22-S1-Q3

Q: (a) state-space model

$$m_1(t) - m_2(t) = \frac{dp(t)}{dt} + k_1 p(t) \qquad (1)$$

$$m_2(t) = k_2 \frac{d\phi(t)}{dt} \qquad (2)$$

$$k_3 p(t) = k_4 \frac{d\phi(t)}{dt} + k_5 \frac{d^2\phi(t)}{dt^2} \qquad (3)$$

$$x_1(t) = \phi(t) \qquad (4)$$

$$x_2(t) = \frac{d\phi(t)}{dt} \qquad (5)$$

$$x_3(t) = p(t) \qquad (6)$$

$$u(t) = m_1(t) \qquad (7)$$

$$y(t) = \phi(t) \qquad (8)$$

Solution

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{k_1}{k_5} & \frac{k_3}{k_5} \\ 0 & -k_2 & -k_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

(4)(5) $\dot{x}_1(t) = \frac{d\phi(t)}{dt} = x_2(t)$

(2) $m_2(t) = k_2 x_2(t)$

①②⑥ $u(t) - k_2 x_2(t) = \dot{x}_3(t) + k_1 x_3(t)$

$$\dot{x}_3(t) = -k_2 x_2(t) - k_1 x_3(t) + u(t)$$

③⑤ $k_3 x_3(t) = k_1 x_2(t) + k_5 \dot{x}_2(t)$

$$\dot{x}_2(t) = -\frac{k_1}{k_5} x_2(t) + \frac{k_3}{k_5} x_3(t)$$

④⑧ $y = x_1(t)$

(b) Q: C → D?

Solution ① $[SI - A]^{-1}$

$$[SI-A]^{-1} = \begin{bmatrix} s+2 & 0 \\ -3 & s+3 \end{bmatrix}^{-1} = \frac{1}{(s+2)(s+3)} \begin{bmatrix} s+3 & 0 \\ 3 & s+2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s+2} & 0 \\ \frac{3}{(s+2)(s+3)} & \frac{1}{s+3} \end{bmatrix}$$

#9 $\frac{1}{(s+2)(s+3)}$ $\quad e^{-2t} - e^{-3t}$

#4 $\frac{1}{s+2}$ $\quad e^{-2t}$

#5 $\frac{1}{s+3}$ $\quad e^{-3t}$

$$\Phi(t) = \mathcal{L}^{-1}\{[sI-A]^{-1}\} = \begin{bmatrix} e^{-2t} & 0 \\ 3e^{-2t}-3e^{-3t} & e^{-3t} \end{bmatrix}$$

$$\Phi(T) = \begin{bmatrix} e^{-2T} & 0 \\ 3e^{-2T}-3e^{-3T} & e^{-3T} \end{bmatrix} \qquad T = 0.1$$

$$= \begin{bmatrix} 0.8187 & 0 \\ 0.2337 & 0.7408 \end{bmatrix}$$

$$\theta(T) = \int_0^T \Phi(t)\,dt\, B$$

$$= \int_0^T \begin{bmatrix} e^{-2t} & 0 \\ 3e^{-2t}-3e^{-3t} & e^{-3t} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} dt$$

$$= \int_0^T \begin{bmatrix} 2e^{-2t} \\ 6(e^{-2t}-e^{-3t}) \end{bmatrix} dt \qquad T = 0.1 \qquad \text{casio}$$

$$= \begin{bmatrix} 0.1813 \\ 0.02544 \end{bmatrix}$$

So $x(k+1) = \begin{bmatrix} 0.8187 & 0 \\ 0.2337 & 0.7408 \end{bmatrix} x(k) + \begin{bmatrix} 0.1813 \\ 0.02544 \end{bmatrix} u(k)$

$$y(k) = [\,1 \;\; 0\,]\, x(k)$$

(ii) Q: poles → stable? T

Solution

$$\Phi(T) = \begin{bmatrix} e^{-2T} & 0 \\ 3e^{-2T}-e^{-3T} & e^{-3T} \end{bmatrix} = A_d$$

$$|zI - A_d| = \begin{vmatrix} z - e^{-2T} & \\ 3e^{-2T}-e^{-3T} & z - e^{-3T} \end{vmatrix} = (z-e^{-2T})(z-e^{-3T})$$

$|z_1| = e^{-2T} < 1 \quad |z_2| = e^{-3T} < 1$

for $T > 0$ $\quad z_{1,2} \in (0,1)$ the system is stable

So the range of T is $T > 0$

(c) state-mode ?

Solution

$$x_1(k+1) = x_1(k) + x_2(k) + u(k) \quad ①$$
$$x_2(k+1) = x_2(k) + u(k) \quad ②$$
$$y(k) = x_1(k) \quad ③$$
$$x_3(k) = u(k) \quad ④$$
$$u(k+1) = -a\,u(k) + r(k) - y(k) \quad ⑤$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & -a \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

③④⑤ $x_3(k+1) = -a\,x_3(k) + r(k) - x_1(k)$

②④ $x_2(k+1) = x_2(k) + x_3(k)$

①④ $x_1(k+1) = x_1(k) + x_2(k) + x_3(k)$