22-S1-Q4

Q: (a) Σ ? λ? ϕ? show $\sigma_i^2 \overset{\text{方差 } \phi_i}{=} \lambda_i$

Solution: $\Sigma = \frac{1}{n} X X^T$ } from Topic 9 page 11
$\Sigma \phi_i = \lambda_i \phi_i$

$$X = \begin{array}{c} \xleftrightarrow{\; n \;} \\ \begin{bmatrix} \text{sample 1} & \text{sample 2} & \cdots \end{bmatrix} \end{array} \; m \downarrow \quad E(\text{row}) = 0$$

$$\phi^T \Sigma \phi = \frac{1}{n} \sum_{i=1}^{n} \phi^T x_i [\phi^T x_i]^T = \sigma_i^2 \text{ 方差定义}$$

$$\phi^T \phi = 1$$

$$\phi^T \Sigma \phi = \phi^T \lambda \phi = \lambda \phi^T \phi = \lambda$$

$$\sigma_i^2 = \lambda$$

$$y_i = \phi_i^T X$$

$$\sigma_i^2 = \frac{1}{n} y_i y_i^T = \frac{1}{n} (\phi_i^T X)(\phi_i^T X)^T = \frac{1}{n} \phi_i^T X X^T \phi_i$$

$$\sigma_i^2 = \phi_i^T \Sigma \phi_i = \phi_i^T \lambda_i \phi_i = \lambda_i$$

(2) Q: λ_k 与 λ_j 不相, 证明正交 $\phi_k^T \phi_j = 0$

$$\Sigma \phi_k = \lambda_k \phi_k$$

$$\Sigma \phi_j = \lambda_j \phi_j$$

$$\phi_k^T \Sigma \phi_j = \phi_k^T (\lambda_j \phi_j) = \lambda_j (\phi_k^T \phi_j)$$

$$= (\Sigma \phi_k)^T \phi_j = (\lambda_k \phi_k)^T \phi_j = \lambda_k \phi_k^T \phi_j$$

So $\lambda_j (\phi_k^T \phi_j) = \lambda_k (\phi_k^T \phi_j)$

$$(\lambda_j - \lambda_k)(\phi_k^T \phi_j) = 0$$

$$\lambda_j \neq \lambda_k \Rightarrow \phi_k^T \phi_j = 0$$

(3) $y_k = \phi_k^T X \quad y_j = \phi_j^T X$

$$\sigma_{kj}^2 = \frac{1}{n} y_k y_j^T$$

$$= \frac{1}{n} (\phi_k^T X)(\phi_j^T X)^T$$

$$= \frac{1}{n} \phi_k^T X X^T \phi_j$$

$$= \phi_k^T \Sigma \phi_j$$

$$= \phi_k^T (\lambda_i \phi_j)$$

$$= \lambda_j (\phi_k^T \phi_j)$$

$$= \lambda_j \times 0$$

$$= 0$$