23- S2-Q3

Q: (a) covariance matrix ?

Solution ① mean of X

$$\overline{X} = \frac{1}{5}\left(\begin{bmatrix}5\\1\end{bmatrix}+\begin{bmatrix}4\\2\end{bmatrix}+\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}2\\4\end{bmatrix}+\begin{bmatrix}1\\5\end{bmatrix}\right)$$

$$= \frac{1}{5}\begin{bmatrix}15\\15\end{bmatrix}$$

$$= \begin{bmatrix}3\\3\end{bmatrix}$$

② normalization

$$\hat{X}_1 = \begin{bmatrix}5\\1\end{bmatrix}-\begin{bmatrix}3\\3\end{bmatrix}=\begin{bmatrix}2\\-2\end{bmatrix} \quad \hat{X}_2 = \begin{bmatrix}1\\-1\end{bmatrix} \quad \hat{X}_3 = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\hat{X}_4 = \begin{bmatrix}-1\\1\end{bmatrix} \quad \hat{X}_5 = \begin{bmatrix}-2\\2\end{bmatrix}$$

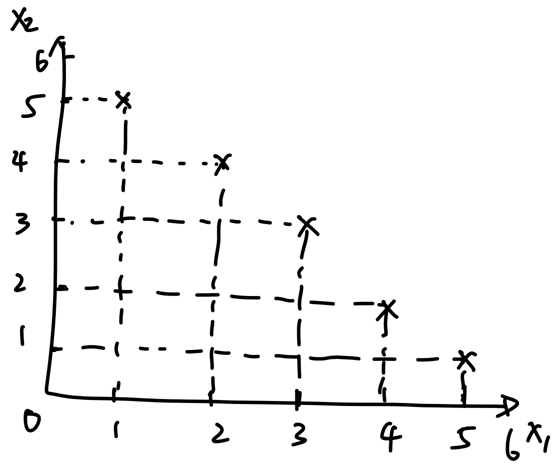
③ covariance

$$\Sigma = \frac{1}{5}\sum_{i=1}^{5}\hat{X}_i\,\hat{X}_i^T$$

$$= \frac{1}{5}\left(\begin{bmatrix}2\\-2\end{bmatrix}\begin{bmatrix}2 & -2\end{bmatrix}+\begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}1 & -1\end{bmatrix}+\begin{bmatrix}0&0\\0&0\end{bmatrix}+\begin{bmatrix}-1\\1\end{bmatrix}\begin{bmatrix}-1 & 1\end{bmatrix}+\begin{bmatrix}-2\\2\end{bmatrix}\begin{bmatrix}-2 & 2\end{bmatrix}\right)$$

$$= \frac{1}{5}\left(\begin{bmatrix}4&-4\\-4&4\end{bmatrix}+\begin{bmatrix}1&-1\\-1&1\end{bmatrix}+\begin{bmatrix}0&0\\0&0\end{bmatrix}+\begin{bmatrix}1&-1\\-1&1\end{bmatrix}+\begin{bmatrix}4&-4\\-4&4\end{bmatrix}\right)$$

$$= \frac{1}{5}\begin{bmatrix}10&-10\\-10&10\end{bmatrix} = \begin{bmatrix}2&-2\\-2&2\end{bmatrix}$$

(b) 

(c) Q eigenvalues eigenvectors ?

Solution

① $\Sigma = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

$$|\lambda I - \Sigma| = \begin{vmatrix} \lambda-2 & 2 \\ 2 & \lambda-2 \end{vmatrix} = (\lambda-2)^2 - 4 = 0$$

$\lambda - 2 = \pm 2 \quad \lambda = 2 \pm 2 \quad \lambda_1 = 4 \quad \lambda_2 = 0$

② when $\lambda_1 = 4$

$$(4I - \Sigma)\phi = 0$$

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = 0 \implies \phi_1 + \phi_2 = 0$$

so $\phi_1' = \begin{bmatrix} k \\ -k \end{bmatrix}, k \in \infty$

translate to unit-length eigenvectors

$$\phi_1 = \frac{\phi_1'}{|\phi_1'|} = \frac{\begin{bmatrix} k \\ -k \end{bmatrix}}{\sqrt{k^2+(-k)^2}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$$

③ when $\lambda_2 = 0$

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = 0 \Rightarrow \begin{cases} -2\phi_1 + 2\phi_2 = 0 \\ 2\phi_1 - 2\phi_2 = 0 \end{cases} \Rightarrow \phi_1 = \phi_2$$

So $\phi_2' = \begin{bmatrix} k \\ k \end{bmatrix}$, $k \in \omega$

translate to unit-length

$$\phi_2 = \frac{\phi_2'}{|\phi_2'|} = \frac{\begin{bmatrix} k \\ k \end{bmatrix}}{\sqrt{k^2+k^2}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

(d) ① eigenvalues and eigenvectors of the covariance matrix reveal the principal variances and directions of the data

② $\lambda = 4$ indicates significant variance along its eigon vector $\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$

③ $\lambda = 0$ indicates zero variances, all data point perfectly aligned without any spread in that direction $\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$