3. (a) An industrial process can be described by the following state-space representation:

$$\begin{bmatrix} \dfrac{dx_1(t)}{dt} \\ \dfrac{dx_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 8 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

where $x_1(t)$ and $x_2(t)$ are the states, $u(t)$ and $y(t)$ are the input and output variables, respectively. The system is sampled with a zero-order hold at a sampling period of T s. Determine a discretised state-space model for the system. Express your answer in terms of T.

(7 Marks)

(b) A discrete-time system has a state-space representation given by

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -6 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

where $\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$, $u(k)$ and $y(k)$ are the states, input and output variables, respectively.

(i) Determine whether the system is controllable and/or observable.

(ii) The control is realised through state-feedback of the following form:

$$u(k) = -\begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + 2r(k)$$

where k_1 and k_2 are real constants and $r(k)$ is the reference input. Determine the values of k_1 and k_2 that must be avoided for the closed-loop system to be completely observable.

(8 Marks)

(c) Consider a system which is described by the following state equation:

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\,u(k)$$

and with an associated performance index given by

$$J = \frac{1}{2}\sum_{k=0}^{\infty}\left(\mathbf{x}^T(k)\mathbf{Q}\mathbf{x}(k) + ru^2(k)\right)$$

The control law that minimises J is of the following form:

$$u^*(k) = -\mathbf{K}\mathbf{x}(k)$$

where the optimal control gain is given by

$$\mathbf{K} = \left(\mathbf{B}^T\mathbf{S}\mathbf{B} + r\right)^{-1}\mathbf{B}^T\mathbf{S}\,\mathbf{A}$$

and $\mathbf{S} > 0$ solves the following equation:

$$\mathbf{S} = \mathbf{A}^T\,\mathbf{S}\,\mathbf{A} + \mathbf{Q} - \mathbf{A}^T\,\mathbf{S}\mathbf{B}\left(r + \mathbf{B}^T\mathbf{S}\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{S}\,\mathbf{A}$$

Now, consider the following system and its associated performance index:

$$x(k+1) = 0.8x(k) + 0.2u(k)$$

$$J = \frac{1}{2}\sum_{k=0}^{\infty}\left(x^2(k) + 6u^2(k)\right)$$

Determine the optimal control law that minimises J and the location of the closed-loop pole.

(5 Marks)

19 - S1 - Q3

Q: (a) discretised ?

Solution ① state transform matrix

$$[sI-A]^{-1} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} = \frac{1}{s(s+3)+2}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$= \frac{1}{s^2+3s+2}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$= \frac{1}{(s+1)(s+2)}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

table #9

$$\frac{b-a}{(s+a)(s+b)} = \frac{1}{(s+1)(s+2)}$$ we get $e^{-t} - e^{-2t}$

$$\frac{s}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$ PFE

$$A(s+2) + B(s+1) = (A+B)s + 2A + B$$

$$\begin{cases} A+B = 1 \\ 2A+B = 0 \end{cases} \Rightarrow \begin{cases} A = -1 \\ B = 2 \end{cases}$$

$$\frac{s}{(s+1)(s+2)} = \frac{-1}{s+1} + \frac{2}{s+2}$$

#4 $\frac{1}{s+a}$, we get $-e^{-t}+2e^{-2t}$

Therefore

$$L^{-1}\left\{\frac{s+3}{(s+1)(s+2)}\right\} = -e^{-t}+2e^{-2t}+3e^{-t}-3e^{-2t} = 2e^{-t}-e^{-2t}$$

$$L^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\} = e^{-t}-e^{-2t}$$

$$L^{-1}\left\{\frac{-2}{(s+1)(s+2)}\right\} = -2e^{-t}+2e^{-2t}$$

$$L^{-1}\left\{\frac{s}{(s+1)(s+2)}\right\} = -e^{-t}+2e^{-2t}$$

$$\Phi(t) = L^{-1}\left\{[sI-A]^{-1}\right\} = \begin{bmatrix} 2e^{-t}-e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t} \end{bmatrix}$$

$$\Phi(T) = L^{-1}\left\{[sI-A]^{-1}\right\}\Big|_{t=T}$$

$$= \begin{bmatrix} 2e^{-T}-e^{-2T} & e^{-T}-e^{-2T} \\ -2e^{-T}+2e^{-2T} & -e^{-T}+2e^{-2T} \end{bmatrix}$$

② input matrix

$$\Theta(T) = \int_0^T \Phi(t)\,dt\, B$$

$$= \int_0^T \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} 0 \\ 8 \end{bmatrix} dt$$

$$= \int_0^T \begin{bmatrix} 8e^{-t} - 8e^{-2t} \\ -8e^{-t} + 16e^{-2t} \end{bmatrix} dt$$

$$\int_0^T 8e^{-t} - 8e^{-2t}\,dt$$

$$= 8\int_0^T e^{-t}\,dt - 8\int_0^T e^{-2t}\,dt$$

$$= 8\left(-e^{-t}\right)\Big|_0^T - 8\left(-\tfrac{1}{2}e^{-2t}\right)\Big|_0^T$$

$$= -8\left(e^{-T} - 1\right) + 4\left(e^{-2T} - 1\right)$$

$$= -8e^{-T} + 4e^{-2T} + 4$$

$$\int_0^T -8e^{-t} + 16e^{-2t}\,dt$$

$$= -8\int_0^T e^{-t} + 16\int_0^T e^{-2t}\,dt$$

$$+\cancel{16}^{\,8} \times \left(-\tfrac{1}{2}\right)$$

$$= -8e^{-T} - 8e^{-2T} + 16$$

$$\Theta(T) = \begin{bmatrix} -8e^{-T} + 4e^{-2T} + 4 \\ -8e^{-T} - 8e^{-2T} + 16 \end{bmatrix}$$

So $x(k+1) = \begin{bmatrix} 2e^{-T} - e^{-2T} & e^{-T} - e^{-2T} \\ -2e^{-T} + 2e^{-2T} & -e^{-T} + 2e^{-2T} \end{bmatrix} x(k) + \begin{bmatrix} -8e^{-T} + 4e^{-2T} + 4 \\ -8e^{-T} - 8e^{-2T} + 16 \end{bmatrix} u(k)$

$y(k) = [1 \quad 0]\, x(k)$

(b) (i) Q C? O?

Solution ① $W_C = [B \quad AB] = \begin{bmatrix} 0 & 1 \\ 1 & -6 \end{bmatrix}$

$AB = \begin{bmatrix} 0 & 1 \\ -4 & -6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}$

$|W_C| = -1 \neq 0$ Controllable ✓

② $W_O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 4 & 7 \end{bmatrix}$

$CA = [1 \quad -1] \begin{bmatrix} 0 & 1 \\ -4 & -6 \end{bmatrix} = [4 \quad 7]$

$|W_O| = 7 + 4 = 13 \neq 0$ observable ✓

(ii) unobservable ?

$x(k+1) = \begin{bmatrix} 0 & 1 \\ -4 & -6 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [-k_1 \quad -k_2]\, x(k) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} r(k)$

$= \left(\begin{bmatrix} 0 & 1 \\ -4 & -6 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -k_1 & -k_2 \end{bmatrix} \right) x(k) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} r(k)$

$= \begin{bmatrix} 0 & 1 \\ -4-k_1 & -6-k_2 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} r(k)$

$$W_o = \begin{bmatrix} C \\ CA_r \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 4+k_1 & 7+k_2 \end{bmatrix}$$

$$CA_r = [1 \ \ -1] \begin{bmatrix} 0 & 1 \\ -4-k_1 & -6-k_2 \end{bmatrix} = [4+k_1 \ \ 7+k_2]$$

$$|W_o| = 7+k_2 + 4+k_1 = 0$$

when $k_1+k_2 = -13$, the system isn't observable

(c) optimal control ?

Solution $A = 0.8$ $B = 0.2$ $Q = \cancel{0}\ 1$ $r = 6$

$$S = 0.8^2 S + \cancel{0}\ 1 - 0.8\, S\, 0.2 (6 + 0.2^2 S)^{-1} 0.2\, S\, 0.8$$

$$S = 0.64S - \frac{0.16^2 S^2}{6 + 0.04S} + 1$$

$$\frac{0.0256S^2}{6+0.04S} = -0.36S + 1$$

$$0.0256S^2 = -0.036S(6+0.04S) + 6 + 0.04S$$

$$= -0.2\cancel{1}56S - 0.00144S^2 + 6$$

$$0.02704S^2 + 0.2\cancel{1}56S - 6 = 0$$

$S_1 = 10.8964$ $S_2 = -20.3639$

choose the positive $S = 10.8964$

$K = (0.2^2 \times 10.8964 + 6)^{-1} \times 0.2 \times 10.8964 \times 0.8$

$= 0.2709$

$u^*(k) = -0.2709\, x(k)$

$x(k+1) = [0.8 + 0.2 \times (-0.2709)]\, x(k)$

$= 0.74582\, x(k)$

poles ?