Problem 33: Double boundary layer

a)

$$\epsilon y'' - x^2 y' - y = 0 \qquad ; \qquad y(0) = y(1) = 1 \tag{1}$$

where: $\epsilon \to 0^+$
To find the outer solution we can solve the unperturbed equation where $\epsilon = 0$.

$$-x^2 y_0' - y_0 = 0 \qquad \Rightarrow \qquad \frac{y_0'}{y_0} = -\frac{1}{x^2} \qquad \Rightarrow \qquad ln(y) = \frac{1}{x} + C$$
$$\Rightarrow y_0 = Ce^{\frac{1}{x}} \tag{2}$$

By letting $\epsilon = 0$, eq.(1) will have one less term and we can see that that the boundary conditions are not fulfilled, so $-x^2 y_0' - y_0 = 0$ cannot be a valid leading order approximation. Also $\epsilon y''$ cannot be small everywhere, and it's presumably important close to $x = 0$, so we must have a boundary layer there. We also have that

$$\lim_{x \to 0} y_0 = \infty \qquad \text{if} \qquad C \neq 0$$

so C must then be zero and $y_0 = 0$.

b)

To find the inner solution, we must determine the boundary layer thickness δ. If $\epsilon y''$ is not small, then ϵ shouldn't be there, meaning that we should rescale the the eq.(1) so ϵ is on the right place.
We can do this by introducing $\xi_L = \frac{x}{\delta_L} \quad \Rightarrow \quad x^2 = \xi_L^2 \delta_L^2$, for the **left boundary layer**.

$$\Rightarrow \frac{\epsilon}{\delta_L^2} \frac{d^2 y}{d\xi_L^2} - \frac{x^2}{\delta_L} \frac{dy}{d\xi_L} - y = 0$$

$$\frac{\epsilon}{\delta_L^2} \frac{d^2 y}{d\xi_L^2} - \frac{\xi_L^2 \delta_L^2}{\delta_L} \frac{dy}{d\xi_L} - y = 0$$

$$\frac{\epsilon}{\delta_L^2} \frac{d^2 y}{d\xi_L^2} - \xi_L^2 \delta_L \frac{dy}{d\xi_L} - y = 0$$
$$①\qquad\qquad ②\qquad\qquad ③$$

We can see that ② is not really an option for dominant balance, because whatever δ is then $\xi_L^2 \delta_L$ is small, so ② is sub-dominant.

$$① \sim ③ \quad \Rightarrow \quad \tfrac{\epsilon}{\delta_L^2} - 1 = 0 \quad \Rightarrow \quad \delta_L = \sqrt{\epsilon}$$

$$\Rightarrow \frac{d^2 y_L}{d\xi_L^2} - y_L = 0 \tag{3}$$

$$y_L = A_L e^{-\xi_L} + B_L e^{\xi_L}$$

The second term in the solution must be rejected since it grows exponentially out of the boundary layer.
We have that $y(0) = 1$

$$y_L = A_L e^{-\xi_L} \quad \Rightarrow \quad y_L(0) = A_L = 1 \quad \Rightarrow \quad y_L = e^{-\xi_L} \tag{4}$$

But again we cannot fulfill both boundary conditions, so we need a boundary layer at $x = 1$ as well.

Right boundary layer:

$$\xi_R = \frac{1-x}{\delta_R} \quad \Rightarrow \quad \underset{①}{\frac{\epsilon}{\delta_R^2}\frac{d^2y}{d\xi_R^2}} + \underset{②}{\frac{1}{\delta_R}(1-\delta_R\xi_R)^2\frac{dy}{d\xi_R}} - \underset{③}{y} = 0$$

$$① \sim ② \quad \Rightarrow \quad \tfrac{\epsilon}{\delta_R^2} = \tfrac{1}{\delta_R} \quad \Rightarrow \quad \delta_R = \epsilon$$

$$\frac{\delta_R}{\delta_R^2}\frac{d^2y}{d\xi_R^2} + \frac{1}{\delta_R}(1-\delta_R\xi_R)^2\frac{dy}{d\xi_R} = 0$$

Where $(1 - 2\epsilon\xi_R + \epsilon^2\xi_R^2) \simeq 1$ since $\epsilon \ll 1$

$$\Rightarrow \quad \frac{d^2y_R}{d\xi_R^2} + \frac{dy_R}{d\xi_R} = 0 \tag{5}$$

We assume the solution will be proportinal to $e^{\lambda\xi}$ for some constant λ. Then we substitute this solution into eq.:5:

$$\begin{aligned}
&\Rightarrow \quad \frac{d^2}{d\xi_R^2}(e^{\lambda\xi}) + \frac{d}{d\xi_R}(e^{\lambda\xi}) = 0 \\
&\Rightarrow \quad \lambda^2 e^{\lambda\xi} + \lambda e^{\lambda\xi} = 0 \quad \Rightarrow \quad (\lambda^2+\lambda)e^{\lambda x} = 0 \quad \Rightarrow \quad \lambda(1+\lambda) = 0 \quad \Rightarrow \quad \lambda = -1 \quad \text{or} \quad \lambda = 0 \\
&\Rightarrow \quad y_R = A_R e^{-\xi_R} + B_R \quad , \quad y_R(0) = 1 \quad \Rightarrow \quad y_R = A_R e^{-\xi_R} + (1 - A_R)
\end{aligned} \tag{6}$$

c)

We now need to find the unified solution by matching.

$$\left.\begin{aligned}
&\lim\nolimits_{x\to 0} Ce^{\frac{1}{x}} \quad \Rightarrow \quad C = 0 \\
&\lim\nolimits_{\xi_L\to\infty} e^{-\xi_L} = 0
\end{aligned}\right\} \quad \Rightarrow \quad y_{match,left} = 0$$

$$\left.\begin{aligned}
&\lim\nolimits_{x\to 1} Ce^{\frac{1}{x}} = 0 \quad \text{since} \quad C = 0 \\
&\lim\nolimits_{\xi_R\to\infty} (1-A_R) + A_R e^{-\xi_R} = 1 - A_R \\
&\Rightarrow \quad A_R = 1
\end{aligned}\right\} \quad \Rightarrow \quad y_{match,Right} = 0$$

$$y_{unified} = y_R + y_L - y_{match,Right} - y_{match,left} = e^{-\xi_R} + e^{-\xi_L} = e^{\frac{-1+x}{\epsilon}} + e^{\frac{-x}{\sqrt{\epsilon}}} \tag{7}$$

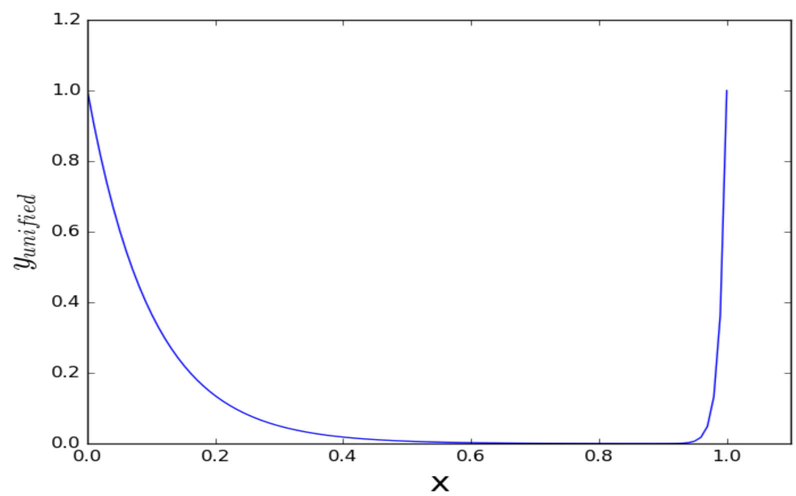

Figure 1: Double boundary layer solution

Problem 50: The perturbed satellite orbit

We seek an approximate solution to:

$$(1+\epsilon z)^3 \frac{d^z}{d\tau^2} + z = 0 \quad , \quad z(0) = 0 \quad , \quad \frac{dz}{d\tau}(0) = 1 \tag{8}$$

by a straightforward perturbation expansion of z in ϵ.

a)

We want to find the first two terms in a series for z.
We can try with a solution of the form:

$$z(\tau) = z_0(\tau) + \epsilon z_1(\tau) + \epsilon^2 z_2(\tau) + ...$$

I choose to do the perturbation expansion for z up to order ϵ^2, because even though I do not need it for the first part of the problem, I will need it for the second part of the problem, where I have to check if the expansion breaks down at order ϵ^2.

$$\begin{aligned}
&(1 + 3\epsilon(z_0 + \epsilon z_1 + ...) + 3\epsilon^2(z_0 + ...)^2)(z_0'' + \epsilon z_1'' + \epsilon^2 z_2'') + z_0 + \epsilon z_1 + \epsilon^2 z_2 = 0 \\
&(1 + 3\epsilon z_0 + 3\epsilon^2 z_1 + 3\epsilon^2 z_0^2)(z_0'' + \epsilon z_1'' + \epsilon^2 z_2'') + z_0 + \epsilon z_1 + \epsilon^2 z_2 = 0 \\
&z_0'' + \epsilon z_1'' + \epsilon^2 z_2'' + 3\epsilon z_0 z_0'' + 3\epsilon^2 z_0 z_1'' + 3\epsilon^2 z_0'' z_1 + 3\epsilon^2 z_0^2 z_0'' + z_0 + \epsilon z_1 + \epsilon^2 z_2 = 0
\end{aligned}$$

For the initial conditions we get:

$$\begin{aligned}
& z_0(0) + \epsilon z_1(0) + \epsilon^2 z_2(0) = 0 \\
\Rightarrow \quad & z_0(0) = 0, \quad z_1(0) = 0, \quad z_2(0) = 0 \\
\\
& z_0'(0) + \epsilon z_1'(0) + \epsilon^2 z_2'(0) = 1 \\
\Rightarrow \quad & z_0'(0) = 1, \quad z_1'(0) = 0, \quad z_2'(0) = 0
\end{aligned}$$

We then have to collect the terms for ϵ^0 and ϵ^1:

$$\begin{aligned}
&\epsilon^0 : z_0 + z_0 = 0 \quad \Rightarrow \quad z_0 = A_0 \sin(\tau) + B_0 \cos(\tau) && (9) \\
&\quad z_0(0) = 0 \quad \Rightarrow \quad B_0 = 0 \quad \Rightarrow \quad z_0 = A_0 \sin(\tau) \\
&\quad z_0'(0) = 1 \quad \Rightarrow \quad A_0 = 1 \\
\Rightarrow \quad & z_0 = \sin(\tau) && (10)
\end{aligned}$$

$$\begin{aligned}
&\epsilon^1 : z_1'' + 3z_0 z_0'' + z_1 = 0 \\
\Rightarrow \quad & z_1'' + z_1 = -3z_0 z_0'' = 3\sin^2(\tau) = \frac{3}{2}(1 - \cos(2\tau)) = \frac{3}{2} - \frac{3}{2}\cos(2\tau) && (11) \\
& z_1(\tau) = z_1^P + z_1^H = z_1^P + A_1 \cos(\tau) + B_1 \sin(\tau)
\end{aligned}$$

Where the particular solution z_1^P, must be of the form:

$$z_1^P = a_1 - b_1 \cos(2\tau) \tag{12}$$

We substitute for this solution into eq.(11) and get:

$$
\begin{aligned}
& 4b_1\cos(2\tau)+a_1-b_1cos(2\tau)=\frac{3}{2}-\frac{3}{2}\cos(2\tau) \quad\Rightarrow\quad a_1=\frac{3}{2}\\
\Rightarrow\quad & 3b_1\cos(2\tau)=-\frac{3}{2}\cos(2\tau) \quad\Rightarrow\quad b_1=-\frac{1}{2}\\
& z_1^P=\frac{3}{2}+\frac{1}{2}\cos(2\tau)\\
\Rightarrow\quad & z_1=\frac{3}{2}+\frac{1}{2}\cos(2\tau)+A_1\cos(\tau)+B_1\sin(\tau)\\
& z_1(0)=\frac{3}{2}+\frac{1}{2}+A_1=0 \quad\Rightarrow\quad A_1=-2\\
& z_1'(0)=-\sin(0)+2\sin(0)+B_1\cos(0)=B_1=0
\end{aligned}
$$

$$
\Rightarrow\quad z_1=\frac{3}{2}+\frac{1}{2}\cos(2\tau)-2\cos(\tau) \tag{13}
$$

The first two terms in a series for z are then:

$$
z=\underbrace{\sin(\tau)}_{z_0(\tau)}+\underbrace{\frac{3\epsilon}{2}+\frac{\epsilon}{2}\cos(2\tau)-2\epsilon\cos(\tau)}_{\epsilon z_1(\tau)} \tag{14}
$$

b)

To find z_2, we need to solve the equation that we will get by collecting all the terms of order ϵ^2 in the perturbation expansion for z.

$$
\begin{aligned}
z_2''+z_2 &= -3(-z_0z_1+3z_0^3-z_0z_1-z_0^3)\\
&= 6\sin(\tau)\left(\frac{1}{2}\cos(2\tau)-2\cos(\tau)+\frac{3}{2}\right)-6\sin^3(\tau)\\
&= 3\sin(\tau)\cos(\tau)-12\sin(\tau)\cos(\tau)+9\sin(\tau)-6sin^3(\tau)
\end{aligned}
$$

This expression can be rewritten into an expression with only first order sinus terms, using trigonometric formulas:

$$
\begin{aligned}
z_2''+z_2 &= -\frac{3}{2}\sin(\tau)+\frac{3}{2}\sin(3\tau)-6\sin(2\tau)+9\sin(\tau)-\frac{9}{2}\sin(\tau)+\frac{3}{2}\sin(3\tau)\\
&= 3\sin(3\tau)-6\sin(2\tau)+3\sin(\tau)
\end{aligned} \tag{15}
$$

$$
z_2(\tau)=z_2^P+z_2^H
$$

We can use the method of undetermined coefficients to find the particular solution, which will be the sum of the particular solutions to:

$$
\begin{aligned}
& z_{2,1}''+z_{2,1}=3\sin(\tau) && \Rightarrow\quad z_{2,1}^P=a_2\tau\cos(\tau)+b_2\tau\sin(\tau) && (16)\\
& z_{2,2}''+z_{2,2}=-6\sin(2\tau) && \Rightarrow\quad z_{2,2}^P=c_2\cos(2\tau)+d_2\sin(2\tau) && (17)\\
& z_{2,3}''+z_{2,3}=3\sin(3\tau) && \Rightarrow\quad z_{2,3}^P=e_2\cos(3\tau)+f_2\sin(3\tau) && (18)
\end{aligned}
$$

We can see from the first part of the particular solution (16), that we have non-periodic terms that will grow in time so $\epsilon^2\tau \gg 1$ and $\epsilon^2 z_2 \gg z_0$. This means that the straightforward perturbation expansion breaks down at order ϵ^2. The remedy to this problem would be the **Poincaré-Lindstedt method**.

We introduce a rescaled time, set $\sigma=\omega\tau$ and seek a solution of period 2π in σ, so $\omega\tau$ will change by 2π for each period.

The rescaled equation is:

$$\frac{d^2z}{d\tau^2} = \omega^2 \frac{d^2z}{d\sigma^2} \quad \Rightarrow \quad \omega^2(1+\epsilon z)^3\ \frac{d^2z}{d\sigma^2} + z = 0 \tag{19}$$

$$z(0) = 0 \qquad , \qquad \omega\frac{dz}{d\sigma}(0) = 1 \tag{20}$$

$$z = z_0 + \epsilon z_1 + \epsilon^2 z_2 \qquad \text{and} \qquad \omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2$$

Since the straightforward perturbation expression worked for the first two terms and we didn't have to use the Poincaré-Lindstedt method for those two terms, we set $\omega_0 = 1$ and $\omega_1 = 0$ so $\mathcal{O}(\epsilon^0)$ and $\mathcal{O}(\epsilon^1)$ remain unaltered.

$$(\omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2)(1 + 3\epsilon z_0 + 3\epsilon^2 z_1 + 3\epsilon^2 z_0^2)(z_0'' + \epsilon z_1'' + \epsilon^2 z_2'') + z_0 + \epsilon z_1 + \epsilon^2 z_2 = 0$$

$$\begin{aligned}\epsilon^2 = z_2'' + z_2 &= -6z_0^3 + 6z_0z_1 - 2\omega_2 z_0'' = -6z_0^3 + 6z_0z_1 + 2\omega_2 z_0\\ &= 3\sin(3\tau) + 3\sin(\tau) - 6\sin(2\tau) - 2\omega_2\sin(\tau)\\ &= 3\sin(3\tau) - 6\sin(2\tau) + (3 - 2\omega_2)\sin(\tau)\end{aligned}$$

$(3 - 2\omega_2)$ must be 0 to avoid secular or non-periodic terms. $\quad\Rightarrow\quad \omega_2 = -\frac{3}{2}$

We have now that:

$$z_2'' + z_2 = 3\sin(3\tau) - 6\sin(2\tau) \tag{21}$$

$$z_2 = z_2^H + z_2^P \quad \text{where} \quad z_2^H = A_2\sin(\tau) + B_2\cos(\tau)$$

$$z_2^P = a_2\cos(2\tau) + b_2\cos(3\tau) + c_2\sin(2\tau) + d_2\sin(3\tau) \tag{22}$$

We then need to substitute (22) into (21).

$$\begin{aligned}z_2'' &= -4a_2\cos(2\tau) - 9b_2\cos(3\tau) - 4c_2\sin(2\tau) - 9d_2\sin(3\tau)\\ z_2'' + z_2 &= -4a_2\cos(2\tau) - 9b_2\cos(3\tau) - 4c_2\sin(2\tau) - 9d_2\sin(3\tau)\\ &\quad + a_2\cos(2\tau) + b_2\cos(3\tau) + c_2\sin(2\tau) + d_2\sin(3\tau)\\ &= -3a_2\cos(2\tau) - 8b_2\cos(3\tau) - 3c_2\sin(2\tau) - 8d_2\sin(3\tau)\\ &= 3\sin(3\tau) - 6\sin(2\tau) \quad \Rightarrow \quad a_2 = b_2 = 0\\ &\Rightarrow \qquad -3c_2 = -6 \qquad \Rightarrow \qquad c_2 = 2\\ &\Rightarrow \qquad -8d_2 = -3 \qquad \Rightarrow \qquad d_2 = -\frac{3}{8}\\ &\Rightarrow \qquad z_2^P = 2\sin(2\tau) - \frac{3}{8}\sin(3\tau)\end{aligned}$$

$$z_2 = z_2^H + z_2^P = A_2\sin(\tau) + B_2\cos(\tau) + 2\sin(2\tau) - \frac{3}{8}\sin(3\tau) \tag{23}$$

The initial conditions will give us the constants A_2 and B_2.

$$\begin{aligned}\omega\frac{dz}{d\sigma}(0) = 1 \quad &\Rightarrow \quad (1 + \epsilon^2\omega_2)(z_0'(0) + \epsilon z_1'(0) + \epsilon^2 z_2'(0)) = 1\\ &\Rightarrow \quad \epsilon^0 : z_0'(0) = 1\\ &\Rightarrow \quad \epsilon^1 : z_1'(0) = 0\\ &\Rightarrow \quad \epsilon^2 : z_2'(0) + \omega_2\end{aligned}$$

$$
\begin{aligned}
&z_0'(0) = 0 \quad \Rightarrow \quad z_2'(0) = -\omega_2 \quad \Rightarrow \quad z_0'(0) = \frac{3}{2} \\
&z_2(0) = B_2 = 0 \quad \Rightarrow \quad z_2 = A_2 \sin(\tau) + 2\sin(2\tau) - \frac{3}{8}\sin(3\tau) \\
&z_2'(0) = A_2 \cos(\tau) + 4\cos(2\tau) - \frac{9}{8}\sin(3\tau) = A_2 + 4 = \frac{3}{2} \quad \Rightarrow \quad A_2 = -\frac{5}{2}
\end{aligned}
$$

$$
z_2 = -\frac{5}{2}\sin(\tau) + 2\sin(2\tau) - \frac{3}{8}\sin(3\tau) \tag{24}
$$

Finally we have the full solution to order ϵ^2:

$$
\begin{aligned}
z &= z_0 + \epsilon z_1 + \epsilon^2 z_2 \\
&= \sin(\tau) + \epsilon\left(\frac{1}{2}\cos(2\tau) - 2\cos(\tau) + \frac{3}{2}\right) + \epsilon^2\left(-\frac{3}{8}\sin(3\tau) + 2\sin(2\tau) - \frac{5}{2}\sin(\tau)\right) \qquad (25)
\end{aligned}
$$