21-S1-Q2

Q(a) Hungarian method?

	Mow	Paint	Wash	Clean
John	1	4	6	3
Karen	9	7	10	9
Terri	4	5	11	7
Jean	8	7	8	5

Solution ① row − min row ② Col − min col

	Mow	Paint	Wash	Clean
John	~~1~~ 0	~~4~~ 3	~~6~~ ~~5~~ 2	~~3~~ 2
Karen	~~9~~ 2	~~7~~ 0	~~10~~ ~~3~~ 0	~~9~~ 2
Terri	~~4~~ 0	~~5~~ 1	~~11~~ ~~7~~ 4	~~7~~ 3
Jean	~~8~~ 3	~~7~~ 2	~~8~~ ~~3~~ 0	~~5~~ 0

③ using line to cover 0

	Mow	Paint	Wash	Clean
John	0	3	2	2
Karen	2	0	0	2
Terri	0	1	4	3
Jean	3	2	0	0

④ no cover — min 1 ⑤ cross position add 1

must be equal

	Mow	Paint	Wash	Clean
John	0	~~3~~ 2	~~2~~ 1	~~2~~ 1
Karen	~~2~~ 3	0	0	2
Terri	0	~~1~~ 0	~~4~~ 3	~~3~~ 2
Jean	~~3~~ 4	2	0	0

① assign cost 0 work, add *

	Mow	Paint	Wash	Clean
John	0*	2	1	1
Karen	3	0	0*	2
Terri	0	0*	3	2
Jean	4	2	0	0*

So John do Mow, $1

Karen do wash, $10

Terri do Paint, $5

Jean do clean, $5

Total = 1+10+5+5 = $21

(b) Q : Lagrange → optimal solution

Solution

① Min $z = \frac{1}{2}[x_1, x_2]\begin{bmatrix}4 & 0\\ 0 & 8\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + [1 \quad 2]\begin{bmatrix}x_1\\ x_2\end{bmatrix} + 2$

$= \frac{1}{2}[4x_1 \quad 8x_2]\begin{bmatrix}x_1\\ x_2\end{bmatrix} + x_1 + 2x_2 + 2$

$= \frac{1}{2}(4x_1^2 + 8x_2^2) + x_1 + 2x_2 + 2$

$= 2x_1^2 + 4x_2^2 + x_1 + 2x_2 + 2$

subject to $x_1 - 2 \le 0$

$x_2 - 2 \le 0$

②

Lagrange multipliers

$L = 2x_1^2 + 4x_2^2 + x_1 + 2x_2 + 2 + \mu_1(x_1 - 2) + \mu_2(x_2 - 2)$

$$\begin{cases} \frac{\partial L}{\partial x_1} = 4x_1 + 1 + \mu_1 = 0 \\ \frac{\partial L}{\partial x_2} = 8x_2 + 2 + \mu_2 = 0 \\ x_1 - 2 \le 0 \\ x_2 - 2 \le 0 \\ \mu_1(x_1 - 2) = 0 \\ \mu_2(x_2 - 2) = 0 \\ \mu_1 \ge 0 \\ \mu_2 \ge 0 \end{cases}$$

③ if $\mu_1 = \mu_2 = 0$, $x_1 = -\frac{1}{4}$, $x_2 = -\frac{1}{4}$ ✓

if $\mu_1 = 0$ $\mu_2 > 0$, $x_2 = 2$, $\mu_2 = -18$, contradiction

if $\mu_1 > 0$ $\mu_2 = 0$, $x_1 = 2$, $\mu_1 = -9$, contradiction

if $\mu_1 > 0$ $\mu_2 > 0$, $x_1 = 2$, $x_2 = 2$, $\mu_1 = -9$, $\mu_2 = -18$

contradiction.

So $x_1^* = -\frac{1}{4}$ $x_2^* = -\frac{1}{4}$ $\mu_1 = \mu_2 = 0$

$$
\begin{aligned}
Z_{min} &= 2x_1^2 + 4x_2^2 + x_1 + 2x_2 + 2 \\
&= 2\times(-\tfrac{1}{4})^2 + 4\times(-\tfrac{1}{4})^2 - \tfrac{1}{4} - \tfrac{1}{2} + 2 \\
&= \tfrac{13}{8} \\
&= 1.625
\end{aligned}
$$

④ $\mu_1 = \mu_2 = 0$, test optimal, can't use Y, use convex to test, h_1 and h_2 are linear ~ref PPT 46

$$\frac{\partial L}{\partial x_1} = 4x_1 + 1 + \mu_1 = 0$$

$$\frac{\partial L}{\partial x_2} = 8x_2 + 2 + \mu_2 = 0$$

$$\frac{\partial L}{\partial x_1 \partial x_1} = 4 \quad \frac{\partial L}{\partial x_1 \partial x_2} = 0 \quad \frac{\partial L}{\partial x_2 \partial x_2} = 8$$

$$\nabla^2_{xx} L(x,\mu) = \begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix} \geqslant 0 \text{ on } R^3$$

Since $|4| > 0$ $\begin{vmatrix} 4 & 0 \\ 0 & 8 \end{vmatrix} > 0$, $\nabla^2_{xx} L(x,\mu)$ is a positive definite

According to PPT 45, $\nabla^2_{xx} L(x,\mu)$ is a positive definite $\Rightarrow$ the function L is a convex function.

According to PPT 45, A linear function is also convex

$\Rightarrow g_1 = x_1 - 2$, $g_2 = x_2 - 2$ are linea function

$\Rightarrow$ So g_1 g_2 are also convex

According to PPT 46. KT sufficient theorem, $f(x)$ is convex, the inequality constrains $g_i(x)$ are all convex functions and equality constraints $h_i(x)$ be linear. If the exists a solution $(X^* \lambda^* \mu^*)$ that satisfies the K-T conditions, then X^* is an optimal solution to the NLP problem.

So $X^* = (-\frac{1}{4}, -\frac{1}{4})$ is a mininun point

$$\boxed{\begin{aligned} Y^T \nabla^2_{xx} L(x,\mu) Y &= \cancel{[y_1 \ y_2] \begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \\ &= \cancel{[4y_1 \ 8y_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \\ &= 4y_1^2 + 8y_2^2 > 0 \text{ fro all } Y \neq 0 \end{aligned}}$$