MEK4300
Mandatory assignment 1

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Problem 1

We are considering a 2D ($\frac{\partial}{\partial z} = 0$), gravity driven, steady ($\frac{\partial u}{\partial t} = 0$), laminar and viscous Poiseuille flow between two parallel inclined plates as shown in figure:2.
The fluid velocity is tangential to the plates, $\vec{\mathbf{v}} = (u = u(y), v = 0, w = 0)$

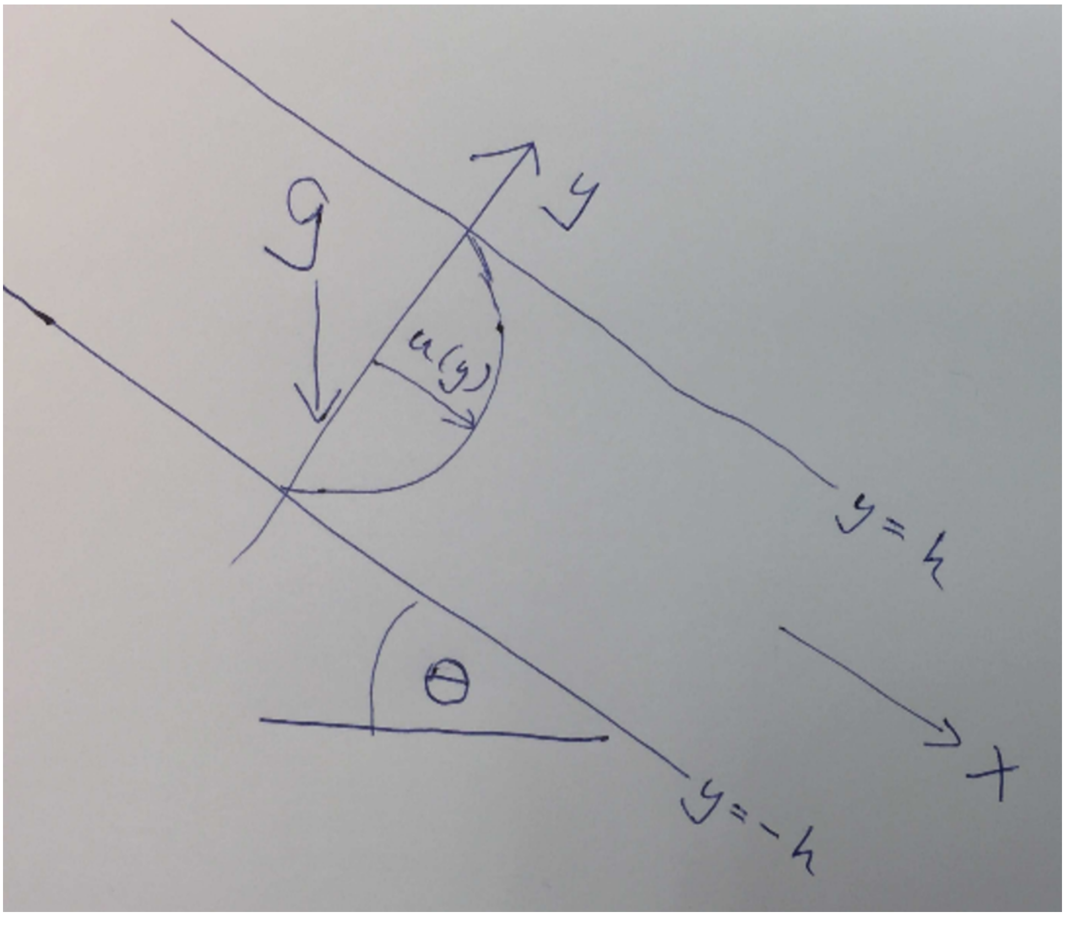

Figure 1: Sketch of flow configuration in problem 1

The fluid in between the plates has following fluid quantities:
ρ : Fluid density
ν : Kinematic viscosity
μ : Dynamic viscosity
g : Acceleration of gravity

a)

Kinematic boundary conditions at $y = \pm h$:
Because of the no-slip condition at the boundaries we have that $u(y = \pm h) = 0$

b)

The x-component of the momentum equation for this steady flow will be:

$$\rho(\cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}} + \cancel{w\frac{\partial u}{\partial z}}) = -\cancel{\frac{\partial P}{\partial x}} + \rho g_x + \mu(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}})$$

$$\Rightarrow 0 = \rho g \sin(\theta) + \mu(\frac{\partial^2 u}{\partial y^2})$$

c)

We now want to find the velocity profile of this flow:

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\rho}{\mu} g \sin(\theta)$$

$$\frac{\partial u}{\partial y} = (-\frac{\rho}{\mu} g \sin(\theta))y + C_1$$

$$u(y) = -\frac{\rho}{2\mu}\rho g \sin(\theta) y^2 + C_1 y + C_2$$

We then find the constants using the boundary conditions at $y = \pm h$

$$u(h) = u(-h) = 0$$

$$\Rightarrow -\frac{1}{2\mu}\rho g \sin(\theta))h^2 + C_1 h + C_2 = -\frac{1}{2\mu}\rho g \sin(\theta))h^2 - C_1 h + C_2 = 0$$

$$\Rightarrow C_1 = 0$$

$$\Rightarrow C_2 = \frac{h^2}{2\mu}\rho g \sin(\theta)$$

$$\Rightarrow u(y) = \frac{\rho g \sin(\theta)}{2\mu}(h^2 - y^2)$$

We can now use the expression for the velocity profile to obtain the volume flux of this flow:

$$Q = \int_{-h}^{h} u dy = \int_{-h}^{h} \frac{h^2}{2\mu} \rho g \sin(\theta) dy - \int_{-h}^{h} \frac{y^2}{2\mu} \rho g \sin(\theta) dy$$

$$= -\frac{\rho g \sin(\theta) h^3}{2\mu} - \frac{\rho g \sin(\theta) h^3}{2\mu} + \frac{\rho g \sin(\theta) h^3}{6\mu} + \frac{\rho g \sin(\theta) h^3}{6\mu}$$

$$= \frac{2\rho g \sin(\theta) h^3}{3\mu}$$

d)

We now want to evaluate the velocity shear and the shear stresses $\tau_{xy} = \tau_{yx} = \mu \frac{du}{dy}$ at both the upper and the lower plate.
For the velocity shear VS we have:

$$VS = \frac{\partial u}{\partial y} = -\frac{\rho g \sin(\theta) y}{\mu}$$

Shear stress on the surface of the upper plate τ_{up}, which is facing the negative y-direction (therefore the minus sign) is:

$$\tau_{\text{ upper plate}} = -\mu \frac{du}{dy}\bigg|_{y=h} = \rho g h \sin(\theta)$$

and we also get the same shear stress on the surface of the lower plate which is facing the positive y-direction:

$$\tau_{\text{ lower plate}} = \mu \frac{du}{dy}\bigg|_{y=-h} = \rho g h \sin(\theta)$$

Problem 2

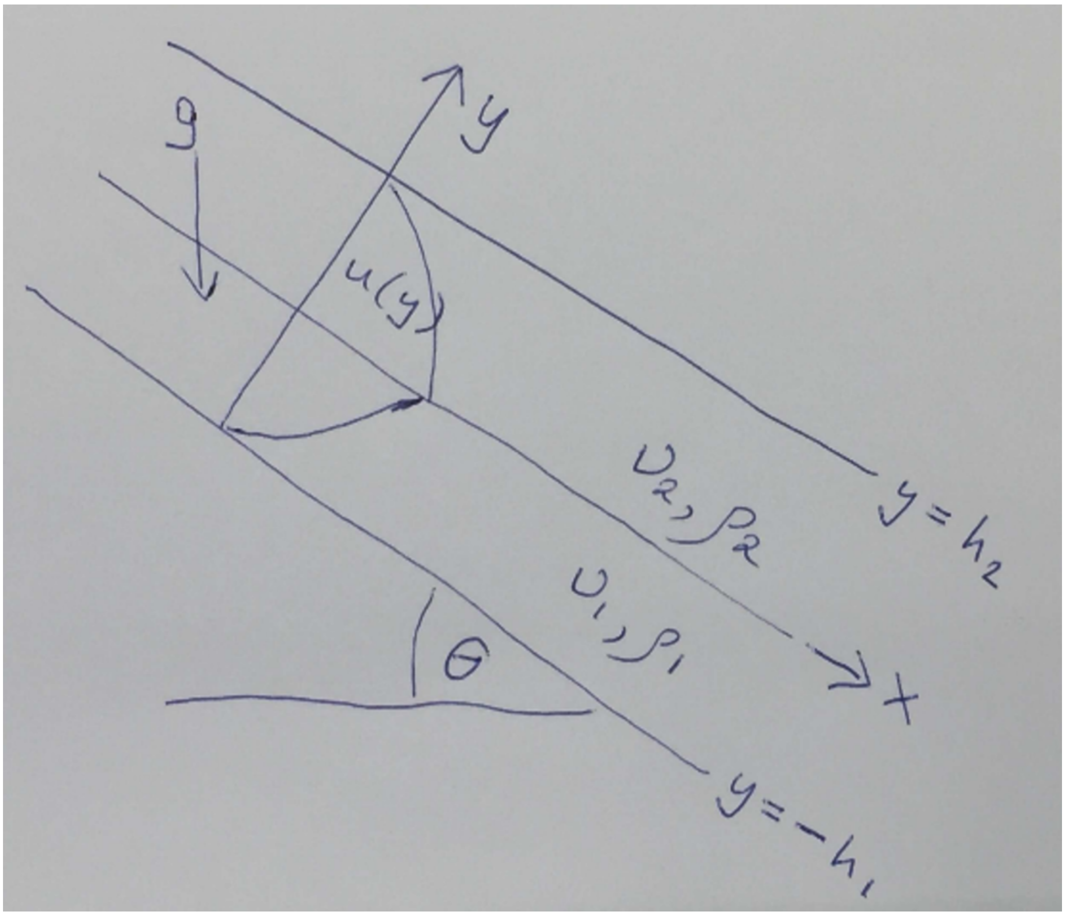

Figure 2: Sketch of flow configuration in problem 2

a)

Because of the No slip boundary condition, the kinematic boundary conditions at the surface of the plates will be:

$$u_1(y = -h_1) = 0 \tag{1}$$

$$u_2(y = h_2) = 0 \tag{2}$$

b)

The velocity is continuous at the interface between the fluid layers where y=0, so the kinematic boundary condition will be:

$$u_1 = u_2 \qquad \text{when} \qquad y = 0 \tag{3}$$

c)

The pressure and the shear stresses are continuous at the fluid interface where y=0, so the dynamic boundary condition at the interface between the fluid layers will be:

$$P_1 = P_2 \tag{4}$$

$$\begin{aligned} \tau_{yx,1} &= \tau_{yx,2} \\ \mu_1 \frac{\partial u_1}{\partial y} &= \mu_2 \frac{\partial u_2}{\partial y} \end{aligned} \tag{5}$$

d)

The flow is steady: $\frac{\partial u}{\partial t} = 0$

2D : $\frac{\partial}{\partial z} = 0$

And the fluid velocity only depends on the y-coordinate: $u = u(y)$ which means $v = w = 0$

the x-component of the momentum equation for fluid layer 1 :

$$0 = \rho_1 g \sin(\theta) + \mu_1 (\frac{\partial^2 u_1}{\partial y^2})$$

the x-component of the momentum equation for fluid layer 2 :

$$0 = \rho_2 g \sin(\theta) + \mu_2 (\frac{\partial^2 u_2}{\partial y^2})$$

e)

$$\begin{aligned} \frac{\partial^2 u_1}{\partial y^2} &= -\frac{\rho_1}{\mu_1} g \sin(\theta) \\ \frac{\partial^2 u_2}{\partial y^2} &= -\frac{\rho_2}{\mu_2} g \sin(\theta) \end{aligned}$$

$$\begin{aligned} \frac{\partial u_1}{\partial y} &= -\frac{\rho_1}{\mu_1} g \sin(\theta) y + C_1 \\ \frac{\partial u_2}{\partial y^2} &= -\frac{\rho_2}{\mu_2} g \sin(\theta) y + D_1 \end{aligned}$$

$$u_1(y) = -\frac{\rho_1}{2\mu_1} g \sin(\theta) y^2 + C_1 y + C_2 \tag{6}$$

$$u_2(y) = -\frac{\rho_2}{2\mu_2} g \sin(\theta) y^2 + D_1 y + D_2 \tag{7}$$

We then need to find the constants using the boundary conditions (1)-(5)
BC (5) gives us :

$$\mu_1 C_1 = \mu_2 D_1 \quad \Rightarrow \quad C_1 = \frac{\mu_2}{\mu_1} D_1 \tag{8}$$

BC (1) and (2) gives us:

$$
\begin{aligned}
u_1(-h_1) &= -\frac{\rho_1}{2\mu_1} g\sin(\theta)h_1^2 - C_1 h_1 + C_2 \\
\Rightarrow C_2 &= \frac{\rho_1}{2\mu_1} g\sin(\theta)h_1^2 + C_1 h_1 \\
&= \frac{\rho_1}{2\mu_1} g\sin(\theta)h_1^2 + \frac{\mu_2}{\mu_1} D_1 h_1
\end{aligned} \tag{9}
$$

where we have used that $C_1 = \frac{\mu_2}{\mu_1} D_1$

$$
\begin{aligned}
u_2(h_2) &= -\frac{\rho_2}{2\mu_2} g\sin(\theta)h_2^2 + D_1 h_2 + D_2 \\
\Rightarrow D_2 &= \frac{\rho_2}{2\mu_2} g\sin(\theta)h_2^2 - D_1 h_2
\end{aligned} \tag{10}
$$

BC (3) gives us that $C_2 = D_2$ also:

$$
\frac{\rho_2}{2\mu_2} g\sin(\theta)h_2^2 - D_1 h_2 = \frac{\rho_1}{2\mu_1} g\sin(\theta)h_1^2 + \frac{\mu_2}{\mu_1} D_1 h_1 \tag{11}
$$

We can solve (11) for D_1, which gives us:

$$
D_1 = \frac{\rho_1 g\sin(\theta)h_1^2}{2(\mu_2 h_1 + \mu_1 h_2)} + \frac{\mu_1 \rho_2 g\sin(\theta)h_2^2}{2\mu_2(\mu_2 h_1 + \mu_1 h_2)} \tag{12}
$$

$$
\tag{13}
$$

By inserting (12) into (8) we get an expression for C_1:

$$
C_1 = \frac{\mu_2 \rho_1 g\sin(\theta)h_1^2}{2\mu_1(\mu_2 h_1 + \mu_1 h_2)} + \frac{\mu_2 \mu_1 \rho_2 g\sin(\theta)h_2^2}{2\mu_1\mu_2(\mu_2 h_1 + \mu_1 h_2)} \tag{14}
$$

$$
\tag{15}
$$

We can now use C_1 and D_1 to complete the expressions for C_2 and D_2. This is done by inserting (12) and (13) into expressions (10) and (9) respectively.

$$
\begin{aligned}
C_2 &= \frac{\rho_1}{2\mu_1} g\sin(\theta)h_1^2 + C_1 h_1 \\
&= \frac{\rho_1}{2\mu_1} g\sin(\theta)h_1^2 \\
&+ \frac{\mu_2 \rho_1 g\sin(\theta)h_1^3}{2\mu_1(\mu_2 h_1 + \mu_1 h_2)} + \frac{\mu_2 \mu_1 \rho_2 g\sin(\theta)h_1 h_2^2}{2\mu_1\mu_2(\mu_2 h_1 + \mu_1 h_2)} \\
&= \frac{h_1^2}{\mu_1}\frac{\rho_1}{2} g\sin(\theta) \\
&+ \frac{h_1^3 \mu_2 \rho_1 g\sin(\theta)}{2\mu_1(\mu_2 h_1 + \mu_1 h_2)} + \frac{h_1 h_2^2 \mu_2 \mu_1 \rho_2 g\sin(\theta)}{2\mu_1\mu_2(\mu_2 h_1 + \mu_1 h_2)}
\end{aligned} \tag{16}
$$

And D_2 will be:

$$\begin{aligned}
D_2 &= \frac{\rho_2}{2\mu_2} g \sin(\theta) h_2^2 - D_1 h_2 \\
&= \frac{\rho_2}{2\mu_2} g \sin(\theta) h_2^2 - \left(\frac{\rho_1 g \sin(\theta) h_1^2 h_2}{2(\mu_2 h_1 + \mu_1 h_2)} + \frac{\mu_1 \rho_2 g \sin(\theta) h_2^3}{2\mu_2(\mu_2 h_1 + \mu_1 h_2)}\right) \\
&= \frac{h_2^2}{2\mu_2} \rho_2 g \sin(\theta) - \frac{h_1^2 h_2 \rho_1 g \sin(\theta)}{2(\mu_2 h_1 + \mu_1 h_2)} - \frac{\mu_1 h_2^3 \rho_2 g \sin(\theta)}{2\mu_2(\mu_2 h_1 + \mu_1 h_2)}
\end{aligned} \tag{17}$$

by putting the expressions we have found for the constants in (6) and (7) we have the expressions for the velocity fields $u_1(y)$ and $u_2(y)$:

$$\begin{aligned}
u_1(y) &= -\frac{\rho_1}{2\mu_1} g \sin(\theta) y^2 \\
&+ \left(\frac{\mu_2 \rho_1 g \sin(\theta) h_1^2}{2\mu_1(\mu_2 h_1 + \mu_1 h_2)} + \frac{\mu_2 \mu_1 \rho_2 g \sin(\theta) h_2^2}{2\mu_1 \mu_2(\mu_2 h_1 + \mu_1 h_2)}\right) y \\
&+ \frac{h_1^2}{\mu_1} \frac{\rho_1}{2} g \sin(\theta) + \frac{h_1^3 \mu_2 \rho_1 g \sin(\theta)}{2\mu_1(\mu_2 h_1 + \mu_1 h_2)} + \frac{h_1 h_2^2 \mu_2 \mu_1 \rho_2 g \sin(\theta)}{2\mu_1 \mu_2(\mu_2 h_1 + \mu_1 h_2)}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad u_1(y) &= \rho_1 g \sin(\theta) \left(\frac{h_1^2 - y^2}{2\mu_1}\right) \\
&+ \frac{\rho_1 g \sin(\theta)(\mu_2 h_1^2 (y + h_1)) + \rho_2 g \sin(\theta)(\mu_1 h_2^2 (y + h_1))}{2\mu_1(\mu_2 h_1 + \mu_1 h_2)}
\end{aligned} \tag{18}$$

$$\begin{aligned}
u_2(y) &= -\frac{\rho_2}{2\mu_2} g \sin(\theta) y^2 \\
&+ \left(\frac{\rho_1 g \sin(\theta) h_1^2}{2(\mu_2 h_1 + \mu_1 h_2)} + \frac{\mu_1 \rho_2 g \sin(\theta) h_2^2}{2\mu_2(\mu_2 h_1 + \mu_1 h_2)}\right) y \\
&+ \frac{h_2^2}{2\mu_2} \rho_2 g \sin(\theta) - \frac{h_1^2 h_2 \rho_1 g \sin(\theta)}{2(\mu_2 h_1 + \mu_1 h_2)} - \frac{\mu_1 h_2^3 \rho_2 g \sin(\theta)}{2\mu_2(\mu_2 h_1 + \mu_1 h_2)}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad u_2(y) &= \rho_2 g \sin(\theta) \left(\frac{h_2^2 - y^2}{2\mu_2}\right) \\
&+ \frac{\rho_1 g \sin(\theta)(\mu_2 h_1^2 (y - h_2)) + \rho_2 g \sin(\theta)(\mu_1 h_2^2 (y - h_2))}{2\mu_2(\mu_2 h_1 + \mu_1 h_2)}
\end{aligned} \tag{19}$$

We can now use the expressions (16) and (17) for the velocity fields to obtain the volume flux in each of the layers.

$$
Q_1 = \int_{-h_1}^{0} u_1 dy = \rho_1 g \sin(\theta)(\frac{h_1^3}{3\mu_1})
$$

$$
- \rho_1 g \sin(\theta)\left(\frac{\mu_2 h_1^4}{4\mu_1(\mu_2 h_1 + \mu_1 h_2)}\right) + \rho_2 g \sin(\theta)\left(\frac{\mu_1 h_2^2 h_1^2}{4\mu_1(\mu_2 h_1 + \mu_1 h_2)}\right)
$$

$$
Q_2 = \int_{0}^{h_2} u_2 dy = \rho_2 g \sin(\theta)(\frac{h_2^3}{3\mu_2})
$$

$$
- \rho_1 g \sin(\theta)\left(\frac{\mu_2 h_1^2 h_2^2}{4\mu_2(\mu_2 h_1 + \mu_1 h_2)}\right) - \rho_2 g \sin(\theta)\left(\frac{\mu_1 h_2^4}{4\mu_2(\mu_2 h_1 + \mu_1 h_2)}\right)
$$

f)

We now want to evaluate the velocity shear VS which is $\frac{du}{dy}$ and the shear stresses at both the upper and the lower plate and also at the interface of the two fluids between the plates.

$$
VS_{\text{ layer1}} = \frac{du_1}{dy} = -\frac{y}{\mu_1}\rho_1 g \sin(\theta) + \frac{\mu_2 \rho_1 g \sin(\theta) h_1^2}{2\mu_1(\mu_2 h_1 + \mu_1 h_2)} + \frac{\mu_1 \rho_2 g \sin(\theta) h_2^2}{2\mu_1(\mu_2 h_1 + \mu_1 h_2)}
$$

$$
\tau_1 = \mu_1 \frac{du_1}{dy} = \mu_1\left(- \frac{y}{\mu_1}\rho_1 g \sin(\theta) + \frac{\mu_2 \rho_1 g \sin(\theta) h_1^2}{2\mu_1(\mu_2 h_1 + \mu_1 h_2)} + \frac{\mu_1 \rho_2 g \sin(\theta) h_2^2}{2\mu_1(\mu_2 h_1 + \mu_1 h_2)}\right)
$$

$$
VS_{\text{ layer2}} = \frac{du_2}{dy} = -\frac{y}{\mu_2}\rho_2 g \sin(\theta) + \frac{\mu_2 \rho_1 g \sin(\theta) h_1^2}{2\mu_2(\mu_2 h_1 + \mu_1 h_2)} + \frac{\mu_1 \rho_2 g \sin(\theta) h_2^2}{2\mu_2(\mu_2 h_1 + \mu_1 h_2)}
$$

$$
\tau_2 = \mu_2 \frac{du_2}{dy} = \mu_2\left(- \frac{y}{\mu_2}\rho_2 g \sin(\theta) + \frac{\mu_2 \rho_1 g \sin(\theta) h_1^2}{2\mu_2(\mu_2 h_1 + \mu_1 h_2)} + \frac{\mu_1 \rho_2 g \sin(\theta) h_2^2}{2\mu_2(\mu_2 h_1 + \mu_1 h_2)}\right)
$$

The upper plate is facing the negative y-direction (therefore the minus sign). so the shear stress at the plates are:

$$\tau_{\text{ upper plate}} = -\mu_2 \frac{du_2}{dy}\bigg|_{y=h_2}$$

$$= -\mu_2\bigg(-\frac{h_2}{\mu_2}\rho_2 g\sin(\theta) + \frac{\mu_2\rho_1 g\sin(\theta)h_1^2}{2\mu_2(\mu_2 h_1+\mu_1 h_2)} + \frac{\mu_1\rho_2 g\sin(\theta)h_2^2}{2\mu_2(\mu_2 h_1+\mu_1 h_2)}\bigg)$$

$$\tau_{\text{ lower plate}} = \mu_1 \frac{du_1}{dy}\bigg|_{y=-h_1}$$

$$= \mu_1\bigg(\frac{h_1}{\mu_1}\rho_1 g\sin(\theta) + \frac{\mu_2\rho_1 g\sin(\theta)h_1^2}{2\mu_1(\mu_2 h_1+\mu_1 h_2)} + \frac{\mu_1\rho_2 g\sin(\theta)h_2^2}{2\mu_1(\mu_2 h_1+\mu_1 h_2)}\bigg)$$

At the interface where $y = 0$ the shear stresses are equal.

$$\tau_{\text{ Interface}} \quad \Rightarrow \quad \mu_2 \frac{du_2}{dy}\bigg|_{y=0} = \mu_1 \frac{du_1}{dy}\bigg|_{y=0}$$

$$\Rightarrow \quad \mu_2\bigg(\frac{\mu_2\rho_1 g\sin(\theta)h_1^2}{2\mu_2(\mu_2 h_1+\mu_1 h_2)} + \frac{\mu_1\rho_2 g\sin(\theta)h_2^2}{2\mu_2(\mu_2 h_1+\mu_1 h_2)}\bigg)$$
$$=$$
$$\mu_1\bigg(\frac{\mu_2\rho_1 g\sin(\theta)h_1^2}{2\mu_1(\mu_2 h_1+\mu_1 h_2)} + \frac{\mu_1\rho_2 g\sin(\theta)h_2^2}{2\mu_1(\mu_2 h_1+\mu_1 h_2)}\bigg)$$
$$=$$
$$\frac{\mu_2\rho_1 g\sin(\theta)h_1^2}{2(\mu_2 h_1+\mu_1 h_2)} + \frac{\mu_1\rho_2 g\sin(\theta)h_2^2}{2(\mu_2 h_1+\mu_1 h_2)}$$