Flat plate integral analysis

We have the following velocity profile:

$$u = U\sin(\frac{\pi y}{2\delta}) \tag{1}$$

To check for how realistic this velocity profile is, we have to check which boundary conditions that will be fulfilled with this profile.

$$u(0) = U\sin(0) = 0 \tag{2}$$
$$u(\delta) = U\sin(\frac{\pi}{2}) = U \tag{3}$$

$$\left.\frac{du}{dy}\right|_{y=\delta} = \frac{\pi}{2\delta}U\cos(\frac{\pi}{2}) = 0 \tag{4}$$

$$\left.\frac{d^2u}{dy^2}\right|_{y=0} = -\frac{\pi^2}{4\delta^2}U\sin(0) = 0 \tag{5}$$

We can conclude that this sinus shaped velocity profile is a more realistic profile than the quadratic one ((4-11) in the book), since this velocity profile also satisfies the zero pressure-gradiet condition (5), whereas the parabolic velocity profile only satisfies (2) - (4).

For the momentum thickness we have:

$$\begin{aligned}
\theta &= \int_0^\delta \frac{u}{U}(1-\frac{u}{U})dy = \delta\int_0^1 \frac{u}{U}(1-\frac{u}{U})d\eta \quad , \quad \text{where: } \eta = \frac{y}{\delta} \\
\theta &= \delta\int_0^1 \sin(\frac{\pi}{2}\eta)(1-\sin(\frac{\pi}{2}\eta))d\eta \\
&= \delta\int_0^1 (\sin(\frac{\pi}{2}\eta) - \sin^2(\frac{\pi}{2}\eta))d\eta \\
&= \delta\Big[-\frac{2}{\pi}cos(\frac{\pi}{2}\eta) - \frac{\eta}{2} + \frac{1}{2\pi}\sin(\pi\eta)\Big]_0^1 \\
&= \delta(-\frac{1}{2}+\frac{2}{\pi}) = \frac{4-\pi}{2\pi}\delta = 0.1366\ \delta
\end{aligned} \tag{6}$$

Displacement thickness:

$$\delta^* = \int_0^\delta (1-\frac{u}{U})dy = \int_0^\delta (1-\sin(\frac{\pi y}{2\delta}))dy \tag{7}$$

$$\begin{aligned}
y &= \eta\delta \quad , \quad dy = \delta d\eta \\
y &= 0 \Rightarrow \eta = 0 \\
y &= \delta \Rightarrow \eta = 1
\end{aligned}$$

$$\Longrightarrow \delta^* = \delta\int_0^1 (1-\sin(\frac{\pi}{2}\eta))d\eta = \Big[\eta + \frac{2}{\pi}\cos(\frac{\pi}{2})\Big]_0^1 = \delta(1-\frac{2}{\pi}) = 0.3634\ \delta \tag{8}$$

For the friction coefficient we have:

$$C_f = \frac{t\tau_w}{\rho U^2} = 2\frac{d\theta}{dx}, \qquad \text{where: } \tau_w = \mu\left.\frac{du}{dy}\right|_{y=0} = \mu\frac{\pi}{2\delta}Ucos(0) = \mu\frac{\pi U}{2\delta}$$

$$\Longrightarrow C_f = \frac{\mu\pi}{\rho U\delta} = 2\frac{d}{dx}(0.1366\delta) = 2\cdot 0.1366\frac{d\delta}{dx} = \frac{\pi\mu}{\rho U\delta} \tag{9}$$

We can then use (9) to find the boundary layer thickness δ:

$$\int \delta d\delta = \int \frac{1}{2 \cdot 0.1366} \frac{\mu\pi}{\rho U} dx \quad \Rightarrow \quad \delta = \sqrt{\frac{\pi\mu x}{0.1366\rho U}} = 4.796\sqrt{\frac{\mu x}{\rho U}} \tag{10}$$

$$\Rightarrow \frac{\delta}{x} = \sqrt{\frac{\pi}{0.1366}}\sqrt{\frac{\mu}{\rho U x}} = \frac{4.796}{\sqrt{Re_x}} \tag{11}$$

$$\text{where: } Re_x = \frac{\rho U x}{\mu}$$

We can then substitute the expression for the boundary layer thickness (10) into (9), which gives us:

$$C_f = \frac{\pi\mu}{\rho U\delta} = \frac{\pi\mu}{\rho U}\frac{1}{\sqrt{\frac{\pi\mu x}{0.1366\rho U}}} = \sqrt{\frac{0.1366\pi^2\mu^2\rho U}{\pi\mu\rho^2U^2x}} = \sqrt{\frac{0.1366\pi}{Re_x}} \tag{12}$$

We can use the expression for friction coefficient (9) to find the drag coefficient.

$$C_D(L) = \frac{1}{L}\int_0^L C_f(x)dx = \sqrt{\frac{0.1366\pi\mu}{L^2\rho U}}\int_0^L \frac{1}{\sqrt{x}}dx$$

$$\Rightarrow C_D(L) = 0.655\sqrt{\frac{\mu}{\rho U L^2}}\left[2\sqrt{x}\right]_0^L = 1.3101\sqrt{\frac{\mu}{\rho U L}} = \frac{1.3101}{\sqrt{Re_L}} \tag{13}$$

$$\text{where: } Re_L = \frac{\rho U L}{\mu}$$

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 L} \quad \Rightarrow \quad D = 0.655\sqrt{\mu\rho U^3 L} \tag{14}$$

We have the following estimates for the boundary layer parameters:

$$\frac{\theta}{x}\sqrt{Re_x} = 0.1366\frac{\delta}{x}\sqrt{Re_x} = \sqrt{\frac{0.1366\pi\mu}{\rho U x}}\sqrt{Re_x} = 0.6550$$

$$\frac{\delta^*}{x}\sqrt{Re_x} = 0.3634\sqrt{\frac{\pi\mu}{0.1366\rho U x}}\sqrt{Re_x} = 1.7427$$

$$\frac{\delta}{x}\sqrt{Re_x} = 4.796$$

$$C_f\sqrt{Re_x} = \sqrt{0.1366\pi} = 0.6550$$

$$C_D\sqrt{Re_L} = 1.3101$$

Comparison to Blasius:

Parameter	u from eq.(1)	Exact from Blasius	Error, %
$\frac{\theta}{x}\sqrt{Re_x}$	0.655	0.664	-1.36
$\frac{\delta^*}{x}\sqrt{Re_x}$	1.743	1.721	+1.3
$\frac{\delta}{x}\sqrt{Re_x}$	4.796	5.0	-4
$C_f\sqrt{Re_x}$	0.655	0.664	-1.36
$C_D\sqrt{Re_L}$	1.310	1.328	-1.35

Figure:(1) shows the drag coefficient of a flat plate as a function of the Reynold's number.

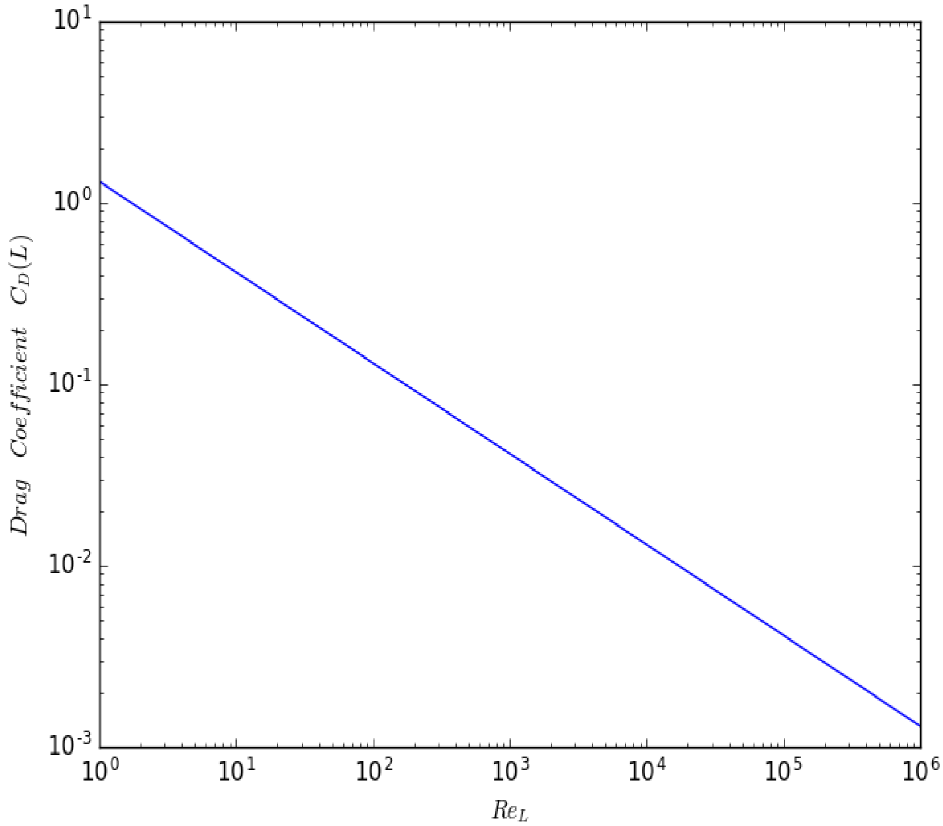

Figure 1: Drag of a plate