The Game of SET and Four Dimensional Toroidal Tic-Tac-Toe

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1 Introduction

The topic of my research proposal is the topic of four dimensional toroidal tic-tac-toe and its relation to the SET card game. The focus of this topic is to understand the SET card game, as well as explore mathematical concepts in four dimensionality and geometry to explain the game of SET.

2 The game of SET

The game of SET is a card game primarily consisting of finding a a group of 3 cards that qualify as a SET. As simple as it sounds, it can be quite complicated to understand at first, but hopefully with some examples, it will be much easier to understand.

Each card in the game of SET consists of four properties: color, shape, pattern, and number. Each one of properties has one of three different variations.

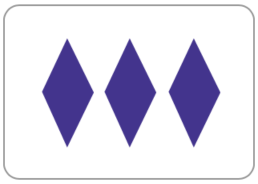
(a) *Color*: Each card is red, purple or green.

(b) *Shape*: Each card has the shape of diamond, oval (or pill), or squiggle.

(c) *Pattern*: Each card has a pattern of solid, stripped or open

(d) *Number*: Each card has one, two or three of the shapes.

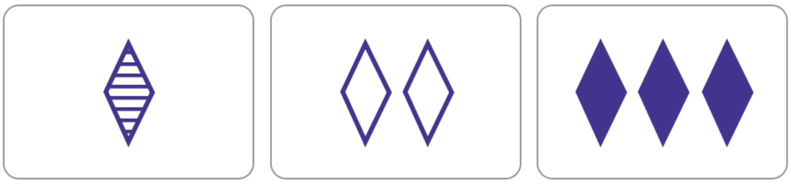
Here is an example of a single SET card.



The SET card above has three (*number*), solid (*pattern*), purple (*color*), diamonds (*shape*). In total there are $3 \times 3 \times 3 \times 3 = 81$ cards in the game of SET. A multiplication of three for each variation of card properties.

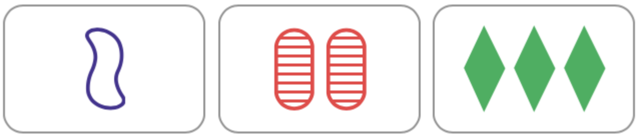
The game of SET is about identifying a group of three cards that make up a SET. So what is a SET? A SET is a group of three cards such that each property of the three cards must all be the same or must all be unique.

Here is an example of a SET. [**?**]

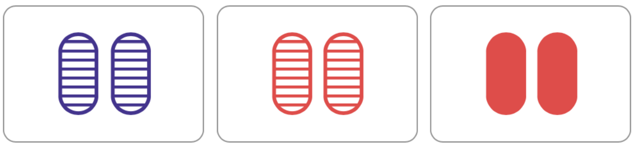


In the above example, the three cards are a SET. All cards have *the same* property of color and shape. All cards also have *different* unique numbers and patterns.

Here are two more examples of three SET cards. [**?**]



Example A [**?**]



Example B [**?**]

Example A is a SET. All cards have *different* shapes, colors, numbers, and patterns. Example B is **not** a set because though all the cards have the *same* shape, not all their colors are the *same* or uniquely *different*. Neither are all their patterns the *same* or *different*. It is highly encouraged to visit the link below and experiment with a game of SET to better understand more examples of the cards needed to make a set.

https://www.nytimes.com/puzzles/set

The game of SET starts with laying down 12 cards face up, and finding three cards that make a SET. However, it is not always guaranteed that it will be the case that a SET can be found. Here is an example:

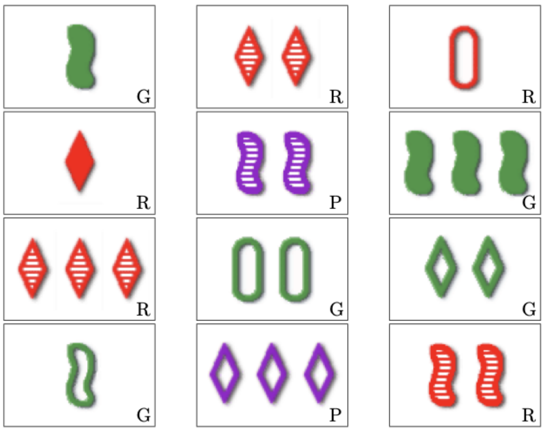


Figure 1: Example of 12 cards without a valid SET. University of Michigan, Chan[**?**]

The example of cards above is a set of cards that do not contain a SET. When this happens, there can be three additional cards placed face up, and there must be another attempt to find a set.

There are many questions about the game of SET that fascinate mathematicians. Questions such as what is the probability of having a SET if only three cards are drawn? Combinatorial questions such as how many SET's exists in the game of SET? The main question to investigate in this paper, is the question: How many cards are required in the game of SET to guarantee a SET?

To answer this question, one can take a combinatorial approach by counting how many possible SET would be possible with x amount of cards, and analyzing the possibilities. However, we can also use a four-dimensional toroidal tic-tac-toe game to answer this question.

3 Four-Dimensional Toroidal Tic-Tac-Toe

Tic-tac-toe is a simple game with a simple concept. Upon an $n \times n$ grid, if there exist n amount of marks in a row, column, or a diagonal orientation, then it is considered a winning state.

Figure 2: Examples of tic-tac-toe winning states

x	.	.
x	.	.
x	.	.

x	x	x
.	.	.
.	.	.

x	.	.
.	x	.
.	.	x

The Four-dimensional toroidal aspect of tic-tac-toe is built upon the simple foundation of tic-tac-toe with two other concepts: Four-Dimensionality and the geometry of a torus.

3.1 Geometry of a Torus in Tic-Tac-Toe

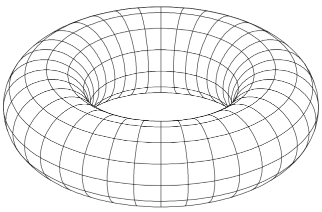


Figure 3: A torus

Geometrically, the shape of a torus can be made by taking a plane, and rolling it to create a cylinder, then taking the top and bottom ends of the cylinder and connecting them to create a torus. (This process is shown in Figure 4.)

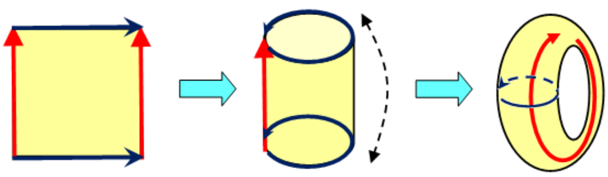


Figure 4: Plane to torus [?]

Tic-tac toe upon a torus can be represented by drawing an $n \times n$ tic-tac-toe board upon a plane and creating a torus as seen in the Figure 5. Therefore tic-tac-toe upon a torus can be represented by unrolling the torus into a plane. The torus reverts back to a plane with different additional winning states of tic-tac-toe. For example, upon a 3×3 tic-tac-toe board the figures below are winning states of a toroidal tic-tac-toe game.

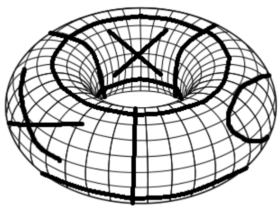


Figure 5: Tic-tac-toe upon a torus [?]

x	.	.
.	.	x
.	x	.

.	x	.
x	.	.
.	.	x

Two additional winning states in toroidal tic-tac-toe. [?]

The figures above are considered a winning state of toroidal tic-tac-toe, in addition to the standard winning states of normal tic-tac-toe. To understand how these two figures are winning states, visualize playing tic-tac-toe upon a cylinder. The left and right edges of the tic-tac-toe board are connected together. For example on the left figure shows the middle-bottom and the rightmost-middle spaces marked. If the board were to be extended further to the right and an additional marking on the upper-right grid existed, that would be considered a winning state in tic-tac-toe. The marking on the upper-left represents the extended upper-right marking in our example.

3.2 Four-Dimensional Toroidal Tic-Tac-Toe

Four-Dimensional tic-tic-toe consists of three instances consecutive instances of three-dimensional tic-tac-toe games, that interact with each other. A single three-dimensional tic-tac-toe board can be represented by three layers of single tic-tac-toe boards, with each layer representing a level of a $n \times n \times n$ cube. All three levels of a three-dimensional tic-tac-toe board interact with each other, by still following the rules of a matching of tic-tac-toe winning. A winning state upon a $n \times n \times n$ board would require n length row upon either a single layer of the $n \times n \times n$ cube or vertically across all layers of the $n \times n \times n$ cube. Figure 6 below shows an example of the winning states. Each pair of colored x when combined with the upper-left x is a winning state.

x	x	x
.	.	.
.	.	.

x	.	.
.	x	.
.	.	.

x	.	.
.	.	.
.	.	x

Figure 6: Three-dimensional tic-tac-toe winning states

As previously stated, four-dimensional tic-tic-toe consists of three instances consecutive instances of three-dimensional tic-tac-toe games, that span laterally. Therefore a four-dimensional tic-tac-toe board would have winning states that span across three three-dimensional tic-tac-toe boards. Figure 7 shows example winning states of four-dimensional tic-tac-toe. Each pair of colored x when combined with the upper-left x is a winning state. There is also an additional matching made of the vertical orange, green and purple coloring.

Four-dimensional toroidal tic-tac-toe follows the properties of four-dimensional tic-tac-toe with the additional property that each grid of the tic-tac-toe board follow the winning states of a torus. Each layer of each three-dimensional $n \times n \times n$ board is a torus. Perceptually, each it is simple to visualize a four-dimensional tic-tac-toe board as three $n \times n \times n$ boards adjacent to each other in a three-dimensional space. For a toroidal visualization of our boards, it is not possible to visualize an $n \times n \times n$ torus in three-dimensional space. Therefore, the best we can do is to create a four-dimensional tic-tac-toe board with the properties of a torus for analysis. The figure below shows examples of winnings states of four-dimensional toroidal tic-tac-toe board

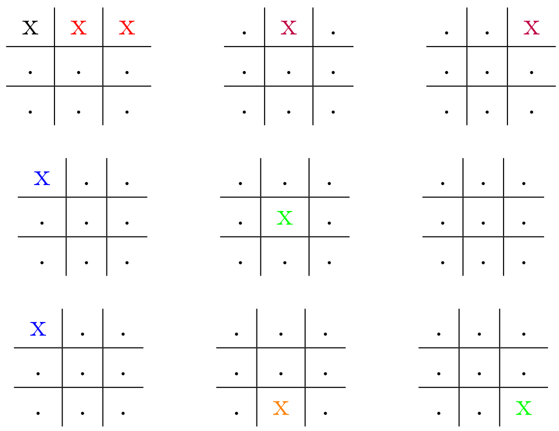


Figure 7: Four-dimensional tic-tac-toe winning states

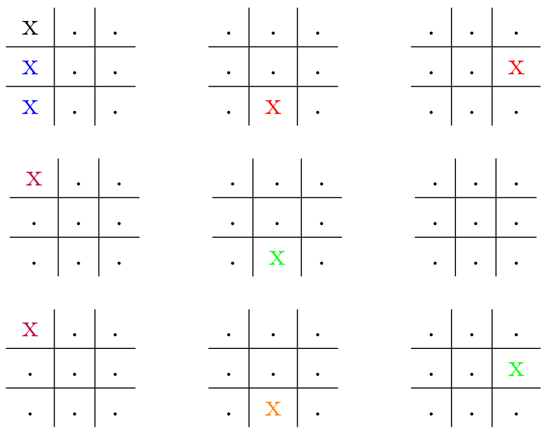


Figure 8: Four-dimensional toroidal tic-tac-toe winning states.

4 Tic-Tac-Toe and SET

The two concepts of four-dimensional toroidal tic-tac-toe and the game of SET are related in fundamental way because they are conceptually the same game. Upon a four-dimensional tic-tac-toe board, we can represent each row as the SET *pattern* property, each column as the SET *number* property, each vertical three-dimensional board as the SET *shape* property, and each horizontal four-dimensional layer as the SET *color* property. This gives us a four-dimensional tic-tac-toe board in which each of its spaces represents a card in the game of SET. Each winning state upon this board is a SET.

This graphical representation of the game of SET makes it easier visually

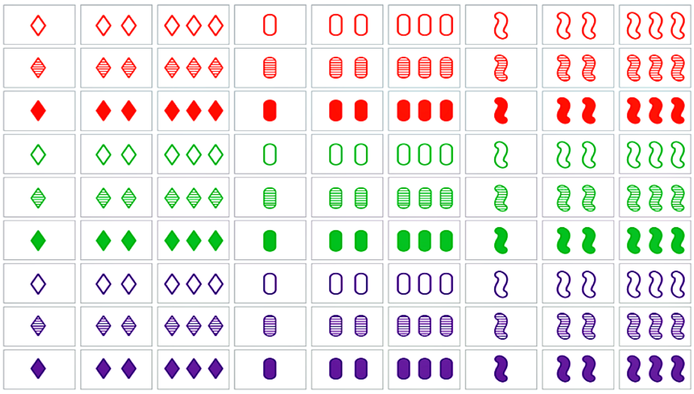


Figure 9: All cards of SET [?]

identify what cards constitute a SET, and to analyze combinatorial questions of SET visually. The relation between SET and four-dimensional tic-tac-toe can be used to analyze a question proposed earlier in this paper: How many cards are required in the game of SET to guarantee a SET? The question can be rephrased into how many markers can be placed upon the four-dimensional toroidal tic-tac-toe board without creating any winning states?

Figure 10 shows that it is at most possible to make 20 markings upon a four-dimensional toroidal tic-tac-toe board without creating any winning states. Therefore, in order to guarantee that a SET is found in a game of SET, 21 cards are needed.

x	.	x
.	.	.
x	.	x

.	x	.
x	.	x
.	x	.

.	.	.
.	x	.
.	.	.

.	x	.
x	.	x
.	x	.

x	.	x
.	.	.
x	.	x

.	.	.
.	x	.
.	.	.

.	.	.
.	x	.
.	.	.

.	.	.
.	x	.
.	.	.

.	.	.
.	.	.
.	.	.

Figure 10: The most amount of possible markings without a winning state. University of Michigan, Chan [?]

5 Conclusion

SET is quite a fun and brain teasing game that has a lot of fun mathematical questions to answer with many different mathematical concepts to help answer those questions. One of those mathematical concepts is four-dimensional toroidal tic-tac-toe. If you know how to play a game of four-dimensional toroidal tic-tac-toe, you are essentially playing the same game as SET and vice versa.