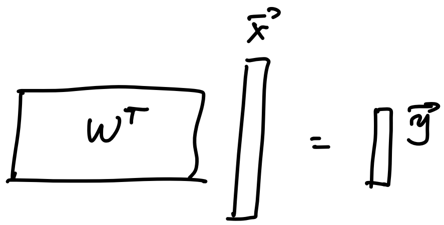
20-S2-Q4

Q: (a) du不变 → 条件?

Solution ① understand

$\vec{y} = W^T \vec{X}$



② $d_u(\vec{x}_1, \vec{x}_2)$

$d_u(\vec{x}_1, \vec{x}_2) = \|\vec{x}_1 - \vec{x}_2\| = \sqrt{(\vec{x}_1 - \vec{x}_2)^T(\vec{x}_1 - \vec{x}_2)}$

② $d_u(\vec{y}_1, \vec{y}_2)$

$\vec{y}_1 - \vec{y}_2 = W^T(\vec{x}_1 - \vec{x}_2)$

$d_u(\vec{y}_1, \vec{y}_2) = \|\vec{y}_1 - \vec{y}_2\| = \sqrt{(\vec{y}_1 - \vec{y}_2)^T(\vec{y}_1 - \vec{y}_2)}$

Since $(AB)^T = B^T A^T$

$d_u(\vec{y}_1, \vec{y}_2) = \sqrt{(\vec{x}_1 - \vec{x}_2)^T W W^T (\vec{x}_1 - \vec{x}_2)}$

③ equality distance

$\sqrt{(\vec{x}_1 - \vec{x}_2)^T(\vec{x}_1 - \vec{x}_2)} = \sqrt{(\vec{x}_1 - \vec{x}_2)^T W W^T (\vec{x}_1 - \vec{x}_2)}$

$(\vec{x}_1 - \vec{x}_2)^T(\vec{x}_1 - \vec{x}_2) = (\vec{x}_1 - \vec{x}_2)^T W W^T (\vec{x}_1 - \vec{x}_2)$

$$(\vec{x}_1-\vec{x}_2)^T W W^T (\vec{x}_1-\vec{x}_2) - (\vec{x}_1-\vec{x}_2)^T(\vec{x}_1-\vec{x}_2) = 0$$

$$(\vec{x}_1-\vec{x}_2)^T (W W^T - I)(\vec{x}_1-\vec{x}_2) = 0$$

For this equation to hold for all $\vec{x}_1$ $\vec{x}_2$

$$W W^T - I = 0$$

$$W W^T = I$$

So W must be orthogonal matrix

Q(b) $d_m = d_m \rightarrow$ condition?

Solution ① Mahalanobis distance before

$$d_m(\vec{x}_1, \vec{x}_2) = (\vec{x}_1-\vec{x}_2)^T \Sigma_x^{-1} (\vec{x}_1-\vec{x}_2)$$

② mean of $\vec{y}$

$$\vec{\mu}_y = E(\vec{y}) = E(W^T\vec{x}) = \underset{\substack{\downarrow \\ \text{constant matrix}}}{W^T} E(\vec{x}) = W^T\vec{\mu}_x$$

③ centered $\vec{y}$

$$\vec{y} - \vec{\mu}_y = W^T\vec{x} - W^T\vec{\mu}_x = W^T(\vec{x} - \vec{\mu}_x)$$

④ Covariance matrix Σ_y

$$\Sigma_y = E[(\vec{y}-\vec{\mu_y})(\vec{y}-\vec{\mu_y})^T]$$

$$= E[W^T(\vec{x}-\vec{\mu_x})(\vec{x}-\vec{\mu_x})^T W]$$

$$= W^T E[(\vec{x}-\vec{\mu_x})(\vec{x}-\vec{\mu_x})^T] W$$

$$= W^T \Sigma_x W$$

⑤ $dm(\vec{y_1}, \vec{y_2})$

$$dm(\vec{y_1}, \vec{y_2}) = (\vec{y_1}-\vec{y_2})^T \Sigma_y^{-1} (\vec{y_1}-\vec{y_2})$$

$$= [W^T(\vec{x_1}-\vec{x_2})]^T [W^T \Sigma_x W]^{-1} [W^T(\vec{x_1}-\vec{x_2})]$$

$$= (\vec{x_1}-\vec{x_2})^T W W^{-1} \Sigma_x^{-1} (W^T)^{-1} W^T (\vec{x_1}-\vec{x_2})$$

⑥ equation

$$(\vec{x_1}-\vec{x_2})^T W W^{-1} \Sigma_x^{-1} (W^T)^{-1} W^T (\vec{x_1}-\vec{x_2}) = (\vec{x_1}-\vec{x_2})^T \Sigma_x^{-1} (\vec{x_1}-\vec{x_2})$$

$$W W^{-1} \Sigma_x^{-1} (W^T)^{-1} W^T = \Sigma_x^{-1}$$

⑦ condition

W^{-1} exist → W is a nonsinglar matrix ← include

$W W^{-1} = I$ → W is a invertible matrix

$(W^T)^{-1} W^T = I$ ↗

⑧ So W is a invertible matrix