MEK4100
Mandatory assignment 1

Farnaz Rezvany Hesary

October 18, 2017

Problem 3

a)

We have the following parameters:

$$c : \left(\frac{L}{T}\right) \qquad \rho : \left(\frac{M}{L^3}\right) \qquad g : \left(\frac{L}{T^2}\right)$$
$$\sigma : \left(\frac{M}{T^2}\right) \qquad \lambda : L \qquad a : L$$

6 parameters - 3 dimensional units = 3 dimensionless numbers.

$$\pi_1 = \frac{\sigma}{\rho g \lambda^2} \quad , \quad \pi_2 = \frac{c^2}{g\lambda} = \frac{c}{\sqrt{g\lambda}} \quad , \quad \pi_3 = \frac{a}{\lambda}$$

b)

$$\text{when} \quad a \rightarrow 0 \qquad \Rightarrow \qquad \pi_3 \rightarrow 0$$

so we have that $G(\pi_1, \pi_2) = 0$

$$\Rightarrow c = \sqrt{g\lambda} f(\pi_1) = \sqrt{g\lambda} f(\frac{\sigma}{\rho g \lambda^2})$$

where π_1 expresses the importance of surface tension in relation to gravity, concerning the wave celerity.

Problem 12

We have the following equation:

$$x^2 - 2x + 1 = \epsilon(1 + 2x)$$

a)

For this part of the exercise we are going to try the straightforward, naive, perturbation scheme.
We assume $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + ...$

$$\Rightarrow (x_0 + \epsilon x_1 + \epsilon^2 x_2)^2 - 2(x_0 + \epsilon x_1 + \epsilon^2 x_2) + 1 - \epsilon - 2\epsilon(x_0 + \epsilon x_1 + \epsilon^2 x_2) = 0$$

$$\Rightarrow x_0^2 + x_0 x_1 \epsilon + x_0 x_2 \epsilon^2 + x_0 x_1 \epsilon + x_1^2 \epsilon^2 + x_1 x_2 \epsilon^3 + x_0 x_2 \epsilon^2 + x_1 x_2 \epsilon^3 + x_2^2 \epsilon^4$$
$$- 2x_0 - 2x_1 \epsilon - 2x_2 \epsilon^2 + 1 - \epsilon - 2\epsilon x_0 - 2\epsilon^2 x_1 - 2x_2 \epsilon^3 = 0$$

$$\Rightarrow x_0^2 - 2x_0 + 1 + \epsilon(2x_0 x_1 - 2x_1 - 1 - 2x_0) + \epsilon^2(2x_0 x_2 + x_1^2 - 2x_1 - 2x_2) = 0$$

We can now collect the terms for ϵ^0 and ϵ^1.

$$\epsilon^0 : x_0^2 - 2x_0 + 1 = 0 \quad \Rightarrow \quad x_0 = 1$$

$$\epsilon^1 : 2x_0 x_1 - 2x_1 = 1 + 2x_0 \quad (\text{ here we can set } x_0 = 1)$$
$$\Rightarrow \quad 2x_1 - 2x_1 = 3$$

This gives us that $0 = 3$ which clearly means that we a get a break down of the direct perturbation technique.
So $x = x_0 + \epsilon x_1$ must be a invalid form for the solution.

b)

In this part of the exercise, we are going to employ a general tye expansion $x = x_0 + x_1 + x_2 + ...$,where $x_0 \gg x_1 \gg x_2 ...$, when $\epsilon \to 0^+$
We try at first a solution of the form: $x = x_0 + x_1$ where $x_0 = 1$

$$\Rightarrow \quad (x_0 + x_1)^2 - 2(x_0 + x_1) + 1 = \epsilon(1 + 2x_0 + 2x_1)$$
$$1 + 2x_1 + x_1^2 - 2 - 2x_1 + 1 = \epsilon(3 + 2x_1)$$
$$\Rightarrow x_1^2 = 3\epsilon + 2\epsilon x_1$$

$$x_1^2 - 2\epsilon x_1 - 3\epsilon = 0$$
$$① \qquad ② \qquad ③$$

Now we can use the method of dominant balancing on these 3 terms to find x_1:

$$① \sim ② \Rightarrow x_1^2 - 2\epsilon x_1 = 0 \quad \Rightarrow x_1 - 2\epsilon = 0 \quad \Rightarrow x_1 = 2\epsilon$$
$$\Rightarrow ③ \gg ① \text{ and} ② \qquad \text{NO!}$$

$$① \sim ③ \Rightarrow x_1^2 - 3\epsilon = 0 \quad \Rightarrow x_1 = \pm\sqrt{3\epsilon} \qquad \text{OK!}$$

$$② \sim ③ \Rightarrow -2\epsilon x_1 - 3\epsilon = 0 \quad \Rightarrow x_1 = \frac{-3}{2} \quad \text{which gives us} \quad \frac{9}{4} = 0 \quad \text{so NO!}$$

Then we have that $x_1 = \pm\sqrt{3\epsilon}$ and $x_0 = 1$. In order to find an expression for x_2 we need to try with a solution of the form $x = x_0 + x_1 + x_2$, which gives us the following expression:

$$(x_0 + x_1 + x_2)^2 - 2(x_0 + x_1 + x_2) + 1 = \epsilon(1 + 2(x_0 + x_1 + x_2))$$

$$\Rightarrow \quad x_0^2 + x_0x_1 + x_0x_2 + x_0x_1 + x_1^2 + x_1x_2 + x_2x_0 + x_2x_1 + x_2^2 - 2x_0 - 2x_1 - 2x_2 + 1$$
$$= \epsilon(1 + 2x_0 + 2x_1 + 2x_2)$$

We can now insert the values we found for x_0 and x_1 into the expression above. This will give us:

$$1 \pm \sqrt{3\epsilon} + x_2 \pm \sqrt{3\epsilon} + 3\epsilon \pm \sqrt{3\epsilon}x_2 + x_2 \pm \sqrt{3\epsilon}x_2 + x_2^2 - 2 \pm 2\sqrt{3\epsilon} - 2x_2 + 1$$
$$= \epsilon(3 \pm 2\sqrt{3\epsilon} + 2x_2)$$

$$\Rightarrow x_2^2 \pm 2x_2\sqrt{3\epsilon} - 2\epsilon x_2 \pm 2\epsilon\sqrt{3\epsilon} = 0$$
$$① \qquad ② \qquad ③ \qquad ④$$

Again we can use the method of dominant balancing on these 4 terms to find x_2

$$① \sim ② \quad \Rightarrow \quad x_2^2 \pm 2x_2\sqrt{3\epsilon} = 0 \quad \Rightarrow \quad x_2 = \pm 2\sqrt{3\epsilon}$$

This gives us $x_2 > x_1$, but we know that $x_0 \gg x_1 \gg x_2$ so NO!

$$① \sim ③ \quad \Rightarrow \quad x_2^2 - 2\epsilon x_2 = 0 \quad \Rightarrow \quad x_2 = 2\epsilon$$
$$\Rightarrow ① \sim \epsilon^2 \quad ② \sim \epsilon^{\frac{3}{2}} \quad ③ \sim \epsilon^2 \quad ④ \sim \epsilon^{\frac{3}{2}}$$

we can see that ① and ③ $<$ ② and ④ so NO!

$$① \sim ④ \quad \Rightarrow \quad x_2^2 \pm 2\epsilon\sqrt{3\epsilon} = 0 \quad \Rightarrow \quad x_2 = \pm\sqrt{2\epsilon}\sqrt[4]{3\epsilon} \sim \epsilon^{\frac{3}{4}}$$
$$\Rightarrow ① \sim \epsilon^{\frac{6}{4}} \quad ② \sim \epsilon^{\frac{5}{4}} \quad ③ \sim \epsilon^{\frac{7}{4}} \quad ④ \sim \epsilon^{\frac{3}{4}}$$

we can see that ① $<$ ② so NO!

$$② \sim ③ \quad \Rightarrow \quad \pm 2x_2\sqrt{3\epsilon} - 2\epsilon x_2 = 0 \quad \Rightarrow \quad x_2 = 0 \quad \text{NO!}$$

$$② \sim ④ \quad \Rightarrow \quad 2x_2\sqrt{3\epsilon} \pm 2\epsilon\sqrt{3\epsilon} = 0 \quad \Rightarrow \quad x_2 = \epsilon$$
$$\Rightarrow ① \sim \epsilon^2 \quad ② \sim \epsilon^{\frac{3}{2}} \quad ③ \sim \epsilon^2 \quad ④ \sim \epsilon^{\frac{3}{2}} \quad \text{OK!}$$

We have now found that

$$x = x_1 + x_1 + x_3 = 1 \pm \sqrt{3\epsilon} + \epsilon$$

c)

In this part of the exercise we are going to solve $(x-1)^n = \epsilon x$. We can start by solving the unpertubed problem to find x_0.

$$(x_0 - 1)^n = 0 \quad \Rightarrow \quad x_0 = 1 \quad \text{when n is a positive integer}$$

Now we can try with a solution of the form $x = x_0 + x_1$ to find x_1

$$((x_0 + x_1) - 1)^n = \epsilon(x_0 + x_1)$$
$$((1 + x_1) - 1)^n = \epsilon + \epsilon x_1$$

$$x_1^n - \epsilon x_1 - \epsilon = 0$$

$$① \quad ② \quad ③$$

Dominant balance method:

$$① \sim ② \quad \Rightarrow \quad x_1^n - \epsilon x_1 = 0 \quad \Rightarrow \quad x_1^{n-1} = \epsilon \quad \Rightarrow \quad x_1 = \epsilon^{\frac{1}{n-1}}$$
$$\Rightarrow ① \sim \epsilon^{\frac{n}{n-1}} \quad ② \sim \epsilon^{\frac{n}{n-1}} \quad ③ \sim \epsilon$$
$$n \neq 1 \quad \Rightarrow \quad n \geq 2 \quad \Rightarrow \quad ① \text{ and} ② \ll ③ \quad \text{NO!}$$

$$① \sim ③ \quad \Rightarrow \quad x_1^n - \epsilon = 0 \quad \Rightarrow \quad x_1 = \epsilon^{\frac{1}{n}}$$
$$\Rightarrow ① \sim \epsilon \quad ② \sim \epsilon^{\frac{n+1}{n}} \quad ③ \sim \epsilon$$
$$① \text{ and } ③ \gg ② \quad \text{OK!}$$

This gives us:

$$x = 1 + \epsilon^{\frac{1}{n}}$$

Problem 14

We have the following first order, non separable differential equation:

$$y' + y + \epsilon x y^2 = 0$$
$$y(0) = 1$$

where ϵ is very small.
We can begin with solving the unpertubed problem to find y_0.

$$y' + y = 0 \qquad \text{This equation is separable}$$
$$\frac{y'}{y} = -1 \quad \Rightarrow \quad ln(y) = -x + C \quad \Rightarrow \quad y = e^{-x+C} = Ce^{-x}$$
$$y(0) = 1 \quad \Rightarrow \quad y(0) = Ce^0 = 1 \quad \Rightarrow \quad C = 1$$

$$\Rightarrow y_0 = e^{-x}$$

In order to find y_1, we can try with a solution of the form: $y = y_0 + \epsilon y_1$

$$(y_0 + \epsilon y_1)' + (y_0 + \epsilon y_1) + \epsilon x (y_0 + \epsilon y_1)^2 = 0$$

The initial condition then becomes:

$$y_0(0) + \epsilon y_1(0) = 1 \quad \text{where:} \quad y_0(0) = 1 \quad \text{and} \quad y_1(0) = 0$$

$$y_0' + \epsilon y_1' + y_0 + \epsilon y_1 + \epsilon x y_0^2 + 2\epsilon^2 x y_0 y_1 + \epsilon^3 x y_1^2 = 0$$

We can then collect all the terms for ϵ^0, ϵ^1, ϵ^2 and ϵ^3:

$$\epsilon^0 : y_0' + y_0 = 0$$
$$\epsilon^1 : y_1' + y_1 + x y_0^2 = 0 \quad \Rightarrow \quad y_1' + y_1 + x e^{-2x} = 0$$
$$\epsilon^2 : 2 x y_0 y_1 = 0$$
$$\epsilon^3 : x y_1^2 = 0$$

We can see that we need to solve $y_1' + y_1 + xe^{-2x} = 0$ with initial condition $y_1(0) = 0$ to find y_1.
The integrating factor is $\mu(x) = e^{\int 1 dx} = e^x$.

$$e^x y_1' + e^x y_1 = e^x(-xe^{-2x}) = -e^{-x} x$$
$$\frac{d}{dx}(e^x y_1(x)) = -e^{-x} x$$
$$\Rightarrow \quad e^x y = \int -e^{-x} x dx = -\int e^{-x} x dx$$

We can then integrate by parts:

$$\Rightarrow \quad -\int e^{-x}dx = e^{-x}x + \int e^{-x}dx$$

We can then use substitution in the integral: $\quad u = -x \quad du = -dx$

$$\Rightarrow \quad e^{-x}x + \int e^{u} = xe^{-x} + e^{u} + C = xe^{-x} + e^{-x} + C = e^{-x}(x+1) + C$$

$$e^{x}y_1 = e^{-x}(x+1) + C = xe^{-x} + e^{-x} + C$$

$$y = xe^{-2x} + e^{-2x} + Ce^{-x} = e^{-2x}(x + 1 + Ce^{x})$$

The initial condition gives:

$$y_1(0) = e^0 + Ce^0 = 0 \quad \Rightarrow \quad 1 + C = 0 \quad \Rightarrow \quad C = -1$$

We will then have:

$$y_1 = e^{-2x}(x - e^{x} + 1) = xe^{-2x} + e^{-2x} - e^{-x}$$

$$\Rightarrow \quad y = y_0 + \epsilon y_1 = e^{-x} + xe^{-2x} + e^{-2x} - e^{-x} = e^{-x} + \epsilon((x+1)e^{-2x} - e^{-x})$$