2. A closed rectangular container is to be designed to have a volume of 98 cm^3. The length of the container must be 2.5 times that of the height. The container is to be made of a special material costing \$10/cm^2. The objective is to achieve the lowest cost of the designed container.

一个封闭的矩形容器的容积设计为98立方厘米。集装箱的长度必须是高度的2.5倍。
这个容器将由一种特殊材料制成，每平方厘米花费10美元。目标是实现设计容器的最低成本。

(a) Formulate a nonlinear program problem to achieve the stated objective. You may let the length of the container be x_1 (cm) and the breadth be x_2 (cm).

制定一个非线性程序问题，以实现既定目标。
你可以设定容器的长度为x（厘米），宽度为x（厘米）。

(5 Marks)

(b) Apply the method of Lagrange multipliers to the nonlinear program in part 2(a) to determine the dimensions of the container (in cm) as well as the optimum material cost. Show all your steps.

将拉格朗日乘数法应用于第2(a)部分的非线性程序，以确定容器的尺寸（以厘米为单位）以及最佳材料成本。
展示你所有的步骤。

(12 Marks)

(c) Determine the approximate change in the cost you found in part 2(b) if the volume of the container is to be reduced by 5%.

如果集装箱的体积要减少5%，请确定您在第2(b)部分中发现的成本的大致变化。

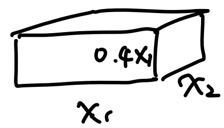
(3 Marks)

23 − S1 − Q2

Q: $V = 98\ cm^3$ $\quad L = 2H$ $\quad$ closed. $\quad \$10/cm^3$

(a) Q: lengh x_1 $\quad$ breadth x_2 $\quad$ cost $Z = ?$

Solution $Z = 10\times[2\times(x_1x_2 + 0.4x_1^2 + 0.4x_1x_2)]$



subject to $\begin{cases} V = x_1 \cdot x_2 \cdot \dfrac{x_1}{2.5} = 0.4x_1^2x_2 = 98 \\ x_1, x_2 \geqslant 0 \end{cases}$

where $Z = 20\times(1.4x_1x_2 + 0.4x_1^2)$

$= 28x_1x_2 + 8x_1^2$

$x_1^2x_2 = 245$

$x_1, x_2 \geqslant 0$

so $Z = 28x_1x_2 + 8x_1^2$

subject to $\begin{cases} x_1^2x_2 = 245 \\ x_1, x_2 \geqslant 0 \end{cases}$

(b) apply Lagrange multipliers

Solution ① $f(x) = 28x_1x_2 + 8x_1^2$ $\qquad h(x) = x_1^2x_2 - 245$

$L = 28x_1x_2 + 8x_1^2 + \lambda(x_1^2x_2 - 245)$

$$
\begin{cases}
\dfrac{\partial L}{\partial x_1} = 28x_2 + 16x_1 + 2\lambda x_1 x_2 = 0 & (1) \\[2ex]
\dfrac{\partial L}{\partial x_2} = 28x_1 + \lambda x_1^2 = 0 & (2) \\[2ex]
\dfrac{\partial L}{\partial \lambda} = x_1^2 x_2 - 245 = 0 & (3)
\end{cases}
$$

from (3) $x_1 \neq 0$

from (2) $\lambda x_1 = -28 \Rightarrow x_1 = -\dfrac{28}{\lambda} \Rightarrow \lambda = \dfrac{-28}{x_1}$

from (3) $x_2 = \dfrac{245}{x_1^2} = \dfrac{245\lambda^2}{28^2} = \dfrac{5}{16}\lambda^2$

into (1) get $x_2 = \dfrac{4x_1}{7}$

from (1) $28 \times \dfrac{5}{16}\lambda^2 + 16\left(-\dfrac{28}{\lambda}\right) + 2\lambda\left(-\dfrac{28}{\lambda}\right)\dfrac{5}{16}\lambda^2 = 0$

from (3) $\dfrac{4}{7}x_1^3 = 245$

$x_1 = 7.5405$

more fast!

$$\frac{35}{4}\lambda^2 - \frac{448}{\lambda} - \frac{35}{2}\lambda^2 = 0$$

$$\boxed{\begin{gathered}
\frac{35}{4}\lambda^2 - \frac{35}{2}\lambda - 448 = 0 \\
\lambda_{1,2} = \frac{5 \pm 3\sqrt{145}}{5} \qquad \times \\
\lambda_1 = 8.2250 \qquad \lambda_2 = -6.2250
\end{gathered}}$$

$$-8.75\lambda^2 - \frac{448}{\lambda} = 0$$

$$-8.75\lambda^3 = 448$$

$$\lambda^3 = -51.2$$

$$\lambda = -3.7133$$

$x_1 = -\frac{28}{\lambda} = 7.5405$

$x_2 = \frac{5}{16}\lambda^2 = 4.3089$

cost $Z = 28x_1x_2 + 8x_1^2 = 1364.6284$

So $x_1 = 7.5405$

$x_2 = 4.3089$

$\lambda = -3.7133$

min cost $Z = 1364.6284$

② Test of optimum

let $h(x) = x_1^2 x_2 - 245 = 0$

$\frac{\partial h(x)}{\partial x_1} = 2x_1x_2$ $\qquad \frac{\partial h(x)}{\partial x_2} = x_1^2$

$\nabla h(x) = [2x_1x_2 \quad x_1^2]^T$

let $M = \{Y : \nabla h(x)^T Y = 0\}$

$\nabla h(x)^T Y = 2x_1x_2 y_1 + x_1^2 y_2 = 0$

$64.9825 y_1 + 56.8591 y_2 = 0$

$y_2 = -1.1429 y_1$

$$\nabla_{xx}^2 L = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} \end{bmatrix}$$

$$\frac{\partial L}{\partial x_1} = 28x_2 + 16x_1 + 2\lambda x_1 x_2 = 0$$

$$\frac{\partial L}{\partial x_2} = 28x_1 + \lambda x_1^2 = 0$$

$x_1 = 7.5405$

$x_2 = 4.3089$

$\lambda = -3.7133$

$$\frac{\partial^2 L}{\partial x_1^2} = 16 + 2\lambda x_2 = -16$$

$$\frac{\partial^2 L}{\partial x_1 \partial x_2} = 28 + 2\lambda x_1 = -28$$

$$\frac{\partial^2 L}{\partial x_2^2} = 0$$

$$\nabla_{xx}^2 L = \begin{bmatrix} -16 & -28 \\ -28 & 0 \end{bmatrix}$$

$A_1 = -16 < 0$

$A_2 = 28^2 > 0$

So $\nabla_{xx}^2 L$ is a negative definite

So the cost z is local optimum solution

and isn't global optimum solution

(c) -5% $\quad z?$

Solution

$\Delta V = -0.05 \times 98 = -4.9\ cm^3$

$V_{new} = 98 - 4.9 = 93.1\ cm^3$

new constraint $\quad x_1^2 x_2 = \dfrac{93.1}{0.4} = 232.75$

constrain change $\quad \Delta B = 232.75 - 245 = -12.25$

$Z_{new} = Z_{old} - \Delta B \lambda$

$= 1364.6284 - (-12.25) \times (-3.7133)$

$= 1319.1405$

Request for Clarification on NLP Problems and Lagrange Multipliers

23-S1-Q2-v3-Q.pdf

Dear Professor Wang Danwei,

I am writing to seek your guidance on a few questions related to solving optimal minimum cost problems in NLP using Lagrange multipliers. Specifically, I have questions regarding the K-T sufficient theorem and Hessian matrix properties as mentioned on slide 84 and 92 of the PPT, which outlines the use of the K-T sufficient theorem to determine whether X* is an optimal solution.

Here are my questions:

1. K-T Sufficient Theorem and Hessian Matrix
   - Does the K-T sufficient theorem inherently include the condition that the Hessian matrix must be positive definite or positive semi-definite?

2. Local vs. Global Minimum with Positive (Semi-)Definiteness
   - When both the K-T sufficient theorem conditions and the positive definite (or positive semi-definite) Hessian matrix condition are satisfied, does this guarantee that Z* is a local minimum or a global minimum?

3. Negative (Semi-)Definiteness of Hessian Matrix
   - If the Hessian matrix is negative definite or negative semi-definite, can X* still be an optimal solution?
   - Under such conditions, is Z* a local minimum, a global minimum, or not a minimum at all?
   - If X* is not the optimal solution but the problem requires finding one, what approach can be used to identify the optimal solution (e.g., referencing 23-S1-Q2)?

4. Indefiniteness of Hessian Matrix
   - If the Hessian matrix is indefinite, can X* be considered an optimal solution?
   - Under these conditions, is Z* a local minimum, a global minimum, or not a minimum at all?

5. Nonlinear h(x)
   - If h(x) is nonlinear and the K-T sufficient theorem cannot be applied to determine optimality:
     - Can X* still be considered an optimal solution?
     - Is Z* a local minimum, a global minimum, or not a minimum at all?
     - If X* is not the optimal solution but the problem requires finding one, how can the optimal solution be determined (e.g., referencing 23-S1-Q2)?

Thank you for your time and guidance on these questions. I greatly appreciate your insights, which would be