21-S1 - Q2

Q (a) Hungarian method?

| | Mow | Paint | Wash | Clean |
|---|---|---|---|---|
| John | 1 | 4 | 6 | 3 |
| karen | 9 | 7 | 10 | 9 |
| Terri | 4 | 5 | 11 | 7 |
| Jean | 8 | 7 | 8 | 5 |

Solution ① row - min row ② Col - min col

| | Mow | Paint | Wash | Clean |
|---|---|---|---|---|
| John | $\cancel{1}$ 0 | $\cancel{4}$ 3 | $\cancel{6}$ $\cancel{5}$ 2 | $\cancel{3}$ 2 |
| karen | $\cancel{9}$ 2 | $\cancel{7}$ 0 | $\cancel{10}$ $\cancel{3}$ 0 | $\cancel{9}$ 2 |
| Terri | $\cancel{4}$ 0 | $\cancel{5}$ 1 | $\cancel{11}$ $\cancel{7}$ 4 | $\cancel{7}$ 3 |
| Jean | $\cancel{8}$ 3 | $\cancel{7}$ 2 | $\cancel{8}$ $\cancel{3}$ 0 | $\cancel{5}$ 0 |

③ using line to cover 0

| | Mow | Paint | Wash | Clean |
|---|---|---|---|---|
| John | 0 | 3 | 2 | 2 |
| karen | 2 | 0 | 0 | 2 |
| Terri | 0 | 1 | 4 | 3 |
| Jean | 3 | 2 | 0 | 0 |

④ no cover − min ⑤ cross position add 4

| | Mow | Paint | Wash | Clean |
|---|---|---|---|---|
| John | 0 | ~~3~~ 2 | ~~2~~ 1 | ~~2~~ 1 |
| Karen | ~~2~~ 6 | 0 | 0 | 2 |
| Terri | 0 | ~~1~~ 0 | ~~4~~ 3 | ~~3~~ 2 |
| Jean | ~~3~~ 7 | 2 | 0 | 0 |

① assign cost 0 work, add *

| | Mow | Paint | Wash | Clean |
|---|---|---|---|---|
| John | 0* | 2 | 1 | 1 |
| Karen | 6 | 0 | 0* | 2 |
| Terri | 0 | 0* | 3 | 2 |
| Jean | 7 | 2 | 0 | 0* |

So John do Mow, $1

Karen do wash, $10

Terri do Paint, $5

Jean do Clean, $5

Total = 1+10+5+5 = $21

(b) Q : Lagrange → optimal solution

Solution

① Min $z = \frac{1}{2}[x_1\ x_2]\begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [1\ \ 2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 2$

$= \frac{1}{2}[4x_1\ \ 8x_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + x_1 + 2x_2 + 2$

$= \frac{1}{2}(4x_1^2 + 8x_2^2) + x_1 + 2x_2 + 2$

$= 2x_1^2 + 4x_2^2 + x_1 + 2x_2 + 2$

subject to $x_1 - 2 \le 0$

$x_2 - 2 \le 0$

②

Lagrange multipliers

$L = 2x_1^2 + 4x_2^2 + x_1 + 2x_2 + 2 + \mu_1(x_1-2) + \mu_2(x_2-2)$

$$\begin{cases} \frac{\partial L}{\partial x_1} = 4x_1 + 1 + \mu_1 = 0 \\ \frac{\partial L}{\partial x_2} = 8x_2 + 2 + \mu_2 = 0 \\ x_1 - 2 \le 0 \\ x_2 - 2 \le 0 \\ \mu_1(x_1-2) = 0 \\ \mu_2(x_2-2) = 0 \\ \mu_1 \ge 0 \\ \mu_2 \ge 0 \end{cases}$$

③ if $\mu_1=\mu_2=0$, $x_1=-\frac{1}{4}$, $x_2=-\frac{1}{4}$ ✓

if $\mu_1=0$ $\mu_2>0$, $x_2=2$, $\mu_2=-18$, contradiction

if $\mu_1>0$ $\mu_2=0$, $x_1=2$, $\mu_1=-9$, contradiction

if $\mu_1>0$ $\mu_2>0$, $x_1=2$, $x_2=2$, $\mu_1=-9$, $\mu_2=-18$, contradiction.

So $x_1^*=-\frac{1}{4}$ $x_2^*=-\frac{1}{4}$ $\mu_1=\mu_2=0$

$$
\begin{aligned}
Z_{min} &= 2x_1^2+4x_2^2+x_1+2x_2+2 \\
&= 2\times(-\tfrac{1}{4})^2+4\times(-\tfrac{1}{4})^2-\tfrac{1}{4}-\tfrac{1}{2}+2 \\
&= \frac{13}{8} \\
&= 1.625
\end{aligned}
$$

要test optimal吗? ✓ $\mu_1=\mu_2=0$, 怎么test? ✓

Don't need to calculate $\nabla g_i^T Y$

$$\frac{\partial L}{\partial x_1}=4x_1+1+\mu_1=0$$

$$\frac{\partial L}{\partial x_2}=8x_2+2+\mu_2=0$$

$$\frac{\partial L}{\partial x_1\partial x_1}=4 \quad \frac{\partial L}{\partial x_1\partial x_2}=0 \quad \frac{\partial L}{\partial x_2\partial x_2}=8$$

$$\nabla_{xx}^2 L(x,\mu) = \begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix} \geqslant 0 \text{ on } R^3$$

Since $|4| > 0$

$\begin{vmatrix} 4 & 0 \\ 0 & 8 \end{vmatrix} > 0$

$$Y^T \nabla_{xx}^2 L(x,\mu) Y = [y_1 \;\; y_2] \begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= [4y_1 \;\; 8y_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= 4y_1^2 + 8y_2^2 > 0 \text{ fro all } Y \neq 0$$

So $X = (-\frac{1}{4}, -\frac{1}{4})$ is a mininun point