22-S2-Q3

Q: $x_1 - x_4$ ϕ_1 ϕ_2

(a) Q project?

Solution ① formula: project = vector$^T \cdot \phi$

$$\mu = \frac{1}{4}(x_1 + x_2 + x_3 + x_4) = \frac{1}{4}\left(\begin{bmatrix}1\\6\end{bmatrix} + \begin{bmatrix}4\\7\end{bmatrix} + \begin{bmatrix}2\\9\end{bmatrix} + \begin{bmatrix}5\\10\end{bmatrix}\right)$$

$$= \frac{1}{4} \times \begin{bmatrix}12\\32\end{bmatrix}$$

$$= \begin{bmatrix}3\\8\end{bmatrix}$$

vector			along ϕ_1		along ϕ_2
$\hat{x}_1$	$\begin{bmatrix}-2\\-2\end{bmatrix}$	a_1	$-2\sqrt{2}$	b_1	0
$\hat{x}_2$	$\begin{bmatrix}1\\-1\end{bmatrix}$	a_2	0	b_2	$-\sqrt{2}$
$\hat{x}_3$	$\begin{bmatrix}-1\\1\end{bmatrix}$	a_3	0	b_3	$\sqrt{2}$
$\hat{x}_4$	$\begin{bmatrix}2\\2\end{bmatrix}$	a_4	$2\sqrt{2}$	b_4	0

$\hat{x}_1^T \phi_1 = [-2,-2]\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix} = -2\sqrt{2}$

$\hat{x}_1^T \phi_2 = [-2,-2]\begin{bmatrix}-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix} = 0$

$\hat{x}_2^T \phi_1 = [1,-1]\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix} = 0$

$\hat{x}_2^T \phi_2 = [1,-1]\begin{bmatrix}-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix} = -\sqrt{2}$

$\hat{x}_3^T \phi_1 = [-1,1]\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix} = 0$

$\hat{x}_3^T \phi_2 = [-1,1]\begin{bmatrix}-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix} = \sqrt{2}$

$\hat{x}_4^T \phi_1 = [2,2]\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix} = 2\sqrt{2}$

$\hat{x}_4^T \phi_2 = [2,2]\begin{bmatrix}-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix} = 0$

(b)a eigenvalues ?

Solution ① get covariance matrix

$$\Sigma = \frac{1}{4}\sum_{i=1}^{4}(X_i-\mu)(X_i-\mu)^T = \frac{1}{4}\sum_{i=1}^{4}\hat{X}_i\hat{X}_i^T$$

$$\Sigma = \frac{1}{4}\left(\begin{bmatrix}-2\\-2\end{bmatrix}\begin{bmatrix}-2 & -2\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}1 & -1\end{bmatrix} + \begin{bmatrix}-1\\1\end{bmatrix}\begin{bmatrix}-1 & 1\end{bmatrix} + \begin{bmatrix}2\\2\end{bmatrix}\begin{bmatrix}2 & 2\end{bmatrix}\right)$$

$$= \frac{1}{4}\left(\begin{bmatrix}4 & 4\\4 & 4\end{bmatrix} + \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} + \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} + \begin{bmatrix}4 & 4\\4 & 4\end{bmatrix}\right)$$

$$= \frac{1}{4}\begin{bmatrix}10 & 6\\6 & 10\end{bmatrix}$$

$$= \begin{bmatrix}\frac{5}{2} & \frac{3}{2}\\ \frac{3}{2} & \frac{5}{2}\end{bmatrix}$$

② eigen value formula $|\lambda I - \Sigma| = 0$

$$\begin{vmatrix}\lambda-\frac{5}{2} & -\frac{3}{2}\\ -\frac{3}{2} & \lambda-\frac{5}{2}\end{vmatrix} = (\lambda-\frac{5}{2})^2 - \frac{9}{4} = 0$$

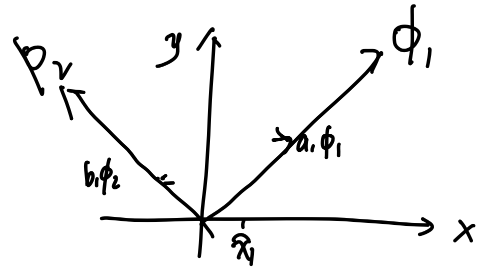
$$\lambda - \frac{5}{2} = \pm\frac{3}{2}$$

$$\lambda = \frac{5\pm3}{2}$$

$$\lambda_1 = 4 \quad \lambda_2 = 1$$

(c) Q: reconstruction min e?

Solution:



① Reconstract using a_i and ϕ_1

	Mean Absulate Error
$\hat{x}_i = \mu + a_i\phi_1$	
$\hat{x}_1 = \begin{bmatrix}3\\8\end{bmatrix} + (-2\sqrt{2}) \times \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix} = \begin{bmatrix}1\\6\end{bmatrix}$	0
$\hat{x}_2 = \begin{bmatrix}3\\8\end{bmatrix} + 0 = \begin{bmatrix}3\\8\end{bmatrix}$	2
$\hat{x}_3 = \begin{bmatrix}3\\8\end{bmatrix} + 0 = \begin{bmatrix}3\\8\end{bmatrix}$	2
$\hat{x}_4 = \begin{bmatrix}3\\8\end{bmatrix} + (2\sqrt{2}) \times \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}5\\10\end{bmatrix}$	0
	sum = 4

② Reconstract using b_i and ϕ_2

$\hat{x}_i = \mu + b_i\phi_2$	
$\hat{x}_1 = \begin{bmatrix}3\\8\end{bmatrix} + 0 = \begin{bmatrix}3\\8\end{bmatrix}$	4
$\hat{x}_2 = \begin{bmatrix}3\\8\end{bmatrix} + (-\sqrt{2}) \times \begin{bmatrix}-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}4\\7\end{bmatrix}$	0
$\hat{x}_3 = \begin{bmatrix}3\\8\end{bmatrix} + (\sqrt{2}) \times \begin{bmatrix}-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}2\\9\end{bmatrix}$	0
$\hat{x}_4 = \begin{bmatrix}3\\8\end{bmatrix} + 0 = \begin{bmatrix}3\\8\end{bmatrix}$	4
	sum = 8

So, Reconstruct using a_i and ϕ_i minimize the error