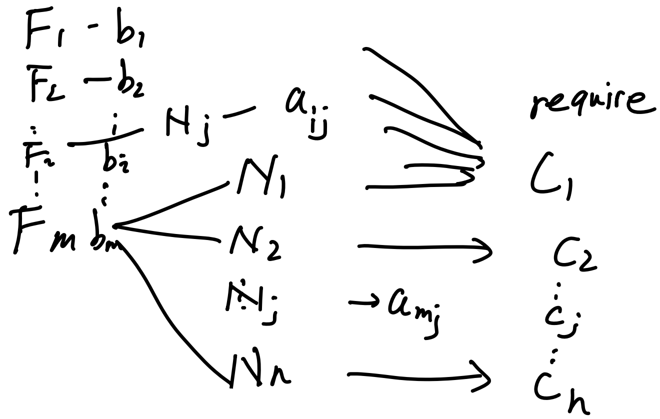
15-S2-Q1

(a)(i)

Solution① formula Linear Programming



let Z denote cost

$$\text{Min } Z = \sum_{i=1}^{m} a_{ij} b_i$$

$$\text{s.t.} \begin{cases} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} = c_j \\ a_{ij},\ c_j \geq 0 \end{cases}$$

(2) Two phase method

$① \min Z = x_1 + 2x_2 + 0 \cdot x_3 + 0 \cdot x_4 + Mx_5$

$$\text{s.t.}\begin{cases} 4x_1+3x_2+x_3 = 12 \\ x_1 - 2x_2 - x_4 + \bar{x}_5 = 1 \\ x_1, x_2, x_3, x_4, \bar{x}_5 \geq 0 \end{cases}$$

② $A=\begin{bmatrix} 4 & 3 & 1 & 0 & 0 \\ 1 & -2 & 0 & -1 & 1 \end{bmatrix}$ $B=\begin{bmatrix} 12 \\ 1 \end{bmatrix}$

$X = [x_1\ x_2\ x_3\ x_4\ \bar{x}_5]^T$ $X_0 = [x_3\ \cancel{x_4}\ x_5]^T$

$C = [1\ 2\ 0\ 0\ M]^T$ $C_0 = [0\ \cancel{0}\ M]^T$

$$C^T - C_0^T A = [1\ 2\ 0\ 0\ M] - [0\ \cancel{0}\ M]\begin{bmatrix} 4 & 3 & 1 & 0 & 0 \\ 1 & -2 & 0 & -1 & 1 \end{bmatrix}$$

$$= [1\ 2\ 0\ 0\ M] - [M\ \ -2M\ \ 0\ \ -M\ \ M]$$

$$= [1-M\ \ \ 2+2M\ \ \ 0\ \ \ M\ \ \ 0]$$

$$-C_0^T B = [0\ \ -M]\begin{bmatrix} 12 \\ 1 \end{bmatrix} = -M$$

③ Iteration 1

	x_1	x_2	x_3	x_4	$\bar{x}_5$		Ratio
x_3	4	3	1	0	0	12	$12/4=3$
$\bar{x}_5$	1^*	-2	0	-1	0	1	$1/1=1$ min
	1	2	0	0	0	0	
	-1	2	0	1	0	-1	

↑
work

④

	x_1	x_2	x_3	x_4	$\bar{x}_5$		
x_3	$\cancel{4} \to 0$	$\cancel{3} \to 11$	1	$\cancel{0}\ 4$	0	$\cancel{12}\ 8$	$-4R_2+R_1$
$\bar{x}_5 \to x_1$	1^*	-2	0	-1	0	1	
	$\cancel{1} \to 0$	$\cancel{2} \to 4$	0	$\cancel{0}\ 1$	0	$\cancel{0}\ -1$	$-R_2+R_3$
	$\cancel{-1} \to 0$	$\cancel{2}\ 0$	0	$\cancel{1}\ 0$	0	$\cancel{-1}\ 0$	R_2+R_4

↑
work

⑤

	x_1	x_2	x_3	x_4	
x_3	0	11	1	4	8
x_1	1	-2	0	-1	1
	0	4	0	1	-1

the solution is

$x_1^* = 1 \quad x_3^* = 8 \qquad x_2^* = x_4^* = \bar{x}_5^* = 0$

$Z_{min} = 1$

(b) ① Minimize

$$z = \frac{1}{2}\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -22 & -14.5 & 13 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 1$$

$$= \frac{1}{2}\begin{bmatrix} 13x_1+12x_2-2x_3 & 12x_1+17x_2+6x_3 & -2x_1+6x_2+12x_3 \end{bmatrix}$$

$$\times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -22 & -14.5 & 13 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 1$$

$$= \frac{1}{2}\Big[13x_1^2 + 12x_1x_2 - 2x_1x_3 + 12x_1x_2 + 17x_2^2 + 6x_2x_3$$

$$-2x_1x_3 + 6x_2x_3 + 12x_3^2\Big] - 22x_1 - 14.5x_2 + 13x_3 + 1$$

$$= \frac{13}{2}x_1^2 + \frac{17}{2}x_2^2 + 6x_3^2 + 12x_1x_2 - 2x_1x_3 + 6x_2x_3$$

$$-22x_1 - 14.5x_2 + 13x_3 + 1$$

$$\text{s.t.}\begin{cases} x_1 - 1 \le 0 \\ -x_1 - 1 \le 0 \\ x_2 - 1 \le 0 \\ -x_2 - 1 \le 0 \\ x_3 - 1 \le 0 \\ -x_3 - 1 \le 0 \end{cases}$$

② $L = \frac{13}{2}x_1^2 + \frac{17}{2}x_2^2 + 6x_3^2 + 12x_1x_2 - 2x_1x_3 + 6x_2x_3$

$-22x_1 - 14.5x_2 + 13x_3 + 1 + \mu_1(x_1 - 1) + \mu_2(-x_1 - 1)$

$+\mu_3(x_2-1)+\mu_4(-x_2-1)+\mu_5(x_3-1)+\mu_6(-x_3-1)$

③ $\frac{\partial L}{\partial x_1} = 13x_1+12x_2-2x_3-22+\mu_1-\mu_2=0$

$\frac{\partial L}{\partial x_2} = 17x_2+12x_1+6x_3-14.5+\mu_3-\mu_4=0$

$\frac{\partial L}{\partial x_3} = 12x_3-2x_1+6x_2+13+\mu_5-\mu_6=0$

$\mu_1(x_1-1)=0$ $\quad g_1(x)= x_1-1\le 0$ $\quad \mu_1\geq 0$

$\mu_2(-x_1-1)=0$ $\quad g_2(x)=-x_1-1\le 0$ $\quad \mu_2\geq 0$

$\mu_3(x_2-1)=0$ $\quad g_3(x)=x_2-1\le 0$ $\quad \mu_3\geq 0$

$\mu_4(-x_2-1)=0$ $\quad g_4(x)=-x_2-1\le 0$ $\quad \mu_4\geq 0$

$\mu_5(x_3-1)=0$ $\quad g_5(x)=x_3-1\le 0$ $\quad \mu_5\geq 0$

$\mu_6(-x_3-1)=0$ $\quad g_6(x)=-x_3-1\le 0$ $\quad \mu_6\geq 0$

④ from the 1.(b)(ii)

$x^*=[1 \quad \frac{1}{2} \quad -1]^T$

$13+12\times\frac{1}{2}+2-22+\mu_1-\mu_2=0$

$17\times\frac{1}{2}+12-6-14.5+\mu_3-\mu_4=0$

$$12 \times (-1) - 2 \times 1 + 3 \quad + 13 + \mu_5 - \mu_6 = 0$$

$\mu_2 = \mu_1 + 1$ let $\mu_1 = 0, \mu_2 = 1$

$\mu_4 = \mu_3$ $\mu_3 = 0, \mu_4 = 0$

$2 + \mu_5 = \mu_6$ $\mu_5 = 0, \mu_6 = 2$

$$X^* = [1 \quad \tfrac{1}{2} \quad -1]^T$$

$$\nabla g_1^T Y = y_1 = 0$$

$$\nabla g_5^T Y = y_3 = 0$$

$$\nabla^2 L_{xx} = \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix}$$

$A_1 = 13 > 0$ $A_2 = \begin{vmatrix} 13 & 12 \\ 12 & 17 \end{vmatrix} = 77 > 0$

$A_3 = 100 > 0$ $\nabla^2 L_{xx}$ is a positive-definite

$$Y^T \nabla^2 L_{xx} Y = [y_1, y_2, y_3] \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 17 y_2^2 > 0 \quad \forall y \neq 0$$