Thought，持续 21 秒

Answer:

(a) To estimate the translational motion $t = \mu_x - R\mu_p$ using Singular Value Decomposition (SVD), follow these steps:

(a)估计平移运动 $t = \mu_x - R\mu_p$ 使用奇异值分解 (SVD)，请按照下列步骤操作：

1. **Compute the Centroids**:
   - Calculate the mean (centroid) of each point set:计算每个点集的平均值（质心）：

$$\mu_x = \frac{1}{N_p}\sum_{i=1}^{N_p} x_i, \quad \mu_p = \frac{1}{N_p}\sum_{i=1}^{N_p} p_i$$

2. **Center the Point Sets:将点集居中**:
   - Subtract the centroid from each point to obtain centered coordinates:
     从每个点减去质心以获得中心坐标:

$$X_c = \{x_i - \mu_x\}, \quad P_c = \{p_i - \mu_p\}$$

3. **Compute the Covariance Matrix:计算协方差矩阵**:
   - Calculate the covariance matrix H:计算协方差矩阵 H：

$$H = \sum_{i=1}^{N_p} P_c^{(i)}(X_c^{(i)})^T$$

4. **Perform SVD on the Covariance Matrix:对协方差矩阵执行 SVD**：
   - Decompose H using SVD:分解 H 使用奇异值分解:

$$H = U\Sigma V^T$$

     where U and V are orthogonal matrices, and Σ is a diagonal matrix.
     在哪里 U 和 V 是正交矩阵，并且 Σ 是对角矩阵。

5. **Compute the Rotation Matrix**:
   - Obtain the rotation matrix R:

$$R = VU^T$$

     - Ensure that $\det(R) = +1$ (proper rotation). If $\det(R) = -1$, adjust U or V to correct for reflection.
       确保 $\det(R) = +1$ （适当旋转）。如果 $\det(R) = -1$ ， 调整 U 或者 V 纠正反射。

6. **Compute the Translation Vector:计算平移向量**:
   - Finally, calculate the translation vector t:最后计算平移向量 t：

$$t = \mu_x - R\mu_p$$

(b)

(i) Recovery of Rotation Angle and Axis(i) 旋转角度和轴的恢复

1. **Show that $\Omega = \arccos\left(\frac{\mathrm{trace}(R)-1}{2}\right)$**:
   - The trace of a rotation matrix R in three dimensions is related to the rotation angle Ω by:旋转矩阵的迹 R 三维度与旋转角度有关 Ω 经过：

$$\mathrm{trace}(R) = r_{11} + r_{22} + r_{33} = 1 + 2\cos(\Omega)$$

   - Rearranging, we get:重新整理一下，我们得到:

$$\cos(\Omega) = \frac{\mathrm{trace}(R) - 1}{2}$$

   - Therefore:

$$\Omega = \arccos\left(\frac{\mathrm{trace}(R) - 1}{2}\right)$$

2. **Show that $l = N\left[\begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix}\right]$**:
   - The skew-symmetric part of R relates to the rotation axis:
     的斜对称部分 R 与旋转轴相关：

$$R - R^T = 2\sin(\Omega)[l]_\times$$

     where $[l]_\times$ is the skew-symmetric matrix of l.在哪里 $[l]_\times$ 是斜对称矩阵 l 。

   - The elements of $R - R^T$ are:的要素 $R - R^T$ 是:

$$R - R^T = \begin{pmatrix} 0 & r_{12} - r_{21} & r_{13} - r_{31} \\ r_{21} - r_{12} & 0 & r_{23} - r_{32} \\ r_{31} - r_{13} & r_{32} - r_{23} & 0 \end{pmatrix}$$

   - Extract the rotation axis components:提取旋转轴分量:

$$l = \frac{1}{2\sin(\Omega)}\begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix}$$

   - Normalize l:

$$l = N\left[\begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix}\right]$$

(ii) Derivation of Equivalent Quaternion from R(ii) 等价四元数的推导 R

1. **Compute the Rotation Angle Ω and Axis l:计算旋转角度 Ω 和轴 l**：
   - As shown in part (i), calculate Ω and l.如第(i)部分所示，计算 Ω 和 l 。

2. **Compute Quaternion Components:计算四元数分量**:
   - The quaternion $q = (q_0, q_1, q_2, q_3)^T$ representing the rotation is given by:
     四元数 $q = (q_0, q_1, q_2, q_3)^T$ 表示旋转的方式由下式给出:

$$q_0 = \cos\left(\frac{\Omega}{2}\right), \quad q = l\sin\left(\frac{\Omega}{2}\right)$$

   - Therefore:

$$q_0 = \cos\left(\frac{\Omega}{2}\right)$$

$$q_1 = l_1\sin\left(\frac{\Omega}{2}\right), \quad q_2 = l_2\sin\left(\frac{\Omega}{2}\right), \quad q_3 = l_3\sin\left(\frac{\Omega}{2}\right)$$

3. **Express Quaternion in Terms of R:用以下形式表达四元数 R**：
   - Use the previously computed Ω and l derived from R to obtain q.
     使用之前计算的 Ω 和 l 源自 R 获得 q 。

Final Expressions:最终表达：

- **Rotation Angle**:

$$\Omega = \arccos\left(\frac{\mathrm{trace}(R) - 1}{2}\right)$$

- **Rotation Axis**:

$$l = N\left[\begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix}\right]$$

- **Quaternion Components:四元数组件**:

$$q_0 = \cos\left(\frac{\Omega}{2}\right)$$

$$q_1 = l_1\sin\left(\frac{\Omega}{2}\right), \quad q_2 = l_2\sin\left(\frac{\Omega}{2}\right), \quad q_3 = l_3\sin\left(\frac{\Omega}{2}\right)$$

By following these steps, you can estimate the translational motion t and derive the equivalent quaternion q from the rotation matrix R.

通过执行以下步骤，您可以估计平移运动 t 并推导等效四元数 q 从旋转矩阵 R 。