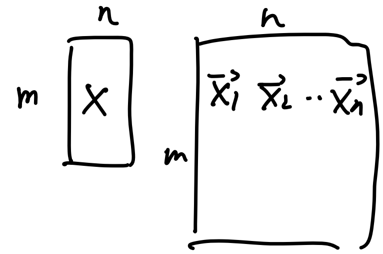
22-S1-Q4

Q: (a) Σ ? λ? ϕ ?



Solution: ① definition of Σ

$$\Sigma = \frac{1}{n} X X^T$$

② definition of λ and ϕ

$$\Sigma \phi_i = \lambda_i \phi_i$$

③ Show projected variance σ_i^2 equals to λ_i

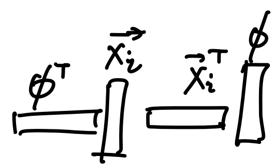
project training data to ϕ_i is produce operation

$$y_i = \phi^T x_i$$

variance σ_i^2 definition

$$\sigma_i^2 = \frac{1}{n}\sum_{i=1}^{n} y_i^2$$

$$= \frac{1}{n}\sum_{i=1}^{n} \phi_i^T x_i\, x_i^T \phi_i$$



num × num

$$= \phi_i^T \left(\frac{1}{n}\sum_{i=1}^{n} x_i x_i^T\right) \phi_i$$

$$= \phi_i^T \Sigma \phi_i$$

$$= \phi_i^T \lambda_i \phi_i$$

$$= \lambda_i \phi_i^T \phi_i$$

from unit-length eigenvector ϕ_i

$$\phi_i^T \phi_i = 1$$

So $\sigma_i^2 = \lambda_i$

(2) Q: λ_k与λ_j不相同,证明正交 $\phi_k^T \phi_j = 0$

$$\Sigma \phi_k = \lambda_k \phi_k$$

$$\Sigma \phi_j = \lambda_j \phi_j$$

$$\phi_k^T \Sigma \phi_j = \phi_k^T (\lambda_j \phi_j) = \lambda_j (\phi_k^T \phi_j)$$

$$= (\Sigma \phi_k)^T \phi_j = (\lambda_k \phi_k)^T \phi_j = \lambda_k \phi_k^T \phi_j$$

So $\lambda_j (\phi_k^T \phi_j) = \lambda_k (\phi_k^T \phi_j)$

$$(\lambda_j - \lambda_k)(\phi_k^T \phi_j) = 0$$

$$\lambda_j \neq \lambda_k \Rightarrow \phi_k^T \phi_j = 0$$

(3) $y_k = \phi_k^T X \quad y_j = \phi_j^T X$

$$\sigma_{kj}^2 = \frac{1}{n} y_k y_j^T$$

$$= \frac{1}{n} (\phi_k^T X)(\phi_j^T X)^T$$

$$= \frac{1}{n} \phi_k^T X X^T \phi_j$$

$$= \phi_k^T \Sigma \phi_j$$

$$= \phi_k^T (\lambda_i \phi_j)$$

$$= \lambda_j (\phi_k^T \phi_j)$$

$$= \lambda_j \times 0$$

$$= 0$$