An O(N) Sorting Algorithm: Machine Learning Sorting

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We propose an O(N) sorting algorithm based on Machine Learning method, which shows a huge potential for sorting big data. This sorting algorithm can be applied to parallel sorting and is suitable for GPU or TPU acceleration. Furthermore, we apply this algorithm to sparse hash table.

INTRODUCTION

Sorting, as a fundamental operation on data, has attracted intensively interest from the beginning of computing [1]. Lots of excellent algorithms have been designed, however, it's been proven that sorting algorithms based on comparison have a fundamental requirement of $\Omega(N\log N)$ comparisons, which means $O(N\log N)$ time complexity. Recent years, with the emergence of big data (even terabytes of data), efficiency becomes more important for data processing, and researchers have put many efforts to make the sorting algorithms more efficient.

Most of the state-of-art sorting algorithms employ parallel computing to handle big datasets and have accomplished outstanding achievements [2–6]. For example [7], in 2015, FuxiSort [8], developed by Alibaba Group, is a distributed sort implementation on top of Apsara. FuxiSort is able to complete the 100TB Daytona GraySort benchmark in 377 seconds on random non-skewed dataset and 510 seconds on skewed dataset, and Indy GraySort benchmark in 329 seconds. Then, in 2016, Tencent Sort [9] has achieved a speed of 60.7 TB/min in sorting 100 TB data for the Indy GraySort, using a cluster of 512 OpenPOWER servers optimized for hyperscale data centers. However, these algorithms are still limited by the lower bound complexity and time-consuming networking [10].

On the other hand, machine learning is a field that has been developing rapidly these years, and it has been applied widely across different areas [11–13]. In 2012, the emergence of ImageNet [14] classification with deep convolutional neural networks was a great breakthrough that almost halve the error rate for object recognition, and precipitated the rapid adoption of deep learning by the computer vision community. In March 2016, AlphaGo [15] utilized neural network to beat the human world champion Lee Sedol in the game of Go, which was a grand challenge of artificial intelligence (AI). The huge success of machine learning shows that computer AI could go beyond human knowledge in complicated tasks, even starting from scratch. After that, machine learning

algorithms had been widely applied to various areas such as human vision, natural language understanding, medical image processing and so on, and had achieved great accomplishments.

Neural network models [16, 17], which are inspired by the biology of human brains, have input layer, output layer and hidden layers. Hidden layers consist of lots of connecting artificial neurons. These neurons are tuned according to the input and output data, to precisely reflect the relationship. The nature of the neural network is a mapping from the input data to output data. Once the training phase is done, we can apply this neuron network to make a prediction of unknown data. This is the so-called inference phase. The precision and efficiency of inference phase inspires us to apply machine learning skills to sorting. In some way, we can treat sorting as a mapping from the data to its position at the data set.

In this paper, we propose a sorting algorithm with a complexity of $O(N \cdot M)$ using machine learning, which works especially well on big data. Here M is a small constant indicating the number of neurons in the hiddenlayers of neural network. We first use a small training dataset to approximate the distribution of the large dataset through a 3-layer neural network [18], then use this neural network to estimate the position of every unknown data's position in the future sorted sequence. In the inference phase here we do not need comparison operations between two data, since we already have the approximate distribution. After the inference phase is done, we get an almost sorted sequence. Hence, we only need to apply an O(N) time operation to get the fully sorted data sequence. Furthermore, this algorithm can be applied to sparse hash table.

ALGORITHM

Now suppose we have a data sequence S of real numbers, with a size of N, upper boundary x_{max} and lower boundary x_{min} . For an effective sorting algorithm, we need to make sure the new sequence S' is sorted, by exchanging the positions of x_i 's. Suppose a real number x_i ranks r_i in S', then we can treat the sorting problem as a bijective function $G(x_i) = r_i$. If we can get this function in advance, then the complexity of sorting would be

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O(N). In fact, if all the numbers in S come from a distribution f(x), when N is large enough, the position r_i of x_i in S' approximately equals to

$$G(x_i) = r_i \approx NF(x_i) = N \int_{x_{min}}^{x_i} f(x)dx$$
 (1)

Where F is the cumulative probability distribution of our data. When N tends to infinity, the equality holds.

The problem is that the function G(x) is usually hard to derive, so is the distribution function f(x). However, when we're dealing with big data sequence, N is large enough for the sequence to possess some kind of statistical properties. Hence, if we can derive the distribution function f(x), then we get the chance to degrade the complexity of sorting algorithm to O(N) by using Eq. (1).

In order to do that, we choose N_0 numbers out of sequence S randomly. We call it the training sequence $A = \{a_1, a_2, \cdots, a_{N_0}\}$. The distribution of this training sequence A should share a similar distribution to that of the original sequence S. We sort A using the conventional sorting algorithm to get A', consequently $a'_i > a'_{i+1}$. If we set a'_i as the x-coordinate, i/N_0 as the y-coordinate (Here a'_i ranks i), the relationship between i/N_0 and a'_i is then a cumulative distribution function $F(x; N_0)$ of sequence A. In fact, $F(x; N_0)$ is an approximate version of distribution function F(x; N) of the original sequence. Up to here, we have transformed the sorting problem into a function fitting problem.

As an example, suppose the fitted distribution function is piecewise linear, which means the data follows a uniform distribution on the interval (a'_i, a'_{i+1}) , then for any $x_i \in (a'_i, a'_{i+1})$ that comes from S,

$$r_j = i \cdot \frac{N}{N_0} + \frac{a_i' - x_j}{a_i' - a_{i+1}'} \cdot \frac{N}{N_0}$$
 (2)

However, through experiments we found out that, this approach of piecewise linear fitting is too rough. When we set $N=10^4$, $N_0=10^3$ and the data follows a normal distribution $x_i \sim \mathcal{N}(0,1)$, it turns out that the estimate ranking might have a deviation larger than 100 compared to the real ranking position. Under the assumption of uniform distribution or some other special cases, the deviation might be smaller, but it's obvious that piecewise linear fitting is not a fair fitting approach. We realize that the key to this problem is a good fitting approach.

A rapid developing approach during recent years is machine learning. Via artificial neural networks, we can make the fitting function much more precise. In this paper, we apply General Vector Machine (GVM), which is a 3-layer neural network and has only 1 hidden layer. The structure of GVM is schematically shown in the Fig. 1. The learning process of GVM is based on Monte-Carlo algorithm rather than back propagation. We find the GVM is particularly suitable for function fitting.

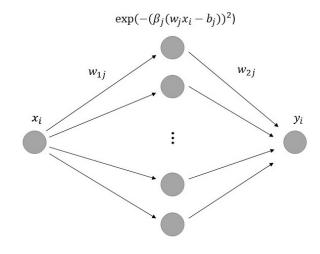


FIG. 1. Schematic representation of the GVM. We fix 100 hidden-layer neurons for each experiment.

In this neural network, the input layer has only one neuron and input is x_i for fitting problem, so is the output layer and output y_i . We fix the number of neurons for the hidden layer to be M = 100. In fact, in some way, the more neurons we have in the hidden layer, the more precise our fitting is, but it comes along with the problem of overfitting, as well as degrading the computational efficiency. After training (a'_i, i) in the training sequence, we could use this neural network as a fitting function. After inputting data x_i , the neural network outputs a real number t_j , which is the estimate ranking of x_j in the data sequence A. Then, $r_j = round(y_j \cdot \frac{N}{N_0})$ is the estimate ranking of x_j in sequence S, given by the machine learning model. We round $y_j \cdot \frac{N}{N_0}$ so that r_j will be an integer in [0, N]. Actually, sometimes r_i could go out of the range [0, N] because of the small deviation, but that doesn't pose a big problem because the value of such data is very small.

The process of estimating ranking of N numbers takes only $O(N \cdot M)$ time. We then put x_i into a quasi-onedimensional array ps (presort) which takes O(N) space. For each x_i and its corresponding estimate ranking r_i , we put x_i into $ps(r_i)$ of the array, but $ps(r_i)$ may have more than one data, because it's possible that we have the same estimate ranking for different data which have a close value. After all the data is stored in ps array, we collapse the ps array into a one dimensional array final. As long as our machine learning model is precise enough, the *final* array here should be an almost sorted sequence, which means that the current ranking of each data is at most L units away from its real ranking. It would be easy to derive the fully sorted sequence now. We set a "comb" of length L, for each interval $[final_i, final_{i+L-1}]$ of length L, find the largest value of that interval and exchange it with $final_i$. In the experiment we found out that when the machine learning works well enough, L is

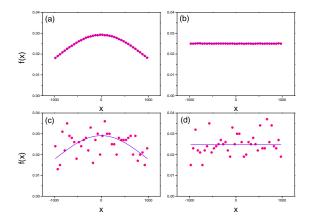


FIG. 2. Distribution of the data. The truncated normal distribution (a) and the uniform distribution (b) for 10^7 data. The training sequence distribution for truncated normal distribution (c) and the uniform distribution (d) for 10^3 data. The blue dash line is the analytical distribution and pink dot is the experiment data.

usually very small. For example, when we take the data distribution as a truncated normal distribution, and the data size as $N=10^7$, with a proper machine learning process, L would be less than 15. All the above array storage and combing operation takes O(N) time. Even in the worst case, the operation only takes $N \cdot L$ time.

The M is independent to N and there is no lower boundary for M analytically such as $\log(N)$. For example, if the distribution of data sequence is Gaussian, the neural network with 1 neuron can achieve the goal in principle. In practical, we may apply more than 1 neuron to fit the Gaussian distribution, the number of neurons depends on the machine learning method. Also, L is independent of the $\log(N)$.

Next we'd like to discuss the potential of parallel computing of this algorithm. Since in the prediction process, it needs no comparison and exchange operation, and the estimation of ranking for each data is independent of each other, it would be efficient to do parallel computing, and it requires very little networking workload. In addition to efficient parallel computing, since the machine learning requires matrix manipulation, it's also suitable for working on GPUs or TPUs for acceleration [19]. It's obvious that we could also apply parallel computing to array storage operation. Especially for the "combing" process, we could divide the whole array into several chunks, and comb each chunk separately. Finally in the merging phase, we only have to apply another combing operation in the boundary of each chunk, which would take constant time.

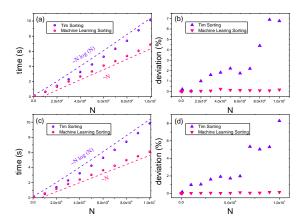


FIG. 3. The relation between time complexity and data number for truncated normal distribution (a) and the time complexity deviation from average (b). The relation between time complexity and number of data for uniform distribution (c) and the time complexity deviation from average (d). For (a) and (c), the purple spheres are the time for the Tim Sorting and pink spheres are the time for the Machine Learning Sorting. For (b) and (d) the purple triangles are for the Tim Sorting while pink triangles are for the Machine Learning Sorting. An ensemble average over 10² realizations has been used for obtaining the results.

EXPERIMENTS

As shown in Fig 2, we select two kinds of distributions to run our experiments: uniform distribution and truncated normal distribution. The size of data ranges from $N=10^3$ to $N=10^7$. All of them are double type data on the interval [-1000, 1000]. We utilize the sorted function in python 3.6 as comparison. Python 3.6's sorted function utilizes Tim sorting, which is a mix sorting of merging sorting and insert sorting, and has a lower complexity boundary of O(N), upper complexity boundary $O(N \log N)$ and average complexity $O(N \log N)$. In fact, if we use a better optimized library, we might have a better result than sorted function in python 3.6, but meanwhile, the machine learning algorithm we present here also has many details that can be optimized. However, as a conceptual experiment, we only select the widely used sorted function in python. All the experiments in this paper are based on single CPU, no parallel computing is applied on neither Tim sorting nor machine learning sorting.

For a machine learning model, no matter how big the test data is, we set the size of training set as 10^3 . For example, Fig. 2b, shows one of the training sets. Based on GVM, if we fix the training time at 10^5 steps, the corresponding training time is 0.904s. In fact, this time is not enough for training. Longer learning time could lead to a more precise prediction. But on the one hand, there is no solid theoretical basis for the training time.

and overlong training time would lead to overfitting. On the other hand, we consider that usually we don't want to spend too much time on preprocessing in real applications.

Figure 3 compares the running time of Tim sorting and machine learning sorting. Here we subtract 0.9s from the machine learning sorting time as it's the time needed for training. As shown in Fig. 3, Tim sorting time grows in $O(N \log N)$ under both distributions. However, machine learning sorting time grows in linear rigidly, which implies the O(N) complexity. Even though Tim sorting is faster when the data size is relatively small, once $N > 10^6$, it's slower than our machine learning sorting. When $N > 5 \cdot 10^6$, machine learning sorting is much better than Tim sorting, even taking into consideration of the training time cost.

Now we turn to the deviation of sorting time. Tim sorting has much larger deviation of time than machine learning sorting. This is because that the complexity of Tim sorting highly depends on the order of data. Different from that, machine learning sorting treats equally of all data, no matter in what order the data holds. Once the learning model is built, the sorting time is determined.

CONCLUSION

We propose a sorting algorithm with complexity of O(N). This algorithm is built upon valid learning, and it works well for big data sets. Because the big data sets usually have fair statistical properties, and its distribution is usually easy to derive and also has some kind of continuity. The distributions we select in this paper is relative smooth. For some distributions with less smoothness, it might be difficult to learn the distribution, and if the training fails, the estimate ranking might be far away from the length of comb in the sorting phase, therefore the algorithm might fail. However, we have some solutions for this situation. First, tune the initial parameters of machine learning model, and pick the best one to proceed to the sorting phase. Second, which is more doable, divide the unsmooth area into relatively smooth area pieces and then apply our algorithm to each piece.

Of course the learning quality also depends on the machine learning models. Here we apply the GVM algorithm, other machine learning algorithms could also fit a fair distribution. However, the advantage of GVM is that it has only 3 neuron layers, and is more efficient when fitting functions with small hidden-layer neurons. GVM also has some theories in parameters tuning. What's more important, for a machine learning model, is that if it does have a topological structure, which implies that if a < b, then r(a) < r(b), then this kind of learning model could be applied to sorting algorithm naturally.

Compared to the conventional sorting algorithms, machine learning sorting has a better potential in parallel

computing. Though the complexity is $O(N \cdot M)$, both N and M can be divided for parallel computing. Unlike the conventional sorting algorithms which mainly utilize a large number of comparison operations, a large portion of operations in machine learning sorting is multiplication of real numbers, which can be done by matrix operation, and is perfect for GPU or TPU acceleration.

Furthermore, this algorithm can be applied to sparse hash table to reduce space complexity. Sort the data first, and then set the data and its ranking as the training dataset, so that the key code is its ranking. When searching for a new data, we put the data into the GVM and get an estimate ranking. The real ranking of this new data is around the estimate ranking with a deviation less than L units. Hence the time complexity is O(L), and the space complexity of the hash table is O(N).

For a lot of websites, they have a need for highly efficient management of big data sets. The statistical properties of these data sets do not change too much in a short period of time, so we don't have to tune the parameters meanwhile when applying machine learning sorting algorithm. For some long-term running models, if the distribution changes slowly over time, we could also consider a machine learning model that changes over time to fit the changing data.

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