Example 6.1

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \qquad Q: \pi(n) = ?$$

case 1 $a = b = 0$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\pi(n) = \pi(0) P^n = \pi(0) I = \pi(0)$$

$X_0 = 0$, $\pi(0) = [1 \ 0] = \pi(n)$

$X_0 = 1$, $\pi(0) = [0 \ 1] = \pi(n)$

case 2. $a = b = 1$

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\pi(n) = \pi(0) P^n$$

$$P^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$n \in$ odd $\quad \pi(n) = \pi(0) P^{n-1} P = \pi(0) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$n \in$ even $\quad \pi(n) = \pi(0)$

$X_0 = 0$, $\pi(0) = [0 \ 1]$ $\quad \pi(n) = \begin{cases} [0 1] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = [1 \ 0] & n \in \text{odd} \\ [0 1] & n \in \text{even} \end{cases}$

$X_0 = 1$, $\pi(0) = [1 \ 0]$ $\quad \pi(n) = \begin{cases} [1 0] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = [0 \ 1] & n \in \text{odd} \\ [1, 0] & n \in \text{even} \end{cases}$

Case 3 $a, b \in (0,1)$

$$\pi(n) = \pi(0) P^n \qquad P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

$$\cancel{P^2} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

$$= \begin{bmatrix} (1-a)^2+ab & (1-a)a^2(1-b) \\ b^2(1-a)(1-b) & ab(1-b)^2 \end{bmatrix}$$

$$\cancel{P^3} = \begin{bmatrix} (1-a)^2+ab & (1-a)a^2(1-b) \\ b^2(1-a)(1-b) & ab(1-b)^2 \end{bmatrix} \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

$$= \begin{bmatrix} (1-a)^3+ab(1-a) + a^2b(1-a)(1-b) & a(1-a)^2+a^2b + a^2(1-a)(1-b)^2 \\ b^2(1-a)^2(1-b) + ab^2(1-b)^2 & ab^2(1-a)(1-b) + ab(1-b)^3 \end{bmatrix}$$

① 特征值

$$|P - \lambda I| = 0$$

$$(1-a-\lambda)(1-b-\lambda) - ab = 0$$

let $x = 1-a-b$

$$(1-a-\lambda) = (x+b-\lambda)$$

$$(1-b-\lambda) = (x+b-\lambda)$$

其中 $(1-a-\lambda)(1-b-\lambda)$

$$= 1-a-b+ab-(1-a+1-b)\lambda+\lambda^2$$

$$= x+ab-[2-a-b]\lambda+\lambda^2$$

$$\lambda^2-(1+x)\lambda+x=0$$

$$\lambda = \frac{1+x \pm (1-x)}{2} \qquad \lambda_1 = 1 \quad \lambda_2 = x$$

② 特征向量 $\lambda_1 = 1$ $(P - I)v = 0$

$$\begin{bmatrix} -a & a \\ b & -b \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 = v_2$$

$$v^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = x, \quad v^{(2)} = \begin{bmatrix} a \\ -b \end{bmatrix}$$

③ 对角化矩阵 P

对角矩阵

$$V = \begin{bmatrix} 1 & a \\ 1 & -b \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix}$$

$$|V| = -(a+b) \qquad V^{-1} = \frac{1}{-(a+b)}\begin{bmatrix} -b & -a \\ -1 & 1 \end{bmatrix}$$

④ P^n

$$P = VDV^{-1}$$

$$P^n = VD^nV^{-1}$$

$$= \begin{bmatrix} 1 & a \\ 1 & -b \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & x^n \end{bmatrix}\frac{1}{-(a+b)}\begin{bmatrix} -b & -a \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{a+b}\begin{bmatrix} b+ax^n & a-ax^n \\ b-bx^n & a+bx^n \end{bmatrix}$$

$$\lim_{n\to\infty} P^n = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}$$

$X_0 = 0 \quad \pi(0) = [1 \quad 0]$

$$\pi(n) = \pi(0) P^n = [1 \quad 0] \frac{1}{a+b} \begin{bmatrix} b+a x^n & a-a x^n \\ b-b x^n & a+b x^n \end{bmatrix}$$

$$= \frac{1}{a+b} [b+a x^n \quad a-a x^n]$$

$X_0 = 1 \quad \pi(0) = [0 \quad 1]$

$$\pi(n) = \frac{1}{a+b} [b-b x^n \quad a+b x^n]$$

$n \to \infty \quad x \in (0,1) \quad x^n \to 0 \quad \pi(n) = \left[\frac{b}{a+b} \quad \frac{a}{a+b} \right]$

steady-state

probabilities