2}e^{-3t} \\ -\frac{3}{2}e^{-t}+\frac{3}{2}e^{-3t} & -\frac{1}{2}e^{-t}+\frac{3}{2}e^{-3t} \end{bmatrix}\Bigg|_{t=T}$$

$$= \left. \begin{bmatrix} \frac{3}{2}e^{-T} - \frac{1}{2}e^{-3T} & \frac{1}{2}e^{-T} - \frac{1}{2}e^{-3T} \\ -\frac{3}{2}e^{-T} + \frac{3}{2}e^{-3T} & -\frac{1}{2}e^{-T} + \frac{3}{2}e^{-3T} \end{bmatrix} \right|_{T=1}$$

$$= \begin{bmatrix} 0.5269 & 0.1590 \\ -0.4771 & -0.1093 \end{bmatrix}$$

④ input matrix

$$\theta(T) = \int_0^T \Phi(\eta) d\eta \, B$$

$$= \int_0^T \begin{bmatrix} \frac{3}{2}e^{-\eta} - \frac{1}{2}e^{-3\eta} & \frac{1}{2}e^{-\eta} - \frac{1}{2}e^{-3\eta} \\ -\frac{3}{2}e^{-\eta} + \frac{3}{2}e^{-3\eta} & -\frac{1}{2}e^{-\eta} + \frac{3}{2}e^{-3\eta} \end{bmatrix} d\eta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \int_0^T \begin{bmatrix} \frac{1}{2}e^{-\eta} - \frac{1}{2}e^{-3\eta} \\ -\frac{1}{2}e^{-\eta} + \frac{3}{2}e^{-3\eta} \end{bmatrix} d\eta$$

where $\int_0^T \frac{1}{2}e^{-\eta} - \frac{1}{2}e^{-3\eta} \, d\eta$

$$= \frac{1}{2}\int_0^T e^{-\eta} d\eta - \frac{1}{2}\int_0^T e^{-3\eta} d\eta$$

$$= \frac{1}{2}\left(-e^{-\eta}\Big|_0^1\right) - \frac{1}{2}\left(-\frac{1}{3}e^{-3\eta}\Big|_0^1\right)$$

$$= \frac{1}{2}(-e^{-1} + e^{0}) + \frac{1}{6}(e^{-3} - e^{0})$$

$= -\frac{1}{2}e^{-1} + \frac{1}{2} + \frac{1}{6}e^{-3} - \frac{1}{6}$ casio

$= 0.1576$

where $\int_0^T -\frac{1}{2}e^{-\eta} + \frac{3}{2}e^{-3\eta}\, d\eta = 0.1590$ casio

$\theta(T) = \begin{bmatrix} 0.1576 \\ 0.1590 \end{bmatrix}$

So the discrete time model

$$x(k+1) = \begin{bmatrix} 0.5269 & 0.1590 \\ -0.4771 & -0.1093 \end{bmatrix} x(k) + \begin{bmatrix} 0.1576 \\ 0.1590 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(k)$$

(b) (i) C? O?

Solution

$W_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 4 & 14 \\ 6 & 2 \end{bmatrix}$

$AB = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 8+6 \\ -4+6 \end{bmatrix} = \begin{bmatrix} 14 \\ 2 \end{bmatrix}$

$|W_c| = 8 - 6 \times 14 \neq 0$, controllable

$W_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$

$CA = [0 \quad 1]\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \end{bmatrix}$

$|W_o| = 1 \neq 0$, observable.

(ii) $u, y \to x$

Solution

let $k = 0$

$$\begin{cases} x(1) = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} x(0) + \begin{bmatrix} 4 \\ 6 \end{bmatrix} u(0) \\ y(0) = [0 \; 1]\, x(0) + 8\, u(0) \end{cases}$$

$$\begin{cases} x(1) = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} x(0) = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \\ -1 = [0 \quad 1]\, x(0) = [0 \; 1] \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \end{cases}$$

$$\begin{cases} x_1(1) = 2x_1(0) + x_2(0) = 2x_1(0) - 1 \\ x_2(1) = -x_1(0) + x_2(0) = -x_1(0) - 1 \\ x_2(0) = -1 \quad ✓ \end{cases}$$

let $k = 1$

$y(1) = [0 \quad 1]\begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} + 8\, u(1)$

$3 = x_2(1) - 8$

$x_2(1) = 3 + 8 = 11$ ✓

$x_2(1) = -x_1(0) - 1 = 11$

$x_1(0) = -12$ ✓

$x_1(1) = 2x_1(0) - 1 = -24 - 1 = -25$

So $x(0) = \begin{bmatrix} -12 \\ -1 \end{bmatrix}$ $x(1) = \begin{bmatrix} -25 \\ 11 \end{bmatrix}$

(c) P ? A B C ? similarity transformation

Solution

$P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

$\hat{x}(k+1) = \hat{A}\,\hat{x}(k) + \hat{B}u(k)$

$y(k) = \hat{C}\,\hat{x}(k) + \hat{d}\,u(k)$

$\hat{A} = PAP^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$

$\hat{B} = PB = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \end{bmatrix}$

$\hat{C} = CP^{-1} = [0\ 1]\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = [-1\ 1]$

$\hat{d} = d = 8$

$\hat{x}(k+1) = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}\hat{x}(k) + \begin{bmatrix} 4 \\ 10 \end{bmatrix}u(k)$

$y(k) = [-1\ 1]\,\hat{x}(k) + 8\,u(k)$