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MEK4100

Mandatory assignment 2

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Problem 33: Double boundary layer

a)

$$\epsilon y'' - x^2 y' - y = 0 \quad ; \quad y(0) = y(1) = 1 \tag{1}$$

where: $\epsilon \to 0^+$
To find the outer solution we can solve the unperturbed equation where $\epsilon = 0$.

$$- x^2 y_0' - y_0 = 0 \quad \Rightarrow \quad \frac{y_0'}{y_0} = -\frac{1}{x^2} \quad \Rightarrow \quad ln(y) = \frac{1}{x} + C$$
$$\Rightarrow y_0 = C e^{\frac{1}{x}} \tag{2}$$

By letting $\epsilon = 0$, eq.(??) will have one less term and we can see that that the boundary conditions are not fulfilled, so $-x^2 y_0' - y_0 = 0$ cannot be a valid leading order approximation. Also $\epsilon y''$ cannot be small everywhere, and it's presumably important close to $x = 0$, so we must have a boundary layer there. We also have that

$$\lim_{x \to 0} y_0 = \infty \quad \text{if} \quad C \neq 0$$

so C must then be zero and $y_0 = 0$.

b)

To find the inner solution, we must determine the boundary layer thickness δ. If $\epsilon y''$ is not small, then ϵ shouldn't be there, meaning that we should rescale the the eq.(??) so ϵ is on the right place.
We can do this by introducing $\xi_L = \frac{x}{\delta_L} \quad \Rightarrow \quad x^2 = \xi_L^2 \delta_L^2$, for the **left boundary layer**.

$$\Rightarrow \frac{\epsilon}{\delta_L^2} \frac{d^2 y}{d\xi_L^2} - \frac{x^2}{\delta_L} \frac{dy}{d\xi_L} - y = 0$$

$$\frac{\epsilon}{\delta_L^2} \frac{d^2 y}{d\xi_L^2} - \frac{\xi_L^2 \delta_L^2}{\delta_L} \frac{dy}{d\xi_L} - y = 0$$

$$\frac{\epsilon}{\delta_L^2} \frac{d^2 y}{d\xi_L^2} - \xi_L^2 \delta_L \frac{dy}{d\xi_L} - y = 0$$
$$\textcircled{1} \qquad \textcircled{2} \qquad \textcircled{3}$$

We can see that $\textcircled{2}$ is not really an option for dominant balance, because whatever δ is then $\xi_L^2 \delta_L$ is small, so $\textcircled{2}$ is sub-dominant.

$$\textcircled{1} \sim \textcircled{3} \quad \Rightarrow \quad \frac{\epsilon}{\delta_L^2} - 1 = 0 \quad \Rightarrow \quad \delta_L = \sqrt{\epsilon}$$

$$\Rightarrow \frac{d^2 y_L}{d\xi_L^2} - y_L = 0 \tag{3}$$

$$y_L = A_L e^{-\xi_L} + B_L e^{\xi_L}$$

The second term in the solution must be rejected since it grows exponentially out of the boundary layer.

We have that $y(0) = 1$

$$y_L = A_L e^{-\xi_L} \quad \Rightarrow \quad y_L(0) = A_L = 1 \quad \Rightarrow \quad y_L = e^{-\xi_L} \tag{4}$$

But again we cannot fulfill both boundary conditions, so we need a boundary layer at $x = 1$ as well.

Right boundary layer:

$$\xi_R = \frac{1-x}{\delta_R} \quad \Rightarrow \quad \frac{\epsilon}{\delta_R^2}\frac{d^2y}{d\xi_R^2} + \frac{1}{\delta_R}(1-\delta_R\xi_R)^2\frac{dy}{d\xi_R} - y = 0$$

$$\text{①} \qquad\qquad \text{②} \qquad\qquad \text{③}$$

$$\text{①} \sim \text{②} \quad \Rightarrow \quad \frac{\epsilon}{\delta_R^2} = \frac{1}{\delta_R} \quad \Rightarrow \quad \delta_R = \epsilon$$

$$\frac{\delta_R}{\delta_R^2}\frac{d^2y}{d\xi_R^2} + \frac{1}{\delta_R}(1-\delta_R\xi_R)^2\frac{dy}{d\xi_R} = 0$$

Where $(1 - 2\epsilon\xi_R + \epsilon^2\xi_R^2) \simeq 1$ since $\epsilon \ll 1$

$$\Rightarrow \quad \frac{d^2y_R}{d\xi_R^2} + \frac{dy_R}{d\xi_R} = 0 \tag{5}$$

We assume the solution will be proportial to $e^{\lambda\xi}$ for some constant λ. Then we substitute this solution into eq.:**??**:

$$\Rightarrow \quad \frac{d^2}{d\xi_R^2}(e^{\lambda\xi}) + \frac{d}{d\xi_R}(e^{\lambda\xi}) = 0$$

$$\Rightarrow \quad \lambda^2 e^{\lambda\xi} + \lambda e^{\lambda\xi} = 0 \quad \Rightarrow \quad (\lambda^2+\lambda)e^{\lambda x} = 0 \quad \Rightarrow \quad \lambda(1+\lambda) = 0 \quad \Rightarrow \quad \lambda = -1 \quad \text{or} \quad \lambda = 0$$

$$\Rightarrow \quad y_R = A_R e^{-\xi_R} + B_R \quad , \quad y_R(0) = 1 \quad \Rightarrow \quad y_R = A_R e^{-\xi_R} + (1-A_R) \tag{6}$$

c)

We now need to find the unified solution by matching.

$$\left.\begin{array}{l} \lim_{x\to 0} C e^{\frac{1}{x}} \quad \Rightarrow \quad C = 0 \\[4pt] \lim_{\xi_L\to\infty} e^{-\xi_L} = 0 \end{array}\right\} \quad \Rightarrow \quad y_{match,left} = 0$$

$$\left.\begin{array}{l} \lim_{x\to 1} C e^{\frac{1}{x}} = 0 \quad \text{since} \quad C = 0 \\[4pt] \lim_{\xi_R\to\infty}(1-A_R) + A_R e^{-\xi_R} = 1 - A_R \\[4pt] \Rightarrow \quad A_R = 1 \end{array}\right\} \quad \Rightarrow \quad y_{match,Right} = 0$$

$$y_{unified} = y_R + y_L - y_{match,Right} - y_{match,left} = e^{-\xi_R} + e^{-\xi_L} = e^{\frac{-1+x}{\epsilon}} + e^{\frac{-x}{\sqrt{\epsilon}}} \tag{7}$$
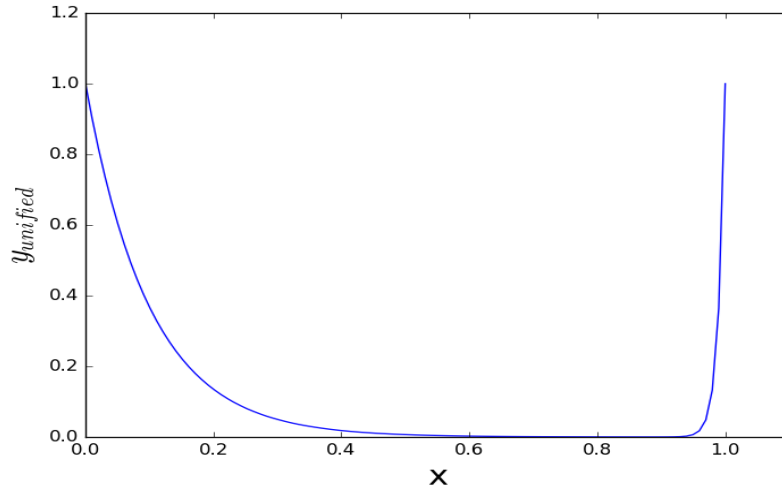

Figure 1: Double boundary layer solution

Problem 50: The perturbed satellite orbit

We seek an approximate solution to:

$$(1 + \epsilon z)^3 \frac{d^z}{d\tau^2} + z = 0 \quad, \quad z(0) = 0 \quad, \quad \frac{dz}{d\tau}(0) = 1 \tag{8}$$

by a straightforward perturbation expansion of z in ϵ.

a)
We want to find the first two terms in a series for z.
We can try with a solution of the form:

$$z(\tau) = z_0(\tau) + \epsilon z_1(\tau) + \epsilon^2 z_2(\tau) + ...$$

I choose to do the perturbation expansion for z up to order ϵ^2, because even though I do not need it for the first part of the problem, I will need it for the second part of the problem, where I have to check if the expansion breaks down at order ϵ^2.

$$(1 + 3\epsilon(z_0 + \epsilon z_1 + ...) + 3\epsilon^2(z_0 + ...)^2)(z_0'' + \epsilon z_1'' + \epsilon^2 z_2'') + z_0 + \epsilon z_1 + \epsilon^2 z_2 = 0$$

$$(1 + 3\epsilon z_0 + 3\epsilon^2 z_1 + 3\epsilon^2 z_0^2)(z_0'' + \epsilon z_1'' + \epsilon^2 z_2'') + z_0 + \epsilon z_1 + \epsilon^2 z_2 = 0$$

$$z_0'' + \epsilon z_1'' + \epsilon^2 z_2'' + 3\epsilon z_0 z_0'' + 3\epsilon^2 z_0 z_1'' + 3\epsilon^2 z_0'' z_1 + 3\epsilon^2 z_0^2 z_0'' + z_0 + \epsilon z_1 + \epsilon^2 z_2 = 0$$

For the initial conditions we get:

$$z_0(0) + \epsilon z_1(0) + \epsilon^2 z_2(0) = 0$$
$$\Rightarrow \quad z_0(0) = 0, \quad z_1(0) = 0, \quad z_2(0) = 0$$

$$z_0'(0) + \epsilon z_1'(0) + \epsilon^2 z_2'(0) = 1$$
$$\Rightarrow \quad z_0'(0) = 1, \quad z_1'(0) = 0, \quad z_2'(0) = 0$$

We then have to collect the terms for ϵ^0 and ϵ^1:

$$\epsilon^0 : z_0 + z_0 = 0 \quad \Rightarrow \quad z_0 = A_0 \sin(\tau) + B_0 \cos(\tau) \tag{9}$$
$$z_0(0) = 0 \quad \Rightarrow \quad B_0 = 0 \quad \Rightarrow \quad z_0 = A_0 \sin(\tau)$$
$$z_0'(0) = 1 \quad \Rightarrow \quad A_0 = 1$$
$$\Rightarrow \quad z_0 = \sin(\tau) \tag{10}$$

$$\epsilon^1 : z_1'' + 3z_0 z_0'' + z_1 = 0$$
$$\Rightarrow \quad z_1'' + z_1 = -3z_0 z_0'' = 3\sin^2(\tau) = \frac{3}{2}(1 - \cos(2\tau)) = \frac{3}{2} - \frac{3}{2}\cos(2\tau) \tag{11}$$
$$z_1(\tau) = z_1^P + z_1^H = z_1^P + A_1 \cos(\tau) + B_1 \sin(\tau)$$

Where the particular solution z_1^P, must be of the form:

$$z_1^P = a_1 - b_1 \cos(2\tau) \tag{12}$$

We substitute for this solution into eq.(??) and get:

$$4b_1 \cos(2\tau) + a_1 - b_1 cos(2\tau) = \frac{3}{2} - \frac{3}{2}\cos(2\tau) \quad \Rightarrow \quad a_1 = \frac{3}{2}$$

$$\Rightarrow \quad 3b_1 \cos(2\tau) = -\frac{3}{2}\cos(2\tau) \quad \Rightarrow \quad b_1 = -\frac{1}{2}$$

$$z_1^P = \frac{3}{2} + \frac{1}{2}\cos(2\tau)$$

$$\Rightarrow \quad z_1 = \frac{3}{2} + \frac{1}{2}\cos(2\tau) + A_1\cos(\tau) + B_1\sin(\tau)$$

$$z_1(0) = \frac{3}{2} + \frac{1}{2} + A_1 = 0 \quad \Rightarrow \quad A_1 = -2$$

$$z_1'(0) = -\sin(0) + 2\sin(0) + B_1\cos(0) = B_1 = 0$$

$$\Rightarrow \quad z_1 = \frac{3}{2} + \frac{1}{2}\cos(2\tau) - 2\cos(\tau) \tag{13}$$

The first two terms in a series for z are then:

$$z = \underbrace{\sin(\tau)}_{z_0(\tau)} + \underbrace{\frac{3\epsilon}{2} + \frac{\epsilon}{2}\cos(2\tau) - 2\epsilon\cos(\tau)}_{\epsilon z_1(\tau)} \tag{14}$$

b)

To find z_2, we need to solve the equation that we will get by collecting all the terms of order ϵ^2 in the perturbation expansion for z.

$$z_2'' + z_2 = -3(-z_0 z_1 + 3z_0^3 - z_0 z_1 - z_0^3)$$

$$= 6\sin(\tau)\left(\frac{1}{2}\cos(2\tau) - 2\cos(\tau) + \frac{3}{2}\right) - 6\sin^3(\tau)$$

$$= 3\sin(\tau)\cos(\tau) - 12\sin(\tau)\cos(\tau) + 9\sin(\tau) - 6sin^3(\tau)$$

This expression can be rewritten into an expression with only first order sinus terms, using trigonometric formulas:

$$z_2'' + z_2 = -\frac{3}{2}\sin(\tau) + \frac{3}{2}\sin(3\tau) - 6\sin(2\tau) + 9\sin(\tau) - \frac{9}{2}\sin(\tau) + \frac{3}{2}\sin(3\tau)$$

$$= 3\sin(3\tau) - 6\sin(2\tau) + 3\sin(\tau) \tag{15}$$

$$z_2(\tau) = z_2^P + z_2^H$$

We can use the method of undetermined coefficients to find the particular solution, which will be the sum of the particular solutions to:

$$z_{2,1}'' + z_{2,1} = 3\sin(\tau) \qquad \Rightarrow \qquad z_{2,1}^P = a_2\tau\cos(\tau) + b_2\tau\sin(\tau) \tag{16}$$

$$z_{2,2}'' + z_{2,2} = -6\sin(2\tau) \qquad \Rightarrow \qquad z_{2,2}^P = c_2\cos(2\tau) + d_2\sin(2\tau) \tag{17}$$

$$z_{2,3}'' + z_{2,3} = 3\sin(3\tau) \qquad \Rightarrow \qquad z_{2,3}^P = e_2\cos(3\tau) + f_2\sin(3\tau) \tag{18}$$

We can see from the first part of the particular solution (**??**), that we have non-periodic terms that will grow in time so $\epsilon^2\tau \gg 1$ and $\epsilon^2 z_2 \gg z_0$. This means that the straightforward perturbation expansion breaks down at order ϵ^2. The remedy to this problem would be the **Poincaré-Lindstedt method**.

We introduce a rescaled time, set $\sigma = \omega\tau$ and seek a solution of period 2π in σ, so $\omega\tau$ will change by 2π for each period.

The rescaled equation is:

$$\frac{d^2 z}{d\tau^2} = \omega^2 \frac{d^2 z}{d\sigma^2} \quad \Rightarrow \quad \omega^2 (1 + \epsilon z)^3 \frac{d^2 z}{d\sigma^2} + z = 0 \tag{19}$$

$$z(0) = 0 \quad , \quad \omega \frac{dz}{d\sigma}(0) = 1 \tag{20}$$

$$z = z_0 + \epsilon z_1 + \epsilon^2 z_2 \quad \text{and} \quad \omega = \omega_0 + \epsilon \omega_1 + \epsilon^2 \omega_2$$

Since the straightforward perturbation expression worked for the first two terms and we didn't have to use the Poincaré-Lindstedt method for those two terms, we set $\omega_0 = 1$ and $\omega_1 = 0$ so $\mathcal{O}(\epsilon^0)$ and $\mathcal{O}(\epsilon^1)$ remain unaltered.

$$(\omega_0 + \epsilon \omega_1 + \epsilon^2 \omega_2)(1 + 3\epsilon z_0 + 3\epsilon^2 z_1 + 3\epsilon^2 z_0^2)(z_0'' + \epsilon z_1'' + \epsilon^2 z_2'') + z_0 + \epsilon z_1 + \epsilon^2 z_2 = 0$$

$$\epsilon^2 = z_2'' + z_2 = -6z_0^3 + 6z_0 z_1 - 2\omega_2 z_0'' = -6z_0^3 + 6z_0 z_1 + 2\omega_2 z_0$$
$$= 3\sin(3\tau) + 3\sin(\tau) - 6\sin(2\tau) - 2\omega_2 \sin(\tau)$$
$$= 3\sin(3\tau) - 6\sin(2\tau) + (3 - 2\omega_2)\sin(\tau)$$

$(3 - 2\omega_2)$ must be 0 to avoid secular or non-periodic terms. $\quad \Rightarrow \quad \omega_2 = -\frac{3}{2}$

We have now that:

$$z_2'' + z_2 = 3\sin(3\tau) - 6\sin(2\tau) \tag{21}$$

$$z_2 = z_2^H + z_2^P \quad \text{where} \quad z_2^H = A_2 \sin(\tau) + B_2 \cos(\tau)$$

$$z_2^P = a_2 \cos(2\tau) + b_2 \cos(3\tau) + c_2 \sin(2\tau) + d_2 \sin(3\tau) \tag{22}$$

We then need to substitute (**??**) into (**??**).

$$z_2'' = -4a_2 \cos(2\tau) - 9b_2 \cos(3\tau) - 4c_2 \sin(2\tau) - 9d_2 \sin(3\tau)$$

$$z_2'' + z_2 = -4a_2 \cos(2\tau) - 9b_2 \cos(3\tau) - 4c_2 \sin(2\tau) - 9d_2 \sin(3\tau)$$
$$+ a_2 \cos(2\tau) + b_2 \cos(3\tau) + c_2 \sin(2\tau) + d_2 \sin(3\tau)$$
$$= -3a_2 \cos(2\tau) - 8b_2 \cos(3\tau) - 3c_2 \sin(2\tau) - 8d_2 \sin(3\tau)$$
$$= 3\sin(3\tau) - 6\sin(2\tau) \quad \Rightarrow \quad a_2 = b_2 = 0$$
$$\Rightarrow \quad -3c_2 = -6 \quad \Rightarrow \quad c_2 = 2$$
$$\Rightarrow \quad -8d_2 = -3 \quad \Rightarrow \quad d_2 = -\frac{3}{8}$$
$$\Rightarrow \quad z_2^P = 2\sin(2\tau) - \frac{3}{8}\sin(3\tau)$$

$$z_2 = z_2^H + z_2^P = A_2 \sin(\tau) + B_2 \cos(\tau) + 2\sin(2\tau) - \frac{3}{8}\sin(3\tau) \tag{23}$$

The initial conditions will give us the constants A_2 and B_2.

$$\omega \frac{dz}{d\sigma}(0) = 1 \quad \Rightarrow \quad (1 + \epsilon^2 \omega_2)(z_0'(0) + \epsilon z_1'(0) + \epsilon^2 z_2'(0)) = 1$$

$$\Rightarrow \quad \epsilon^0 : z_0'(0) = 1$$

$$\Rightarrow \quad \epsilon^1 : z_1'(0) = 0$$

$$\Rightarrow \quad \epsilon^2 : z_2'(0) + \omega_2$$

$$z_0'(0) = 0 \quad \Rightarrow \quad z_2'(0) = -\omega_2 \quad \Rightarrow \quad z_0'(0) = \frac{3}{2}$$

$$z_2(0) = B_2 = 0 \quad \Rightarrow \quad z_2 = A_2 \sin(\tau) + 2\sin(2\tau) - \frac{3}{8}\sin(3\tau)$$

$$z_2'(0) = A_2 \cos(\tau) + 4\cos(2\tau) - \frac{9}{8}\sin(3\tau) = A_2 + 4 = \frac{3}{2} \quad \Rightarrow \quad A_2 = -\frac{5}{2}$$

$$z_2 = -\frac{5}{2}\sin(\tau) + 2\sin(2\tau) - \frac{3}{8}\sin(3\tau) \tag{24}$$

Finally we have the full solution to order ϵ^2:

$$z = z_0 + \epsilon z_1 + \epsilon^2 z_2$$

$$= \sin(\tau) + \epsilon\left(\frac{1}{2}\cos(2\tau) - 2\cos(\tau) + \frac{3}{2}\right) + \epsilon^2\left(-\frac{3}{8}\sin(3\tau) + 2\sin(2\tau) - \frac{5}{2}\sin(\tau)\right) \tag{25}$$