

2. A closed rectangular container is to be designed to have a volume of  $98 \text{ cm}^3$ . The length of the container must be 2.5 times that of the height. The container is to be made of a special material costing  $\$10/\text{cm}^2$ . The objective is to achieve the lowest cost of the designed container. 一个封闭的矩形容器的容积设计为98立方厘米。集装箱的长度必须是高度的2.5倍。这个容器将由一种特殊材料制成，每平方厘米花费10美元。目标是实现设计容器的最低成本。

- (a) Formulate a nonlinear program problem to achieve the stated objective. You may let the length of the container be  $x_1$ (cm) and the breadth be  $x_2$ (cm).

制定一个非线性程序问题，以实现既定目标。  
你可以设定容器的长度为 $x$ （厘米），宽度为 $x$ （厘米）。 (5 Marks)

- (b) Apply the method of Lagrange multipliers to the nonlinear program in part 2(a) to determine the dimensions of the container (in cm) as well as the optimum material cost. Show all your steps.

将拉格朗日乘数法应用于第2(a)部分的非线性程序，以确定容器的尺寸（以厘米为单位）以及最佳材料成本。  
展示你所有的步骤。 (12 Marks)

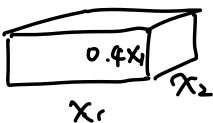
- (c) Determine the approximate change in the cost you found in part 2(b) if the volume of the container is to be reduced by 5%.

如果集装箱的体积要减少5%，请确定您在第2(b)部分中发现的成本的大致变化。 (3 Marks)

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Q:  $V = 98 \text{ cm}^3$      $L = 2H$     closed.     $\$10/\text{cm}^3$

(a) Q: length  $x_1$     breadth  $x_2$     cost  $Z = ?$

Solution  $Z = 10 \times [2 \times (x_1 x_2 + 0.4 x_1^2 + 0.4 x_1 x_2)]$  

$$\text{subject to } \begin{cases} V = x_1 \cdot x_2 \cdot \frac{x_1}{2.5} = 0.4 x_1^2 x_2 = 98 \\ x_1, x_2 \geq 0 \end{cases}$$

where  $Z = 20 \times (1.4 x_1 x_2 + 0.4 x_1^2)$

$$= 28 x_1 x_2 + 8 x_1^2$$

$$x_1^2 x_2 = 245$$

$$x_1, x_2 \geq 0$$

$$\text{so } Z = 28 x_1 x_2 + 8 x_1^2$$

$$\text{subject to } \begin{cases} x_1^2 x_2 = 245 \\ x_1, x_2 \geq 0 \end{cases}$$

(b) apply Lagrange multipliers

Solution ①  $f(x) = 28 x_1 x_2 + 8 x_1^2$      $h(x) = x_1^2 x_2 - 245$

$$L = 28 x_1 x_2 + 8 x_1^2 + \lambda (x_1^2 x_2 - 245)$$

$$\begin{cases} \frac{\partial L}{\partial x_1} = 28x_2 + 16x_1 + 2\lambda x_1 x_2 = 0 & (1) \\ \frac{\partial L}{\partial x_2} = 28x_1 + \lambda x_1^2 = 0 & (2) \\ \frac{\partial L}{\partial \lambda} = x_1^2 x_2 - 245 = 0 & (3) \end{cases}$$

from (3)  $x_1 \neq 0$

from (2)  $\lambda x_1 = -28 \Rightarrow x_1 = -\frac{28}{\lambda} \Rightarrow \lambda = \frac{-28}{x_1}$

from (3)  $x_2 = \frac{245}{x_1^2} = \frac{245\lambda^2}{28^2} = \frac{5}{16}\lambda^2$

from (1)  $28 \times \frac{5}{16}\lambda^2 + 16(-\frac{28}{\lambda}) + 2\lambda(-\frac{28}{\lambda}) \frac{5}{16}\lambda^2 = 0$

$$\frac{35}{4}\lambda^2 - \frac{448}{\lambda} - \frac{35}{2}\lambda^2 = 0$$

$$\frac{35}{4}\lambda^2 - \frac{35}{2}\lambda - 448 = 0$$

$$\lambda_{1,2} = \frac{5 \pm 3\sqrt{145}}{5}$$

$$\lambda_1 = 8.2250$$

$$\lambda_2 = -6.2250$$

$$-8.75\lambda^2 - \frac{448}{\lambda} = 0$$

$$-8.75\lambda^3 = 448$$

$$\lambda^3 = -51.2$$

$$\lambda = -3.7133$$

into (1) get

$$x_2 = \frac{4x_1}{7}$$

from (3)

$$\frac{4}{7}x_1^3 = 245$$

$$x_1 = 7.5405$$

more fast!

$$x_1 = -\frac{28}{\lambda} = 7.5405$$

$$x_2 = \frac{5}{16} \lambda^2 = 4.3089$$

$$\text{cost } Z = 28x_1x_2 + 8x_1^2 = 1364.6284$$

$$\text{So } x_1 = 7.5405$$

$$x_2 = 4.3089$$

$$\lambda = -3.7133$$

$$\text{min cost } Z = 1364.6284$$

② Test of Optimum

$$\text{let } h(x) = x_1^2 x_2 - 245 = 0$$

$$\frac{\partial h(x)}{\partial x_1} = 2x_1x_2 \quad \frac{\partial h(x)}{\partial x_2} = x_1^2$$

$$\nabla h(x) = [2x_1x_2 \quad x_1^2]^T$$

$$\text{let } \mu = \{ \gamma : \nabla h(x)^T \gamma = 0 \}$$

$$\nabla h(x)^T \gamma = 2x_1x_2\gamma_1 + x_1^2\gamma_2 = 0$$

$$64.9825\gamma_1 + 56.8591\gamma_2 = 0$$

$$\gamma_2 = -1.1429\gamma_1$$

$$\nabla_{xx}^2 L = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} \end{bmatrix}$$

$$\frac{\partial L}{\partial x_1} = 28x_2 + 16x_1 + 2\lambda x_1 x_2 = 0 \quad x_1 = 7.5405$$

$$x_2 = 4.3089$$

$$\frac{\partial L}{\partial x_2} = 28x_1 + \lambda x_1^2 = 0$$

$$\lambda = -3.7133$$

$$\frac{\partial^2 L}{\partial x_1^2} = 16 + 2\lambda x_2 = -16$$

$$\frac{\partial^2 L}{\partial x_1 \partial x_2} = 28 + 2\lambda x_1 = -28$$

$$\frac{\partial^2 L}{\partial x_2^2} = 0$$

$$\nabla_{xx}^2 L = \begin{bmatrix} -16 & -28 \\ -28 & 0 \end{bmatrix} \quad \begin{aligned} A_1 &= -16 < 0 \\ A_2 &= 28^2 > 0 \end{aligned}$$

So  $\nabla_{xx}^2 L$  is a negative definite

So the cost  $Z$  is local optimum solution  
and isn't global optimum solution

(c) -5%,  $z$ ?

Solution

$$\Delta V = -0.05 \times 98 = -4.9 \text{ cm}^3$$

$$V_{\text{new}} = 98 - 4.9 = 93.1 \text{ cm}^3$$

new constraint  $x_1^2 x_2 = \frac{93.1}{0.4} = 232.75$

constraint change  $\Delta B = 232.75 - 245 = -12.25$

$$Z_{\text{new}} = Z_{\text{old}} - \Delta B \lambda$$

$$= 1364.6284 - (-12.25) \times (-3.7133)$$

$$= 1319.1405$$