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MEK4100

Mandatory assignment 2

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Problem 33: Double boundary layer

a)

$$\epsilon y'' - x^2 y' - y = 0 \quad ; \quad y(0) = y(1) = 1 \quad (1)$$

where: $\epsilon \rightarrow 0^+$

To find the outer solution we can solve the unperturbed equation where $\epsilon = 0$.

$$\begin{aligned} -x^2 y'_0 - y_0 = 0 & \Rightarrow \frac{y'_0}{y_0} = -\frac{1}{x^2} \Rightarrow \ln(y) = \frac{1}{x} + C \\ \Rightarrow y_0 = C e^{\frac{1}{x}} \end{aligned} \quad (2)$$

By letting $\epsilon = 0$, eq.(1) will have one less term and we can see that the boundary conditions are not fulfilled, so $-x^2 y'_0 - y_0 = 0$ cannot be a valid leading order approximation. Also $\epsilon y''$ cannot be small everywhere, and it's presumably important close to $x = 0$, so we must have a boundary layer there. We also have that

$$\lim_{x \rightarrow 0} y_0 = \infty \quad \text{if} \quad C \neq 0$$

so C must then be zero and $y_0 = 0$.

b)

To find the inner solution, we must determine the boundary layer thickness δ . If $\epsilon y''$ is not small, then ϵ shouldn't be there, meaning that we should rescale the eq.(1) so ϵ is on the right place.

We can do this by introducing $\xi_L = \frac{x}{\delta_L} \Rightarrow x^2 = \xi_L^2 \delta_L^2$, for the **left boundary layer**.

$$\begin{aligned} \Rightarrow \frac{\epsilon}{\delta_L^2} \frac{d^2 y}{d\xi_L^2} - \frac{x^2}{\delta_L} \frac{dy}{d\xi_L} - y &= 0 \\ \frac{\epsilon}{\delta_L^2} \frac{d^2 y}{d\xi_L^2} - \frac{\xi_L^2 \delta_L^2}{\delta_L} \frac{dy}{d\xi_L} - y &= 0 \\ \frac{\epsilon}{\delta_L^2} \frac{d^2 y}{d\xi_L^2} - \xi_L^2 \delta_L \frac{dy}{d\xi_L} - y &= 0 \\ \textcircled{1} \quad \quad \quad \textcircled{2} \quad \quad \quad \textcircled{3} \end{aligned}$$

We can see that $\textcircled{2}$ is not really an option for dominant balance, because whatever δ is then $\xi_L^2 \delta_L$ is small, so $\textcircled{2}$ is sub-dominant.

$$\begin{aligned} \textcircled{1} \sim \textcircled{3} \quad \Rightarrow \quad \frac{\epsilon}{\delta_L^2} - 1 = 0 \quad \Rightarrow \quad \delta_L = \sqrt{\epsilon} \\ \Rightarrow \frac{d^2 y_L}{d\xi_L^2} - y_L = 0 \end{aligned} \quad (3)$$

$$y_L = A_L e^{-\xi_L} + B_L e^{\xi_L}$$

The second term in the solution must be rejected since it grows exponentially out of the boundary layer.

We have that $y(0) = 1$

$$y_L = A_L e^{-\xi_L} \Rightarrow y_L(0) = A_L = 1 \Rightarrow y_L = e^{-\xi_L} \quad (4)$$

But again we cannot fulfill both boundary conditions, so we need a boundary layer at $x = 1$ as well.

Right boundary layer:

$$\xi_R = \frac{1-x}{\delta_R} \Rightarrow \frac{\epsilon}{\delta_R^2} \frac{d^2 y}{d\xi_R^2} + \frac{1}{\delta_R} (1 - \delta_R \xi_R)^2 \frac{dy}{d\xi_R} - y = 0$$

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$$\textcircled{1} \sim \textcircled{2} \Rightarrow \frac{\epsilon}{\delta_R^2} = \frac{1}{\delta_R} \Rightarrow \delta_R = \epsilon$$

$$\frac{\delta_R}{\delta_R^2} \frac{d^2 y}{d\xi_R^2} + \frac{1}{\delta_R} (1 - \delta_R \xi_R)^2 \frac{dy}{d\xi_R} = 0$$

Where $(1 - 2\epsilon\xi_R + \epsilon^2\xi_R^2) \simeq 1$ since $\epsilon \ll 1$

$$\Rightarrow \frac{d^2 y_R}{d\xi_R^2} + \frac{dy_R}{d\xi_R} = 0 \quad (5)$$

We assume the solution will be proportional to $e^{\lambda\xi}$ for some constant λ . Then we substitute this solution into eq.:5:

$$\begin{aligned} \Rightarrow \frac{d^2}{d\xi_R^2}(e^{\lambda\xi}) + \frac{d}{d\xi_R}(e^{\lambda\xi}) &= 0 \\ \Rightarrow \lambda^2 e^{\lambda\xi} + \lambda e^{\lambda\xi} &= 0 \Rightarrow (\lambda^2 + \lambda)e^{\lambda\xi} = 0 \Rightarrow \lambda(1 + \lambda) = 0 \Rightarrow \lambda = -1 \text{ or } \lambda = 0 \\ \Rightarrow y_R = A_R e^{-\xi_R} + B_R, \quad y_R(0) = 1 &\Rightarrow y_R = A_R e^{-\xi_R} + (1 - A_R) \end{aligned} \quad (6)$$

c)

We now need to find the unified solution by matching.

$$\left. \begin{aligned} \lim_{x \rightarrow 0} C e^{\frac{1}{x}} &\Rightarrow C = 0 \\ \lim_{\xi_L \rightarrow \infty} e^{-\xi_L} &= 0 \end{aligned} \right\} \Rightarrow y_{match, left} = 0$$

$$\left. \begin{aligned} \lim_{x \rightarrow 1} C e^{\frac{1}{x}} &= 0 \text{ since } C = 0 \\ \lim_{\xi_R \rightarrow \infty} (1 - A_R) + A_R e^{-\xi_R} &= 1 - A_R \\ \Rightarrow A_R &= 1 \end{aligned} \right\} \Rightarrow y_{match, Right} = 0$$

$$y_{unified} = y_R + y_L - y_{match, Right} - y_{match, left} = e^{-\xi_R} + e^{-\xi_L} = e^{\frac{-1+x}{\epsilon}} + e^{\frac{-x}{\sqrt{\epsilon}}} \quad (7)$$

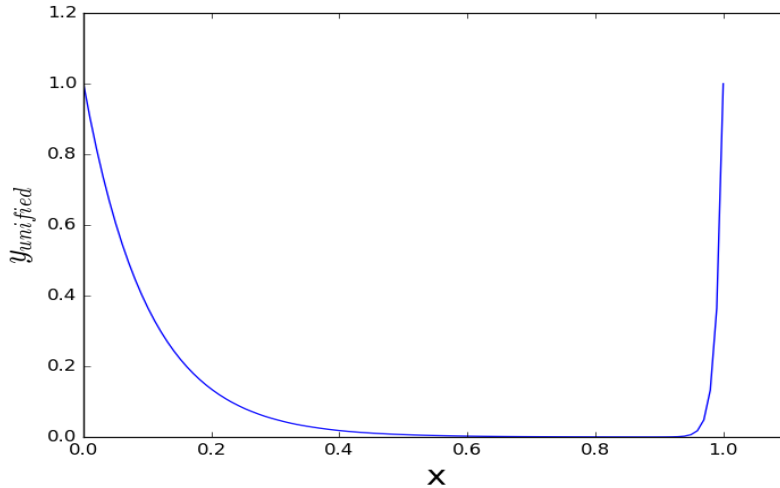


Figure 1: Double boundary layer solution

Problem 50: The perturbed satellite orbit

We seek an approximate solution to:

$$(1 + \epsilon z)^3 \frac{dz}{d\tau^2} + z = 0 \quad , \quad z(0) = 0 \quad , \quad \frac{dz}{d\tau}(0) = 1 \quad (8)$$

by a straightforward perturbation expansion of z in ϵ .

a)

We want to find the first two terms in a series for z .

We can try with a solution of the form:

$$z(\tau) = z_0(\tau) + \epsilon z_1(\tau) + \epsilon^2 z_2(\tau) + \dots$$

I choose to do the perturbation expansion for z up to order ϵ^2 , because even though I do not need it for the first part of the problem, I will need it for the second part of the problem, where I have to check if the expansion breaks down at order ϵ^2 .

$$(1 + 3\epsilon(z_0 + \epsilon z_1 + \dots) + 3\epsilon^2(z_0 + \dots)^2)(z_0'' + \epsilon z_1'' + \epsilon^2 z_2'') + z_0 + \epsilon z_1 + \epsilon^2 z_2 = 0$$

$$(1 + 3\epsilon z_0 + 3\epsilon^2 z_1 + 3\epsilon^2 z_0^2)(z_0'' + \epsilon z_1'' + \epsilon^2 z_2'') + z_0 + \epsilon z_1 + \epsilon^2 z_2 = 0$$

$$z_0'' + \epsilon z_1'' + \epsilon^2 z_2'' + 3\epsilon z_0 z_0'' + 3\epsilon^2 z_0 z_1'' + 3\epsilon^2 z_0^2 z_1' + 3\epsilon^2 z_0^2 z_0'' + z_0 + \epsilon z_1 + \epsilon^2 z_2 = 0$$

For the initial conditions we get:

$$\begin{aligned} z_0(0) + \epsilon z_1(0) + \epsilon^2 z_2(0) &= 0 \\ \Rightarrow z_0(0) &= 0, \quad z_1(0) = 0, \quad z_2(0) = 0 \end{aligned}$$

$$\begin{aligned} z_0'(0) + \epsilon z_1'(0) + \epsilon^2 z_2'(0) &= 1 \\ \Rightarrow z_0'(0) &= 1, \quad z_1'(0) = 0, \quad z_2'(0) = 0 \end{aligned}$$

We then have to collect the terms for ϵ^0 and ϵ^1 :

$$\epsilon^0 : z_0 + z_0 = 0 \quad \Rightarrow \quad z_0 = A_0 \sin(\tau) + B_0 \cos(\tau) \quad (9)$$

$$z_0(0) = 0 \quad \Rightarrow \quad B_0 = 0 \quad \Rightarrow \quad z_0 = A_0 \sin(\tau)$$

$$z_0'(0) = 1 \quad \Rightarrow \quad A_0 = 1$$

$$\Rightarrow z_0 = \sin(\tau) \quad (10)$$

$$\epsilon^1 : z_1'' + 3z_0 z_0'' + z_1 = 0$$

$$\Rightarrow z_1'' + z_1 = -3z_0 z_0'' = 3\sin^2(\tau) = \frac{3}{2}(1 - \cos(2\tau)) = \frac{3}{2} - \frac{3}{2}\cos(2\tau) \quad (11)$$

$$z_1(\tau) = z_1^P + z_1^H = z_1^P + A_1 \cos(\tau) + B_1 \sin(\tau)$$

Where the particular solution z_1^P , must be of the form:

$$z_1^P = a_1 - b_1 \cos(2\tau) \quad (12)$$

We substitute for this solution into eq.(11) and get:

$$\begin{aligned}
4b_1 \cos(2\tau) + a_1 - b_1 \cos(2\tau) &= \frac{3}{2} - \frac{3}{2} \cos(2\tau) \Rightarrow a_1 = \frac{3}{2} \\
\Rightarrow 3b_1 \cos(2\tau) &= -\frac{3}{2} \cos(2\tau) \Rightarrow b_1 = -\frac{1}{2} \\
z_1^P &= \frac{3}{2} + \frac{1}{2} \cos(2\tau) \\
\Rightarrow z_1 &= \frac{3}{2} + \frac{1}{2} \cos(2\tau) + A_1 \cos(\tau) + B_1 \sin(\tau) \\
z_1(0) &= \frac{3}{2} + \frac{1}{2} + A_1 = 0 \Rightarrow A_1 = -2 \\
z_1'(0) &= -\sin(0) + 2\sin(0) + B_1 \cos(0) = B_1 = 0 \\
\Rightarrow z_1 &= \frac{3}{2} + \frac{1}{2} \cos(2\tau) - 2\cos(\tau)
\end{aligned} \tag{13}$$

The first two terms in a series for z are then:

$$z = \underbrace{\sin(\tau)}_{z_0(\tau)} + \underbrace{\frac{3\epsilon}{2} + \frac{\epsilon}{2} \cos(2\tau) - 2\epsilon \cos(\tau)}_{\epsilon z_1(\tau)} \tag{14}$$

b)

To find z_2 , we need to solve the equation that we will get by collecting all the terms of order ϵ^2 in the perturbation expansion for z .

$$\begin{aligned}
z_2'' + z_2 &= -3(-z_0 z_1 + 3z_0^3 - z_0 z_1 - z_0^3) \\
&= 6\sin(\tau) \left(\frac{1}{2} \cos(2\tau) - 2\cos(\tau) + \frac{3}{2} \right) - 6\sin^3(\tau) \\
&= 3\sin(\tau) \cos(\tau) - 12\sin(\tau) \cos(\tau) + 9\sin(\tau) - 6\sin^3(\tau)
\end{aligned}$$

This expression can be rewritten into an expression with only first order sinus terms, using trigonometric formulas:

$$\begin{aligned}
z_2'' + z_2 &= -\frac{3}{2} \sin(\tau) + \frac{3}{2} \sin(3\tau) - 6\sin(2\tau) + 9\sin(\tau) - \frac{9}{2} \sin(\tau) + \frac{3}{2} \sin(3\tau) \\
&= 3\sin(3\tau) - 6\sin(2\tau) + 3\sin(\tau)
\end{aligned} \tag{15}$$

$$z_2(\tau) = z_2^P + z_2^H$$

We can use the method of undetermined coefficients to find the particular solution, which will be the sum of the particular solutions to:

$$z_{2,1}'' + z_{2,1} = 3\sin(\tau) \Rightarrow z_{2,1}^P = a_2 \tau \cos(\tau) + b_2 \tau \sin(\tau) \tag{16}$$

$$z_{2,2}'' + z_{2,2} = -6\sin(2\tau) \Rightarrow z_{2,2}^P = c_2 \cos(2\tau) + d_2 \sin(2\tau) \tag{17}$$

$$z_{2,3}'' + z_{2,3} = 3\sin(3\tau) \Rightarrow z_{2,3}^P = e_2 \cos(3\tau) + f_2 \sin(3\tau) \tag{18}$$

We can see from the first part of the particular solution (16), that we have non-periodic terms that will grow in time so $\epsilon^2 \tau \gg 1$ and $\epsilon^2 z_2 \gg z_0$. This means that the straightforward perturbation expansion breaks down at order ϵ^2 . The remedy to this problem would be the **Poincaré-Lindstedt method**.

We introduce a rescaled time, set $\sigma = \omega \tau$ and seek a solution of period 2π in σ , so $\omega \tau$ will change by 2π for each period.

The rescaled equation is:

$$\frac{d^2 z}{d\tau^2} = \omega^2 \frac{d^2 z}{d\sigma^2} \Rightarrow \omega^2 (1 + \epsilon z)^3 \frac{d^2 z}{d\sigma^2} + z = 0 \quad (19)$$

$$z(0) = 0, \quad \omega \frac{dz}{d\sigma}(0) = 1 \quad (20)$$

$$z = z_0 + \epsilon z_1 + \epsilon^2 z_2 \quad \text{and} \quad \omega = \omega_0 + \epsilon \omega_1 + \epsilon^2 \omega_2$$

Since the straightforward perturbation expression worked for the first two terms and we didn't have to use the Poincaré-Lindstedt method for those two terms, we set $\omega_0 = 1$ and $\omega_1 = 0$ so $\mathcal{O}(\epsilon^0)$ and $\mathcal{O}(\epsilon^1)$ remain unaltered.

$$(\omega_0 + \epsilon \omega_1 + \epsilon^2 \omega_2)(1 + 3\epsilon z_0 + 3\epsilon^2 z_1 + 3\epsilon^2 z_0^2)(z_0'' + \epsilon z_1'' + \epsilon^2 z_2'') + z_0 + \epsilon z_1 + \epsilon^2 z_2 = 0$$

$$\begin{aligned} \epsilon^2 = z_2'' + z_2 &= -6z_0^3 + 6z_0 z_1 - 2\omega_2 z_0'' = -6z_0^3 + 6z_0 z_1 + 2\omega_2 z_0 \\ &= 3\sin(3\tau) + 3\sin(\tau) - 6\sin(2\tau) - 2\omega_2 \sin(\tau) \\ &= 3\sin(3\tau) - 6\sin(2\tau) + (3 - 2\omega_2)\sin(\tau) \end{aligned}$$

$$(3 - 2\omega_2) \text{ must be 0 to avoid secular or non-periodic terms.} \Rightarrow \omega_2 = -\frac{3}{2}$$

We have now that:

$$z_2'' + z_2 = 3\sin(3\tau) - 6\sin(2\tau) \quad (21)$$

$$\begin{aligned} z_2 &= z_2^H + z_2^P \quad \text{where} \quad z_2^H = A_2 \sin(\tau) + B_2 \cos(\tau) \\ z_2^P &= a_2 \cos(2\tau) + b_2 \cos(3\tau) + c_2 \sin(2\tau) + d_2 \sin(3\tau) \end{aligned} \quad (22)$$

We then need to substitute (22) into (21).

$$\begin{aligned} z_2'' &= -4a_2 \cos(2\tau) - 9b_2 \cos(3\tau) - 4c_2 \sin(2\tau) - 9d_2 \sin(3\tau) \\ z_2'' + z_2 &= -4a_2 \cos(2\tau) - 9b_2 \cos(3\tau) - 4c_2 \sin(2\tau) - 9d_2 \sin(3\tau) \\ &\quad + a_2 \cos(2\tau) + b_2 \cos(3\tau) + c_2 \sin(2\tau) + d_2 \sin(3\tau) \\ &= -3a_2 \cos(2\tau) - 8b_2 \cos(3\tau) - 3c_2 \sin(2\tau) - 8d_2 \sin(3\tau) \\ &= 3\sin(3\tau) - 6\sin(2\tau) \Rightarrow a_2 = b_2 = 0 \\ &\Rightarrow -3c_2 = -6 \Rightarrow c_2 = 2 \\ &\Rightarrow -8d_2 = -3 \Rightarrow d_2 = -\frac{3}{8} \\ &\Rightarrow z_2^P = 2\sin(2\tau) - \frac{3}{8}\sin(3\tau) \\ z_2 &= z_2^H + z_2^P = A_2 \sin(\tau) + B_2 \cos(\tau) + 2\sin(2\tau) - \frac{3}{8}\sin(3\tau) \end{aligned} \quad (23)$$

The initial conditions will give us the constants A_2 and B_2 .

$$\begin{aligned} \omega \frac{dz}{d\sigma}(0) = 1 &\Rightarrow (1 + \epsilon^2 \omega_2)(z_0'(0) + \epsilon z_1'(0) + \epsilon^2 z_2'(0)) = 1 \\ &\Rightarrow \epsilon^0 : z_0'(0) = 1 \\ &\Rightarrow \epsilon^1 : z_1'(0) = 0 \\ &\Rightarrow \epsilon^2 : z_2'(0) + \omega_2 \end{aligned}$$

$$\begin{aligned}
z'_0(0) = 0 &\Rightarrow z'_2(0) = -\omega_2 \Rightarrow z'_0(0) = \frac{3}{2} \\
z_2(0) = B_2 = 0 &\Rightarrow z_2 = A_2 \sin(\tau) + 2 \sin(2\tau) - \frac{3}{8} \sin(3\tau) \\
z'_2(0) = A_2 \cos(\tau) + 4 \cos(2\tau) - \frac{9}{8} \sin(3\tau) = A_2 + 4 = \frac{3}{2} &\Rightarrow A_2 = -\frac{5}{2} \\
z_2 = -\frac{5}{2} \sin(\tau) + 2 \sin(2\tau) - \frac{3}{8} \sin(3\tau) & \tag{24}
\end{aligned}$$

Finally we have the full solution to order ϵ^2 :

$$\begin{aligned}
z &= z_0 + \epsilon z_1 + \epsilon^2 z_2 \\
&= \sin(\tau) + \epsilon \left(\frac{1}{2} \cos(2\tau) - 2 \cos(\tau) + \frac{3}{2} \right) + \epsilon^2 \left(-\frac{3}{8} \sin(3\tau) + 2 \sin(2\tau) - \frac{5}{2} \sin(\tau) \right) \tag{25}
\end{aligned}$$