Fibonacci numbers

Definition

The Fibonacci sequence is defined as follows:

$F_0 = 0,$

$F_1 = 1,$

$F_n = F_{n-1} + F_{n-2}.$

The first few of its members:

$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ ,\ 55,\ 89,\ \ldots$

History

These numbers are introduced in 1202 by Leonardo Fibonacci (Leonardo Fibonacci) (also known as Leonardo of Pisa (Leonardo Pisano)). However, thanks to a 19th-century mathematician Luca (Lucas) the name "Fibonacci numbers" became common.

However, the number of Indian mathematicians mentioned earlier in this sequence: Gopal (Gopala) until 1135, Hemachandra (Hemachandra) - in 1150

Fibonacci numbers in nature

Fibonacci himself mentioned these numbers in connection with this challenge: "A man planted a couple of rabbits in the paddock, surrounded on all sides by a wall. How many pairs of rabbits can produce per year to light the couple, if you know that every month, starting from the second, each pair rabbits produces one pair of light? ". The solution to this problem and will be the number sequence, now called in his honor. However, the situation described by Fibonacci - more mind game than real nature.

Indian mathematicians Gopala and Hemachandra mentioned this number sequence in relation to the number of rhythmic patterns, resulting from the alternation of long and short syllables in verse or strong and weak beats in music. The number of patterns having generally $n$shares power F_n.

Fibonacci numbers appear in the work of Kepler in 1611, which reflected on the numbers found in nature (work "On the hexagonal flakes").

An interesting example of a plant - yarrow, in which the number of stems (and hence flowers) is always a Fibonacci number. The reason is simple: while initially with a single stem, this stem is then divided by two, and then branches off from the main stem of another, then the first two stem branch again, then all the stems, but the last two, branch, and so on. Thus, each stalk after his appearance "misses" one branch, and then begins to divide at each level of branching, which results in a Fibonacci number.

Generally speaking, many colors (eg, lilies), the number of petals is a way or another Fibonacci number.

Also botanically known phenomenon phyllotaxis"". As an example, the location of sunflower seeds: if you look down on their location, you can see two simultaneous series of spirals (like overlapping): some twisted clockwise, the other - against. It turns out that the number of these spirals is roughly equal to two consecutive Fibonacci numbers: 34 and 55 or 89 and 144. Similar facts are true for some other colors, as well as pine cones, broccoli, pineapple, etc.

For many plants (according to some sources, 90% of them) are true and an interesting fact. Consider any leaf, and will descend from him down until until we reach the sheet disposed on the stem in the same way (ie, directed exactly in the same direction). Along the way, we assume that all the leaves, gets us (ie, located at an altitude between the start and end sheet), but arranged differently. Numbering them, we will gradually make the turns around the stem (as the leaves are arranged on the stem in a spiral). Depending on whether the windings perform clockwise or counterclockwise will receive a different number of turns. But it turns out that the number of turns made by us in a clockwise direction, the number of turns made by counterclockwise, and the number of leaves encountered form three consecutive Fibonacci numbers.

However, it should be noted that there are plants for which the above calculations give the number of very different sequences and therefore can not be said that the phenomenon of phyllotaxis is the law - it is rather entertaining trend.

Properties

Fibonacci numbers have many interesting mathematical properties.

Here are some of them:

- Cassini ratio:
- $F_{n+1} F_{n-1} - F_n^2 = (-1)^n.$
- Rule of "addition":
- $F_{n+k} = F_k F_{n+1} + F_{k-1} F_n.$
- From the previous equality at $k = n$follows:

- $F_{2n} = F_n(F_{n+1} + F_{n-1})$.
- From the previous equality can be obtained by induction that
- F_{nk}always divisible F_n.
- The opposite is true of the previous statement:
- if $F_m$fold F_n, then $m$fold n.
- GCD-equality:
- $\gcd(F_m, F_n) = F_{\gcd(m,n)}$.
- With respect to the Euclidean algorithm Fibonacci numbers have the remarkable property that they are the worst input data for this algorithm (see "Theorem Lame" in Euclid's algorithm).

Fibonaccimal value

Zeckendorf's theorem states that every positive integer $n$can be written uniquely as a sum of Fibonacci numbers:

$$N = F_{k_1} + F_{k_2} + \ldots + F_{k_r}$$

where $k_1 \geq k_2 + 2$, $k_2 \geq k_3 + 2$, $\ldots$, $k_r \geq 2$(ie can not be used in recording two consecutive Fibonacci numbers).

It follows that any number can be written uniquely in **the Fibonacci number system** , for example:

$$9 = 8 + 1 = F_6 + F_1 = (10001)_F,$$

$$6 = 5 + 1 = F_5 + F_1 = (1001)_F,$$

$$19 = 13 + 5 + 1 = F_7 + F_5 + F_1 = (101001)_F,$$

and in any number can not go two units in a row.

It is easy to get and usually adding one to the number in the Fibonacci number system, if the minor number is 0, it is replaced by 1, and if equal to 1 (ie at the end worth 01), then 01 is replaced by 10. Then "fix" record sequentially correcting all 011 100. As a result, the linear time is obtained by recording a new number.

Translation numbers in the Fibonacci number system is as simple as "greedy" algorithm: just iterate through the Fibonacci numbers from larger to smaller and if some $F_k \leq n$, it $F_k$is included in the record number n, and we subtract $F_k$from n, and continue to search.

The formula for the n-th Fibonacci number

Formula radicals through

There is a wonderful formula, called after the French mathematician Binet (Binet), although it was known to him Moivre (Moivre):

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

This formula is easily proved by induction, but you can bring it by the concept of forming functions or with a solution of the functional equation.

Immediately you will notice that the second term is always less than 1 in absolute value, and furthermore, decreases very rapidly (exponentially). This implies that the value of the first term gives the "almost" value F_n. This can be written simply as:

$$F_n = \left[\frac{\left(\frac{1+\sqrt{5}}{2}\right)^n}{\sqrt{5}}\right],$$

where the square brackets denote rounding to the nearest integer.

However, for practical use in the calculation of these formulas hardly suitable, because they require very high precision work with fractional numbers.

Array formula for the Fibonacci numbers

It is easy to prove the following matrix equation:

$$\begin{pmatrix} F_{n-2} & F_{n-1} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} F_{n-1} & F_n \end{pmatrix}.$$

But then, denoting

$$P \equiv \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

obtain

$$\begin{pmatrix} F_0 & F_1 \end{pmatrix} \cdot P^n = \begin{pmatrix} F_n & F_{n+1} \end{pmatrix}.$$

Thus, to find nth Fibonacci number you want to raise a matrix $P$to the power n.

Remembering that the construction of the matrix in the n-th power can be accomplished $O(\log n)$(see binary exponentiation), it turns out that n-th Fibonacci number can be easily calculated for $O(\log n)$c using only integer arithmetic.

Periodicity of the Fibonacci sequence modulo

Consider the Fibonacci sequence $F_i$by some module p. We prove that it is periodic, with period and begins with $F_1 = 1$(ie preperiod contains only F_0).

We prove this by contradiction. Consider $p^2 + 1$pairs of Fibonacci numbers taken modulo p:

$$(F_1, F_2),\ (F_2, F_3),\ \ldots,\ (F_{p^2+1}, F_{p^2+2}).$$

Since modulo $p$can only be p^2different pairs, among the sequence there are at least two identical pairs. This already means that the sequence is periodic.

We now choose among all such identical pairs of two identical pairs with the lowest numbers. Let this pair with some numbers $(F_a, F_{a+1})$and (F_b, F_{b+1}). We will prove that $a = 1$. Indeed, otherwise they will have for the previous couple $(F_{a-1}, F_a)$and (F_{b-1}, F_b)that, by the property of Fibonacci numbers will also be equal to each other. However, this contradicts the fact that we chose the matching pairs with the lowest numbers, as required.

Literature

- Ronald Graham, Donald Knuth, and Oren Patashnik. **Concrete Mathematics** [1998]