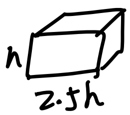
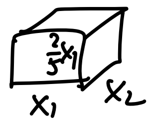
18 – S2 – Q 2

Q $98 cm^3$ 

$\$10/cm^2$ least cost

Solution (a) Formulate a nonlinear program

① length x_1 breadth x_2 let z denote the cost

height $= \dfrac{x_1}{2.5} = \dfrac{2}{5}x_1$ 

$$\text{Min } z = 10\left[2x_1x_2 + 2\tfrac{2}{5}x_1^2 + 2\tfrac{2}{5}x_1x_2\right]$$

$$= 20x_1x_2 + 8x_1^2 + 8x_1x_2$$

$$= 8x_1^2 + 28x_1x_2$$

subject to $\begin{cases} x_1 x_2 \dfrac{x_1}{2.5} = 98 \Rightarrow x_1^2 x_2 = 245 \\ x_1, x_2 \geqslant 0 \end{cases}$

② formulate

$$\text{Min } z = 8x_1^2 + 28x_1x_2$$

s.t. $\begin{cases} x_1^2x_2 = 245 \\ x_1, x_2 \geqslant 0 \end{cases}$

(b) ① Lagrange multipliers

$$L = 8x_1^2 + 28x_1x_2 + \lambda_1(x_1^2x_2 - 245)$$

$$\begin{cases} \dfrac{\partial L}{\partial x_1} = 16x_1 + 28x_2 + 2\lambda_1 x_1 x_2 = 0 & (1) \\ \dfrac{\partial L}{\partial x_2} = 28x_1 + \lambda_1 x_1^2 = 0 & (2) \\ \dfrac{\partial L}{\partial L} = x_1^2 x_2 - 245 = 0 & (3) \end{cases}$$

from(2) $\lambda_1 = \dfrac{-28x_1}{x_1^2} = \dfrac{-28}{x_1}$ (4)

(4) → (1) $16x_1 + 28x_2 + 2x_1 x_2 \left(\dfrac{-28}{x_1}\right)$

$= 16x_1 - 28x_2 = 0$

So $x_2 = \dfrac{4}{7} x_1$ (5)

(5) → (3) $x_1^2 \dfrac{4}{7} x_1 = 245$

So $x_1 = \sqrt[3]{\dfrac{245 \times 7}{4}}$

$= 7.5405$

$x_2 = \dfrac{4}{7} x_1 = 4.3089$

$\lambda_1 = -\dfrac{28}{x_1} = -3.713$

② $\begin{cases} x_1 = 7.5405 \\ x_2 = 4.3089 \\ \lambda_1 = -3.713 \end{cases}$

demensions of the container

length : 7.5405 (cm)

breadth : 4.3089 (cm)

height : 3.0162 (cm)

$$cost = 8x_1^2 + 28x_1x_2 = 1364.6284$$

③ optimum material cost ?

$$h(x) = x_1^2 x_2 - 245$$

$$\frac{\partial h(x)}{\partial x_1} = 2x_1x_2 \qquad \frac{\partial h(x)}{\partial x_2} = x_1^2$$

$$\nabla h(x)^T = \left[\frac{\partial h(x)}{\partial x_1} \quad \frac{\partial h(x)}{\partial x_2} \right]^T = [2x_1x_2 \quad x_1^2]^T$$

Constrain $\nabla h(x)^T Y$

$$= [2x_1x_2 \quad x_1^2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= 2x_1x_2 y_1 + x_1^2 y_2 = 0$$

$$y_1 = \frac{-x_1^2}{2x_1x_2} y_2 = \frac{-x_1}{2x_2} y_2 = -0.8750 y_2 \qquad (6)$$

④ Hessian matrix

$$\nabla_{xx}^2 L = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} \end{bmatrix}$$

$$\frac{\partial L}{\partial x_1} = 16x_1 + 28x_2 + 2\lambda_1 x_1 x_2 = 0$$

$$\frac{\partial L}{\partial x_2} = 28x_1 + \lambda_1 x_1^2 = 0$$

$$= \begin{bmatrix} 16+2\lambda_1 x_2 & 28+2\lambda_1 x_1 \\ 28+2\lambda_1 x_1 & 0 \end{bmatrix}$$

$$Y^T \nabla_{xx}^2 L Y = [y_1\ y_2] \begin{bmatrix} 16+2\lambda_1 x_2 & 28+2\lambda_1 x_1 \\ 28+2\lambda_1 x_1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \begin{bmatrix} (16+2\lambda_1 x_2)y_1 + (28+2\lambda_1 x_1)y_2 & (28+2\lambda_1 x_1)y_1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= (16+2\lambda_1 x_2)y_1^2 + (28+2\lambda_1 x_1)y_1 y_2 + (28+2\lambda_1 x_1)y_1 y_2$$

$$= (16+2\lambda_1 x_2)y_1^2 + 2(28+2\lambda_1 x_1)y_1 y_2$$

$$= (16+2\times(-3.713)\times 4.3089)y_1^2 + 2(28+2\times(-3.713)\times 7.5405)y_1 y_2$$

$$= -15.9979 y_1^2 - 55.9915 y_1 y_2$$

$$\begin{cases} x_1 = 7.5405 \\ x_2 = 4.3089 \\ \lambda_1 = -3.713 \end{cases}$$

from (6) $y_1 = -0.8750 y_2$

$$y_2 = -1.1429\, y_1$$

$$Y^T \nabla_{xx}^2 L\, Y = -15.9979\, y_1^2 - 55.9915\, y_1 y_2$$

$$= 47.9924\, y_1^2$$

So for $\forall y \neq 0$, $Y^T \nabla_{xx}^2 L\, Y > 0$

So the result is optimal, the z is the minimum cost