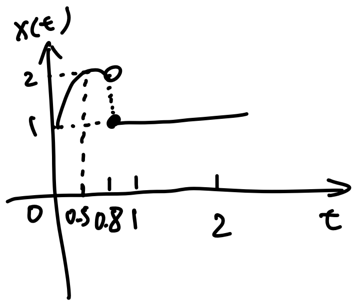
23 - S1 - Q1

Q(a) 连? Z? 0 start $T = 0.5$

Solution

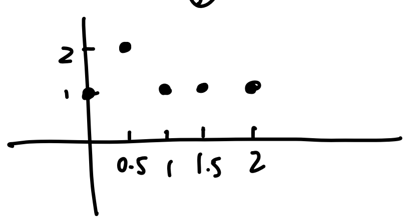
① Yes, $x(t)$ is a continuous-time signal because it is defined for all real values of t. The function $x(t)$ provides a specific value for every time t in the real number domain, fulfilling the definition of a continuous-time signal.

②



$T = \frac{2\pi}{\pi} = 2$

↓ T=0.5 sampling



according to the definition of Z transform

$$X(z) = Z[x(t)] = \sum_{k=0}^{\infty} x(kT) z^{-k}$$

$$= x(0) z^{-0} + X(0.5) z^{-1} + X(1) z^{-2} + X(1.5) z^{-3} + \cdots$$

$$= 1 + 2z^{-1} + z^{-2} + z^{-3} + \cdots$$

So $\sum_{n=2}^{\infty} z^{-n} = \frac{z^{-2}}{1 - z^{-1}}$

$$X(z) = 1 + 2z^{-1} + \frac{z^{-2}}{1 - z^{-1}}$$

$$= \frac{(1+2z^{-1})(1-z^{-1}) + z^{-2}}{1 - z^{-1}}$$

$$= \frac{1 - z^{-1} + 2z^{-1} - 2z^{-2} + z^{-2}}{1 - z^{-1}}$$

$$= \frac{1 + z^{-1} - z^{-2}}{1 - z^{-1}}$$

(b) Q: difference equation

Solution apply z transform to

$$y(k+2) + (\beta - 1)\, y(k+1) - \beta y(k) = \delta(k-1)$$

$$z^2 Y(z) - z^2 y(0) - z y(1) + (\beta-1)[zY(z) - zy(0)] - \beta Y(z) = z^{-1}$$

let $k = -2$, $y(0) + (\beta - 1)\, y(-1) - \beta y(-2) = \delta(-3)$

So $y(0) = 0$

let $k = -1$, $y(1) + (\beta-1)\, y(0) - \beta y(-1) = \delta(-2)$

So $y(1) = 0$

So $z^2 Y(z) + (\beta - 1) z Y(z) - \beta Y(z) = z^{-1}$

$$(z^2+\beta z-z-\beta)Y(z)=z^{-1}$$

$$Y(z)=\frac{z^{-1}}{z^2+(\beta-1)z-\beta}$$

$$\frac{Y(z)}{z}=\frac{1}{z^2(z-1)(z+\beta)}$$

$$=\frac{A}{z^2}+\frac{B}{z-1}+\frac{C}{z+\beta}$$

重根应该是 $Az+B$

错误做法 跳过去

$$A(z-1)(z+\beta)+Bz^2(z+\beta)+Cz^2(z-1)$$

$$A(z^2+\beta z-z+\beta)+B(z^3+\beta z^2)+C(z^3-z^2)$$

$$(B+C)z^3+(A+B\beta-C)z^2+(A\beta-A)z+A\beta$$

$$\begin{cases} B+C=0 \\ A+B\beta-C=0 \\ A\beta-A=0 \\ A\beta=1 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=-\frac{1}{2} \\ C=\frac{1}{2} \\ \beta=1 \end{cases}$$

$$B+C=0$$
$$1+B-C=0$$
$$1+2B=0$$
$$B=-\frac{1}{2}$$

$$\frac{Y(z)}{z}=\frac{1}{z^2}+\frac{-\frac{1}{2}}{z-1}+\frac{\frac{1}{2}}{z+1}$$

$$Y(z)=z^{-1}-\frac{1}{2}\frac{1}{1-z^{-1}}+\frac{1}{2}\frac{1}{1+z^{-1}}$$

$$y(k)=\delta_0(n-1)-\frac{1}{2}1(k)+\frac{1}{2}\cos k\pi$$

(c) (b)部分不含 β, so (b)过程有误

×

$$\frac{Y(z)}{z} = \frac{1}{z^2(z-1)(z+\beta)}$$

$$\frac{A}{z} + \frac{B}{z^2} = \frac{Az+B}{z^2} + \frac{C}{z-1} + \frac{D}{z+\beta}$$

下次写待定的

改进：试根法 ↓ 较快

$$(Az+B)(z-1)(z+\beta) + Cz^2(z+\beta) + Dz^2(z-1)$$

$$= (Az+B)(z^2+\beta z - z - \beta) + C(z^3+\beta z^2) + D(z^3 - z^2)$$

$$= (Az^3 + A\beta z^2 - Az^2 - A\beta z + Bz^2 + B\beta z - Bz - B\beta)$$

$$+ C(z^3 + \beta z^2) + D(z^3 - z^2)$$

$$\begin{cases} A + C + D = 0 & (1) \\ A\beta - A + B + C\beta - D = 0 & (2) \\ -A\beta + B\beta - B = 0 & (3) \\ -B\beta = 1 & (4) \end{cases}$$

$$\begin{cases} A = \frac{1-\beta}{\beta^2} \\ B = -\frac{1}{\beta} \\ C = \frac{1}{1+\beta} \\ D = -\frac{1}{\beta^2(\beta+1)} \end{cases}$$

from (4) $B = -\frac{1}{\beta}$

from (3) $-A = \frac{B - B\beta}{\beta} = -\frac{B(1-\beta)}{\beta} = \frac{1-\beta}{\beta^2} = \cancel{\frac{\beta-1}{\beta^2}}$

from (1)

$$C+D = -A = \frac{\beta-1}{\beta^2} \qquad (5)$$

from (2) $\quad A\beta - A + B + C\beta - D = 0$

$$\frac{1-\beta}{\beta} - \frac{1-\beta}{\beta^2} - \frac{1}{\beta} + C\beta - D = 0$$

$$C\beta - D = \frac{\beta^2+1-\beta}{\beta^2} \qquad (6)$$

(5)+(6) $\quad (1+\beta)C = \frac{\beta-1+\beta^2+1-\beta}{\beta^2} = 1 \quad \Rightarrow C = \frac{1}{1+\beta}$

from (5) $\quad D = \frac{\beta-1}{\beta^2} - C = \frac{\beta-1}{\beta^2} - \frac{1}{1+\beta} = \frac{\beta^2-1-\beta^2}{\beta^2(1+\beta)}$

So $\quad D = \frac{-1}{\beta^2(1+\beta)}$

$$\frac{Y(z)}{z} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-1} + \frac{D}{z+\beta}$$

$$Y(z) = \frac{1-\beta}{\beta^2} - \frac{1}{\beta}z^{-1} + \frac{1}{1+\beta}\frac{1}{1-z^{-1}} - \frac{1}{\beta^2(\beta+1)}\frac{1}{1+\beta z^{-1}}$$

$$y(k) = \frac{1-\beta}{\beta^2}\delta_0(k) - \frac{1}{\beta}\delta_0(k-1) + \frac{1}{1+\beta}1(k) - \frac{1}{\beta^2(\beta+1)}(-\beta)^k.$$

$$y(k) = \begin{cases} \frac{1-\beta}{\beta^2} + \frac{1}{1+\beta} - \frac{(-\beta)^k}{\beta^2(\beta+1)} & k=0 \\ -\frac{1}{\beta} + \frac{1}{1+\beta} - \frac{(-\beta)^k}{\beta^2(\beta+1)} & k=1 \end{cases}$$

$y(0)=0$

$y(1)=0$

$y(2)=0$

$$y(k) = \begin{cases} 0 & k = 0, 1, 2 \\ \dfrac{1}{1+\beta} - \dfrac{(-\beta)^k}{\beta^2(\beta+1)} & k = 3, 4, 5, \cdots \end{cases}$$

$$\left| \quad \frac{1}{1+\beta} - \frac{(-\beta)^k}{\beta^2(\beta+1)} \qquad k \geq 2\right.$$

(c) $\beta \to y(k)$ convergence ?

Solution

① when $|\beta| < 1$, $k \to \infty$, $(-\beta)^k \to 0$

$$y(k) \to \frac{1}{1+\beta} \qquad |z_1| = 1 \quad |z_2| = |-\beta| = |\beta| < 1$$

The Final Value Theorem applies

all poles of $Y(z)$ lie inside the unit circle, with the possible exception of a simple pole at $z=1$

Verification : $\lim_{k\to\infty} y(k) = \lim_{z\to 1} (1-z^{-1}) Y(z) = \dfrac{1}{\beta+1}$

so confirm theorem acceptable.

② when $|\beta| = 1$ $\quad (-\beta)^k$ oscillates ~~between~~ ± 1

$y(k)$ doesn't converge, it oscillates

The Final value Theorem does not apply

③ when $|\beta| > 1$, $(-\beta)^k$ grows without bound

$y(k)$ diverges as $k \to \infty$

The Final value Theorem does not apply

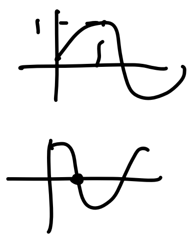
关于(a) Z transform 的思考，前面用定义求的 Z-transform
现在试用表求，

Solution

$$Z\{1+\sin(\pi t)\} \qquad \#3$$

$$= \frac{1}{1-z^{-1}} + Z\{\sin(\pi t)\} \qquad \#14$$

$$= \frac{1}{1-z^{-1}} + \frac{z^{-1}\sin \pi T}{1-2z^{-1}\cos \pi T + z^{-2}} \qquad T=\frac{1}{2}$$



$$= \frac{1}{1-z^{-1}} + \frac{z^{-1}}{1+z^{-2}}$$

$$= \frac{(1+z^{-2}) + z^{-1}(1-z^{-1})}{(1-z^{-1})(1+z^{-2})}$$

$$= \frac{1+z^{-2}+z^{-1}-z^{-2}}{(1-z^{-1})(1+z^{-2})}$$

$$= \frac{1+z^{-1}}{(1-z^{-1})(1+z^{-2})} \qquad 0\le t<0.8$$

结论：分段用定义
连续用查表
查不出来用定义

$$Z\{1\} = \frac{1}{1-z^{-1}} \qquad t\ge 0.8$$

不会对

关于 23 - Q1 - 6 对 ABCD 的优化：试根法

Solution

$$\frac{Y(z)}{z} = \frac{1}{z^2(z-1)(z+\beta)}$$

$$= \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-1} + \frac{D}{z+\beta}$$

$$A z(z-1)(z+\beta) + B(z-1)(z+\beta) + C z^2(z+\beta) + D z^2(z-1) = 1$$

由于等式过于复杂，所以可以采用试根法

A B C D 对 $z \in R$ 都成立，所以代入 z 值得 ABCD

① $z=0$, $-\beta B = 1 \Rightarrow B = -\frac{1}{\beta}$

② $z=1$, $(\beta+1)C = 1 \Rightarrow C = \frac{1}{\beta+1}$

③ $z=-\beta$, $\beta^2(-\beta-1)D = 1 \Rightarrow D = \frac{-1}{\beta^2(\beta+1)}$

④ 考虑 z^3 项求 A

$$Az^3 + Cz^3 + Dz^3 = (A+C+D)z^3$$

$$A+C+D=0$$

$$\Rightarrow A = -C-D = -\frac{1}{\beta+1} + \frac{1}{\beta^2(\beta+1)}$$

$$= \frac{1-\beta^2}{\beta^2(\beta+1)}$$

$$= \frac{(1-\beta)(1+\beta)}{\beta^2(\beta+1)}$$

$$= \frac{1-\beta}{\beta^2}$$