2. You are required to design a closed rectangular container to carry 98 cm^3 of a certain type of powder. The length of the container must be 2.5 times that of the height. The container is to be made of a special material costing \$10/cm^2. The aim is to design the container with the least material cost possible.

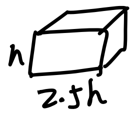
(a) Formulate a nonlinear program for the above problem. You may let the length of the container be x_1 (cm) and the breadth be x_2 (cm).

(5 Marks)

(b) Apply the method of Lagrange multipliers to the nonlinear program in part 2(a) to determine the dimensions of the container (in cm) as well as the optimum material cost. Show all your steps.

(15 Marks)

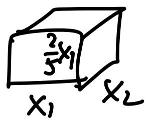
18 − S2 − Q2

Q $98 cm^3$ 

$\$10/cm^2$ least cost

Solution (a) Formulate a nonlinear program

① length x_1 breadth x_2 let z denote the cost

height $= \frac{x_1}{2.5} = \frac{2}{5}x_1$ 

$$\text{Min } z = 10\left[2x_1x_2 + 2\cdot\frac{2}{5}x_1^2 + 2\cdot\frac{2}{5}x_1x_2\right]$$

$$= 20x_1x_2 + 8x_1^2 + 8x_1x_2$$

$$= 8x_1^2 + 28x_1x_2$$

subject to
$$\begin{cases} x_1 x_2 \frac{x_1}{2.5} = 98 \Rightarrow x_1^2 x_2 = 245 \\ x_1, x_2 \geqslant 0 \end{cases}$$

② formulate

$$\text{Min } z = 8x_1^2 + 28x_1x_2$$

$$\text{s.t.} \begin{cases} x_1^2 x_2 = 245 \\ x_1, x_2 \geqslant 0 \end{cases}$$

(b) ① Lagrange multipliers

$$L = 8x_1^2 + 28x_1x_2 + \lambda_1(x_1^2x_2 - 245)$$

$$\begin{cases} \dfrac{\partial L}{\partial x_1} = 16x_1 + 28x_2 + 2\lambda_1 x_1 x_2 = 0 & (1) \\ \dfrac{\partial L}{\partial x_2} = 28x_1 + \lambda_1 x_1^2 = 0 & (2) \\ \dfrac{\partial L}{\partial L} = x_1^2 x_2 - 245 = 0 & (3) \end{cases}$$

from (2) $\lambda_1 = \dfrac{-28x_1}{x_1^2} = \dfrac{-28}{x_1}$ (4)

(4) → (1) $16x_1 + 28x_2 + 2x_1x_2\left(\dfrac{-28}{x_1}\right)$

$= 16x_1 - 28x_2 = 0$

So $x_2 = \dfrac{4}{7}x_1$ (5)

(5) → (3) $x_1^2 \dfrac{4}{7} x_1 = 245$

So $x_1 = \sqrt[3]{\dfrac{245 \times 7}{4}}$

$= 7.5405$

$x_2 = \dfrac{4}{7} x_1 = 4.3089$

$\lambda_1 = -\dfrac{28}{x_1} = -3.713$

② $\begin{cases} x_1 = 7.5405 \\ x_2 = 4.3089 \\ \lambda_1 = -3.713 \end{cases}$

demensions of the container

length : 7.5405 (cm)

breadth : 4.3089 (cm)

height : 3.0162 (cm)

$$cost = 8x_1^2 + 28x_1x_2 = 1364.6284$$

③ optimum material cost ?

$$h(x) = x_1^2 x_2 - 245$$

$$\frac{\partial h(x)}{\partial x_1} = 2x_1x_2 \qquad \frac{\partial h(x)}{\partial x_2} = x_1^2$$

$$\nabla h(x)^T = \left[\frac{\partial h(x)}{\partial x_1} \quad \frac{\partial h(x)}{\partial x_2} \right]^T = [2x_1x_2 \quad x_1^2]^T$$

Constrain $\nabla h(x)^T Y$

$$= [2x_1x_2 \quad x_1^2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= 2x_1x_2 y_1 + x_1^2 y_2 = 0$$

$$y_1 = \frac{-x_1^2}{2x_1x_2} y_2 = \frac{-x_1}{2x_2} y_2 = -0.8750 y_2 \qquad (6)$$

④ Hessian matrix

$$\nabla_{xx}^2 L = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} \end{bmatrix}$$

$$\frac{\partial L}{\partial x_1} = 16x_1 + 28x_2 + 2\lambda_1 x_1 x_2 = 0 \quad 0$$

$$\frac{\partial L}{\partial x_2} = 28x_1 + \lambda_1 x_1^2 = 0$$

$$= \begin{bmatrix} 16+2\lambda_1 x_2 & 28+2\lambda_1 x_1 \\ 28+2\lambda_1 x_1 & 0 \end{bmatrix} = \begin{bmatrix} -15.9979 & -27.9958 \\ -27.9958 & 0 \end{bmatrix}$$

$A_1 = -15.9979 < 0$

$A_2 = -27.9958^2 < 0$

So $\nabla_{xx}^2 L$ is a negative-definite

$$Y^T \nabla_{xx}^2 L \, Y = [y_1 \; y_2] \begin{bmatrix} 16+2\lambda_1 x_2 & 28+2\lambda_1 x_1 \\ 28+2\lambda_1 x_1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \begin{bmatrix} (16+2\lambda_1 x_2) y_1 + (28+2\lambda_1 x_1) y_2 & (28+2\lambda_1 x_1) y_1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= (16+2\lambda_1 x_2) y_1^2 + (28+2\lambda_1 x_1) y_1 y_2 + (28+2\lambda_1 x_1) y_1 y_2$$

$$= (16+2\lambda_1 x_2) y_1^2 + 2(28+2\lambda_1 x_1) y_1 y_2$$

$$= (16+2\times(-3.713)\times4.3089)y_1^2 + 2(28+2\times(-3.713)\times7.5405)y_1y_2$$

$$= -15.9979y_1^2 - 55.9915y_1y_2$$

$$\begin{cases} x_1 = 7.5405 \\ x_2 = 4.3089 \\ \lambda_1 = -3.713 \end{cases}$$

from (6) $y_1 = -0.8750y_2$

$$y_2 = -1.1429y_1$$

$$Y^T\nabla_{xx}^2 LY = -15.9979y_1^2 - 55.9915y_1y_2$$

$$= 47.9924y_1^2$$

So for $\forall y \neq 0$, $Y^T\nabla_{xx}^2 LY > 0$

So the result is optimal, the z is the minimum cost