Grades (3=perfect, 0=horrible): Clear:___, Concrete:___, Complete:___, Correct:___, Convincing:___, Concise:___, Clever:___.

1 Problem 1: Four-Arc Caplets

1.1 Input

Four points S, L, E, R on a 2-D plane. S and L define a circle with S being the center and L on its circumference. E and R define another circle with E being the center and R on its circumference.

We assume that these two circles separate with each other or partly overlap. Besides, point L cannot be inside circle E and point R cannot be inside circle S. Note that we refer to a circle by its center. Another restriction is that L and R should be in different sides of line SE.

1.2 Objectives

We want to compute four smoothly connected circular arcs and display the region W bounded by these arcs. We should first find two missing arcs, starting from L and R respectively, that are tangent to both circle S and circle E. To get the boundary of W, we should trim circle S and circle E properly, which give another two arcs.

1.3 Output

The boundary is represented as a sequence of points and will be visualized as a polygon defined by these points. Region W will be filled by one color.

1.4 Validation

The boundary should be made of 4 circular arcs. The two missing arcs do not intersect with each other. Moreover, circle S and circle E should be in region W. If a result satisfies these requirements, we regard it as a good result.

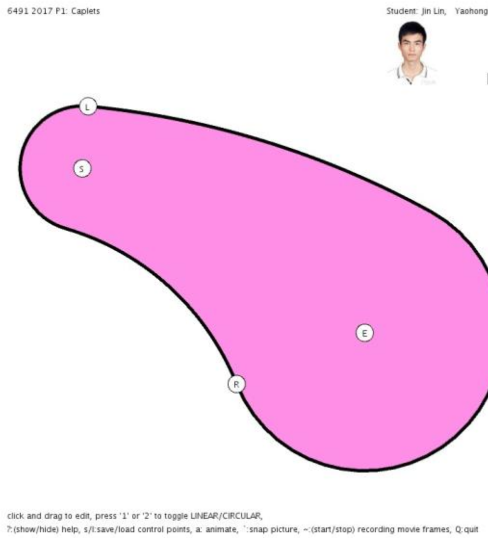
1.5 Outline of your approach

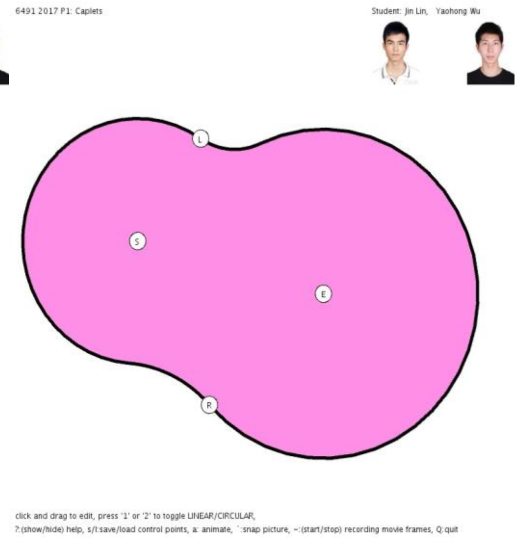
For finding a missing arc, the key is to find a "hat". For example, assume that the missing arc starting from L is tangent to circle E at point L', if we draw a tangent line at L and a tangent line at L', these two lines will intersect at point H in general cases. Our approach first computes this point H and then computes point L'.

To trim original circles, we just need to discard the inner arcs and choose the outer arcs. For example, after finding missing arcs, we now have point L and another point R' on circle S. There are two possible arcs defined by point L and point R'. We need to choose the one that does not intersect with line segment SE.

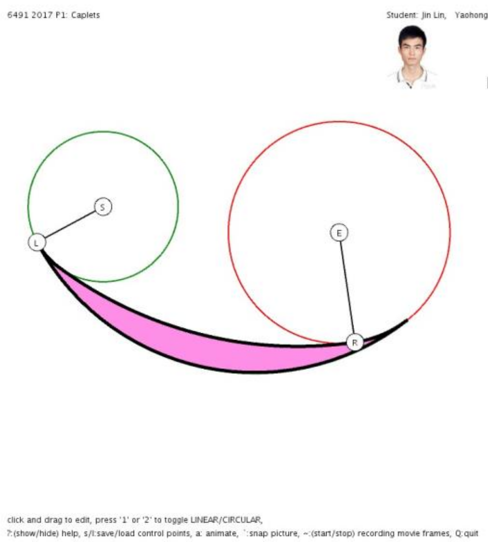
1.6 Results

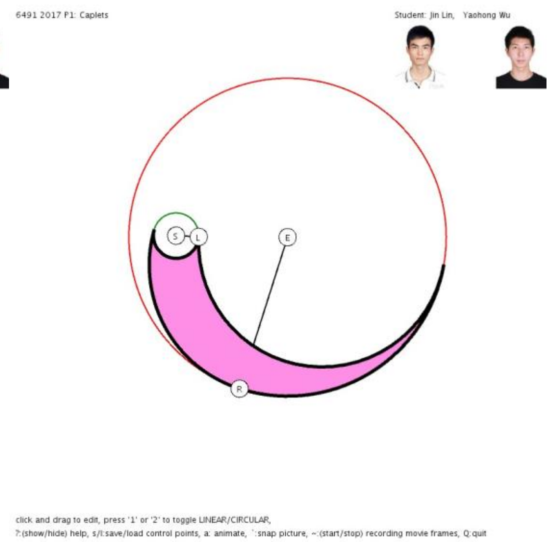
When circle S and circle E separate with each other, or partly overlap with L outside circle E and R outside circle S, our method gives satisfying results given that L and R are on different sides of line SE.





When L and R are on the same side of line SE or they both on line SE, results are not good. This is mainly because of our construction method. Our method assumes that L and R are on different sides of line SE such that we can construct an arc starting at L as an "upper" part of the boundary and another arc starting at R as a "lower" part of the boundary. Results are worse when one circle is inside the other circle.

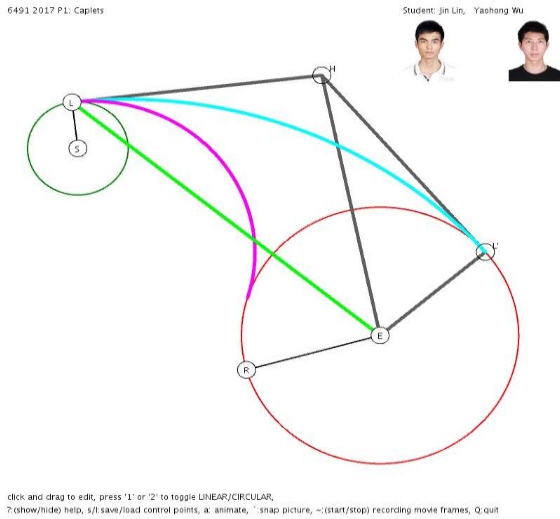


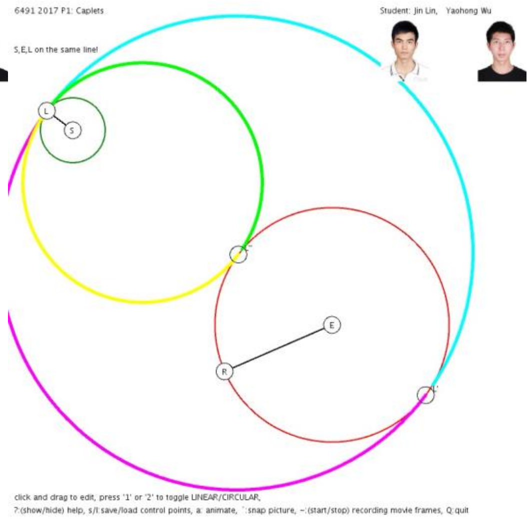


1.7 Details

Step 1. Find missing arcs.

Take L for example. We denote b as the radius of circle E. Find the middle point H of the "hat". Let $H = L + xT$. Here T is the tangent vector at L and x is an unknown. Note that $|HE|^2 = |HL'|^2 + |EL'|^2$. We can solve x as $\frac{b^2-|EL|^2}{2EL\cdot T}$. From H, we can find that there are two possible locations of L'. That is, we can draw two possible arcs from L which are the cyan arc and magenta arc in the following left picture. We can find that $L' = E + b(\frac{EH}{|EH|})^\circ(\pm\cos^{-1}\frac{b}{|EH|})$. We choose the point that on the different side from S w.r.t line LE. Thus, we prefer the cyan one.

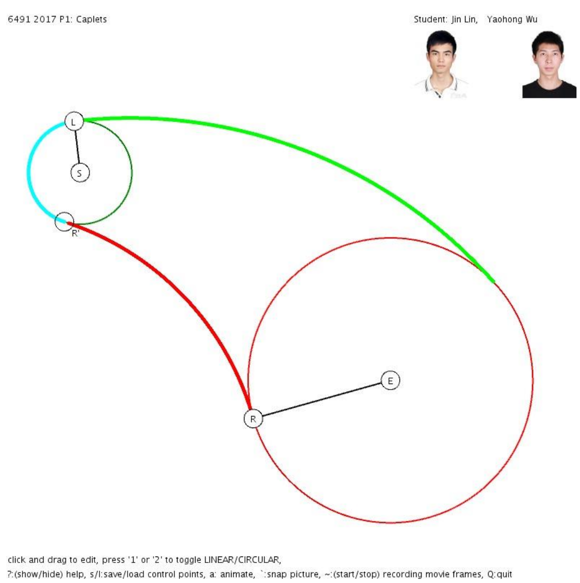




We should point out that there is a corner case in which L, S, E are collinear. In such a case, we cannot compute H which is at infinity. But we can compute L' directly. It is easy to know that L' should also be on line SE, which gives $L' = E \pm b\,\frac{SE}{|SE|}$. We prefer the one on the "right" of E. That is, $L' = E + b\,\frac{SE}{|SE|}$. Also, there are two possible arcs. We prefer the one on the different side from R w.r.t line SE. Thus, we would choose the cyan arc in the above right picture.

Step 2. Trim circle S and circle E.

Take circle S for example. After we compute R' on circle S, we have two possible arcs to choose from. One is cyan and the other is green as in following picture. We prefer the one that does not face point E. Mathematically, we compute $SL + SR'$ and $(SL + SR') \cdot SE$. If it isn't positive, we choose the minor arc defined by L and R'. Otherwise, we choose the other. In the following picture, we choose the cyan arc.



Step 3. Generate a sequence of Points and display the region.

To get points on circular arcs of circle S or E is easy. We just need to compute the angle of our preferred arcs correctly. To get points on two missing arcs, we need to compute the center of the two potential circles. Take the circle related to L and L' as an example. Using properties of similar triangles, we can compute the center $U = H + dW$, where $d = \frac{HL \cdot HL}{HL \cdot W}$ and $W = HL + HL'$. After we collect all the points we compute, we draw them as a polygon, which gives us a boundary of region W. Then we fill the region with a color.

1.8 References

None.

1.9 Further research

We may try to avoid assuming that L and R are on different sides of line SE. According to the approach we described above, computing the missing arc starting from L doesn't depend on the exact location of point R, though we need to know the radius of circle E. When L and R are on the same side of line SE, it is very likely that point R will have an effect when we choose from two possible arcs starting from L.

2 Problem 2: Exact Point-Membership Classifier

2.1 Input

Region W in Problem 1. Assumptions are the same as those in problem 1.

2.2 Objectives

Compute a Point-Membership Classifier that can determine if a point in 2-D plane is in this region.

2.3 Output

A classifier.

2.4 Validation

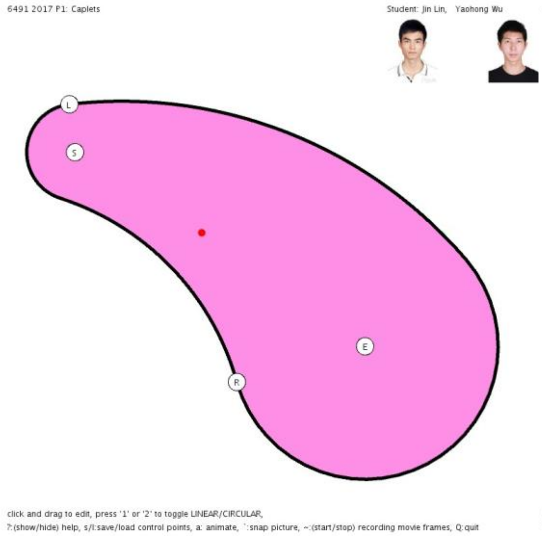
When the point we check belongs to region W, we draw it with color A, otherwise with color B.

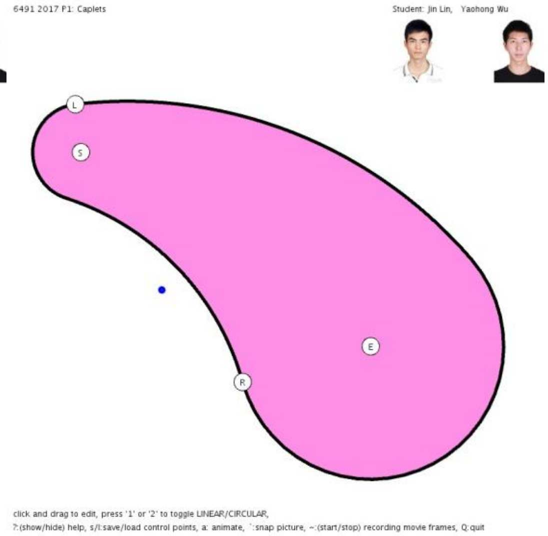
2.5 Outline of your approach

First we check if the point is inside circle S or E. If yes, we output true. Otherwise, we need to do more tests. More specifically, if the point is in region W, it must be between line LR' and line RL'. Furthermore, it is also surrounded by arc LL' and arc RR'.

2.6 Results

Under our assumptions in Problem 1, the classifier we compute can produce accurate results. As in problem 1, results could be very bad when we discard some of the assumptions. Following pictures show the results of our classifier. When the test point is in region W, we draw it with red color otherwise with blue color.





2.7 Details

Step 1. Check whether the test point P is inside circle S or circle E.

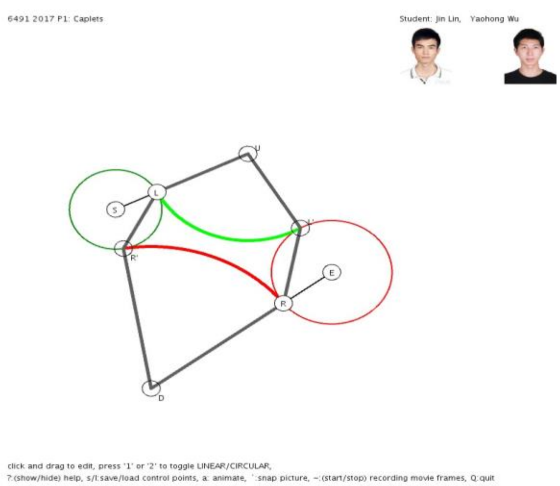
If yes, output true. Otherwise, move to Step 2.

Step 2. Check whether point P is between line LR' and line RL'.

If P is on the same side as E w.r.t. line LR' and on the same side as S w.r.t. line RL', then P is between line LR' and line RL'. To check if P is on the same side as E w.r.t. line LR', we compute $LE \cdot (LR')^{\circ}$ and $LP \cdot (LR')^{\circ}$. If they have the same sign, then P is on the same side as E w.r.t. line LR'. Check the other part is similar. If we find that P is not between line LR' and RL', then output false otherwise we go to Step 3.

Step 3. Check where point P is between arc LL' and arc RR'.

Use U to denote the circle that arc LL' belongs to and D to denote the circle that arc RR' belongs to. We want to know the relationship between region W and circle U. When region W is inside circle U and P is outside circle U, we can output false. The relationship between region W and circle U can be determined by the relationship between point E and circle U. That is, if E is outside circle U, then region W is also outside circle U. Otherwise, region W is inside circle U. Thus, point P should act like E if P is in region W, which means P and E should be outside or inside circle U at the same time. For arc RR' and circle D, we can do similar things. If any requirement doesn't meet, we can output false. Otherwise, we output true.



2.8 References

None.

2.9 Further research

As in problem 1, we would like to try under circumstance that L and R are on the same side of line SE.

3 References & resources

None.