4.2.2.4 Example

Q $3^d - 2^{d+1} + 1$

$d = 6$

Solution ① counting possible item subsets

non-empty subsets of a set with d items

$2^d - 1$

②

general proof using induction 数学归纳法证明

(1) $d=0$ $2^0=1$ ϕ the formula holds

(2) Assume $d=k$ formula holds.

(3) $d=k+1$

1. take any of the 2^k subsets from the first k elements.

2. add the new $(k+1)^{th}$ element to each of these 2^k subsets forming another 2^k new

3. Thus, the total number of subsets for $d=k+1$ is $2^k+2^k=2^{k+1}$

(4) So A set with d elemets has 2^d subsets.

② S contains at least two elements

$S \to X,\ Y$ in $2^{|S|}-2$ ways

similary, we can use induction to proof.

eg. $S=\{a,b\}$ $|S|=2$ $2^2-2=2$

X	Y	
$X=\{a,b\}$	$Y=\emptyset$	✗
a	b	✓
b	a	✓
$\emptyset$	$\{a,b\}$	✗

③ Total Rule Count computation

$$\underbrace{\sum_{k=2}^{d}}_{\text{summing over all possible subset size } k}\ \underbrace{\binom{d}{k}}_{d\to k \text{ ways}}\ \underbrace{(2^k-2)}_{S \to X,\,Y \text{ ways}}$$

$$= \sum_{k=2}^{d} \binom{d}{k} 2^k - \sum_{k=2}^{d} \binom{d}{k} 2$$

④ Take the first part

Binomial Theorem 二项式定理

$$(a+b)^d = \sum_{k=0}^{d} \binom{d}{k} a^{d-k} b^k$$

we can proof by induction

$$\sum_{k=0}^{d} \binom{d}{k} 2^k = (1+2)^d = 3^d$$

$$\sum_{k=2}^{d} \binom{d}{k} 2^k = 3^d - \binom{d}{0} 2^0 - \binom{d}{1} 2^1$$

$$= 3^d - 1 - 2d$$

⑤ Take the second part

$$\sum_{k=2}^{d} \binom{d}{k} = 2^d - 1 - d$$

Due to bionomial identity 二项恒等式

$$\sum_{k=0}^{d} \binom{d}{k} = 2^d$$

we can proof by binomial theorem

$$2^d = (1+1)^d = \sum_{k=0}^{d} \binom{d}{k} 1^{d-k} 1^k = \sum_{k=0}^{d} \binom{d}{k}$$

⑥ $\sum_{k=2}^{d} \binom{d}{k} 2^k - \sum_{k=2}^{d} \binom{d}{k} 2$

$= 3^d - 1 - 2d - 2 \times (2^d - 1 - d)$

$= 3^d - 1 - 2d - 2^{d+1} + 2 + 2d$

$= 3^d - 2^{d+1} + 1$

⑦ So

$3^6 - 2^{6+1} + 1 = 3^6 - 2^7 + 1 = 602$