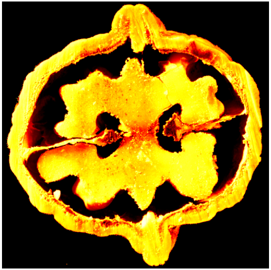
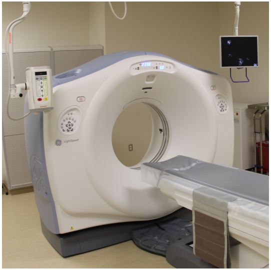
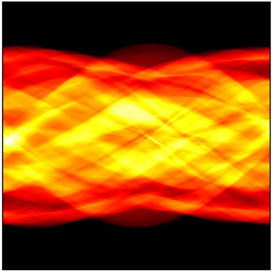
Mathematics of Tomography







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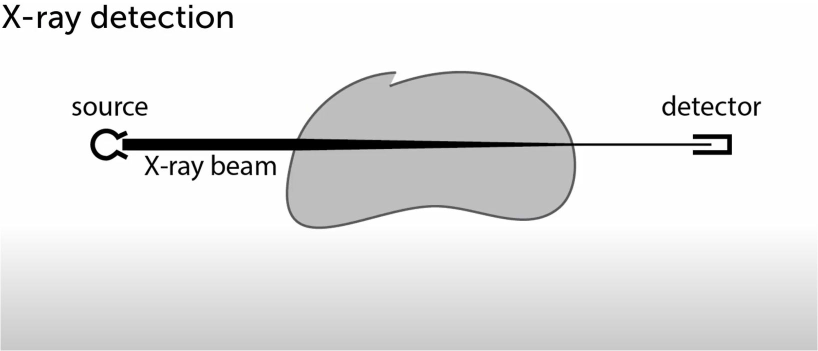
xCTing training

March 29, 2022

Mathematics of CT 1: Beer's Law



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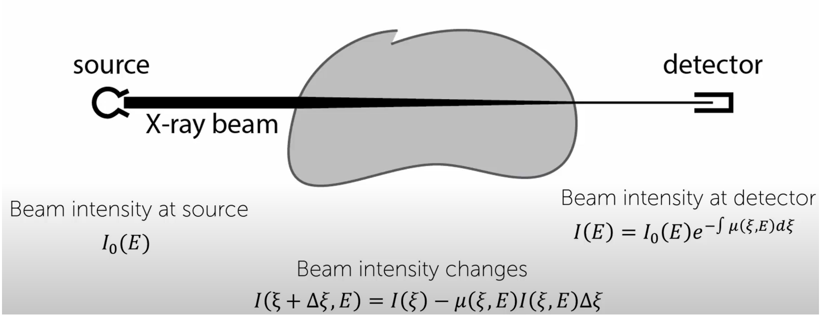
Mathematics of CT 1: Beer's Law

X-ray detection



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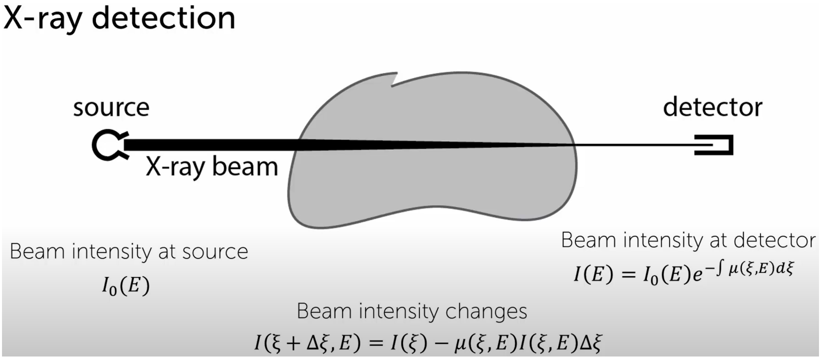
Mathematics of CT 1: Beer's Law



$$\Rightarrow P := -\log\left(\frac{I(E)}{I_0(E)}\right) = \int_{beam} \mu(\xi, E) d\xi$$

Taken from corresponding video by the ASTRA toolbox team YouTube

Radiography

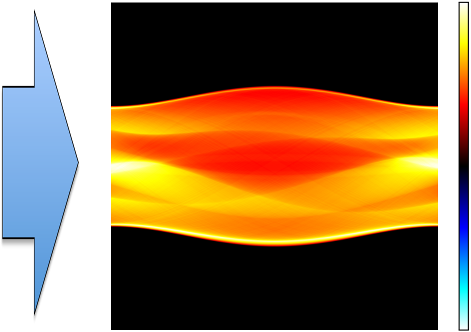
An excellent video by Samuli Siltanen:

source: Wikimedia Commons

From projections to sinograms

Another excellent video by Samuli Siltanen: YouTube

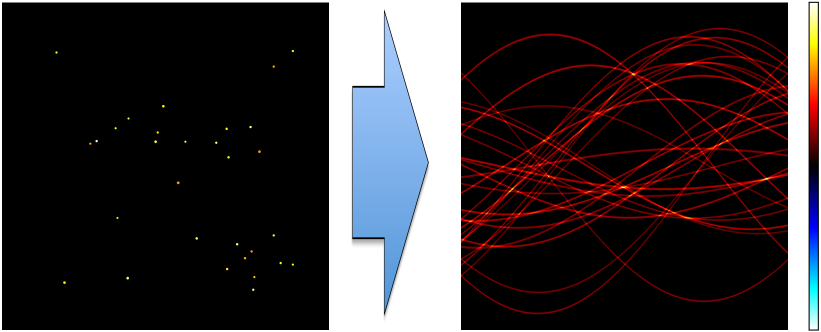
(a) image



(b) sinogram

From projections to sinograms

Another excellent video by Samuli Siltanen:



(a) image (b) sinogram

From projections to sinograms

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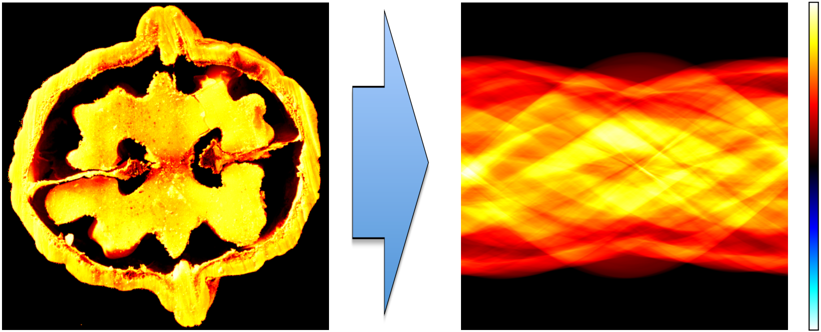
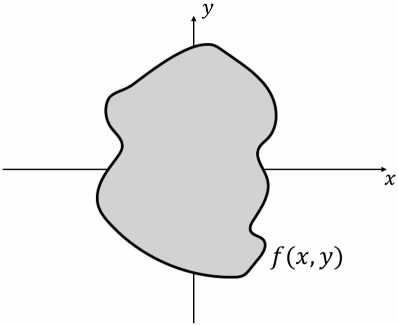


(a) image (b) sinogram

Radon transform



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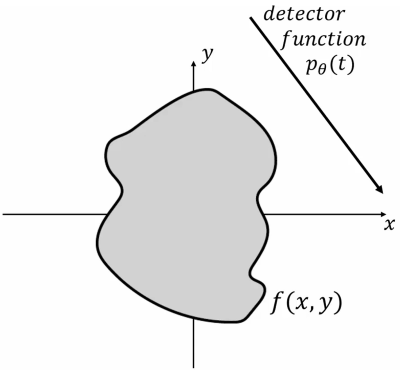
Radon transform



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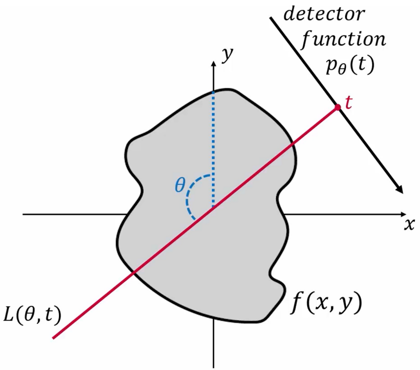
Radon transform



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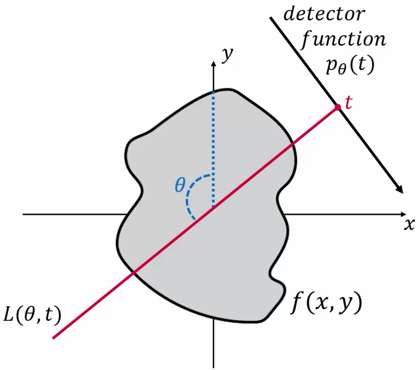
Radon transform

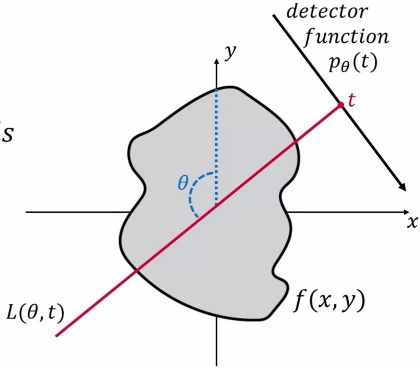


$$L(\theta, t) = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \cos\theta + y \sin\theta = t\}$$

Taken from corresponding video by the ASTRA toolbox team YouTube

Radon transform

$$p_\theta(t) = \int_{L(\theta,t)} f(x,y)ds$$

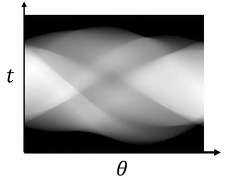


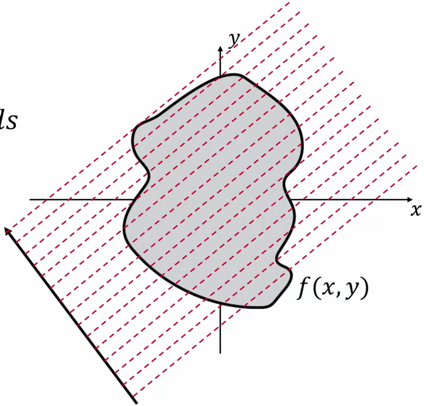
$$L(\theta,t) = \{(x,y) \in \mathbb{R} \times \mathbb{R}: x\cos\theta + y\sin\theta = t\}$$

Taken from corresponding video by the ASTRA toolbox team YouTube

Radon transform

$$\mathcal{R}f(\theta, t) = \int_{L(\theta,t)} f(x,y)\,ds$$





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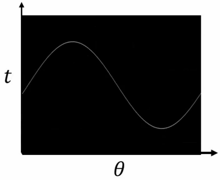
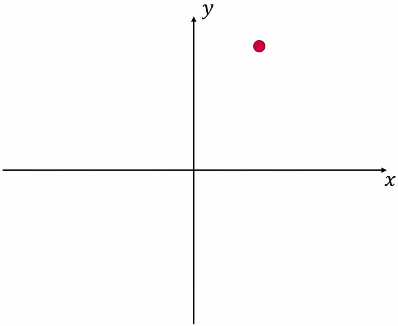
Radon transform

$$\boxed{\mathcal{R}f(\theta,t)} = \int_{L(\theta,t)} f(x,y)\,ds$$

Sinogram





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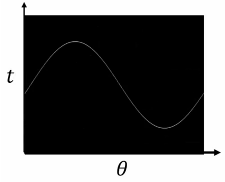
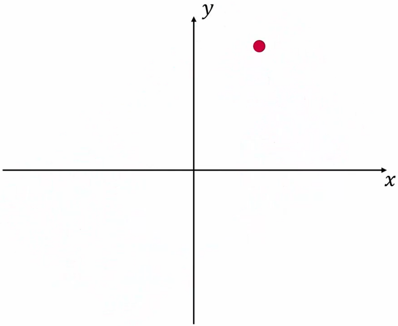
CWI

Radon transform

$$\boxed{\mathcal{R}f(\theta,t)} = \int_{L(\theta,t)} f(x,y)\,ds$$

Sinogram

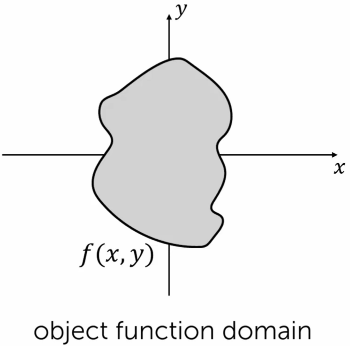




R is a linear operator, but is it invertible?

Taken from corresponding video by the ASTRA toolbox team YouTube

Fourier Slice Theorem



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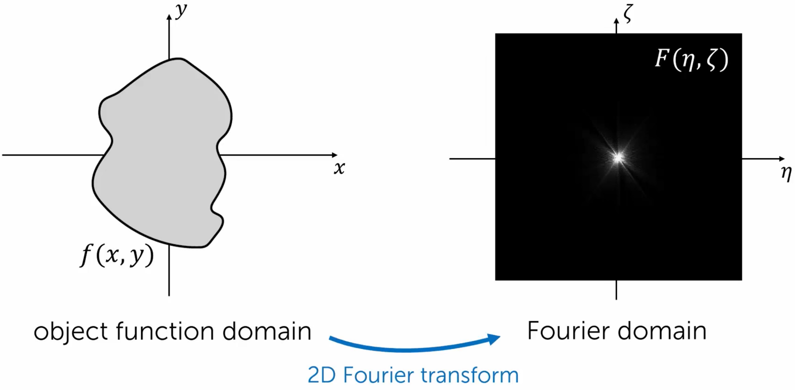
Fourier Slice Theorem



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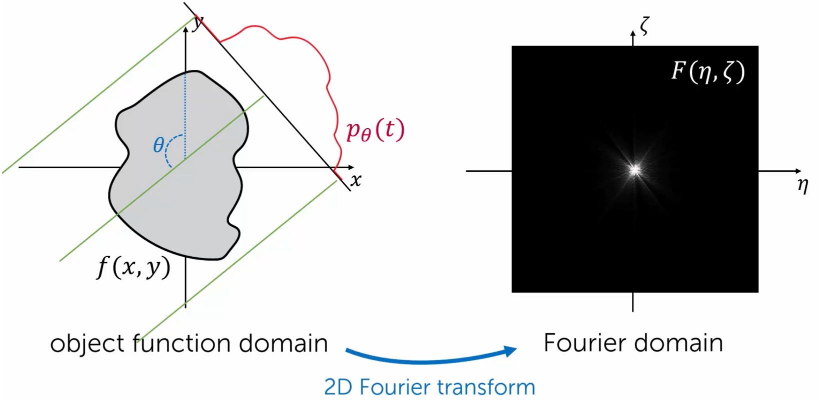
Fourier Slice Theorem



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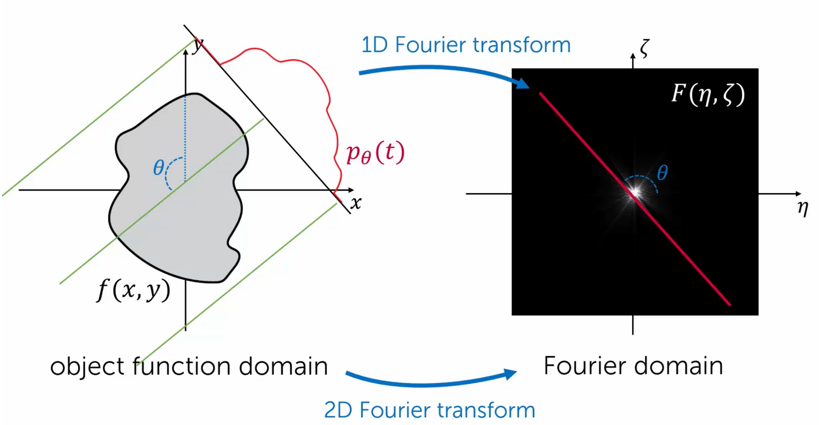
Fourier Slice Theorem



$$\mathcal{F}_1\left[\mathcal{R}_\theta f\right](\omega) = \mathcal{F}_2\left[f\right](\omega\nu), \qquad \nu = (\cos\theta, \sin\theta)$$

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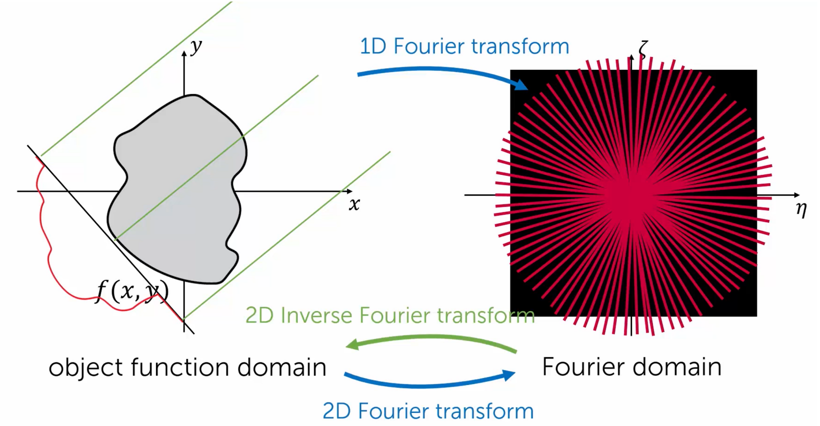
Fourier Slice Theorem



$$\mathcal{F}_1\left[\mathcal{R}_\theta f\right](\omega) = \mathcal{F}_2\left[f\right](\omega\nu), \qquad \nu = (\cos\theta, \sin\theta)$$

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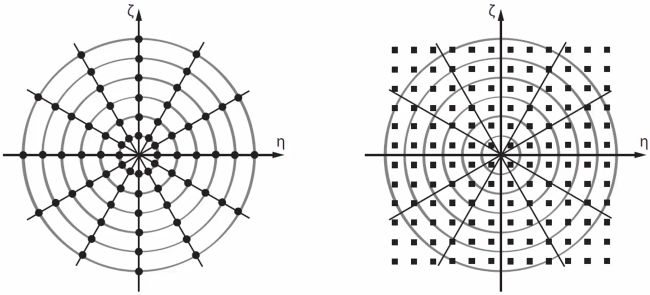
Non-uniform Fourier sampling



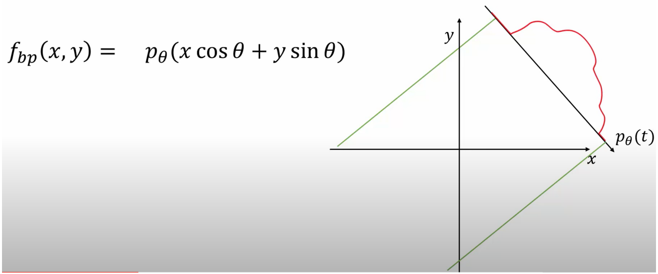
Fourier sampling with Fourier Slice Theorem

Sampling required by Fast Fourier Transform (FFT)

- ! high frequencies (= high resolution details) undersampled
- ! sampling non-uniform

Taken from corresponding video by the ASTRA toolbox team

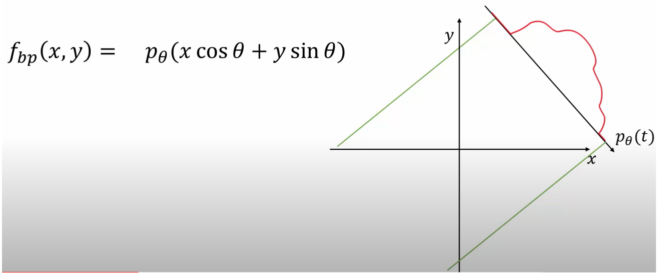
Backprojection



$$BP\left[p(\theta, t)\right](x, y) := \int p_\theta\left(x \cos\theta + y \sin\theta\right) d\theta$$

CWI

Backprojection



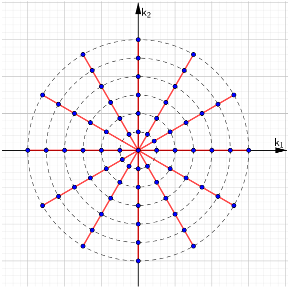
$$BP\left[p(\theta,t)\right](x,y) := \int p_\theta\left(x\cos\theta + y\sin\theta\right)d\theta$$

✓ $BP = \mathcal{R}^*$ and computationally efficient

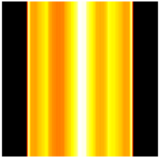
! $\mathcal{R}^*\mathcal{R}\left[f\right] \propto \frac{1}{\|x\|} * f$

Backprojection in action

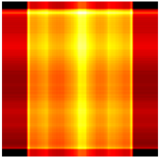
(a) true image



(b) Fourier sampling



(c) 1 angle



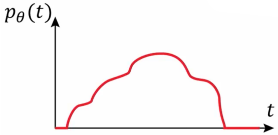
(d) 2 angles

(e) 8 angles

(f) 64 angles

(g) 256 angles

Filtered backprojection



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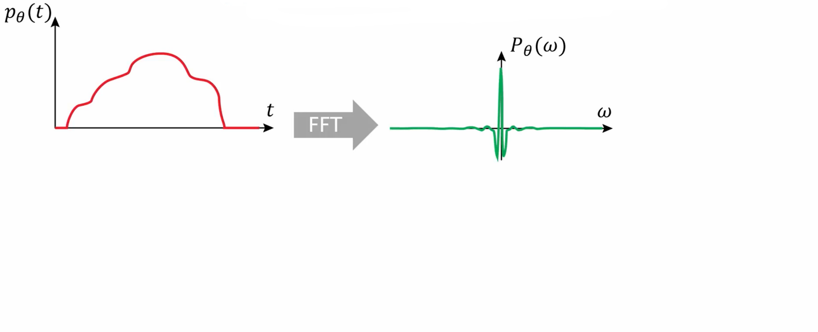
Filtered backprojection



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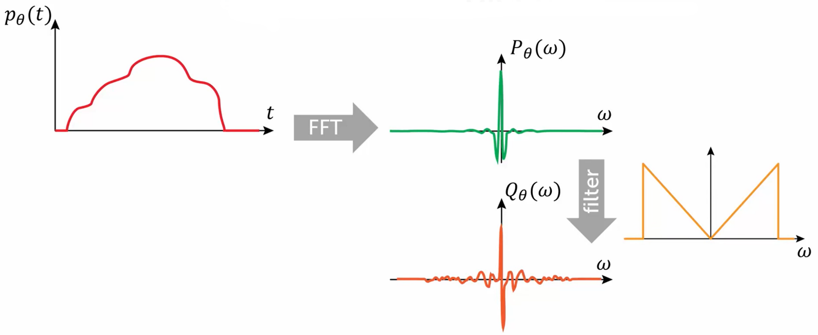
Filtered backprojection



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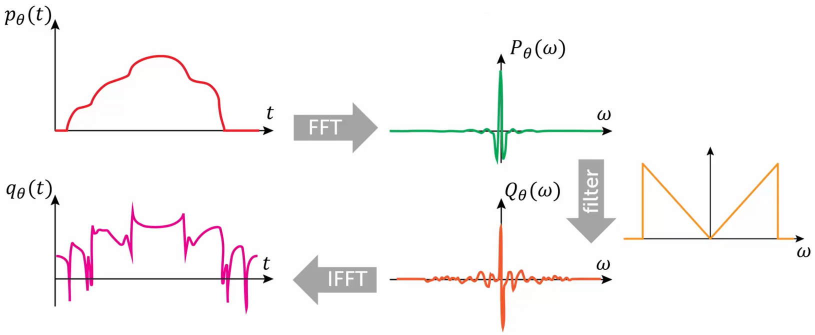
Filtered backprojection



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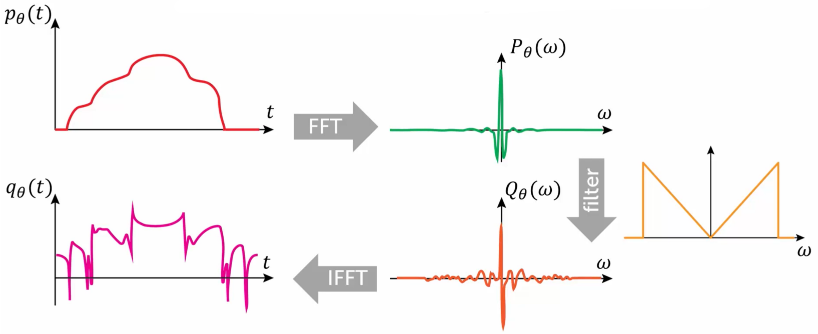
Filtered backprojection



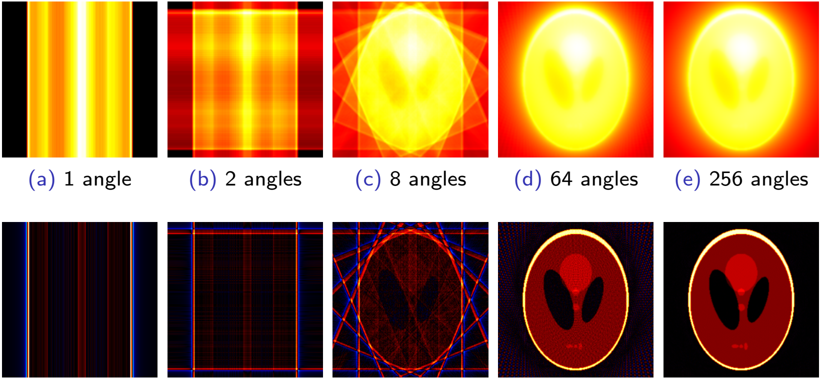
$$FBP\left[p(\theta,t)\right](x,y) := BP\left[q(\theta,t)\right](x,y) := \int q_\theta\left(x\cos\theta + y\sin\theta\right)d\theta$$

$$q_\theta(t) := \int \mathcal{F}[p_\theta](\omega)|\omega|e^{i2\pi\omega t}d\omega$$

Taken from corresponding video by the ASTRA toolbox team YouTube

Filtered backprojection in action

It turns out that $FBP(\mathcal{R}f) = \mathcal{R}^* \mathcal{H} \mathcal{R} f = f$



(a) 1 angle (b) 2 angles (c) 8 angles (d) 64 angles (e) 256 angles

(f) 1 angle (g) 2 angles (h) 8 angles (i) 64 angles (j) 256 angles

Just one more video by Samuli Siltanen: YouTube

CWI

CT reconstruction methods

Analytical (or direct) methods a la filtered backprojection:

- ✓ efficient to implement and execute
- ! lack of flexibility for unconventional scanning set-ups
- ! severe artifacts for limited / sparse projection data
- ! hard to introduce a-priori knowledge

Algebraic and variational methods (iterative methods):

- ! higher computational cost
- ✓ highly flexible, arbitrary geometries
- ✓ less artifacts for limited / sparse projection data
- ✓ introduction of a-priori knowledge possible

Algebraic Reconstructions

Idea: Find $f \in \mathcal{C}$ with $p \approx \mathcal{R} f$ as

$$f = \operatorname*{argmin}_{f \in \mathcal{C}} \|\mathcal{R} f - p\|_2^2 \quad ,$$

for instance via projected gradient descent:

$$f^{k+1} = P_{\mathcal{C}} \left(f^k - \nu \mathcal{R}^* \left(\mathcal{R} f^k - p \right) \right)$$

Many variants of this exist such as ART, SART, SIRT, ...

Variational Reconstructions

$$f = \operatorname*{argmin}_{f \in \mathcal{C}} \mathcal{D}(\mathcal{R}f, p) + \lambda \mathcal{J}(f) \quad ,$$

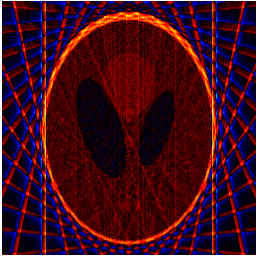
where $\mathcal{D}$ and $\mathcal{J}$ are derived from **probabilistic models** for data generation (*likelihood*) and typical images (*prior*), for instance

$$\mathcal{D}(\mathcal{R}f, g) := \left\| M^{-1/2} \left(\mathcal{R}f - p\right) \right\|_2^2, \qquad \mathcal{J}(f) := \|\nabla f\|_1$$

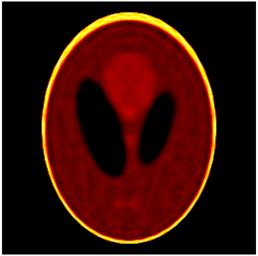
Solution via **iterative optimization schemes** such as proximal gradient descent, primal-dual hybrid gradient, alternating direction method of multipliers, ...

Iterative methods in action: 15 angles

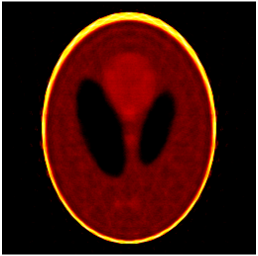
(a) true image



(b) FBP



(c) ART



(d) SIRT

(e) TV regularization

Further reading

- **P.C. Hansen, W.R.B. Lionheart, 2021.** Computed Tomography: Algorithms, Insight, and Just Enough Theory, *Society for Industrial and Applied Mathematics.*

- **T. M. Buzug, 2008.** Computed Tomography - From Photon Statistics to Modern Cone-Beam CT, *Springer-Verlag Berlin Heidelberg.*

- **G. T. Herman, 2009.** Fundamentals of Computerized Tomography Image Reconstruction from Projections, *Springer-Verlag London.*

- **F. Natterer, 2001.** The Mathematics of Computerized Tomography, *Society for Industrial and Applied Mathematics.*