Thought，持续 56 秒

Answer to Question 5:对问题5的回答：

(a) Proof that the cross-ratio is projective invariant:(a) 交比射影不变的证明：

The cross-ratio of four collinear points A, B, C, D is defined as:
四个共线点的交比 A, B, C, D 定义为：

$$(A, B; C, D) = \frac{\|AC\|}{\|BC\|} \Big/ \frac{\|AD\|}{\|BD\|} = \frac{\|AC\| \cdot \|BD\|}{\|AD\| \cdot \|BC\|}$$

Under a projective transformation (including projection onto an image plane), points along a line are transformed by a linear fractional (projective) function of the form:
在射影变换（包括投影到图像平面上）下，沿直线的点通过以下形式的线性分数（射影）函数进行变换：

$$x' = \frac{ax + b}{cx + d}$$

where x and x' are the coordinates before and after the transformation.
在哪里 x 和 x' 分别是变换前和变换后的坐标。

Calculating the cross-ratio after transformation, we have:计算变换后的交叉比，我们有：

$$(A', B'; C', D') = \frac{(x'_A - x'_C)(x'_B - x'_D)}{(x'_A - x'_D)(x'_B - x'_C)}$$

Due to the properties of linear fractional transformations, the cross-ratio remains unchanged:由于线性分数变换的特性，交比保持不变：

$$(A', B'; C', D') = (A, B; C, D)$$

Conclusion: Since projections are projective transformations, the cross-ratio of four collinear points is invariant under projection. Thus, the cross-ratio computed from the projected points a, b, c, d is equal to that of the original points A, B, C, D.
结论：由于投影是射影变换，所以四个共线点的交比在投影下不变。因此，根据投影点计算出交叉比 a, b, c, d 等于原始点的值 A, B, C, D 。

(b) Derivation of the epipolar constraint with the essential and fundamental matrices:
(b) 用本质矩阵和基本矩阵推导对极约束：

Epipolar Constraint with the Essential Matrix:基本矩阵的极线约束：

1. **Normalized Image Coordinates:标准化图像坐标：**

$$\tilde{p} = K^{-1}p, \quad \tilde{p}' = K^{-1}p'$$

   where K is the intrinsic matrix, and p, p' are homogeneous image coordinates.
   在哪里 K 是内在矩阵，并且 p, p' 是齐次图像坐标。

2. **Essential Matrix E:**

$$E = [t]_\times R$$

   where $[t]_\times$ is the skew-symmetric matrix of translation vector t, and R is the rotation matrix.在哪里 $[t]_\times$ 是平移向量的斜对称矩阵 t ， 和 R 是旋转矩阵。

3. **Epipolar Constraint:**

$$\tilde{p}'^\top E \tilde{p} = 0$$

Epipolar Constraint with the Fundamental Matrix:基本矩阵的极线约束：

1. **Fundamental Matrix F:基本矩阵 F :**

$$F = K^{-\top} E K^{-1}$$

2. **Epipolar Constraint:**

$$p'^\top F p = 0$$

Mathematical Relationship Between E and F:之间的数学关系 E 和 F :

- The essential matrix E encapsulates the extrinsic parameters (R and t) and operates on normalized coordinates.基本矩阵 E 封装外部参数（ R 和 t ）并在标准化坐标上进行操作。
- The fundamental matrix F incorporates both intrinsic and extrinsic parameters and operates on pixel coordinates.基本矩阵 F 结合了内在和外在参数并在像素坐标上运行。
- Relationship:

$$F = K^{-\top} E K^{-1}$$

Constraints When Using the Essential Matrix:使用基本矩阵时的限制：

- **Calibration Required:** The intrinsic matrix K must be known for both cameras to compute normalized coordinates.
  需要校准：固有矩阵 K 必须知道两个相机才能计算标准化坐标。
- **Scale Ambiguity:** The essential matrix is defined up to a scale factor; the translation t cannot be recovered uniquely without additional information.
  尺度模糊性：本质矩阵被定义为一个尺度因子；翻译 t 如果没有附加信息，则无法唯一恢复。
- **Assumption of Identical Intrinsics:** Since both cameras share the same K, simplifying computations; differences in intrinsic parameters require adjustments.
  相同本质的假设：由于两个相机共享相同的 K ，简化计算；内在参数的差异需要调整。

(c) Deriving Depth Z and Depth Error ΔZ:(c) 导出深度 Z 和深度误差 ΔZ :

Depth from Disparity:

$$Z = \frac{fB}{d}$$

where f is the focal length, B is the baseline, and d is the disparity.
在哪里 f 是焦距， B 是基线，并且 d 是差距。

Estimating Depth Error ΔZ:估计深度误差 ΔZ :

1. **Estimated Disparity:估计差异：**

$$\hat{d} = d + \Delta d$$

2. **Estimated Depth:预计深度：**

$$\hat{Z} = \frac{fB}{\hat{d}} = \frac{fB}{d + \Delta d}$$

3. **Depth Error:**

$$\Delta Z = \hat{Z} - Z = \frac{fB}{d + \Delta d} - \frac{fB}{d}$$

4. **Using Taylor Series Approximation (Assuming Δd is small):**
   使用泰勒级数逼近（假设 Δd 很小）：

$$\frac{1}{d + \Delta d} \approx \frac{1}{d} - \frac{\Delta d}{d^2}$$

5. **Simplifying ΔZ:**

$$\Delta Z \approx \left(\frac{fB}{d} - \frac{fB\Delta d}{d^2}\right) - \frac{fB}{d} = -\frac{fB}{d^2}\Delta d$$

6. **Expressing in Terms of Z:表达为 Z :**

$$\Delta Z = -Z\left(\frac{\Delta d}{d}\right)$$

Conclusion: The estimated depth error ΔZ is proportional to the true depth Z and the relative disparity error $\frac{\Delta d}{d}$:**结论：**估计深度误差 ΔZ 与真实深度成正比 Z 和相对视差误差 $\frac{\Delta d}{d}$:

$$\Delta Z = -Z\left(\frac{\Delta d}{d}\right)$$

(d) Deriving the Brightness Constancy Constraint Equation and Discussion:
(d) 亮度恒定性约束方程的推导及讨论：

Derivation:推导：

1. **Brightness Constancy Assumption:亮度恒定假设：**

$$I(x, y, t) = I(x + u, y + v, t + 1)$$

2. **Taylor Series Expansion (for small u, v):泰勒级数展开（对于小 u, v ）：**

$$I(x + u, y + v, t + 1) \approx I(x, y, t) + I_x u + I_y v + I_t$$

   where I_x, I_y, I_t are partial derivatives of intensity.在哪里 I_x, I_y, I_t 是强度的偏导数。

3. **Subtracting $I(x, y, t)$:**

$$I(x + u, y + v, t + 1) - I(x, y, t) = I_x u + I_y v + I_t = 0$$

4. **Brightness Constancy Constraint Equation:亮度恒定性约束方程：**

$$I_x u + I_y v + I_t = 0$$

Limitations of the Brightness Constancy Constraint:亮度恒定性约束的局限性：

- **Lighting Changes:** Assumes constant illumination; fails under varying lighting conditions.**光照变化：**假设光照恒定；在不同的照明条件下会失败。
- **Surface Reflectance Changes:** Does not account for changes due to specular reflections or shadows.**表面反射率变化：**不考虑镜面反射或阴影引起的变化。
- **Small Motion Assumption:** Accurate only for small displacements; large motions invalidate the linear approximation.
  小运动假设：仅对小位移准确；大的运动会使线性近似无效。
- **Textureless Regions:** In areas with little intensity variation, gradients I_x, I_y are near zero, making flow estimation unreliable.
  无纹理区域：在强度变化很小的区域，梯度 I_x, I_y 接近于零，使得流量估计不可靠。

Other Key Assumptions for Estimating Optical Flow:估计光流的其他关键假设：

- **Spatial Smoothness:** The flow field is assumed to be smooth, except at motion boundaries.**空间平滑度：**假设流场是平滑的，运动边界除外。
- **Lambertian Surfaces:** Surfaces reflect light uniformly; no specular highlights.
  朗伯表面：表面均匀地反射光；没有镜面高光。
- **Temporal Consistency:** The scene brightness does not change rapidly over time.
  时间一致性：场景亮度不会随时间快速变化。

Mathematical Conditions for Successful Optical Flow Estimation:成功光流估计的数学条件：

- **Well-Conditioned Gradient Matrix:** The matrix**良条件梯度矩阵：**矩阵

$$\begin{pmatrix} I_x^2 & I_x I_y \\ I_x I_y & I_y^2 \end{pmatrix}$$

  should have significant eigenvalues for reliable flow computation.
  应具有可靠的流量计算的显着特征值。
- **Adequate Texture:** Sufficient intensity gradients in the image region to avoid the aperture problem.**足够的纹理：**图像区域有足够的强度梯度以避免孔径问题。
- **Validity of Linear Approximation:** Motion should be small enough for the Taylor series expansion to hold.**线性近似的有效性：**运动应该足够小以使泰勒级数展开式成立。

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