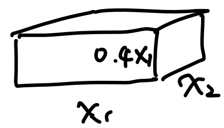
23 - S1 - Q2

Q: $V = 98\ cm^3$ $\quad L = 2H \quad$ closed. $\quad \$10/cm^3$

(a) Q: lengh x_1 breadth x_2 cost $Z = ?$

Solution $Z = 10 \times [2 \times (x_1 x_2 + 0.4 x_1^2 + 0.4 x_1 x_2)]$



subject to $\begin{cases} V = x_1 \cdot x_2 \cdot \frac{x_1}{2.5} = 0.4 x_1^2 x_2 = 98 \\ x_1, x_2 \geq 0 \end{cases}$

where $Z = 20 \times (1.4 x_1 x_2 + 0.4 x_1^2)$

$= 28 x_1 x_2 + 8 x_1^2$

$x_1^2 x_2 = 245$

$x_1, x_2 \geq 0$

so $Z = 28 x_1 x_2 + 8 x_1^2$

subject to $\begin{cases} x_1^2 x_2 = 245 \\ x_1, x_2 \geq 0 \end{cases}$

(b) apply Lagrange multipliers

Solution ① $f(x) = 28 x_1 x_2 + 8 x_1^2 \qquad h(x) = x_1^2 x_2 - 245$

$L = 28 x_1 x_2 + 8 x_1^2 + \lambda (x_1^2 x_2 - 245)$

$$\begin{cases} \dfrac{\partial L}{\partial x_1} = 28x_2 + 16x_1 + 2\lambda x_1 x_2 = 0 & (1) \\ \dfrac{\partial L}{\partial x_2} = 28x_1 + \lambda x_1^2 = 0 & (2) \\ \dfrac{\partial L}{\partial \lambda} = x_1^2 x_2 - 245 = 0 & (3) \end{cases}$$

from (3) $x_1 \neq 0$

from (2) $\lambda x_1 = -28 \quad \Rightarrow x_1 = -\dfrac{28}{\lambda}$ $\Rightarrow \lambda = \dfrac{-28}{x_1}$

into (1) get

$x_2 = \dfrac{4x_1}{7}$

from (3) $x_2 = \dfrac{245}{x_1^2} = \dfrac{245\lambda^2}{28^2} = \dfrac{5}{16}\lambda^2$

from (1) $28 \times \dfrac{5}{16}\lambda^2 + 16\left(-\dfrac{28}{\lambda}\right) + 2\lambda\left(-\dfrac{28}{\lambda}\right)\dfrac{5}{16}\lambda^2 = 0$

$$\frac{35}{4}\lambda^2 - \frac{448}{\lambda} - \frac{35}{2}\lambda^2 = 0$$

from (3)

$\dfrac{4}{7}x_1^3 = 245$

$x_1 = 7.5405$

more fast!

$$\frac{35}{4}\lambda^2 - \frac{35}{2}\lambda - 448 = 0$$

$$\lambda_{1,2} = \frac{5 \pm 3\sqrt{145}}{5} \quad \times$$

$\lambda_1 = 8.2250 \qquad \lambda_2 = -6.2250$

$$-8.75\lambda^2 - \frac{448}{\lambda} = 0$$

$$-8.75\lambda^3 = 448$$

$$\lambda^3 = -51.2$$

$$\lambda = -3.7133$$

$x_1 = -\frac{28}{\lambda} = 7.5405$

$x_2 = \frac{5}{16}\lambda^2 = 4.3089$

cost $Z = 28x_1x_2 + 8x_1^2 = 1364.6284$

So $x_1 = 7.5405$

$x_2 = 4.3089$

$\lambda = -3.7133$

min cost $Z = 1364.6284$

② Test of optimum

let $h(x) = x_1^2 x_2 - 245 = 0$

$\frac{\partial h(x)}{\partial x_1} = 2x_1x_2 \qquad \frac{\partial h(x)}{\partial x_2} = x_1^2$

$\nabla h(x) = [2x_1x_2 \quad x_1^2]^T$

let $M = \{Y : \nabla h(x)^T Y = 0\}$

$\nabla h(x)^T Y = 2x_1x_2 y_1 + x_1^2 y_2 = 0$

$64.9825 y_1 + 56.8591 y_2 = 0$

$y_2 = -1.1429 y_1$

$$\nabla_{xx}^2 L = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} \end{bmatrix}$$

$$\frac{\partial L}{\partial x_1} = 28x_2 + 16x_1 + 2\lambda x_1 x_2 = 0$$

$$\frac{\partial L}{\partial x_2} = 28x_1 + \lambda x_1^2 = 0$$

$x_1 = 7.5405$

$x_2 = 4.3089$

$\lambda = -3.7133$

$$\frac{\partial^2 L}{\partial x_1^2} = 16 + 2\lambda x_2 = -16$$

$$\frac{\partial^2 L}{\partial x_1 \partial x_2} = 28 + 2\lambda x_1 = -28$$

$$\frac{\partial^2 L}{\partial x_2^2} = 0$$

>0 <0 16000 >0

$$\nabla_{xx}^2 L = \begin{bmatrix} -16 & -28 \\ -28 & 0 \end{bmatrix}$$

$A_1 = -16 < 0$

$A_2 = 0 - 28^2 < 0$

So $\nabla_{xx}^2 L$ is a indefinite

Professor Wang

the z isn't optimal minimum ref → 3. understand → 2

JIL_WANG: Don't need $\nabla_{xx}^2 L$ positive-definite

Don't need $\nabla_{xx}^2 h$ positive-definite

only need $Y^T \nabla_{xx}^2 L Y > 0 \quad \forall y \neq 0$

$\nabla h(x) = [2x_1x_2 \quad x_1^2]^T$ $\qquad x_1 = 7.5405$

$$\nabla_{xx}^2 h = \begin{bmatrix} 2x_2 & 2x_1 \\ 2x_1 & 0 \end{bmatrix} = \begin{bmatrix} 8.6178 & 15.0810 \\ 15.0810 & 0 \end{bmatrix}$$

$x_2 = 4.3089$

$\lambda = -3.7133$

$A_1 = 8.6178 > 0 \quad A_2 = -15.081^2 < 0$

So $\nabla_{xx}^2 h$ is indefinite

$$Y^T \nabla_{xx}^2 L\, Y = [y_1\ y_2] \begin{bmatrix} -16 & -28 \\ -28 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= [-16y_1 - 28y_2 \quad -28y_1] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= -16y_1^2 - 28y_1y_2 - 28y_1y_2$$

$$= -16y_1^2 - 56y_1y_2 \qquad y_2 = -1.1429y_1$$

$$= -16y_1^2 - 56y_1(-1.1429y_1)$$

$$= 48.0024y_1^2 > 0$$

for $\forall y \neq 0$

So, $x_1 = 7.5405 \quad x_2 = 4.3089$

cost $z = 1364.6284$ is a optimal minimum

(c) -5% $\quad Z?$

Solution

$\Delta V = -0.05 \times 98 = -4.9\ cm^3$

$V_{new} = 98 - 4.9 = 93.1\ cm^3$

new constraint $\quad x_1^2 x_2 = \dfrac{93.1}{0.4} = 232.75$

constrain change $\quad \Delta B = 232.75 - 245 = -12.25$

$Z_{new} = Z_{old} - \Delta B \lambda$

$= 1364.6284 - (-12.25) \times (-3.7133)$

$= 1319.1405$