3. A dataset contains 4 samples $\mathbf{x}_1 = (1, 6)^T$, $\mathbf{x}_2 = (4, 7)^T$, $\mathbf{x}_3 = (2, 9)^T$, and $\mathbf{x}_4 = (5, 10)^T$. Its covariance matrix has eigenvectors $\phi_1 = (\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}})^T$ and $\phi_2 = (-\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}})^T$.

(a) Compute the one-dimensional representations a_i and b_i, i=1, 2, 3, 4, of these 4 samples along each of the two eigenvectors.

(8 Marks)

(b) Compute the two eigenvalues.

(7 Marks)

(c) Compute the two-dimensional samples $\hat{\mathbf{x}}_i$ reconstructed from single one-dimensional representation (a_i or b_i) which minimize the reconstruction error.

(10 Marks)

22-S2-Q3

Q: $x_1 - x_4$ ϕ_1 ϕ_2

(a) Q project?

Solution ① formula: project = $\text{vector}^T \cdot \phi$

vector	along ϕ_1	along ϕ_2
x_1	$\frac{7\sqrt{2}}{2}$	$\frac{5\sqrt{2}}{2}$
x_2	$\frac{11\sqrt{2}}{2}$	$\frac{3\sqrt{2}}{2}$
x_3	$\frac{11\sqrt{2}}{2}$	$\frac{7\sqrt{2}}{2}$
x_4	$\frac{15\sqrt{2}}{2}$	$\frac{5\sqrt{2}}{2}$

$$x_1^T\phi_1 = [1,6]\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} = \frac{1+6}{\sqrt{2}} = \frac{7}{\sqrt{2}} \quad \frac{7\sqrt{2}}{2}$$

$$x_1^T\phi_2 = [1,6]\begin{bmatrix}-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} = \frac{-1+6}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$$

(b) Q eigenvalues?

Solution ① get covariance matrix

$$\Sigma = \frac{1}{4}\sum_{i=1}^{4}(X_i-\mu)(X_i-\mu)^T$$

$$\mu = \frac{1}{4}(x_1+x_2+x_3+x_4) = \frac{1}{4}\left(\begin{bmatrix}1\\6\end{bmatrix}+\begin{bmatrix}4\\7\end{bmatrix}+\begin{bmatrix}2\\9\end{bmatrix}+\begin{bmatrix}5\\10\end{bmatrix}\right)$$

$$= \frac{1}{4}\times\begin{bmatrix}12\\32\end{bmatrix}$$

$$= \begin{bmatrix}3\\8\end{bmatrix}$$

$$\Sigma = \frac{1}{4}\left(\begin{bmatrix}-2\\-2\end{bmatrix}\begin{bmatrix}-2 & -2\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}1 & -1\end{bmatrix} + \begin{bmatrix}-1\\1\end{bmatrix}\begin{bmatrix}-1 & 1\end{bmatrix} + \begin{bmatrix}2\\2\end{bmatrix}\begin{bmatrix}2 & 2\end{bmatrix}\right)$$

$$= \frac{1}{4}\left(\begin{bmatrix}4 & 4\\4 & 4\end{bmatrix} + \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} + \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} + \begin{bmatrix}4 & 4\\4 & 4\end{bmatrix}\right)$$

$$= \frac{1}{4}\begin{bmatrix}10 & 6\\6 & 10\end{bmatrix}$$

$$= \begin{bmatrix}\frac{5}{2} & \frac{3}{2}\\ \frac{3}{2} & \frac{5}{2}\end{bmatrix}$$

② eigen value formula $|\lambda I - \Sigma| = 0$

$$\begin{vmatrix}\lambda - \frac{5}{2} & -\frac{3}{2}\\ -\frac{3}{2} & \lambda - \frac{5}{2}\end{vmatrix} = (\lambda - \frac{5}{2})^2 - \frac{9}{4} = 0$$

$$\lambda - \frac{5}{2} = \pm\frac{3}{2}$$

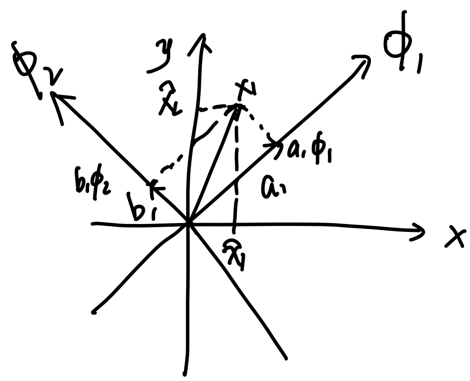
$$\lambda = \frac{5 \pm 3}{2}$$

$$\lambda_1 = 4 \quad \lambda_2 = 1$$

(c) Q: reconstruction ?

Solution: ① formula

$$\hat{x}_i = a_i\phi_1 + b_i\phi_2$$



$$\frac{7\sqrt{2}}{2}\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}+\frac{5\sqrt{2}}{2}\begin{bmatrix}-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{7}{2}\\\frac{7}{2}\end{bmatrix}+\begin{bmatrix}-\frac{5}{2}\\\frac{5}{2}\end{bmatrix}$$

$$=\begin{bmatrix}1\\6\end{bmatrix}$$