2. You are required to design a closed rectangular container to carry 98 cm^3 of a certain type of powder. The length of the container must be 2.5 times that of the height. The container is to be made of a special material costing \$10/cm^2. The aim is to design the container with the least material cost possible.

(a) Formulate a nonlinear program for the above problem. You may let the length of the container be x_1 (cm) and the breadth be x_2 (cm).

(5 Marks)

(b) Apply the method of Lagrange multipliers to the nonlinear program in part 2(a) to determine the dimensions of the container (in cm) as well as the optimum material cost. Show all your steps.

(15 Marks)

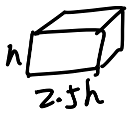
2. Local vs. Global Minimum with Positive (Semi-)Definiteness 当K-T充分定理条件和正定（或正半定）Hessian矩阵条件同时满足时，是否保证Z*是局部极小值还是全局极小值？

- When both the K-T sufficient theorem conditions and the positive definite (or positive semi-definite) Hessian matrix condition are satisfied, does this guarantee that Z* is a local minimum or a global minimum?

If there are multiple stationary points, then the obtained minimum must be a local. If there is only one stationary point and Hessian matrix is globally PD (ND) on M space, then the stationary point is a global optimum.

如果有多个平稳点，那么得到的最小值必须是局部的。
如果只有一个驻点，且Hessian矩阵在M空间上是全局PD（ND），则驻点是全局最优

18 − S2 − Q2

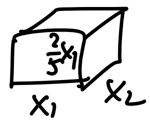
Q $98cm^3$ 

\$10/cm^2 least cost

Solution (a) Formulate a nonlinear program

① length x_1 breadth x_2 let z denote the cost

$$\text{height} = \frac{x_1}{2.5} = \frac{2}{5}x_1$$



$$\text{Min } z = 10\left[2x_1x_2 + 2\,\tfrac{2}{5}x_1^2 + 2\,\tfrac{2}{5}x_1x_2\right]$$

$$= 20x_1x_2 + 8x_1^2 + 8x_1x_2$$

$$= 8x_1^2 + 28x_1x_2$$

subject to
$$\begin{cases} x_1 x_2 \dfrac{x_1}{2.5} = 98 \Rightarrow x_1^2 x_2 = 245 \\ x_1, x_2 \geq 0 \end{cases}$$

② formulate

$$\text{Min } z = 8x_1^2 + 28x_1x_2$$

s.t.
$$\begin{cases} x_1^2 x_2 = 245 \\ x_1, x_2 \geq 0 \end{cases}$$

(b) ① Lagrange multipliers

$$L = 8x_1^2 + 28x_1x_2 + \lambda_1(x_1^2x_2 - 245)$$

$$\begin{cases} \dfrac{\partial L}{\partial x_1} = 16x_1 + 28x_2 + 2\lambda_1 x_1 x_2 = 0 & (1) \\ \dfrac{\partial L}{\partial x_2} = 28x_1 + \lambda_1 x_1^2 = 0 & (2) \\ \dfrac{\partial L}{\partial L} = x_1^2 x_2 - 245 = 0 & (3) \end{cases}$$

from(2) $\lambda_1 = \dfrac{-28x_1}{x_1^2} = \dfrac{-28}{x_1}$ (4)

(4) → (1) $16x_1 + 28x_2 + 2x_1 x_2 \left(\dfrac{-28}{x_1}\right)$

$= 16x_1 - 28x_2 = 0$

So $x_2 = \dfrac{4}{7}x_1$ (5)

(5) → (3) $x_1^2 \dfrac{4}{7}x_1 = 245$

So $x_1 = \sqrt[3]{\dfrac{245 \times 7}{4}}$

$= 7.5405$

$x_2 = \dfrac{4}{7}x_1 = 4.3089$

$\lambda_1 = -\dfrac{28}{x_1} = -3.713$

② $\begin{cases} x_1 = 7.5405 \\ x_2 = 4.3089 \\ \lambda_1 = -3.713 \end{cases}$

demensions of the container

length : 7.5405 (cm)

breadth : 4.3089 (cm)

height : 3.0162 (cm)

$$\text{cost} = 8x_1^2 + 28x_1x_2 = 1364.6284$$

③ optimum material cost ?

$$h(x) = x_1^2 x_2 - 245$$

$$\frac{\partial h(x)}{\partial x_1} = 2x_1x_2 \qquad \frac{\partial h(x)}{\partial x_2} = x_1^2$$

$$\nabla h(x)^T = \left[\frac{\partial h(x)}{\partial x_1} \quad \frac{\partial h(x)}{\partial x_2} \right]^T = [2x_1x_2 \quad x_1^2]^T$$

Constrain $\nabla h(x)^T y$

$$= [2x_1x_2 \quad x_1^2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= 2x_1x_2 y_1 + x_1^2 y_2 = 0$$

$$y_1 = \frac{-x_1^2}{2x_1x_2} y_2 = \frac{-x_1}{2x_2} y_2 = -0.8750 y_2 \qquad (6)$$

④ Hessian matrix

$$\nabla_{xx}^2 L = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} \end{bmatrix}$$

$$\frac{\partial L}{\partial x_1} = 16x_1 + 28x_2 + 2\lambda_1 x_1 x_2 = 0$$

$$\frac{\partial L}{\partial x_2} = 28x_1 + \lambda_1 x_1^2 = 0$$

$$= \begin{bmatrix} 16+2\lambda_1 x_2 & 28+2\lambda_1 x_1 \\ 28+2\lambda_1 x_1 & 0 \end{bmatrix} = \begin{bmatrix} -15.9979 & -27.9958 \\ -27.9958 & 0 \end{bmatrix}$$

$A_1 = -15.9979 < 0$

$A_2 = -27.9958^2 < 0$

So $\nabla_{xx}^2 L$ is a negative-definite

$$Y^T \nabla_{xx}^2 L Y = [y_1\ y_2] \begin{bmatrix} 16+2\lambda_1 x_2 & 28+2\lambda_1 x_1 \\ 28+2\lambda_1 x_1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \begin{bmatrix} (16+2\lambda_1 x_2)y_1 + (28+2\lambda_1 x_1)y_2 & (28+2\lambda_1 x_1)y_1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= (16+2\lambda_1 x_2)y_1^2 + (28+2\lambda_1 x_1)y_1 y_2 + (28+2\lambda_1 x_1)y_1 y_2$$

$$= (16+2\lambda_1 x_2)y_1^2 + 2(28+2\lambda_1 x_1)y_1 y_2$$

$= (16+2\times(-3.713)\times 4.3089)y_1^2 + 2(28+2\times(-3.713)\times 7.5405)y_1 y_2$

$= -15.9979y_1^2 - 55.9915y_1y_2$

$\begin{cases} x_1 = 7.5405 \\ x_2 = 4.3089 \\ \lambda_1 = -3.713 \end{cases}$

from (6) $y_1 = -0.8750y_2$

$y_2 = -1.1429y_1$

$Y^T \nabla_{xx}^2 L Y = -15.9979y_1^2 - 55.9915y_1y_2$

$= 47.9924y_1^2$

So for $\forall y \neq 0$, $Y^T \nabla_{xx}^2 L Y > 0$

So the result is optimal, the z is the minimum cost