23-S2-Q3

Q: (a) covariance matrix?

Solution ① mean of X

$$\bar{X} = \frac{1}{5}\left(\begin{bmatrix}5\\1\end{bmatrix}+\begin{bmatrix}4\\2\end{bmatrix}+\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}2\\4\end{bmatrix}+\begin{bmatrix}1\\5\end{bmatrix}\right)$$

$$= \frac{1}{5}\begin{bmatrix}15\\15\end{bmatrix}$$

$$= \begin{bmatrix}3\\3\end{bmatrix}$$

② normalization

$$\hat{X}_1 = \begin{bmatrix}5\\1\end{bmatrix} - \begin{bmatrix}3\\3\end{bmatrix} = \begin{bmatrix}2\\-2\end{bmatrix} \qquad \hat{X}_2 = \begin{bmatrix}1\\-1\end{bmatrix} \qquad \hat{X}_3 = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\hat{X}_4 = \begin{bmatrix}-1\\1\end{bmatrix} \qquad \hat{X}_5 = \begin{bmatrix}-2\\2\end{bmatrix}$$

③ covariance

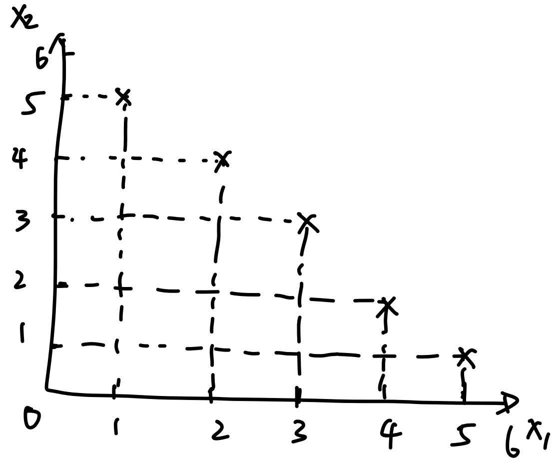
$$\Sigma = \frac{1}{5}\sum_{i=1}^{5} \hat{X}_i \hat{X}_i^T$$

$$= \frac{1}{5}\left(\begin{bmatrix}2\\-2\end{bmatrix}\begin{bmatrix}2 & -2\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}1 & -1\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} + \begin{bmatrix}-1\\1\end{bmatrix}\begin{bmatrix}-1 & 1\end{bmatrix} + \begin{bmatrix}-2\\2\end{bmatrix}\begin{bmatrix}-2 & 2\end{bmatrix}\right)$$

$$= \frac{1}{5}\left(\begin{bmatrix}4 & -4\\-4 & 4\end{bmatrix} + \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} + \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} + \begin{bmatrix}4 & -4\\-4 & 4\end{bmatrix}\right)$$

$$= \frac{1}{5}\begin{bmatrix}10 & -10\\-10 & 10\end{bmatrix} = \begin{bmatrix}2 & -2\\-2 & 2\end{bmatrix}$$

(b)



(c) Q eigenvalues eigenvectors ?

Solution

① $\Sigma = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

$$|\lambda I - \Sigma| = \begin{vmatrix} \lambda-2 & 2 \\ 2 & \lambda-2 \end{vmatrix} = (\lambda-2)^2 - 4 = 0$$

$\lambda - 2 = \pm 2 \qquad \lambda = 2 \pm 2 \qquad \lambda_1 = 4 \quad \lambda_2 = 0$

② when $\lambda_1 = 4$

$$(4I - \Sigma)\phi = 0$$

$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = 0 \quad \Rightarrow \phi_1 + \phi_2 = 0 \qquad$ so $\phi_1 = \begin{bmatrix} k \\ -k \end{bmatrix}, k \in \infty$

whe $\lambda_2 = 0$

$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = 0 \quad \Rightarrow \begin{cases} -2\phi_1 + 2\phi_2 = 0 & \Rightarrow \phi_1 = \phi_2 \\ 2\phi_1 - 2\phi_2 = 0 \end{cases}$

So $\phi_2 = \begin{bmatrix} k \\ k \end{bmatrix}, k \in \infty$

cd) ① eigenvalues and eigenvectors of the covariance matrix reveal the principal variances and directions of the data

② $\lambda = 4$ indicates significant variance along its eigon vector $\begin{bmatrix} -k \\ -k \end{bmatrix}$

③ $\lambda = 0$ indicates zero variances, all data point perfectly aligned without any spread in that direction $\begin{bmatrix} k \\ k \end{bmatrix}$