Hawking radiation as Seen by Observers

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Ich versichere, dass ich diese Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt sowie die Zitate kenntlich gemacht habe.

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CHAPTER 1

Quantum field theory in spacetimes

1.1 Klein-Gordon-Field

(Quelle?)Consider a massless real Klein-Gordon-field in a curved spacetime with metric $g_{\mu\nu}$ given by the lagrangian:

$$\mathcal{L} = -\frac{1}{2}\sqrt{|g|}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi \tag{1.1}$$

The equation of motion is given by the Klein-Gordon-equation:

$$\sqrt{|g|}\nabla_\mu\nabla^\mu\phi = \partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\partial_\nu\phi\right) = 0 \tag{1.2}$$

For solutions we also require to drop to zero at the boundary. Define a scalar product of two such solutions ϕ, ψ over a Cauchysurface Σ via:

$$(\phi|\psi) := i\int_\Sigma \mathrm{d}S^\mu\;\phi^*\nabla_\mu\psi - \psi\nabla_\mu\phi^* = i\int_\Sigma \mathrm{d}S^\mu\;\phi^*\overleftrightarrow{\nabla}_\mu\psi \tag{1.3}$$

The scalar product is independent of the choice of Σ[**Townsend**]: Assume two Cauchysurfaces Σ, Σ' and denote the 'sandwiched' region between them by A. Then by Gauss' law

$$\begin{aligned}(\phi|\psi) - (\phi|\psi)' &= i\int_\Sigma \mathrm{d}S^\mu\;\phi^*\nabla_\mu\psi - \psi\nabla_\mu\phi^* - \int_{\Sigma'} \mathrm{d}S^\mu\;\phi^*\nabla_\mu\psi - \psi\nabla_\mu\phi^* &(1.4)\\ &= i\int_A \sqrt{|g|}\mathrm{d}^4x\;\nabla^\mu\left(\phi^*\nabla_\mu\psi - \psi\nabla_\mu\phi^*\right) &(1.5)\\ &= i\int_A \sqrt{|g|}\mathrm{d}^4x\;\phi^*\nabla^\mu\nabla_\mu\psi - \psi\nabla^\mu\nabla_\mu\phi^* = 0. &(1.6)\end{aligned}$$

Note $(\phi^*|\psi^*) = -(\phi|\psi)^*$ and $(\phi|\psi)^* = (\psi|\phi)$.

Now choose a complete set of solutions $\{u_i\}$:

$$(u_i|u_j) = \delta_{ij},\; (u_i^*|u_j^*) = -\delta_{ij} \text{ and } (u_i^*|u_j) = 0 \tag{1.7}$$

The completeness of the modes implies $(\phi|\psi) = \sum_i (\phi|u_i)(u_i|\psi) - (\phi|u_i^*)(u_i^*|\psi)$.

1.2 Quantisation, Bogolyubov Transformations and Vacua

(Quelle ?)We can quantize the field by introducing the canonical commutation relations CCR on a cauchysurface Σ with (future directed) normal vector S^μ:

$$[\phi(x), \phi(x')]_\Sigma = 0 \tag{1.8}$$
$$[\phi(x), \nabla_S \phi(x')]_\Sigma = i\delta(x - x') \tag{1.9}$$
$$[\nabla_S \phi(x), \nabla_S \phi(x')]_\Sigma = 0 \tag{1.10}$$

One can show that if they hold on one cauchysurface they hold on every cauchysurface[**krishnan1011.5875**]. Given a complete set of modes this leads to $\phi = \sum_i u_i a_i + u_i^* a_i^\dagger$, with a a bosonic annihilation operator satisfying $[a_i, a_j^\dagger] = \delta_{ij}$.

Of course there are many different complete sets. One could also expand it in a different set $\{v_j\}$: $\phi = \sum_j v_j b_j + v_j^* b_j^\dagger$. The b's are then given by

$$b_j = \sum_i (v_j|u_i)a_i + (v_j|u_i*)a_i^\dagger \tag{1.11}$$

This is called a Bogolyubov transformation.

So far everything was in complete analogy to quantisation in Minkowskispace. However problems arise when one tries to define the ground state of the system which is defined as the state with the lowest energy. The notion of energy (and thus the hamiltonian) depends on the notion of time. Therefore different coordinate systems will have different hamiltonians and thus different ground states. Since on a manifold there is no preferred coordinate system as in flat space we will have to guess the state of the field. This state may appear as the vacuum to some observers but will appear as an excited state to others (this is for example the reason why an eternal black hole seems to be thermal for an observer outside.).

In a static spacetime one usually chooses the state given by $a_i |0\rangle = 0$, where a_i are annihilation operators for positive frequency modes.

For the collapsing star we will choose the groundstate of the (static) spacetime before the collapse (which will then eventually convert into an excited state).

1.3 Greens functions

(Quelle?)

1.3.1 Vacuum Greens function

After defining the groundstate of the QFT on can define several Greens functions (there are many more, but we will only need those):

- The Wightman function $D^+(x,x') := \langle 0|\, \phi(x)\phi(x')\, |0\rangle$
- Expectationvalue of the commutator: $iD(x,x') := [\phi(x),\phi(x')] = 2i\,\mathrm{Im}\, D^+(x,x')$
- Expectationvalue of the anticommutator $D^{(1)}(x,x') := \langle 0|\, \{\phi(x),\phi(x')\}\, |0\rangle = 2\,\mathrm{Re}\, D^+(x,x')$

One does not need to take the expectationvalue of the commutator since (using the commutation relations) it is a c-number.

$$iD(x,x') = \sum_{i,j}[u_i(x)a_i + u_i^*(x)a_i^\dagger, u_j(x')a_j + u_j^*(x')a_j^\dagger] \tag{1.12}$$

$$= \sum_i u_i(x)u_i^*(x') - u_i^*(x)u_i(x') \tag{1.13}$$

Since $\nabla^\mu\nabla_\mu\phi(x) = 0$ this also holds for all Greensfunctions, i.e $\nabla^\mu\nabla_\mu D^+(x,x') = 0$.

If the ground state is defined as $a_i\,|0\rangle = 0$ for a complete set of modes u_i (as for example for positiv frequency modes in a static spacetime) we can calculate $D^+(\mathbf{x},\mathbf{x}')$ by summing over all modes:

$$D^+(\mathbf{x},\mathbf{x}') = \langle 0|\, \phi(\mathbf{x})\phi(\mathbf{x}')\, |0\rangle = \sum_i u_i(\mathbf{x})u_i^*(\mathbf{x}') \tag{1.14}$$

1.3.2 Thermal Greens function

Later we will also need thermal greens function. These are given by replacing the vacuum expectation value $\langle 0|\ldots|0\rangle$ by the thermal expectation value $\langle\ldots\rangle_\beta = \frac{1}{Z}\mathrm{Tr} e^{-\beta H}\ldots$ with $\beta = \frac{1}{k_B T}$, the hamiltonian H and $Z = Tre^{-\beta H}$.

It can be shown [**davies**] that $D^{(1)}_\beta$ is given by shifting the time by $i\beta n$ and then summing over n

$$D^{(1)}_\beta(t,\vec{x};t',\vec{x}') = \sum_n D^{(1)}(t - i\beta n, \vec{x}; t', \vec{x}') \tag{1.15}$$

To find D^+_β we can use that D (which is the imaginary part of D^+ and D^+_β) is just a c-number and therefore independent of the state of the field. If one is only interested in points where D^+ is real (as we will) one can replace $D^{(1)}$ by D^+ in the above formula since both greens functions are then proportional.

1.4 Particle Detectors

We have already seen that there is no suitable definition of vacuum in a spacetime. This implies that in the rest frame of an observer the vacuum state could differ from the vacuum state we defined. Therefore also the notion of what a particle will be different for different observers. To analyse what particles a specific observer sees, Unruh and DeWitt invented a model for a particle detector which measures the energy excitations (particles) of the field along a specific trajectory.

The calculations are done in the appendix A.3. The important result is that one can split the result in a contribution from the detector and one from the field given by an excitation rate at energy E:

$$\frac{\mathrm{d}F_E}{\mathrm{d}\tau} = 2\mathrm{Re}\int_{-\infty}^{0} \mathrm{d}\tau'\, e^{-iE\tau'} D^{+}(x(\tau+\tau'), x(\tau)) \tag{1.16}$$

This excitation rate is considered as energy distribution of particles an observer E will measure or see. In case the Wightmanfunction only depends on the difference $\tau - \tau'$ one can simplify this further to achieve:

$$\frac{\mathrm{d}F_E}{\mathrm{d}\tau} = \int_{-\infty}^{\infty} \mathrm{d}\tau'\, e^{-iE\tau'} D^{+}(x(\tau'), x(0)) \tag{1.17}$$

The excitation rate is constant and given by the fouriertransform of the Wightmanfunction evaluated along the curve.

1.5 QFT in Minkowskispace and the Unruh effect

In order to get a feeling for the calculations in the last section it is useful to consider a simple example in Minkowskispace, the Unruh effect: An observer with constant proper acceleration observes a heat bath when moving through Minkowski vacuum.

1.5.1 Solutions of the Klein-Gordon-Equation

The Klein-Gordon-Equation in Minkowskispace is the normal wave equation:

$$\partial_\mu \partial^\mu \phi = 0. \tag{1.18}$$

The solutions are given by plane waves:

$$u_{\vec{k}}(x) = \frac{1}{\sqrt{2|k|}} \frac{e^{ikx}}{\sqrt{2\pi}^3}, \text{ with} k^0 = |k| \tag{1.19}$$

The prefactor $\frac{1}{\sqrt{2|k|}}$ is required for normalisation $(u_{\vec{k}}|u_{\vec{k}'}) = \delta^3(\vec{k} - \vec{k}')$. It is natural to define the vacuum in Minkowskispace by $a_{\vec{k}} \left|0_M\right\rangle = 0$. Throughout this paper we will always exclude the mode $\omega = |\vec{\mathbf{k}}| = 0$[1].

Since the Minkowskispace is also spherical symmetric one can also choose spherical modes

$$u^{\mathrm{M}}_{\omega,l,m} = \frac{\sqrt{\omega}}{\sqrt{\pi}} e^{-i\omega t} j_l(\omega r) Y_l^m(\theta, \phi), \tag{1.20}$$

where j_l is a spherical Bessel function. The prefactor is again due to normalisation and can be achieved using the completeness relation for spherical Bessels $\int_0^\infty r^2\, \mathrm{d}r\, j_l(wr) j_l(w'r) = \frac{\pi}{2\omega^2}\delta(\omega - \omega')$.

[1] This might seem a bit ad hoc first but it is mainly to exclude some $\delta(\omega)$ terms which lead to clearly unphysical behaviour, e.g. an infinite transition rate to the groundstate of our detector.

For great distances from the origin one can approximate the Bessels by their asymptotic behaviour $j_l(x) \overset{x\gg 1}{\longrightarrow} \frac{\sin\left(x-l\frac{\pi}{2}\right)}{x}$ and achieves:

$$u^{\mathrm{M}}_{\omega,l,m} \approx \frac{1}{\sqrt{\pi\omega}} e^{-i\omega t} \frac{\sin\left(\omega r - l\frac{\pi}{2}\right)}{r} Y_l^m(\theta,\phi) \tag{1.21}$$

It is important to note that for this approximation it is necessary to have $r \gg 1/\omega$. So if one fixes r than the approximation will break down for small ω.

1.5.2 The Wightmanfunction

The Wightmanfunction is calculated in the appendix A.1:

$$D^+(x,x') = -\frac{1}{4\pi^2} \frac{1}{(t-t'-i\varepsilon)^2 - |\vec{x}-\vec{x}'|^2} \tag{1.22}$$

Up to the small imaginary number $i\varepsilon$ this is a real function. This means that the imaginary part (or $D(x,x')$) can only be non-vanishing if the denominator goes to 0 for $\varepsilon \to 0$. This is only the case for lightlike seperated x and x' or equivalently if x is on the lightcone of x'. This means when computing $D^+(x(\tau), x(0))$ on a trajectory (like for a detector in eq. 1.17) the imaginary part will always vanish, since our detector stays strictly inside the lightcone[2].

Since we will later use spherical modes in order to calculate D^+ in the Schwarzschildmetric it is useful to have an expression for the Wightmanfunction in terms of the spherical modes. This is given by:

$$D^+(\mathbf{x},\mathbf{x}') = \int_0^\infty \frac{\omega\,\mathrm{d}\omega}{\pi} \sum_{l,m} e^{-i\omega(t-t')} j_l(\omega r) j_l(\omega r') Y_l^m(\theta,\phi) Y_l^{m*}(\theta',\phi') \tag{1.23}$$

or when using the approximate forms for great r

$$D^+(\mathbf{x},\mathbf{x}') = \int_0^\infty \frac{\mathrm{d}\omega}{\pi\omega} \sum_{l,m} e^{-i\omega(t-t')} \frac{\sin\left(\omega r - l\frac{\pi}{2}\right)}{r} \frac{\sin\left(\omega r' - l\frac{\pi}{2}\right)}{r'} Y_l^m(\theta,\phi) Y_l^{m*}(\theta',\phi') \tag{1.24}$$

When using the approximate form two main problems occur. They can be seen by expanding $\sin\left(\omega r - l\frac{\pi}{2}\right) = \frac{e^{i\omega r} i^{-l} - e^{-i\omega r} i^{l}}{2i}$.

1. After expanding one has to integrate $\int_0^\infty \frac{\mathrm{d}\omega}{\omega} e^{i\omega\dots}$. This integral does not converge since the real part has a (logarithmic) divergence. This IR-divergence is due to the fact that the asymptotic approximation is only true for $r \gg 1/\omega$ and therefore fails for small ω. The spherical Bessel functions remain finite when approaching $\omega \to 0$, while in the asymptotic form $\frac{\cos\omega\dots}{\omega}$ diverges.

2. Instead of integrating over ω one could also do the summation over l,m first. Taking care of the $i^{\pm l}$ one either has to sum $\sum_{l,m} Y_l^m(\theta,\phi) Y_l^{m*}(\theta',\phi') \sim \delta(\phi-\phi')\delta(\theta-\theta')$ or

[2] Of course there's also the case $x = x'$ which we have to treat seperatly. But from the CCR we can conclude $2i\,\mathrm{Im}\,D^+(x,x) = iD(x,x) = [\phi(x),\phi(x)] = 0$.

$\sum_{l,m}(-1)^l Y_l^m(\theta,\phi)Y_l^{m*}(\theta',\phi') = \sum_{l,m} Y_l^m(\pi-\theta,\pi+\phi)Y_l^{m*}(\theta',\phi') \sim \delta(\phi-\phi'+\pi)\delta(\theta-\pi+\theta')$ which means that there's only a contribution in two directions namely the direction of the detector and the opposite direction. This is again an artefact of the asymptotic form but in this case it is due to the assumption that r is big. Since $D^+(\mathbf{x},\mathbf{x}')$ has the same spatial size independent of $\mathbf{x}$, moving it far away from the origin will shrink the corresponding solid angle. For r quite big all the contribution will appear only in one direction.

In asymptotic flat spacetimes (as the Schwarzschild metric) one often cannot find exact solutions but rather asymptotic forms of them. Therefore these two effects can (and will in our case) occur when calculating D^+ in such spacetimes.

1.5.3 Inertial Observer

We can now calculate the excitation rate for observers in Minkowskispace. Let's start with an steady observer $t(\tau)=\tau, \vec{x}(\tau)=0$:

$$D^+(x(\tau),x(\tau')) = -\frac{1}{4\pi^2}\frac{1}{(\tau-\tau'-i\varepsilon)^2} \tag{1.25}$$

Clearly $D^+(x(\tau),x(\tau'))$ only depends on $\Delta\tau=\tau-\tau'$. So we can use eq. 1.17 to obtain:

$$\frac{\mathrm{d}F_E(\tau)}{\mathrm{d}\tau} = \int_{-\infty}^{\infty}\mathrm{d}\tau e^{-iE\tau}D^+(x(\tau),x(0)) \tag{1.26}$$

$$= -\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}\tau e^{-iE\tau}\frac{1}{(\tau-i\varepsilon)^2} = 0 \tag{1.27}$$

For the last step use contour integration and close the contour in the lower half plane. Since $e^{-iE\tau}$ drops to 0 for large τ with negative imaginary part the integral is given by the sum over all residuals in the lower half plane. Because the ε moves the pole at $\tau=0$ into the upper half plane there are no poles in the lower half plane and therefore the integral vanishes. So there are no particle excitations for an observer on a inertial worldline[3] which simply means that he treats the Minkowski vacuum as a state with no particles.

1.5.4 The Unruh Effect

Another interesting observer is an observer which is accelerating with an constant proper acceleration $\alpha>0$, i.e. $t(\tau)=1/\alpha\sinh\alpha\tau$, $x(\tau)=1/\alpha\cosh\alpha\tau$, $y(\tau)=z(\tau)=0$. Define $\lambda=\alpha\tau$. First calculate D^+ for $\varepsilon=0$

[3] Note that by poincare invariance of $D^+(x,x')$ one can always choose a frame in which an inertial observer does not move.

$$-\frac{\alpha^2}{4\pi^2 D^+(x(\lambda/\alpha), x(\lambda'/\alpha))} = (\sinh\lambda - \sinh\lambda')^2 - (\cosh\lambda - \cosh\lambda')^2 \tag{1.28}$$
$$= -2 - 2\sinh\lambda\sinh\lambda' + 2\cosh\lambda\cosh\lambda' \tag{1.29}$$
$$= -2\big(1 - \cosh(\lambda - \lambda')\big) \tag{1.30}$$
$$= 4\sinh^2\frac{\lambda - \lambda'}{2} \tag{1.31}$$
$$D^+(x(\tau), x(\tau')) = -\frac{\alpha^2}{16\pi^2}\frac{1}{\sinh^2\frac{\alpha(\tau-\tau')}{2}} \tag{1.32}$$

So again D^+ only depends on $\Delta\tau = \tau - \tau'$. So we will fouriertransform D^+ according to eq. 1.17. Again we will close the contour in the lower half plane. Therefore we only need the residues in the lower half. We will later (see section 2.3.2) find that in general the pole at $\tau = \tau'$ will be moved to the upper half[4]. So we can ignore the pole at 0 for the contourintegration. It is useful to use the expansion $\frac{1}{\sin^2 \pi x} = \pi^{-2}\sum_k (x-k)^{-2}$ to obtain

$$D^+(x(\tau), x(0)) = -\frac{1}{4\beta^2\pi^2}\sum_k\left(\frac{\tau}{\beta} - ik\right)^{-2} = -\frac{1}{4\pi^2}\sum_k(\tau + ik\beta)^{-2} \tag{1.33}$$

where $\beta = \frac{2\pi}{\alpha}$:

$$\frac{\mathrm{d}F_E(\tau)}{\mathrm{d}\tau} = \int_{-\infty}^{\infty}\mathrm{d}\tau e^{-iE\tau}D^+(x(\tau), x(0)) \tag{1.34}$$
$$= -\frac{1}{4\pi^2}\sum_{k>0}\int_{-\infty}^{\infty}\mathrm{d}\tau e^{-iE\tau}\frac{1}{(\tau + ik\beta)^2} \tag{1.35}$$
$$= \frac{2\pi i}{4\pi^2}\sum_{k>0}\mathrm{Res}\left(e^{-iE\tau}\frac{1}{(\tau + ik\beta)^2}, \tau = -ik\beta\right) \tag{1.36}$$
$$= \frac{2\pi E}{4\pi^2}\sum_{k>0}e^{-\beta kE} \tag{1.37}$$
$$= \frac{1}{2\pi E}\left(\frac{1}{1 - e^{-\beta E}} - 1\right) \tag{1.38}$$
$$= \frac{1}{2\pi}\left(\frac{E}{e^{\beta E} - 1}\right) \tag{1.39}$$

So the detector detects particles. The distribution contains the typical Bose-Einstein factor $\frac{1}{e^{\beta E}-1}$. Actually this is the same distribution as an inertial observer would observe in a heat bath with temperature $T = \frac{1}{k_B\beta} = \frac{\alpha}{2\pi k_B}$. This can be seen by computing the thermal greens function in Minkowskispace:

[4] Alternatively one can explicitly calculate this by absorbing positive functions into ε and achieves $D^+(x(\tau), x(0)) \sim \frac{1}{\sinh^2(\frac{\tau\alpha}{2} - i\varepsilon)}$. So indeed the pole is shifted to the upper half

$$D^+_\beta(x,x') = -\frac{1}{4\beta^2}\frac{1}{\sinh^2\left(\frac{\pi}{\beta}\sqrt{(t-t')^2-|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|^2}\right)} \tag{1.40}$$

Evaluating this on a inertial trajectory, i.e. $t=\tau, \vec{x}=\vec{x}'=t'=0$ leads to the same formula as in eq. 1.32

$$D^+_\beta(\mathbf{x}(\tau),0) = -\frac{1}{4\beta^2}\frac{1}{\sinh^2\left(\frac{\pi}{\beta}\tau\right)} \tag{1.41}$$

So an accelerating observer actually sees the Minkowski vacuum as a heat bath. So indeed the notion of a particle is observer dependent.

CHAPTER 2

Unruh-Detector in static Spacetimes

Static spacetimes have a metric that looks like

$$\mathrm{d}s^2 = -\beta(\vec{\mathbf{x}})\,\mathrm{d}t^2 + g_{ij}(\vec{\mathbf{x}})\,\mathrm{d}x^i\,\mathrm{d}x^j \tag{2.1}$$

The metric only depends on the spatial coordinates and therefore ∂_t is a global timelike killing vector. For simplicity we will denote $g = \det\left(g_{ij}\right)$ instead of the four dimensional determinant.

2.1 Positive frequency modes

A solution u is called positive frequency if

$$i\partial_t u = \omega u, \omega > 0 \tag{2.2}$$

In case of a static metric it is possible to find a complete set of positive frequency solutions [**Townsend**]

$$u(t, x^i) \sim e^{-i\omega t} A(x^i) \tag{2.3}$$

Using the normalisation condition on a cauchysurface $t = \text{const.}$ one finds that

$$u_i(t, \vec{\mathbf{x}}) = \frac{1}{\sqrt{2\omega_i}} e^{-i\omega_i t} A_i(\vec{\mathbf{x}}) \tag{2.4}$$

$$\delta_{ij} = \int_\Sigma \mathrm{d}^3x\, \frac{\sqrt{g}}{\sqrt{\beta}} A_i^*(\vec{\mathbf{x}}) A_j(\vec{\mathbf{x}}) \tag{2.5}$$

$$\sum_k A_k(\vec{\mathbf{x}}) A_k^*(\vec{\mathbf{x}}') = \frac{\sqrt{\beta}}{\sqrt{g}} \delta^3(\vec{\mathbf{x}} - \vec{\mathbf{x}}') \tag{2.6}$$

2.2 Groundstate of the field

As discussed in before it is suitable to define the groundstate as the state of lowest energy on a surface $t = \text{const.}$

The momentum and the hamiltonian density are given by

$$\pi = \frac{\partial \mathcal{L}}{\partial \partial_t \phi} = \sqrt{g\beta}\beta^{-1}\partial_t \phi = \frac{\sqrt{g}}{\sqrt{\beta}}\partial_t \phi \tag{2.7}$$

$$\mathcal{H} = \pi \partial_t \phi - \mathcal{L} = \frac{\sqrt{g}}{\sqrt{\beta}}\partial_t \phi \partial_t \phi - \frac{1}{2}\frac{\sqrt{g}}{\sqrt{\beta}}\partial_t \phi \partial_t \phi + \frac{1}{2}\sqrt{g\beta}g^{ij}\partial_i \phi \partial_j \phi \tag{2.8}$$

$$= \frac{1}{2}\frac{\sqrt{g}}{\sqrt{\beta}}\partial_t \phi \partial_t \phi + \frac{1}{2}\sqrt{g\beta}g^{ij}\partial_i \phi \partial_j \phi \tag{2.9}$$

To get the Hamiltonian we need to integrate this over the whole space. Then we can perform a integration by parts and insert the Klein-Gordon-equation:

$$H = \int \mathrm{d}^3 x\, \mathcal{H} = \int \mathrm{d}^3 x\, \frac{1}{2}\frac{\sqrt{g}}{\sqrt{\beta}}\partial_t \phi \partial_t \phi + \frac{1}{2}\sqrt{g\beta}g^{ij}\partial_i \phi \partial_j \phi \tag{2.10}$$

$$\stackrel{\mathrm{PI}}{=} \int \mathrm{d}^3 x\, \frac{1}{2}\frac{\sqrt{g}}{\sqrt{\beta}}\partial_t \phi \partial_t \phi - \frac{1}{2}\phi \partial_i \sqrt{g\beta}g^{ij}\partial_j \phi \tag{2.11}$$

$$= \int \mathrm{d}^3 x\, \frac{1}{2}\frac{\sqrt{g}}{\sqrt{\beta}}\partial_t \phi \partial_t \phi + \frac{1}{2}\phi \partial_t \sqrt{g\beta}g^{tt}\partial_t \phi \tag{2.12}$$

$$= \int \mathrm{d}^3 x\, \frac{1}{2}\frac{\sqrt{g}}{\sqrt{\beta}}\partial_t \phi \partial_t \phi - \frac{1}{2}\frac{\sqrt{g}}{\beta}\phi \partial_t \partial_t \phi \tag{2.13}$$

$$= \frac{1}{2}i(\phi|\partial_t \phi) = \frac{1}{2}(\phi|i\partial_t \phi) \tag{2.14}$$

Here we used the definition of the scalarproduct. Inserting the definition of ϕ and using the orthogonality of the modes yields

$$H = \frac{1}{2}(\phi|i\partial_t \phi) = \frac{1}{2}\sum_{ij}(u_i a_i + u_i^* a_i^\dagger | \omega_j u_j a_j - \omega_j u_j^* a_j^\dagger) \tag{2.15}$$

$$= \frac{1}{2}\sum_i \omega_i a_i^\dagger a_i + \omega_i a_i a_i^\dagger \tag{2.16}$$

$$:H: = \sum_i \omega_i a_i^\dagger a_i \tag{2.17}$$

The normal ordering is as always applied to get rid of an infinite offset energy. Written in this way it is clear that the vacuum state corresponding to our modes, i.e. $a_i |0\rangle = 0$ is the state with the lowest energy and therefore the groundstate.

2.3 Properties of the Wightman function

The Wightman function is given by

$$D^+(\mathbf{x}, \mathbf{x}') = \sum_i \frac{1}{2\omega_i} e^{-i\omega_i(t-t')} A_i(\vec{\mathbf{x}}) A_i(\vec{\mathbf{x}}') \tag{2.18}$$

Actually to make this convergent we will replace $t \longrightarrow t - i\varepsilon$ and treat D^+ as a distribution[1]. For convenience we will set $\mathbf{x}' = 0$.

A first property of D^+ is obtained by derivating w.r.t t and then setting $t = 0$

$$i\partial_t\, D^+(\mathbf{x}, 0)\bigg|_{t=0} = \frac{1}{2}\sum_i A_i(\vec{\mathbf{x}})A_i(\vec{\mathbf{x}}') \tag{2.19}$$

$$= \frac{1}{2}\frac{\sqrt{\beta}}{\sqrt{g}}\delta^3(\vec{\mathbf{x}}) \tag{2.20}$$

2.3.1 Wightman function in normal coordinates

We can choose any coordinate system we like for $g_{ij}(\vec{\mathbf{x}})$. It will be useful to have the Wightmanfunction in normal coordinates around a point $\mathbf{x}' = 0$ (see appendix A.4):

$$D^+(\mathbf{x}, 0) = -\frac{1}{4\pi^2}\frac{1}{a(t - i\varepsilon)^2 - |\vec{\mathbf{x}}|^2} + O\left(x^2\right) \tag{2.21}$$

which is basically the same as in Minkowskispace up to a prefactor $a = \beta(0)$.

2.3.2 The pole at the origin

When calculating the excitation rate we will first evaluate D^+ on a timelike trajectory $\mathbf{x}(\tau)$ with $\mathbf{x}(0) = 0$ and $\dot{\mathbf{x}}^2 = -\beta\dot{t}^2 + g_{ij}\dot{x}^i\dot{x}^j = -1$ and then integrate over τ. Thereby we encounter a pole on the real axis at $\tau = 0$. Due to the ε this (second order) pole will move either in upper or in the lower half or could even split into two poles. To examine the behaviour of this pole define τ_ε as the position of the pole at $\tau = 0$ for a non vanishing ε, i.e. τ_ε satisfies

$$a(t(\tau_\varepsilon) - i\varepsilon)^2 - |\vec{\mathbf{x}}(\tau_\varepsilon)|^2 = 0 \tag{2.22}$$

Differentiate this twice with respect to ε and then setting $\varepsilon \longrightarrow 0$ yields (Note that $\tau_0 = 0$)

$$a(\dot{t}(0)\delta\tau - i)^2 - |\dot{\vec{\mathbf{x}}}(0)|^2\delta\tau^2 = 0 \tag{2.23}$$

where we defined $\delta\tau = \frac{\mathrm{d}\tau_\varepsilon}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}$. Noting that $a\dot{t}(0)^2 - |\dot{\vec{\mathbf{x}}}(0)|^2 = 1$ one finds

$$0 = \delta\tau^2 - 2ia\dot{t}(0)\delta\tau - 1 \tag{2.24}$$

$$\delta\tau = ia\dot{t}(0) \pm \sqrt{-a^2\dot{t}(0)^2 + 1} \tag{2.25}$$

$\delta\tau$ has two solutions and therefore the pole will split into two poles. But we know that $a\dot{t}(0) > 0$ and so will both values of $\delta\tau$ have positive imaginary part. Recall that $\delta\tau = \frac{\mathrm{d}\tau_\varepsilon}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}$ and so $\tau_\varepsilon = \delta\tau\varepsilon + O\left(\varepsilon^2\right)$ which means that for a sufficient small value of ε both poles will lie in the upper half of the complex plane.

[1] We will from now on assume that this replacement is enough to make the integral convergent.

2.3.3 The pole structure of the Wightman function

We know that the Wightman function solves the Klein-Gordon-Equation, i.e.

$$\nabla_\mu \nabla^\mu D^+(\mathbf{x}, \mathbf{x}') = 0 \tag{2.26}$$

Now again fix $\mathbf{x}'$ and define $A(\mathbf{x}) = \frac{1}{D^+(\mathbf{x},\mathbf{x}')}$.

$$\begin{aligned}
0 &= \nabla_\mu \nabla^\mu \frac{1}{A} && (2.27)\\
&= -\nabla_\mu \frac{\nabla^\mu A}{A^2} && (2.28)\\
&= -\frac{A^2 \nabla_\mu \nabla^\mu A - 2A\nabla_\mu A \nabla^\mu A}{A^4} && (2.29)\\
&= -\frac{A\nabla_\mu \nabla^\mu A - 2\nabla_\mu A \nabla^\mu A}{A^3} && (2.30)\\
0 &= A\nabla_\mu \nabla^\mu A - 2\nabla_\mu A \nabla^\mu A && (2.31)\\
\nabla_\mu A \nabla^\mu A &= \frac{A}{2} \nabla_\mu \nabla^\mu A && (2.32)
\end{aligned}$$

This must be also the case for points where the Wightman function has a pole, i.e. $A = 0$. In this case we conclude that at such a point $\nabla_\mu A \nabla^\mu A = 0$ which means that ∇A is a lightlike vector. Poles can now have two different behaviours: either they are an isolated singularity or they are part of a hypersurface on which $D^+ = \infty$. We will exclude the first type by the following handwaving argument: since D^+ solves the Klein-Gordon-equation with well we don't expect such solutions to create isolated poles. The second type appears for example in Minkowskispace on the light cone. We know that such a hypersurface is given by $A = 0$ and since ∇A is a lightlike vector it is a null hypersurface. Since the $t = t'$ plane is cauchy this hypersurfaces will it[2]. So if D^+ stays finite on $t = t'$ except for $\mathbf{x} = \mathbf{x}'$ (which will be the case for our examples) we can conclude that there will be no singular behaviour of D^+ apart from the lightcone of $\mathbf{x}'$. Since observers stay always inside the lightcone this implies that it will not encounter any pole on its trajectory except for $\mathbf{x}'$.

Note that we can repeat the same argumentation for $D = 2\mathrm{Im}D^+$. Around the origin D vanishes except for being singular on the lightcone. It will therefore remain singular there. Apart from that we know that by causality outside the lightcone, $iD = [\phi(\mathbf{x}), \phi(\mathbf{x}')] = 0$ which implies that there are no more hypersurfaces with $D = \infty$. So inside the lightcone D is nonsingular. If we assume that D is analytically in a region without singularities we can conclude $D = 0$ inside the lightcone (since this is true in a small region around the origin). So D^+ is a real function inside the lightcone.

With this argumentation (although it is not a proof) we will assume from now on that D^+ is real and finite on all trajectories (except for $\mathbf{x} = \mathbf{x}'$).

[2] We will exclude spacetimes with closed null curves

2.4 Observers on Trajectories

2.4.1 Static observers

We will start by showing the following important lemma:

Lemma 2.4.1 *In a static spacetime a static observer does not observe any particles.*

Proof: Recall that the modes in a static spacetime (see eq. 2.5) can be written as $u_i = \frac{1}{\sqrt{2\omega_i}} e^{-i\omega_i t} A_i(\vec{\mathbf{x}})$. Since the observer moves only along ∂_t he will have four-velocity $\dot{\mathbf{x}} = \frac{1}{\sqrt{\beta(\vec{\mathbf{x}})}}\partial_t$. Note that $\beta(\vec{\mathbf{x}})$ is independent of τ because the spatial coordinates stay constant $\vec{\mathbf{x}}(\tau) = \vec{\mathbf{x}}_0$. Integrating yields $t(\tau) = \frac{1}{\sqrt{\beta(\vec{\mathbf{x}})}}\tau + t_0$. Now we can evaluate $D^+(\mathbf{x}(\tau), \mathbf{x}(\tau'))$:

$$D^+(\mathbf{x}(\tau), \mathbf{x}(\tau')) = \langle 0|\, \phi(\mathbf{x}(\tau))\phi(\mathbf{x}(\tau'))\, |0\rangle = \sum_i u_i(\mathbf{x}(\tau)) u_i^*(\mathbf{x}(\tau')) \tag{2.33}$$

$$= \sum_i \frac{1}{2\omega_i} e^{-i\omega_i (t(\tau) - t(\tau'))} A_i(\vec{\mathbf{x}}_0) A_i^*(\vec{\mathbf{x}}_0) \tag{2.34}$$

$$= \sum_i \frac{1}{2\omega_i} e^{-i\omega_i (\tau - \tau')/\sqrt{\beta(\vec{\mathbf{x}}_0)}} A_i(\vec{\mathbf{x}}_0) A_i^*(\vec{\mathbf{x}}_0) \tag{2.35}$$

So D^+ only depends on the difference $\tau - \tau'$. Therefore we can apply eq. **??**:

$$\frac{\mathrm{d}F_E}{\mathrm{d}\tau} = \int_{-\infty}^{\infty} \mathrm{d}\tau\, e^{-iE\tau} D^+(\mathbf{x}(\tau), \mathbf{x}(0)) \tag{2.36}$$

$$= \sum_i \frac{1}{2\omega_i} A_i(\vec{\mathbf{x}}_0) A_i^*(\vec{\mathbf{x}}_0) \left(\int_{-\infty}^{\infty} \mathrm{d}\tau\, e^{-iE\tau} e^{-i\omega\tau//\sqrt{\beta(\vec{\mathbf{x}}_0)}} \right) \tag{2.37}$$

$$= \sum_i \frac{1}{2\omega_i} A_i(\vec{\mathbf{x}}_0) A_i^*(\vec{\mathbf{x}}_0) \delta\left(E + \omega/\sqrt{\beta(\vec{\mathbf{x}}_0)} \right) = 0 \tag{2.38}$$

The deltafunction is always zero because $E \leq 0$, $\omega > 0$, and $\beta(\vec{\mathbf{x}}_0) > 0$[3]. So the observer does not detect any particles. QED.

This exact result will be later used to show whether the approximate form of the Wightmanfunction in the Schwarzschild metric is applicable.

2.4.2 Detector on general Trajectories

In this section we will use the information we gathered about the Wightman function to tackle the problem how to calculate the detector response function on a general trajectory. In particular we have the problem that in general we cannot integrate from $-\infty$ to ∞ and use the residue theorem but we would rather have to integrate from $-\infty$ to 0. Since there's a singularity at 0 we will have in all cases a diverging integral.

[3] Here one can see why it is sensible to exclude $\omega = 0$ because it would lead to an infinite transition rate to the groundstate $E = 0$ which is clearly not physical

Vacuum case

The last sentence is not really true. There is the small ε which removes the singularity from the real axis. We will assume that the integral over the trajectory will remain finite for $\varepsilon \to 0$[4]. Apart from keeping this in mind we will drop the ε during this section.

Recall that by eq. 1.16 the excitation rate is given by

$$\frac{\mathrm{d}F_E}{\mathrm{d}\tau} = 2\mathrm{Re} \int_{-\infty}^{0} \mathrm{d}\tau' \, e^{-iE\tau'} D^+(\mathbf{x}(\tau + \tau'), \mathbf{x}(\tau)) \tag{2.39}$$

Without loss of generality we can consider the current proper time $\tau = 0$ and set $\mathbf{x}(\tau) = 0$. We will now do a series expansion of the Wightmanfunction around $\tau' = 0$, i.e.

$$D^+(x(\tau'), 0) = \frac{a_{-2}}{\tau'^2} + W(\tau') \tag{2.40}$$

where $W(\tau')$ is finite at $\tau' = 0$. We can find a_{-2} by the following limit

$$\begin{aligned} a_{-2} &= \lim_{\tau' \to 0} \tau'^2 \cdot D^+(x(\tau'), 0) && (2.41)\\ &= -\frac{1}{4\pi^2} \lim_{\tau' \to 0} \frac{\tau'^2}{at(\tau')^2 - |\vec{\mathbf{x}}(\tau)|^2} + O\left(x^2\right) && (2.42)\\ &= -\frac{1}{4\pi^2} \lim_{\tau' \to 0} \frac{\tau'}{at\dot{t} - \vec{\mathbf{x}}\dot{\vec{\mathbf{x}}}} && (2.43)\\ &= -\frac{1}{4\pi^2} \frac{1}{a\dot{t}^2 - \dot{\vec{\mathbf{x}}}^2} && (2.44)\\ &= -\frac{1}{4\pi^2} && (2.45) \end{aligned}$$

Where we have used $t(0) = \vec{\mathbf{x}}(0) = 0$ and $a\dot{t}^2 - \dot{\vec{\mathbf{x}}}^2 = 1$. So the singular part of the Wightmanfunction does neither depend on the specific trajectory nor on the geometry of the spacetime at all. So to calculate this we can take any trajectory we like, for example a static trajectory for which the remaining part vanishes $W(\tau') = 0$. Since the rate for this trajectory is zero the $\frac{1}{\tau'^2}$ term will not contribute in general.

This means that instead of integrating over $D^+(x(\tau'), 0)$ we can equivalently integrate over $W(\tau') = D^+(x(\tau'), 0) + \frac{1}{4\pi^2\tau'^2}$ which is well defined.

Thermal case

In a thermal field we can also extract the contribution from an inertial observer to be left with a non singular function. To do this plug the expansion of $D^+(x(\tau'), 0) = -\frac{1}{4\pi^2\tau'^2} + W(\tau')$ into the formula

[4] One can for example proof this by an explicit calculation for an inertial trajectory.

1.15 for D^+_β

$$D^+_\beta(t(\tau), x(\tau); 0) = \sum_{n=-\infty}^{\infty} D^+(t(\tau) - i\beta n, x(\tau); 0) \tag{2.46}$$

$$= \sum_{n=-\infty}^{\infty} -\frac{1}{4\pi^2 \sqrt{(\tau' - i\beta\sqrt{a}n)}^2} + \sum_{n=-\infty}^{\infty} W(\tau(t - i\beta n)) \tag{2.47}$$

$$= -\frac{1}{4\beta^2 a} \frac{1}{\sinh^2\left(\frac{\pi}{\beta\sqrt{a}}\tau\right)} + W_\beta(\tau) \tag{2.48}$$

So the observer will see a thermal spectrum of temperature $T_{\text{static}} = \frac{T}{\sqrt{a}}$ (compare with eq. 1.41) plus some corrections coming from $W_\beta(\tau)$. These corrections will vanish for a static observer, be small for slow observers and might become the dominating spectrum for fast observers. The relation $T_{\text{static}} = \frac{T}{\sqrt{a}}$ is also known from other analysis of thermal systems in general relativity and is called Tolman relation (Quelle?). The origin of this effect is the observer dependent time dilation in the spacetime.

2.4.3 Equivalence principle?

The statement of lemma 2.4.1 that a static observer does not recognize particles might seem surprising as in many spacetimes a static observer needs to accelerate in order to stay at his position (take the schwarzschild metric for example). As a first guess on could think that by equivalence principle an proper accelerating observer would see a heat bath as given by the unruh effect. This would also imply that a freely falling observer does not detect any particles. In order to show that this assumption is misleading we will first analyse the properties of trajectories on which no particles will be detected in general. To conclude the discussion we will show that on circular geodesics in the Schwarzschildmetric one actually detects something.

Recall that transition probability (not the rate) for a detector proportional to the square of the following state

$$|\psi\rangle = \int_{-\infty}^{\tau} \mathrm{d}\tau'\, e^{iE\tau'} \phi(\mathbf{x}(\tau'))\, |0\rangle \tag{2.49}$$

$$= \sum_i \frac{1}{\sqrt{2\omega_i}} \int_{-\infty}^{\tau} \mathrm{d}\tau'\, e^{iE\tau'} e^{+i\omega_i t(\tau')} A_i(\vec{\mathbf{x}}(\tau'))^*\, |\mathbf{1}_i\rangle \tag{2.50}$$

Since we would like to have no transitions at all the transition probability has to be zero and (note that the scalarproduct of states is positive definite) therefore the state $|\psi\rangle = 0$. But this implies since the one particle states $|\mathbf{1}_i\rangle$ are linear independent that all

$$A_i := \int_{-\infty}^{\tau} \mathrm{d}\tau'\, e^{iE\tau'} e^{+i\omega_i t(\tau')} A_i(\vec{\mathbf{x}}(\tau'))^* \stackrel{!}{=} 0 \tag{2.51}$$

have to vanish. If we were not dealing with distributions but rather with functions this would imply (since we need this for all τ):

$$\forall\tau : e^{iE\tau} e^{+i\omega_i t(\tau)} A_i(\vec{\mathbf{x}}(\tau))^* \stackrel{!}{=} 0 \tag{2.52}$$

$$\Rightarrow A_i(\vec{\mathbf{x}}(\tau)) \stackrel{!}{=} 0 \tag{2.53}$$

However this is impossible since the A_i are supposed to form a complete basis over the full space.

We cannot apply this argument directly since D^+ is a distribution. But apart from the $-\frac{1}{4\pi^2\tau^2}$ term the Wightmanfunction behaves like a function and therefore the $W(\tau)$ has to vanish in order to see no excitations. Therefore D^+ evaluated on the trajectory can only be given by $-\frac{1}{4\pi^2\tau^2}$[5]. It is clear that it will be quite hard to figure out a trajectory that satisfies that apart from static trajectories. This argument shows that it is very unlikely that all free falling observers will not recognize any particles.

We will conclude the argumentation with giving an explicit geodesic on which the detector response function is non zero. To do this we will need the following lemma for observers moving along killing vectors:

Lemma 2.4.2 *In a static spacetime an observer moving with constant velocity $\dot{\mathbf{x}} = A\partial_t + B\mathbf{k}$ along a spatial killingvector $\mathbf{k}$ will see excitations if there exists at least one eigenfunction u[6] to $\mathbf{k}$ with eigenvalue 'im' such that $\frac{A}{B} < \frac{m}{\omega_m}$.*

Proof: Choose a coordinate system for the spatial metric such that it has a coordinate ϕ with $\partial_\phi = \mathbf{k}$. Since ∂_ϕ is killing the metric will not depend on ϕ which implies that an eigenfunction of ∂_ϕ is also an eigenfunction to $\nabla_\mu\nabla^\mu$ i.e. we can find a complete set of solutions such that

$$u_{m,i} \sim e^{-i\omega_m t} \cdot e^{im\phi} \tag{2.54}$$

In order to show that an observer will see excitations it is enough to show that he won't see nothing at $\tau \to \infty$:

$$A_m \sim \int_{-\infty}^{\infty} \mathrm{d}\tau'\, e^{iE\tau'} e^{+i\omega_m t(\tau')} e^{-im\phi(\tau')} = \int_{-\infty}^{\infty} \mathrm{d}\tau'\, e^{iE\tau'} e^{+i\omega_m A\tau'} e^{-imB\tau'} \tag{2.55}$$

$$= \delta(E + \omega_m A - mB) \tag{2.56}$$

This will be non zero at least for one energy if $\omega_m A - mB < 0$ or $\frac{A}{B} < \frac{m}{\omega_m}$[7]. Using the assumption completes the proof. QED.

We can now apply this to a circular geodesic in the Schwarzschild metric. Since the metric is spherically symmetric the trajectory is along a killing vector. Also the values of ω are continuous from 0 to ∞ and are especially independent of m (this will be derived in the next chapter). This means no matter how big $\frac{A}{B}$ is we will always find a combination of m and ω that fulfils the second condition. So we have found one explicit example for a geodesic on which particle excitations occur.

But how does this work with the equivalence principle. This simply resolved by recalling that the equivalence principle only states that it is impossible to distinguish between flat space and curved

[5] This is for example the case for inertial trajectories in Minkowskispace

[6] Of course it is implied that u is a solution to the Klein-Gordon-equation with $\omega = \omega_m$

[7] If either $m < 0$ or $B < 0$ just switch the sign of B and redo the proof.

space using only local measurements. But the calculations above required integration over the whole worldline of the particle. Therefore this is a non local effect and we cannot apply the equivalence principle here.

CHAPTER 3

Hawking radiation as seen by Observers

3.1 The metric

In this thesis we will only use the Schwarzschild-metric to describe stars and black holes (which means that they have no charge and no angular momentum). The metric for an spherical symmetric object with mass M is given by

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + \frac{1}{f(r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega \qquad \mathrm{d}\Omega = \mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\phi^2 \tag{3.1}$$

where $f(r) = 1 - \frac{2M}{r}$. The metric is only valid outside the boundary of the star or for $r > 2M$. The two vector fields ∂_t and ∂_ϕ are killing.

3.2 The Klein-Gordon-Equation in the Schwarzschild metric

Consider a static spherical star where the outer metric is the Schwarzschild metric (i.e. non rotating and uncharged). It's important that we are considering a star because a black hole does not provide a global timelike killing vector field (analytic extension of ∂_t leads to a spacelike vectorfield inside the black hole). From now on we will only consider the outer region. To achieve a global solution one need to match inner with outer solutions.

The Klein-Gordon-Equation $\nabla_\mu \nabla^\mu \phi = 0$ can be written as:

$$-\frac{r^2}{f(r)}\partial_t^2\phi + \left(\partial_r r^2 f(r)\partial_r\right)\phi - L^2\phi = 0 \tag{3.2}$$

where L^2 is the usual angular momentum operator.

3.2.1 Spherical Modes

Since the spacetime is spherical symmetric and has the killing vector field ∂_t we can do the following ansatz for the modes

$$u_{\omega lm} = Ae^{-i\omega t}\frac{R_{\omega l}}{r}Y_l^m(\theta,\phi) \tag{3.3}$$

Plugging this into the Klein-Gordon-Equation $\nabla_\mu \nabla^\mu u_{\omega lm} = 0$ yields to the following equation for $R_{\omega l}$ (see for example (Birell Davies)):

$$\frac{\mathrm{d}^2 R_{\omega l}}{\mathrm{d}r_*^2} + \omega^2 R_{\omega l} - \left(\frac{l(l+1)}{r^2} + \frac{f'(r)}{r}\right) f(r) R_{\omega l} = 0 \tag{3.4}$$

$$\frac{\mathrm{d}^2 R_{\omega l}}{\mathrm{d}r_*^2} + \omega^2 R_{\omega l} - O\left(r^{-2}\right) R_{\omega l} = 0 \tag{3.5}$$

So for $\omega r \gg l$ one can neglect the r dependent part. In this case we find the asymptotic solutions

$$R_{\omega l} = e^{\pm i\omega r_*} \tag{3.6}$$

$$u_{\omega lm} = \frac{A_{\omega lm}}{r}e^{-i\omega t + i\omega r_*}Y_l^m(\theta,\phi) + \frac{B_{\omega lm}}{r}e^{-i\omega t - i\omega r_*}Y_l^m(\theta,\phi) \tag{3.7}$$

$$= \frac{A_{\omega lm}}{r}e^{-i\omega u}Y_l^m(\theta,\phi) + \frac{B_{\omega lm}}{r}e^{-i\omega v}Y_l^m(\theta,\phi) \tag{3.8}$$

Unfortunately we either cannot determine the (quite important) phase between $A_{\omega lm}$ and $B_{\omega lm}$ nor can we normalise the modes by integrating over all space. Instead I will impose that very far away from the star (where $r_* \approx r$) the field behaves as in Minkowskispace (which means that e.g. all experiments give the same results). This implies that $D^+(\mathbf{x},\mathbf{x}')$ is the same as in Minkowskispace. Comparing with the asymptotic spherical modes in Minkowskispace (see eq. 1.21) yields to:

$$u_{\omega lm} = \frac{1}{\sqrt{\pi\omega}}e^{-i\omega t}\frac{\sin\left(\omega r_* - l\frac{\pi}{2}\right)}{r}Y_l^m(\theta,\phi) = \frac{r_*}{r}u^{\mathrm{M}}_{\omega lm}(r_*,\theta,\phi) \tag{3.9}$$

$$= \frac{i^{-l}}{2i\sqrt{\pi\omega}r}e^{-i\omega u}Y_l^m(\theta,\phi) - \frac{i^l}{2i\sqrt{\pi\omega}r}e^{-i\omega v}Y_l^m(\theta,\phi) \tag{3.10}$$

3.2.2 The Wightman Function

The next step would be to calculate the Wightman function $D^+(\mathbf{x},\mathbf{x}')$. Note that since ∂_t is a timelike killing vector we define the ground state $|0\rangle$ by $a_{\omega lm}|0\rangle = 0$ and use eq. 1.14.

$$D^+(\mathbf{x},\mathbf{x}') = \int_0^\infty \frac{\mathrm{d}\omega}{\pi\omega}\sum_{l,m} e^{-i\omega(t-t')}\frac{\sin\left(\omega r_* - l\frac{\pi}{2}\right)}{r}\frac{\sin\left(\omega r_*' - l\frac{\pi}{2}\right)}{r'}Y_l^m(\theta,\phi)Y_l^{m*}(\theta',\phi') \tag{3.11}$$

Since this integral is nearly the same as in Minkowskispace (see eq. 1.24) we also encounter the same problems, namely the IR divergence and the fact that it is non-zero only in two directions. This is due to the fact that the approximation $\omega r \gg l$ breaks down for small ω and for large l. Recall that the problems in the non approximate calculation in Minkowskispace didn't occur because the $j_l(\omega r)$ remain finite at $\omega \to 0$. In other words the essential feature of the $j_l(\omega r)$ is that they let all the $\cos(\omega \ldots)$ terms in the integral drop to zero for $\omega \to 0$ instead of $\cos 0 = 1$ in the approximate case. Therefore we can assume that the same happens for exact solutions around the star.

Unfortunately the exact behaviour for small ω is the same as for small r which will depend on the specific geometry of the star. But the metric of a star is almost flat (since the radius of a star R_0 is much bigger than its Schwarzschildradius R_S). Therefore we will approximate the global mode by replacing the sine with the spherical bessel function:

$$u_{\omega lm} = \frac{\sqrt{\omega}}{\sqrt{\pi}} e^{-i\omega t} \frac{r_*}{r} F(r) j_l(\omega r_*) Y_l^m(\theta, \phi) = \frac{r_*}{r} F(r) u^{\mathrm{M}}_{\omega lm}(r_*, \theta, \phi) \tag{3.12}$$

We introduced an extra factor $F(r)$ to correct the r dependence again. This factor must approach 1 at infinity. Because the modes now are the same (up to a prefactor and replacing $r \to r_*$) as in Minkowskispace we can find the Wightman function by adjusting the Wightman function of Minkowskispace:

$$D^+(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\pi^2} \frac{r_* r'_* F(r) F(r')}{r r'} \frac{1}{(t - t' - i\varepsilon)^2 - |\vec{\mathbf{x}}_* - \vec{\mathbf{x}}'_*|^2} \tag{3.13}$$

$$= -\frac{1}{4\pi^2} \frac{r_* r'_* F(r) F(r')}{r r'} \frac{1}{(t - t' - i\varepsilon)^2 - r_*^2 - r_*'^2 + 2 r_* r'_* \cos\alpha} \tag{3.14}$$

where $\vec{\mathbf{x}}_*$ is obtained by replacing $r \to r_*$ in $\vec{\mathbf{x}}$ and α is the angle between the two vectors. We still have the factor $F(r)$. This can be fixed because we know the behaviour around $\mathbf{x} \approx \mathbf{x}'$ by eq. 2.21. First we know exact that for $\vec{\mathbf{x}} = \vec{\mathbf{x}}'$ it should look like $-\frac{1}{4\pi^2} \frac{1}{f(r)(t-t')^2}$. This implies $F(r) = \frac{r}{r_* \sqrt{f(r)}}$. Second for $t = t'$ the denominator is given by the spatial distance $|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|^2$. The distance between two infinitesimal separated r and $r + \mathrm{d}r$ is given by $|\vec{\mathbf{x}}(r + \mathrm{d}r) - \vec{\mathbf{x}}(r)| = \sqrt{g_{rr}}\, \mathrm{d}r = \frac{\mathrm{d}r}{\sqrt{f(r)}}$. But $|r_*(r + \mathrm{d}r) - r_*(r)| = \frac{\mathrm{d}r_*}{\mathrm{d}r} \mathrm{d}r = \frac{\mathrm{d}r}{f(r)}$. Again we can correct this by setting $F(r) = \frac{r}{r_* \sqrt{f(r)}}$. Since both independent arguments lead to the same result we conclude that the asymptotic form of the Wightmanfunction is given by:

$$D^+(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\pi^2 \sqrt{f(r) f(r')}} \frac{1}{(t - t' - i\varepsilon)^2 - |\vec{\mathbf{x}}_* - \vec{\mathbf{x}}'_*|^2} \tag{3.15}$$

$$= -\frac{1}{4\pi^2 \sqrt{f(r) f(r')}} \frac{1}{(t - t' - i\varepsilon)^2 - r_*^2 - r_*'^2 + 2 r_* r'_* \cos\alpha} \tag{3.16}$$

3.2.3 The Wightmanfunction after the collapse

To calculate the Wightmanfunction after the collapse we will follow the treatment of Hawking (see Appendix A.7) to find that the expectation value of two operators thereafter is given by the thermal expectation value with $\beta = 8\pi M$ in the spacetime before the collapse. The corresponding temperature $T_H = \frac{1}{8\pi M k_{\mathrm{B}}}$ is called Hawking temperature.

Therefore the (vacuum) Wightmanfunction at late times is given by the thermal Wightmanfunction at early (it can be computed analogously to the one in Minkowskispace)

$$D^+_\beta(\mathbf{x},\mathbf{x}') = -\frac{1}{4\beta^2\sqrt{f(r)f(r')}}\frac{1}{\sinh^2\left(\frac{\pi}{\beta}\sqrt{(t-t')^2-|\vec{\mathbf{x}}_*-\vec{\mathbf{x}}'_*|^2}\right)} \tag{3.17}$$

3.3 Observers in the Schwarzschildmetric

3.3.1 Static observer

Before the collapse

We already know by lemma 2.4.1 that a static observer will not see any excitations. We will now show that this is also true when using our approximate form of D^+. A static observer is given by $t = \frac{\tau}{\sqrt{f(r)}}$ and all other coordinates constant. Inserting this into the Wightmanfunction yields

$$D^+(\mathbf{x}(\tau),\mathbf{x}(\tau')) = -\frac{1}{4\pi^2}\frac{1}{(\tau-\tau'-i\varepsilon)^2} \tag{3.18}$$

Up to a prefactor this is the same as for an inertial observer in Minkowskispace (see eq. 1.25). So a static observer indeed does not recognize any particles.

After the collapse

The thermal Wightmanfunction is given by

$$D^+_\beta(\mathbf{x}(\tau),0) = -\frac{1}{4\beta^2 f(r)}\frac{1}{\sinh^2\left(\frac{\pi}{\beta\sqrt{f(r)}}\tau\right)} \tag{3.19}$$

which results in an observed temperature of $T = f(r)^{-1/2}T_\mathrm{H}$ which agrees with the Tolman effect. So a static observer will see a slightly higher temperature than the Hawking temperature.

3.3.2 Circular observer

Before the collapse

By Lemma 2.4.2 we know that a circular observer should see some excitations. Using our approximate form we can now calculate what he will actually measure. A circular observer is given by $t = a\tau$, $\phi = B\tau$, $r = \text{const.}$, $\theta = \frac{\pi}{2}$[1]. The geodesic equation and $\dot{\mathbf{x}}^2 = 0$ give constrains on the constants

$$A^2 = \frac{r}{r-3M} \tag{3.20}$$

$$B^2 = \frac{1}{r^2}\frac{M}{r-3M} \tag{3.21}$$

[1] Since the spacetime is spherically symmetric we can choose a coordinate system such that $\theta = \frac{\pi}{2}$ and $B > 0$

The Wightman function evaluated on the curve is

$$D^+(\mathbf{x}(\tau), \mathbf{x}(\tau')) = -\frac{1}{4\pi^2 f(r)} \frac{1}{(A(\tau-\tau') - i\varepsilon)^2 - 2r_*^2(1-\cos B(\tau-\tau'))} \tag{3.22}$$

We see that D^+ only depends on $\tau - \tau'$. So we can use the simplified formular (1.17). For this we need to find the poles in the lower half of $D^+(\mathbf{x})(\tau), 0$ which means finding the roots of

$$0 = A^2\tau^2 - 2r_*^2(1-\cos B\tau) \tag{3.23}$$
$$0 = \xi^2 x^2 - 2(1-\cos x) \tag{3.24}$$

where $x = B\tau$ and $\xi = \frac{A}{Br_*}$ [2]. Clearly $\tau = 0$ is a root (which will lie in the upper half). Apart from that we have to differentiate between two cases:

- $\xi < 1$: In this case there are another two roots on the real axis. The reason is that for $\xi < 1$ (which is a very fast circular motion) the trajectory hits the Minkowski-lightcone. But we know that this is not possible in the Schwarzschild-spacetime by the argumentation in section 2.3.3. So we assume that this behaviour is due to our approximation and therefore exclude this case[3].

- $\xi > 1$: This case represents slower motions. There are two more (first order) poles at $\pm i x_0$ of whom one is in the lower half.

The rate is given by eq. 1.17

$$\frac{\mathrm{d}F_E}{\mathrm{d}\tau} = \int_{-\infty}^{\infty} \mathrm{d}\tau\, e^{-iE\tau} D^+(\mathbf{x}(\tau), \mathbf{x}(0)) \tag{3.25}$$
$$= -\frac{1}{4\pi^2 f(r)} \int_{-\infty}^{\infty} \mathrm{d}\tau\, e^{-iE\tau} \frac{1}{A^2\tau^2 - 2r_*^2(1-\cos B(\tau))} \tag{3.26}$$
$$= -\frac{1}{4\pi^2 r_*^2 f(r) B} \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-iEx/B} \frac{1}{\xi^2 x^2 - 2(1-\cos x)} \tag{3.27}$$
$$= \frac{i}{2\pi r_*^2 f(r) B} \mathrm{Res}\left(e^{-iEx/B} \frac{1}{\xi^2 x^2 - 2(1-\cos x)}, -ix_0\right) \tag{3.28}$$
$$= \frac{i}{2\pi r_*^2 f(r) B} e^{-E/Bx_0} \lim_{x \to -ix_0} \frac{x + ix_0}{\xi^2 x^2 - 2(1-\cos x)} \tag{3.29}$$
$$= \frac{1}{2\pi r_*^2 f(r) B} e^{-E/Bx_0} \frac{1}{-2\xi^2 x_0 + 2\sinh x_0} \tag{3.30}$$

So basically a circular observer sees a exponentially falling energy distribution.

[2] We exclude the case that $r_* = 0$ since this corresponds to $r \approx 1.4 \cdot 2M$ which is not far away from the horizon
[3] Note that $\xi < 1$ for a geodesic can only happen for $r \lesssim 1.1 R_S$ which is definitely not far away from the black hole

After the collapse

An observer on a circular orbit $t = A\tau$ and $\phi = B\tau$ has the following thermal Wightman function:

$$D^{+}_{\beta}(\mathbf{x}(\tau), \mathbf{x}(0)) = -\frac{1}{4\beta^2 f(r)} \frac{1}{\sinh^2\left(\frac{\pi}{\beta}\sqrt{A^2\tau^2 - 2r_*^2(1 - \cos B\tau)}\right)} \tag{3.31}$$

$$= -\frac{1}{4\beta^2 f(r)} \frac{1}{\sinh^2\left(\frac{\pi}{\beta} r_* \sqrt{\xi^2 x^2 - 2(1 - \cos x)}\right)} \tag{3.32}$$

where we replaced $x = B\tau$ and $\xi = \frac{A}{Br_*}$ as before.

APPENDIX A

Appendix

A.1 Wightmanfunction in Minkowskispace

Using eq. 1.14 we can calculate the Wightman function:

$$D^+(x,x') = \int \mathrm{d}^3k\, u_{\vec{k}}(x) u^*_{\vec{k}}(x') \tag{A.1}$$

$$= \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{1}{2|k|} e^{-i|k|(t-t')+i\vec{k}(\vec{x}-\vec{x}')} \tag{A.2}$$

$$\overset{\omega=|k|}{=} \int_0^\infty \int_{-1}^1 \frac{\omega^2 \mathrm{d}\omega\, \mathrm{d}\cos\theta}{(2\pi)^2} \frac{1}{2\omega} e^{-i\omega(t-t')+i\omega|\vec{x}-\vec{x}'|\cos\theta} \tag{A.3}$$

$$= \frac{1}{2i|\vec{x}-\vec{x}'|} \int_0^\infty \frac{\mathrm{d}\omega}{(2\pi)^2} e^{-i\omega(t-t')} \left(e^{i\omega|\vec{x}-\vec{x}'|} - e^{-i\omega|\vec{x}-\vec{x}'|} \right) \tag{A.4}$$

$$\tag{A.5}$$

This oscillating integral does not converge. Therefore we will first calculate $D^+(x,x')$ for complex times by setting $t \longrightarrow t - i\varepsilon, \varepsilon > 0$ and then treating $D^+(x,x')$ as a distribution when setting $\varepsilon \longrightarrow 0$.

$$D^+(x,x') = \frac{1}{2i|\vec{x}-\vec{x}'|} \int_0^\infty \frac{\mathrm{d}\omega}{(2\pi)^2} e^{-i\omega(t-t'-i\varepsilon-|\vec{x}-\vec{x}'|)} - e^{-i\omega(t-t'-i\varepsilon+|\vec{x}-\vec{x}'|)} \tag{A.6}$$

$$= -\frac{1}{2i|\vec{x}-\vec{x}'|} \frac{1}{(2\pi)^2} \left(\frac{i}{t-t'-i\varepsilon-|\vec{x}-\vec{x}'|} - \frac{i}{t-t'-i\varepsilon+|\vec{x}-\vec{x}'|} \right) \tag{A.7}$$

$$= -\frac{1}{2|\vec{x}-\vec{x}'|} \frac{1}{(2\pi)^2} \frac{(t-t'-i\varepsilon+|\vec{x}-\vec{x}'|)-(t-t'-i\varepsilon-|\vec{x}-\vec{x}'|)}{(t-t'-i\varepsilon)^2-|\vec{x}-\vec{x}'|^2} \tag{A.8}$$

$$= -\frac{1}{4\pi^2} \frac{1}{(t-t'-i\varepsilon)^2-|\vec{x}-\vec{x}'|^2} \tag{A.9}$$

Alternatively one can achieve the form by first choosing a coordinate system such that $\mathbf{x} = \tilde{t}\partial_0$ (as we will do for the thermal case) and then transforming it back (one has to be very careful with the ε

here).

A.2 Thermal Wigthmanfunction in Minkowskispace

To calculate the thermal Wightmanfunction we will restrict to the interior of the lightcone (we will only need that). At a specific point choose the following coordinate system: First we will set $\mathbf{x}' = 0$ and then choose a coordinate system such that $\mathbf{x} = \tilde{t}\partial_0$. Here $\tilde{t} = \pm\sqrt{-\mathbf{x}^2}$ with the upper sign for $t > 0$ and the lower for $t < 0$. In this coordinate system we can redo the calculation of the Wightmanfunction and achieve

$$D^+(\mathbf{x}, 0) = -\frac{1}{4\pi^2}\frac{1}{(\tilde{t} - i\varepsilon)^2} \tag{A.10}$$

To find the thermal Wightmanfunction we to replace $\tilde{t} \longrightarrow \tilde{t} - i\beta n$and add all the contributions (note that inside the lightcone $D^+ = D^{(1)}$)

$$\begin{aligned}
D^+_\beta(\mathbf{x}, 0) &= -\frac{1}{4\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{(\tilde{t} - i\beta n - i\varepsilon)^2} && \text{(A.11)}\\
&= \frac{1}{4\pi^2\beta^2}\sum_{n=-\infty}^{\infty}\frac{1}{(-i\frac{\tilde{t}}{\beta} - n - \varepsilon)^2} && \text{(A.12)}\\
&= \frac{1}{4\beta^2}\frac{1}{\sin^2\left(-i\pi\frac{\tilde{t}}{\beta} - \varepsilon\right)} && \text{(A.13)}\\
&= -\frac{1}{4\beta^2}\frac{1}{\sinh^2\left(\frac{\pi}{\beta}(\tilde{t} - i\varepsilon)\right)} && \text{(A.14)}
\end{aligned}$$

Now go back to the old frame and replace $\tilde{t} = \pm\sqrt{-\mathbf{x}^2}$

$$\begin{aligned}
D^+_\beta(\mathbf{x}, \mathbf{x}') &= -\frac{1}{4\beta^2}\frac{1}{\sinh^2\left(\frac{\pi}{\beta}\left(\pm\sqrt{(t-t')^2 - |\vec{\mathbf{x}} - \vec{\mathbf{x}}'|^2} - i\varepsilon\right)\right)} && \text{(A.15)}\\
&= -\frac{1}{4\beta^2}\frac{1}{\sinh^2\left(\frac{\pi}{\beta}\sqrt{(t-t')^2 - |\vec{\mathbf{x}} - \vec{\mathbf{x}}'|^2}\right)} && \text{(A.16)}
\end{aligned}$$

In the last step we set $\varepsilon \longrightarrow 0$ to get a closed form. One needs to be careful because the ε will move poles in different directions for $t > 0$ and $t < 0$. However normally we only have to worry about the pole at zero which will as in the vacuum case will be moved to the upper half[1].

[1] To show this recall that the pole at zero is due to the pole of the vacuum Wightmanfunction which contributes to sum as $n = 0$

A.3 The Unruh-Detector

Our treatment of such a detector will follow (Quelle: Birell Davies). One describes a detector by a operator $m(\tau)$ which couples to the field via a interaction term $c \cdot m(\tau)\phi(x(\tau))$, where c is small and $x(\tau)$ is the trajectory of the detector. For $\tau \to -\infty$ the detector is in the groundstate $\left|E_0\right\rangle$ and the field is in the vacuum state $|0\rangle$. The detector develops with time according to $m(\tau) = e^{iH_0\tau} m(0) e^{-iH_0\tau}$ with $H_0 |E\rangle = E |E\rangle$.
We would like to calculate the probability that the detector detects a particle with energy E. Since c is small one can use first order perturbation theory where the transition amplitude to another state $|E, \psi\rangle$ at time τ is given by

$$A_{|E_0,0\rangle\to|E,\psi\rangle}(\tau) = ic \langle E, \psi| \int_{-\infty}^{\tau} m(\tau')\phi(x(\tau'))\mathrm{d}\tau' \left|E_0, 0\right\rangle \tag{A.17}$$

$$= ic \langle E, \psi| \int_{-\infty}^{\tau} e^{iH_0\tau'} m(0) e^{-iH_0\tau'} \phi(x(\tau'))\mathrm{d}\tau' \left|E_0, 0\right\rangle \tag{A.18}$$

$$= ic \langle \psi| \int_{-\infty}^{\tau} e^{iE\tau'} \langle E| m(0) \left|E_0\right\rangle e^{-iE_0\tau'} \phi(x(\tau'))\mathrm{d}\tau' |0\rangle \tag{A.19}$$

$$= ic \langle E| m(0) \left|E_0\right\rangle \int_{-\infty}^{\tau} e^{i(E-E_0)\tau'} \langle \psi| \phi(x(\tau')) |0\rangle \,\mathrm{d}\tau' \tag{A.20}$$

$$\tag{A.21}$$

The transition probability is $P_{|E_0,0\rangle\to|E,\psi\rangle}(\tau) = |A_{|E_0,0\rangle\to|E,\psi\rangle}(\tau)|^2$. But since we are only interested in the state of the detector we sum over all field configurations:

$$P_E(\tau) := \sum_i P_{|E_0,0\rangle\to|E,\psi_i\rangle}(\tau) = \sum_i |A_{|E_0,0\rangle\to|E,\psi\rangle}(\tau)|^2 \tag{A.22}$$

$$= c^2 | \langle E| m(0) \left|E_0\right\rangle |^2 F_{E-E_0}(\tau) \tag{A.23}$$

$$\text{with } F_E(\tau) = \sum_i \left| \int_{-\infty}^{\tau} e^{i(E-E_0)\tau} \left\langle \psi_i \right| \phi(x(\tau')) |0\rangle \,\mathrm{d}\tau' \right|^2 \tag{A.24}$$

$$= \sum_i \int_{-\infty}^{\tau} e^{-iE\tau''} \langle 0| \phi(x(\tau''))\mathrm{d}\tau'' \left|\psi_i\right\rangle \left\langle \psi_i \right| \int_{-\infty}^{\tau} e^{iE\tau'} \phi(x(\tau')) |0\rangle \,\mathrm{d}\tau' \tag{A.25}$$

$$= \int_{-\infty}^{\tau} \mathrm{d}\tau' \int_{-\infty}^{\tau} \mathrm{d}\tau'' e^{-iE(\tau''-\tau')} \langle 0| \phi(x(\tau''))\phi(x(\tau')) |0\rangle := \int_{-\infty}^{\tau} \mathrm{d}\tau' \int_{-\infty}^{\tau} \mathrm{d}\tau'' e^{-iE(\tau''-\tau')} D^+(x(\tau''), x(\tau')) \tag{A.26}$$

There we introduced the Wightman function $D^+(x, x') = \langle 0| \phi(x)\phi(x') |0\rangle$. The probability splits in a product of two parts. The first one only depends on the model of the detector while the second part only depends on the trajectory. We will therefore interpret the (so called detector response) function $F_E(\tau)$ as the distribution of energy excitations as been 'seen' by an observer on the trajectory $x(\tau)$.

The transition rate is then given by:

$$\frac{\mathrm{d}F_E(\tau)}{\mathrm{d}\tau} = \int_{-\infty}^{\tau} \mathrm{d}\tau'' e^{-iE(\tau''-\tau)} D^+(x(\tau''), x(\tau))\phi(x(\tau))\,|0\rangle + \int_{-\infty}^{\tau} \mathrm{d}\tau' e^{-iE(\tau-\tau')} D^+(x(\tau), x(\tau')) \tag{A.27}$$

$$= \int_{-\infty}^{\tau} \mathrm{d}\tau' e^{-iE(\tau'-\tau)} D^+(x(\tau'), x(\tau)) + e^{-iE(\tau-\tau')} D^+(x(\tau), x(\tau')) \tag{A.28}$$

$$\stackrel{\tilde{\tau}=\tau'-\tau}{=} \int_{-\infty}^{0} \mathrm{d}\tilde{\tau} e^{-iE\tilde{\tau}} D^+(x(\tilde{\tau}+\tau), x(\tau)) + e^{iE\tilde{\tau}} D^+(x(\tau), x(\tilde{\tau}+\tau)) \tag{A.29}$$

$$= 2\mathrm{Re} \int_{-\infty}^{0} \mathrm{d}\tilde{\tau} e^{-iE\tilde{\tau}} D^+(x(\tilde{\tau}+\tau), x(\tau)) \tag{A.30}$$

since $D^+(x, x')^* = D^+(x', x)$. For the special case that the Wightman function does only depend on the difference of the τ's, i.e. $D^+(x(\tau_1 + \tau'), x(\tau_2 + \tau')) = D^+(x(\tau_1), x(\tau_2))$ one can simplify this further:

$$\frac{\mathrm{d}F_E(\tau)}{\mathrm{d}\tau} = \int_{-\infty}^{0} \mathrm{d}\tilde{\tau} e^{-iE\tilde{\tau}} D^+(x(\tilde{\tau}+\tau), x(\tau)) + \int_{0}^{\infty} \mathrm{d}\tilde{\tau} e^{-iE\tilde{\tau}} D^+(x(\tau), x(\tau-\tilde{\tau})) \tag{A.31}$$

$$= \int_{-\infty}^{0} \mathrm{d}\tilde{\tau} e^{-iE\tilde{\tau}} D^+(x(\tilde{\tau}+\tau), x(\tau)) + \int_{0}^{\infty} \mathrm{d}\tilde{\tau} e^{-iE\tilde{\tau}} D^+(x(\tau+\tilde{\tau}), x(\tau)) \tag{A.32}$$

$$= \int_{-\infty}^{\infty} \mathrm{d}\tilde{\tau} e^{-iE\tilde{\tau}} D^+(x(\tilde{\tau}+\tau), x(\tau)) = \int_{-\infty}^{\infty} \mathrm{d}\tilde{\tau} e^{-iE\tilde{\tau}} D^+(x(\tilde{\tau}), x(0)) \tag{A.33}$$

The rate is the fouriertransform of the Wightman function and is independent of τ.

A.4 Wightman function in normal coordinates

Fix $\mathbf{x}' = 0$. The metric in normal coordinates then looks like (Quelle?):

$$g_{ij} = \delta_{ij} - \frac{1}{3} R_{iajb} x^a x^b + O\left(x^3\right) \tag{A.34}$$

Since the metric is given to the second order we will also expand other quantities[2]

[2] Note that one can raise and lower indices with δ_{ij} if one is neglecting $O\left(x^2\right)$.

$$g^{ij} = \delta^{ij} + \frac{1}{3}R^{i\ j}_{\ a\ b}x^a x^b + O\left(x^3\right) \tag{A.35}$$

$$\partial_i g^{ij} = -\frac{1}{3}R^j_{\ i}x^i + O\left(x^2\right) = -\frac{1}{3}R_{ji}x^i + O\left(x^2\right) \tag{A.36}$$

$$g = \det g_{ij} = 1 - \frac{1}{3}R_{ij}x^i x^j + O\left(x^3\right) \tag{A.37}$$

$$\frac{1}{g}\partial_i g = -\frac{2}{3}R_{ij}x^j + O\left(x^2\right) \tag{A.38}$$

$$\beta = a + b_i x^i + \frac{1}{2}c_{ij}x^i x^j + O\left(x^3\right) \tag{A.39}$$

A.4.1 Solutions of the Klein-Gordon-equation

The Klein-Gordon-equation is given by

$$0 = \nabla_\mu \nabla^\mu \phi = -\frac{1}{\sqrt{\beta g}}\partial_t\left(\sqrt{\beta g}\frac{1}{\beta}\partial_t\phi\right) + \frac{1}{\sqrt{\beta g}}\partial_i\left(\sqrt{\beta g}g^{ij}\partial_j\phi\right) \tag{A.40}$$

$$= -\frac{1}{\beta}\partial_t^2\phi + \frac{1}{\sqrt{\beta g}}\partial_i\left(\sqrt{\beta g}\right)g^{ij}\partial_j\phi + \left(\partial_i g^{ij}\right)\partial_j\phi + g^{ij}\partial_i\partial_j\phi \tag{A.41}$$

$$= -\frac{1}{\beta}\partial_t^2\phi + \frac{1}{2\beta g}\partial_i(\beta g)g^{ij}\partial_j\phi + \left(\partial_i g^{ij}\right)\partial_j\phi + g^{ij}\partial_i\partial_j\phi \tag{A.42}$$

$$\partial_t^2\phi = \frac{\partial_i\beta}{2}g^{ij}\partial_j\phi + \frac{\beta\partial_i g}{2g}g^{ij}\partial_j\phi + \beta\left(\partial_i g^{ij}\right)\partial_j\phi + \beta g^{ij}\partial_i\partial_j\phi \tag{A.43}$$

$$= \frac{1}{2}\left(b_i + c_{ik}x^k\right)\partial_i\phi - \frac{a}{3}R_{ik}x^k\partial_i\phi - \frac{a}{3}R_{ik}x^k\partial_i\phi + \left(a + b_k x^k\right)\partial_i\partial_i\phi + O\left(x^2\right) \tag{A.44}$$

$$= \frac{1}{2}\left(b_i + c_{ik}x^k\right)\partial_i\phi - \frac{2a}{3}R_{ik}x^k\partial_i\phi + \left(a + b_k x^k\right)\partial_i\partial_i\phi + O\left(x^2\right) \tag{A.45}$$

To solve these equations make the ansatz $\tilde{u}_{\vec{\mathbf{k}}}(\mathbf{x}) = \exp\left(-i\omega t + ik_i x^i + i\frac{1}{2}k_a B^a_{ij}\left(\vec{\mathbf{k}}\right)x^i x^j + iO\left(x^3\right)\right)$ and separate the different orders:

$$-\omega^2 = \frac{1}{2}\left(b_i + c_{ik}x^k\right)i\left(k_i + k_a B^a_{ik}x^k\right) - \frac{2a}{3}R_{ik}x^k ik_i + \left(a + b_k x^k\right)\left(ik_a B^a_{ii} - \left(k_i + k_a B^a_{ik}x^k\right)\left(k_i + k_b B^b_{il}x^l\right)\right) + O\left(x^2\right) \tag{A.46}$$

$$-\omega^2 = \frac{1}{2}ib_i k_i + aik_a B^a_{ii} - ak_i k_i \tag{A.47}$$

$$0 = ik_a\left(\frac{1}{2}b_i B^a_{ik} + \frac{1}{2}c_{ak} - \frac{2a}{3}R_{ak} + b_k B^a_{ii}\right) - 2ak_i k_a B^a_{ik} - b_k k_i k_i \tag{A.48}$$

If we demand that all parameters should be real then we have 8 equations for 18 free parameters of B^a_{ij}[3]. We could now fix some more properties of B but it is not necessary for our argumentation. Note

[3] Note that B^a_{ij} is symmetric in i, j.

that the dispersion relation now reads $\omega = \sqrt{a}\left|\vec{\mathbf{k}}\right|$.

A.4.2 Normalising the modes

Next we need to find the right normalisation of the modes. Since we can't integrate our modes over the whole spacetime we will use the CCR to find the right normalisation[4]. So expand $\phi(\mathbf{x}) = \int \frac{\mathrm{d}^3k}{\sqrt{2\pi}^3}\frac{1}{\sqrt{2\omega N_{\vec{\mathbf{k}}}}}\tilde{u}_{\vec{\mathbf{k}}}(\mathbf{x})a_{\vec{\mathbf{k}}} + \frac{1}{\sqrt{2\omega N_{\vec{\mathbf{k}}}}}\tilde{u}_{\vec{\mathbf{k}}}(\mathbf{x})^* a^\dagger_{\vec{\mathbf{k}}}$ and calculate the CCR for a surface t = const.:

$$[\phi(\mathbf{x}), \phi(0)] = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2\omega N_{\vec{\mathbf{k}}}}\tilde{u}_{\vec{\mathbf{k}}}(\mathbf{x})\tilde{u}_{\vec{\mathbf{k}}}(0)^* - \frac{1}{2\omega N_{\vec{\mathbf{k}}}}\tilde{u}_{\vec{\mathbf{k}}}(\mathbf{x})^*\tilde{u}_{\vec{\mathbf{k}}}(0) \tag{A.49}$$

$$= i\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2\omega N_{\vec{\mathbf{k}}}}e^{i\vec{\mathbf{k}}\vec{\mathbf{x}}+O(x^2)} - \frac{1}{2\omega N_{\vec{\mathbf{k}}}}e^{-i\vec{\mathbf{k}}\vec{\mathbf{x}}+O(x^2)} \stackrel{!}{=} 0 \tag{A.50}$$

$$[\phi(\mathbf{x}), \sqrt{g}\partial_0\phi(0)] = i\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2N_{\vec{\mathbf{k}}}}\tilde{u}_{\vec{\mathbf{k}}}(\mathbf{x})\tilde{u}_{\vec{\mathbf{k}}}(0)^* + \frac{1}{2N_{\vec{\mathbf{k}}}}\tilde{u}_{\vec{\mathbf{k}}}(\mathbf{x})^*\tilde{u}_{\vec{\mathbf{k}}}(0) + O\left(x^2\right) \tag{A.51}$$

$$= i\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2N_{\vec{\mathbf{k}}}}e^{i\vec{\mathbf{k}}\vec{\mathbf{x}}+O(x^2)} + \frac{1}{2N_{\vec{\mathbf{k}}}}e^{-i\vec{\mathbf{k}}\vec{\mathbf{x}}+O(x^2)} + O\left(x^2\right) \tag{A.52}$$

$$\stackrel{!}{=} i\delta^3(\vec{\mathbf{x}}) = i\int \frac{\mathrm{d}^3k}{(2\pi)^3}e^{i\vec{\mathbf{k}}\vec{\mathbf{x}}} \tag{A.53}$$

$$[\sqrt{g}\partial_0\phi(\mathbf{x}), \sqrt{g}\partial_0\phi(0)] = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{\omega}{2N_{\vec{\mathbf{k}}}}\tilde{u}_{\vec{\mathbf{k}}}(\mathbf{x})\tilde{u}_{\vec{\mathbf{k}}}(0)^* - \frac{\omega}{2N_{\vec{\mathbf{k}}}}\tilde{u}_{\vec{\mathbf{k}}}(\mathbf{x})^*\tilde{u}_{\vec{\mathbf{k}}}(0) + O\left(x^2\right) \tag{A.54}$$

$$= i\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{\omega}{2N_{\vec{\mathbf{k}}}}e^{i\vec{\mathbf{k}}\vec{\mathbf{x}}+O(x^2)} - \frac{\omega}{2N_{\vec{\mathbf{k}}}}e^{-i\vec{\mathbf{k}}\vec{\mathbf{x}}+O(x^2)} + O\left(x^2\right) \stackrel{!}{=} 0 \tag{A.55}$$

$$\tag{A.56}$$

Since $e^{i\vec{\mathbf{k}}\vec{\mathbf{x}}}$ is a basis we find:

$$\frac{1}{2\omega N_{\vec{\mathbf{k}}}} - \frac{1}{2\omega N_{-\vec{\mathbf{k}}}} = 0 \tag{A.57}$$

$$\frac{1}{2N_{\vec{\mathbf{k}}}} + \frac{1}{2N_{-\vec{\mathbf{k}}}} = 1 \tag{A.58}$$

$$\frac{\omega}{2N_{\vec{\mathbf{k}}}} - \frac{\omega}{2N_{-\vec{\mathbf{k}}}} = 0 \tag{A.59}$$

This system of equations is only solved for $N_{\vec{\mathbf{k}}} = 1$ and so the normalised modes are

$$u_{\vec{\mathbf{k}}} = \frac{1}{\sqrt{2\pi}^3\sqrt{2\omega}}e^{-i\omega t+i\vec{\mathbf{k}}\vec{\mathbf{x}}+O(x^2)} \tag{A.60}$$

$$= \frac{1}{\sqrt{2\pi}^3\sqrt{2\omega}}e^{-i\omega t+i\vec{\mathbf{k}}\vec{\mathbf{x}}}\left(1+O\left(x^2\right)\right) \tag{A.61}$$

[4] This works because the CCR are only valid in the right normalisation

So up to linear order we achieve the plane wave modes as in Minkowskispace. Therefore the Wightman function is also the equivalent and given by

$$D^+(\mathbf{x},0) = -\frac{1}{4\pi^2}\frac{1}{a(t-i\varepsilon)^2 - |\vec{\mathbf{x}}|^2} + O\left(x^2\right) \tag{A.62}$$

A.5 Radial null geodesics in the Schwarzschildmetric

The radial null geodesics are defined by $\dot{x}^2 = 0$, $\dot{\theta} = \dot{\phi} = 0$, where the dot means derivative w.r.t the affine parameter λ. Because ∂_t is killing we know that $E := f(r)\dot{t}$ is constant along the geodesic. This yields to

$$0 = -f(r)\dot{t}^2 + \frac{1}{f(r)}\dot{r}^2 \tag{A.63}$$

$$= -\frac{1}{f(r)}(E^2 - \dot{r}^2) \tag{A.64}$$

Since $f(r) \neq 0$ we conclude:

$$\dot{r} = \pm E \tag{A.65}$$

$$r = \pm E\lambda + r_0 \tag{A.66}$$

Inserting this into the definition of E yields to

$$\dot{r} = \pm f(r)\dot{t} \tag{A.67}$$

$$0 = \dot{t} \mp \frac{\dot{r}}{f(r)} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(t \mp r_*\right) \tag{A.68}$$

where the tortoise coordinate $r_* = r + 2M\ln\left(\frac{r}{2M} - 1\right)$ has the property $\frac{\mathrm{d}r_*}{\mathrm{d}r} = \frac{1}{f(r)}$. So either $u := t - r_*$ (outgoing geodesic) or $v := t + r_*$ (ingoing geodesic) is constant.

Consider an ingoing null geodesic $v = \text{const}, r = -E\lambda + r_0$ (from $\dot{r} = -E$) and compute

$$\frac{\mathrm{d}u}{\mathrm{d}\lambda} = \frac{\mathrm{d}}{\mathrm{d}2t - v} = 2\dot{t} = 2\frac{E}{f(r)} \tag{A.69}$$

$$= 2\frac{Er}{r-2M} = 2E\frac{r-2M+2M}{r-2M} = 2E\left(1 + \frac{2M}{r-2M}\right) \tag{A.70}$$

$$= 2E\left(1 + \frac{2M}{r_0 - 2M - E\lambda}\right) \tag{A.71}$$

$$u(\lambda) = 2E\lambda - 4M\ln\left(r_0 - 2M - E\lambda\right) + 4M\ln\left(r_0 - 2M\right) + u_0 \tag{A.72}$$

$$= 2E\lambda - 4M\ln\left(1 - \frac{E\lambda}{r_0 - 2M}\right) + u_0 \tag{A.73}$$

where $u_0 = u(0)$. For an outgoing geodesic one finds analogously:

$$v(\lambda) = 2E\lambda + 4M \ln\left(1 + \frac{E\lambda}{r_0 - 2M}\right) + v_0 \tag{A.74}$$

We will later need the affine separation between null geodesics, i.e. the value λ on a lightlike geodesic running from one geodesic to the other. For two ingoing geodesics (characterised by v, v') far away from the black hole (i.e. $E\lambda \ll r_0 - 2M$) one can neglect the logarithm and the distance from v_0 to v is given by:

$$\lambda = \frac{v - v_0}{2E} \tag{A.75}$$

The other case which will be important is the distance of two outgoing geodesics u_0, u where $u_0 = \infty$ is the event horizon $r_0 \to 2M$. Now one can neglect the linear term and the distance (note that $\lambda < 0$) is given by:

$$\begin{aligned}
u &= -4M \ln\left(1 - \frac{E\lambda}{r_0 - 2M}\right) + u_0 && \text{(A.76)}\\
&= -4M \ln\left(1 - \frac{E\lambda}{r_0 - 2M}\right) + v_0 - 2r_{0*} && \text{(A.77)}\\
&= -4M \ln\left(1 - \frac{E\lambda}{r_0 - 2M}\right) + v_0 - 2r_0 - 4M \ln\left(\frac{r_0 - 2M}{2M}\right) && \text{(A.78)}\\
&= -4M \ln\left(\left(1 - \frac{E\lambda}{r_0 - 2M}\right)\frac{r_0 - 2M}{2M}\right) + v_0 - 2r_0 && \text{(A.79)}\\
&= -4M \ln\left(\frac{r_0 - 2M - E\lambda}{2M}\right) + v_0 - 2r_0 && \text{(A.80)}\\
&= -4M \ln\left(-\frac{E\lambda}{2M}\right) + v_0 - 4M && \text{(A.81)}\\
-\lambda &= \frac{2M}{E} e^{1+\frac{v_0}{4M}} e^{-\frac{u}{4M}} && \text{(A.82)}
\end{aligned}$$

A.6 Complete solution of the geodesic equation in a two dimensions

Let us first assume we already solved the geodesic equation over the whole two dimensional spacetime (call it $\mathcal{M}$). Then we can construct lightcone coordinates: Fix a point $\mathbf{x}_0$ in the spacetime. In this point there exist two linear independent null vector namely $\mathbf{t}$ and another one defined by $\mathbf{n}^2 = 0$ and $\mathbf{t} \cdot \mathbf{n} = -1$. Note that since we are in a two dimensional spacetime $\mathbf{n}$ is uniquely defined. Then solve for the geodesic starting at $\mathbf{x}_0$ with tangent vector $\mathbf{n}$. Associate a point on the geodesic with the corresponding value of the affine parameter λ. Then starting at such a point solve the geodesic (call it Σ) with tangent vector $\mathbf{t}$ and associate every point on this geodesic with the value of the affine parameter τ. By this we can find a map $\mathbf{x}(\tau, \lambda)$. We can use this map for a coordinate transformation which yields to the coordinate system $\partial_\tau = \mathbf{t}$ and $\partial_\lambda = \mathbf{n}$.

For the boundary condition take $\mathbf{tn} = -1$ on Σ. Before we start let us summarize the known properties of $\mathbf{t}$ and $\mathbf{n}$: Clearly on the whole spacetime $\nabla_\mathbf{t}\mathbf{t} = 0$ and $\mathbf{t}^2 = 0$. Also since they are a coordinate system $[\mathbf{t}, \mathbf{n}] = 0$ which means $\nabla_{\mathbf{t}}\mathbf{n} = \nabla_{\mathbf{n}}\mathbf{t}$. On Σ we also know that $\mathbf{tn} = -1$ and $\mathbf{n}^2 = 0$.

First observe that

$$\mathbf{t}\nabla_{\mathbf{t}}\mathbf{n} = \mathbf{t}\nabla_{\mathbf{n}}\mathbf{t} = \frac{1}{2}\nabla_{\mathbf{n}}\mathbf{t}^2 = 0 \tag{A.83}$$

in the whole spacetime. Use this to calculate $\nabla_{\mathbf{t}}(\mathbf{tn}) = \mathbf{t}\nabla_{\mathbf{t}}\mathbf{n} = 0$ which means that

$$\mathbf{tn} = \text{const.} = -1 \tag{A.84}$$

This also implies that $\nabla_{\mathbf{t}}\mathbf{n} = -\mathbf{t}(\mathbf{n}\nabla_{\mathbf{t}}\mathbf{n})$ (since $\mathbf{t}\nabla_{\mathbf{t}}\mathbf{n} = 0$ and therefore $\nabla_{\mathbf{t}}\mathbf{n} \sim \mathbf{t}$).

Next derive

$$\mathbf{n}\nabla_{\mathbf{n}}\mathbf{t} = \nabla_{\mathbf{n}}(\mathbf{tn} - \mathbf{t}\nabla_{\mathbf{n}}\mathbf{n} = \nabla_{\mathbf{n}}(-1) - \mathbf{t}\nabla_{\mathbf{n}}\mathbf{n}) = -\mathbf{t}\nabla_{\mathbf{n}}\mathbf{n} \tag{A.85}$$

Note that on Σ: $\nabla_{\mathbf{n}}\mathbf{n} = 0$ which means $\mathbf{n}\nabla_{\mathbf{n}}\mathbf{t} = 0$ and (by eq. A.83)$\nabla_{\mathbf{n}}\mathbf{t} = \nabla_{\mathbf{t}}\mathbf{n} = 0$. Unfortunately this parallel transport condition is only satisfied on Σ not on $\mathcal{M}$. Therefore two things will happen to $\mathbf{n}$: it won't remain a null vector and it will not solve the geodesic equation outside of Σ. To see this calculate

$$\nabla_{\mathbf{t}}\nabla_{\mathbf{t}}\mathbf{n}^2 = 2\nabla_{\mathbf{t}}(\mathbf{n}\nabla_{\mathbf{t}}\mathbf{n}) = 2(\nabla_{\mathbf{t}}\mathbf{n})^2 + 2\mathbf{n}\nabla_{\mathbf{t}}\nabla_{\mathbf{t}}\mathbf{n} \tag{A.86}$$

$$\overset{\nabla_{\mathbf{t}}\mathbf{n}\sim\mathbf{t}}{=} 2\mathbf{n}\nabla_{\mathbf{t}}\nabla_{\mathbf{n}}\mathbf{t} = 2\mathbf{n}R(\mathbf{t}, \mathbf{n})\mathbf{t} + 2\mathbf{n}\nabla_{\mathbf{n}}\nabla_{\mathbf{t}}\mathbf{t} \tag{A.87}$$

$$\overset{\nabla_{\mathbf{t}}\mathbf{t}=0}{=} 2\mathbf{n}\mathrm{R}(\mathbf{t}, \mathbf{n})\mathbf{t} \tag{A.88}$$

where $\mathrm{R}(\mathbf{a}, \mathbf{b}) = \nabla_{\mathbf{a}}\nabla_{\mathbf{b}} - \nabla_{\mathbf{b}}\nabla_{\mathbf{a}} - \nabla_{[\mathbf{a},\mathbf{b}]}$ is the curvature tensor. If it vanishes $\frac{\mathrm{d}^2\mathbf{n}^2}{\mathrm{d}\tau^2} = 0$ and so $\mathbf{n}^2 = a\tau + b$. But from the boundary condition on Σ follows that $a = b = 0$ and so $\mathbf{n}^2 = 0$. However when there is curvature (as in our case) $\mathbf{n}^2$ will differ from 0. Since the behaviour of $\mathbf{n}^2$ fully determines $\nabla_{\mathbf{n}}\mathbf{n}$

$$\mathbf{t}\nabla_{\mathbf{n}}\mathbf{n} = \nabla_{\mathbf{n}}(\mathbf{tn}) - \mathbf{n}\nabla_{\mathbf{n}}\mathbf{t} = -\mathbf{n}\nabla_{\mathbf{t}}\mathbf{n} = -\frac{1}{2}\nabla_{\mathbf{t}}\mathbf{n}^2 \tag{A.89}$$

$$\mathbf{n}\nabla_{\mathbf{n}}\mathbf{n} = \frac{1}{2}\nabla_{\mathbf{n}}\mathbf{n}^2 \tag{A.90}$$

this also means that $\nabla_{\mathbf{n}}\mathbf{n} \neq 0$. However is the curvature is small one can neglect this change and so $\mathbf{n}^2 \approx 0$ and $\nabla_{\mathbf{n}}\mathbf{n} \approx 0$. One may then keep track of two neighbouring geodesics by computing the null geodesic between them and evaluating it at the corresponding λ value. Frankly speaking this means that the (null geodesic) distance λ between null geodesics will remain constant.

A.7 The Hawking Effect

In this secton we will follow the treatment of Hawking to show that the quantum field after the collapse is equivalent to a thermal state in the former spacetime.

A.7.1 Geometric Optics Approximation

We need to solve the Klein-Gordon-Equation in this time dependent metric. Hawking realised that modes after leaving the star shortly before it forms a horizon will become highly redshifted. This means their frequency was much larger inside the star so one can apply a geometric optics approximation. The derivation here is similar to a analogously derivation for light in (Stephani). The ansatz for the wavefunction is $\phi = A(\mathbf{x})e^{-i\omega S(\mathbf{x})}$ where ω is large. Plugging this into the Klein-Gordon-Equation gives

$$
\begin{aligned}
0 &= \nabla_\mu \nabla^\mu \left(Ae^{i\omega S}\right) && \text{(A.91)}\\
&= \nabla_\mu \left(\partial^\mu A e^{i\omega S} + i\omega \partial^\mu S A e^{i\omega S}\right) && \text{(A.92)}\\
&= \nabla_\mu \partial^\mu A e^{i\omega S} + 2i\omega \partial^\mu S \partial_\mu A e^{i\omega S} + i\omega \nabla_\mu \partial^\mu S A e^{-i\omega S} - \omega^2 A \partial_\mu S \partial^\mu S e^{-i\omega S} && \text{(A.93)}
\end{aligned}
$$

For ω quite large we can treat every order of ω separately and then neglect the low order terms. The quadratic term in ω is given by $\partial_\mu S \partial^\mu S = 0$ which means that $\partial^\mu S$ is a null vector. By differentiating this we find that $\partial_\mu S$ actually solves the geodesic equation:

$$
\begin{aligned}
0 &= \nabla_\nu \left(\partial_\mu S \partial^\mu S\right) && \text{(A.94)}\\
&= 2\partial^\mu S \nabla_\nu \partial_\mu S && \text{(A.95)}\\
&= 2\partial^\mu S \nabla_\mu \partial_\nu S && \text{(A.96)}\\
&= 2\nabla_{\nabla S} \partial_\nu S && \text{(A.97)}
\end{aligned}
$$

Therefore $\partial^\mu S$ is the tangent vector to a null geodesic. Since we know that for early times S only depends on t and r $\partial^\mu S$ actually describes radial null geodesics.

In order to determine also the amplitude A we need to take into account the linear order of ω.

$$
\begin{aligned}
0 &= 2\partial^\mu S \partial_\mu A + \nabla_\mu \partial^\mu S A \qquad | \cdot A 0 = 2A\partial_\mu A \partial^\mu S + A^2 \nabla_\mu \partial^\mu S && \text{(A.98)}\\
&= \nabla_\mu \left(A^2 \partial^\mu S\right) && \text{(A.99)}
\end{aligned}
$$

Expanding this again and note that $\partial^\mu S \partial_\mu = \frac{\mathrm{d}}{\mathrm{d}\lambda}$ yields to the following useful formula:

$$
\begin{aligned}
0 &= \partial_\mu \left(A^2\right) \partial^\mu S + A^2 \nabla_\mu \partial^\mu S && \text{(A.100)}\\
\frac{\mathrm{d}A^2}{\mathrm{d}\lambda} &= -A^2 \nabla_\mu \partial^\mu S && \text{(A.101)}
\end{aligned}
$$

We could now solve this equation in general for $A^2(\lambda)$. However we would then have to replace λ by some $\lambda(t,r)$. Since we will later be given a relation $\lambda(r)$ we can solve for $A^2(r)$ instead:

$$
\begin{aligned}
\frac{\mathrm{d}A^2}{\mathrm{d}r} &= -A^2 \frac{\mathrm{d}\lambda}{\mathrm{d}r} \nabla_\mu \partial^\mu S && \text{(A.102)}\\
A^2 &= A_0^2 e^{-\int \mathrm{d}r \frac{\mathrm{d}\lambda}{\mathrm{d}r} \nabla_\mu \partial^\mu S} && \text{(A.103)}
\end{aligned}
$$

A_0 can be found by comparing the result to the modes in early times.

Solving the geodesic equation

We need to solve the geodesic equation for $\mathbf{t} = \nabla S$ namely $\nabla_{\mathbf{t}}\mathbf{t} = 0$ and $\mathbf{t}^2 = 0$. Since the spacetime is spherical symmetric for all times the angular coordinates will stay constant and the resulting geodesics are the same as in the corresponding two dimensional spacetime (e.g. with outer metric $\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + \frac{1}{f(r)}\,\mathrm{d}r^2$. For this section we can therefore consider the two dimensional spacetime which is much easier to handle. We will also assume that the radius of the star is much bigger than $2M$ which means we can neglect the curvature in the early spacetime and that the modes $u_{\omega lm}$ in eq. 3.10 are still a good approximation.

Since we have a two dimensional spacetime we can use the result of section A.6 which basically says that if one can neglect the curvature along the path of some neighbouring geodesics their affine distance along a null geodesic will remain constant. Since the outer spacetime does not change outgoing waves will not be affected by the collapse. Therefore consider ingoing waves i.e. $\sim e^{-i\omega v}$ or $S = -v$. Calculating $\mathbf{t} = \nabla S$ and the corresponding $\mathbf{n}$ gives:

$$\mathbf{t} = \nabla S = -\nabla v = \frac{1}{f(r)}\partial_t - \partial_r = \frac{1}{f(r)}\partial_u \tag{A.104}$$

$$\mathbf{n} = \frac{1}{2}(\partial_t + f(r)\partial_r) = \frac{1}{2}\partial_v \tag{A.105}$$

One can easily show that $\mathbf{n}^2 = 0$ and $\mathbf{tn} = -1$ independent of the position. Note that the second equation is a requirement for the calculations in section A.6.

TODO: BILD

Call the last geodesic that will later form the event horizon γ_0. Now we would like to identify an ingoing geodesic slightly before the formation of a horizon with an outgoing geodesic after the formation if both have the same (null geodesic) distance from γ_0. Therefore we need to check if $\mathbf{n}\mathrm{R}(\mathbf{t},\mathbf{n})\mathbf{t} = -\frac{2M}{r^3}$ is actually small. Since the radius of a star is much bigger than its Schwarzschildradius $2M$ we can neglect the curvature in the early spacetime. However during the collapse the geodesics will be near the black hole horizon so we need the curvature at $r = 2M$ which is $\frac{1}{4M^2}$. The larger the mass of the object is the lower the curvature. So for a sufficiently heavy star we can neglect the curvature[5].

In order to keep the differences as small as possible we will only handle the part of the geodesic which is inside the star by this method. Let's say that the ingoing part of γ_0 is given by $v_0 = 0$. Take the value u_0 such that $r(u_0, v_0) \geq R_0$ is outside the star before the collapse[6]. The distance between another ingoing ray γ with $v < v_0 = 0$ and γ_0 is given by the length of a null geodesic running from v_0

[5] Actually one has to be very careful here since the curvature is not dimensionless. The change of $\mathbf{n}^2$ can be achieved by integrating the curvature twice over the geodesic. But since the length of the geodesic inside the star is $\sim 2M$ we will get a dimensionless change of order $O(1)$

[6] Note that since the star starts to collapse before the event horizon appears (u_0, v_0) is not on the surface of the star at this time. So there will be also some smaller $v < 0$ with (u_0, v) also outside the star. This is important since we need a family of geodesics which start outside the star

to v. Since $R_0 \gg r$ we can use the simplified formula A.75 to find[7]

$$-\lambda = \frac{-v}{2E} = -v \tag{A.106}$$

Now both ingoing geodesics will pass through the star, will become outgoing geodesics at $r = 0$ and then γ_0 will form the horizon and γ will escape. We know that after the star collapsed $\mathbf{t} = a\partial_u$ but we don't know the prefactor a. $\mathbf{n}$ is given by $\mathbf{n} = \frac{1}{2af(r)}\partial_v$. The geodesic from γ_0 to γ is given by $v_A = \text{const.}$ and has the same form as in eq. A.82. Since $\mathbf{n}^t = \frac{1}{2af(r)}$ one finds that $E = \frac{1}{2a}$. So by eq. A.82 and eq. A.106 we can conclude

$$-\lambda = \frac{2M}{E} e^{1+\frac{v_A}{4M}} e^{-\frac{u}{4M}} \tag{A.107}$$

This means (since before $S = -v$) after the collapse

$$S = \frac{-\lambda}{2} = 2Mae^{1+\frac{v_A}{4M}} e^{-\frac{u}{4M}} \tag{A.108}$$

We conclude that the outgoing modes at late times are given by $\sim e^{i\omega \cdot 2Mae^{1+\frac{v_A}{4M}} e^{-\frac{u}{4M}}}$.

Calculating the Amplitude

Now we need to keep track of the Amplitude of the wave. According to eq. A.103 we need to integrate $\nabla_\mu \nabla^\mu S = \nabla_\mu \mathbf{t}^\mu = \frac{\sqrt{|g|}}{\partial}_\mu \left(\sqrt{|g|}\mathbf{t}^\mu\right)$.

For ingoing waves

$$\mathbf{t} = \frac{1}{f(r)}\partial_t - \partial_r \tag{A.109}$$

$$\frac{1}{\sqrt{|g|}}\partial_\mu \left(\sqrt{|g|}\mathbf{t}^\mu\right) = 0 - \frac{1}{r^2}\partial_r\left(r^2\right) = -\frac{2}{r} \tag{A.110}$$

Also since $\frac{\mathrm{d}\tau}{\mathrm{d}r} = -\frac{1}{E} = -\frac{1}{f(r)\mathbf{t}^t} = -1$ by eq. A.66 we find

$$\int \mathrm{d}r \frac{\mathrm{d}\tau}{\mathrm{d}r} \nabla_\mu \partial^\mu S = 2 \int \mathrm{d}r \frac{1}{r} = 2\ln\left(\frac{r}{r_0}\right) = \ln\left(\frac{r^2}{r_0^2}\right) A^2 \qquad = A_0^2 \frac{r_0^2}{r^2} \tag{A.111}$$

$$A = A_0 \frac{r_0}{r} \tag{A.112}$$

So A scales like r^{-1} which is conform with the modes in eq. 3.10.

After the collapse $S(u) = Be^{-\frac{u}{4M}}$ where $B := 2Mae^{1+\frac{v_A}{4M}}$ (compare to eq. A.108).

[7] $E = f(r)\dot{t} = \frac{f(r)}{2} \approx \frac{1}{2}$

$$\mathbf{t} = \frac{S(u)}{4M}\left(\frac{1}{f(r)}\partial_t + \partial_r\right) \tag{A.113}$$

$$\frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}\mathbf{t}^\mu\right) = \frac{2}{r}\frac{S(u)}{4M} \tag{A.114}$$

$$\frac{\mathrm{d}\tau}{\mathrm{d}r} = \frac{1}{E} = \frac{1}{f(r)\mathbf{t}^t} = \frac{4M}{S(u)} \tag{A.115}$$

$$\int \mathrm{d}r\,\frac{\mathrm{d}\tau}{\mathrm{d}r}\nabla_\mu\partial^\mu S = 2\int \mathrm{d}r\,\frac{1}{r} = 2\ln\left(\frac{r}{r_0}\right) = \ln\left(\frac{r^2}{r_0^2}\right) \tag{A.116}$$

$$A^2 = A_0^2\frac{r_0^2}{r^2} \tag{A.117}$$

$$A = A_0\frac{r_0}{r} \tag{A.118}$$

The outgoing modes also scale with r^{-1}.

Since ingoing and outgoing waves scale the same way with r^{-1} before and after the collapse we will assume this also for the collapse. So we conclude $A \cdot r = \text{const.}$. However when comparing with spherical modes in Minkowskispace or in the early spacetime (see eq. 3.10) we see that outgoing modes have an extra factor $-(-1)^l$ as compared to ingoing modes. This factor is caused by a phaseshift when passing $r = 0$[8]. Since at the moment when the geodesics pass $r = 0$ the star is not collapsed yet we can infer that there should be also such a phaseshift for the outgoing modes.

We conclude with stating the form of the modes before and after the collapse. We will call the later modes $\psi_{\omega lm}$ instead of $u_{\omega lm}$ to distinguish them.

$$u_{\omega lm} = \frac{i^{-l}}{2i\sqrt{\pi\omega}r}e^{-i\omega u}Y_l^m(\theta,\phi) - \frac{i^l}{2i\sqrt{\pi\omega}r}e^{-i\omega v}Y_l^m(\theta,\phi) \tag{A.119}$$

$$\psi_{\omega lm} = \frac{i^{-l}}{2i\sqrt{\pi\omega}r}e^{i\omega Be^{-\frac{u}{4M}}}Y_l^m(\theta,\phi) - \frac{i^l}{2i\sqrt{\pi\omega}r}e^{-i\omega v}Y_l^m(\theta,\phi) \tag{A.120}$$

A.7.2 The Wigthman function of the Field

After the collapse the field is given by (i stands for (ωlm))

$$\phi(\mathbf{x}) = \sum_i \psi_i(\mathbf{x})a_i + \psi_i(\mathbf{x})^* a_i^\dagger \tag{A.121}$$

One could now calculate D^+ directly from our new functions. However it gives more insight to do a Bogoliubov transformation and express $\phi(\mathbf{x})$ in terms of the old modes u_i but with different annihilation operator b_i:

$$\phi(\mathbf{x}) = \sum_j u_j(\mathbf{x})b_j + u_j(\mathbf{x})^* b_j^\dagger \tag{A.122}$$

[8] This is because the wavefunction must remain finite at $r = 0$, i.e. one must create a sine instead of a cosine

By formula 1.11 the b_i are given through

$$b_j = \sum_i (u_j|\psi_i)a_i + (u_j|\psi_i^*)a_i^\dagger \tag{A.123}$$

Calculating the scalarproduct

We would like to calculate the scalarproduct between the following modes in the later spacetime:

$$u_{\omega lm} = \frac{i^{-l}}{2i\sqrt{\pi\omega}r}e^{-i\omega u}Y_l^m(\theta,\phi) - \frac{i^l}{2i\sqrt{\pi\omega}r}e^{-i\omega v}Y_l^m(\theta,\phi) \tag{A.124}$$

$$\psi_{\omega lm} = \frac{i^{-l}}{2i\sqrt{\pi\omega}r}e^{i\omega Be^{-\frac{u}{4M}}}Y_l^m(\theta,\phi) - \frac{i^l}{2i\sqrt{\pi\omega}r}e^{-i\omega v}Y_l^m(\theta,\phi) \tag{A.125}$$

To do so we choose as hypersurface a lightlike surface with $v = \text{const.}$ together with some spacelike surface that captures the complete interior region (we need both surfaces since Im^+ is only a partial Cauchysurface [9]). However the the u modes are 0 inside the black hole and therefore the integral over the interior will vanish.

So we will only integrate over the lightlike surface. The normal vector of the surface is given by $S = \frac{r^2 \sin\theta}{f(r)}\partial_u$. We will neglect the factor $f(r) \approx 1$ since the bigger part of the hypersurface will be far away from the black hole and we assume that the non approximate wave functions will drop to zero at the event horizon[10].

Before evaluating the integral let us rewrite the modes. First define the prefactor as $A = \frac{i^{-l}}{2i\sqrt{\pi\omega}}$ and then

$$u_{\omega lm} = \frac{\tilde{u}_\omega}{r}Y_l^m(\theta,\phi) \tag{A.126}$$

$$\psi_{\omega lm} = \frac{\tilde{\psi}_\omega}{r}Y_l^m(\theta,\phi) \tag{A.127}$$

$$\tilde{u}_\omega = Ae^{-i\omega u} + A^*e^{-i\omega v} \tag{A.128}$$

$$\tilde{\psi}_\omega = Ae^{i\omega Be^{-\frac{u}{4M}}} + A^*e^{-i\omega v} \tag{A.129}$$

Using this we can simplify the scalarproduct (we will drop the indices l, m because the angular

[9] Note that $v = \text{const.}$ is a lightlike surface but partial Cauchysurfaces need to be spacelike. So it is not a partial Cauchysurface. However as eq. 1.6 is also true for lightlike surfaces the value of the scalarproduct will not change.

[10] Actually Hawking and all other authors I encountered so far didn't mentioned this factor.

integral will just give $\delta_{ll'}\delta_{mm'}$):

$$(u_{\omega'}|\psi_\omega) = i\int_{-\infty}^{\infty} r^2\,\mathrm{d}u\,\frac{\tilde{u}^*_{\omega'}}{r}\partial_u\frac{\tilde{\psi}_\omega}{r} - \frac{\tilde{\psi}_\omega}{r}\partial_u\frac{\tilde{u}^*_{\omega'}}{r} \tag{A.130}$$

$$= i\int_{-\infty}^{\infty} r^2\,\mathrm{d}u\,\frac{\tilde{u}^*_{\omega'}}{r}\frac{\partial_u\tilde{\psi}_\omega}{r} - \frac{\tilde{\psi}_\omega}{r}\frac{\partial_u\tilde{u}^*_{\omega'}}{r} - \frac{\tilde{u}^*_{\omega'}}{r}\frac{\tilde{\psi}_\omega}{r^2}\partial_u r + \frac{\tilde{\psi}_\omega}{r}\frac{\tilde{u}^*_{\omega'}}{r^2}\partial_u r \tag{A.131}$$

$$= i\int_{-\infty}^{\infty} \mathrm{d}u\,\tilde{u}^*_{\omega'}\partial_u\tilde{\psi}_\omega - \tilde{\psi}_\omega\partial_u\tilde{u}^*_{\omega'} \tag{A.132}$$

$$= -2i\int_{-\infty}^{\infty} \mathrm{d}u\,\tilde{\psi}_\omega\partial_u\tilde{u}^*_{\omega'} \tag{A.133}$$

In the last step we integrated by parts and assume that the boundary terms vanish (We know that the later modes drop to zero at the horizon $u = \infty$. For $u = -\infty$ we have a rapidly oscillating function which is zero at average).

$$(u_{\omega'}|\psi_\omega) = -2i\int_{-\infty}^{\infty} \mathrm{d}u\,\tilde{\psi}_\omega\partial_u\tilde{u}^*_{\omega'} \tag{A.134}$$

$$= 2\omega'\int_{-\infty}^{\infty} \mathrm{d}u\left(Ae^{i\omega Be^{-\frac{u}{4M}}} + A^*e^{-i\omega v}\right)A'^*e^{i\omega' u} \tag{A.135}$$

$$= 2\omega' AA'^*\int_{-\infty}^{\infty} \mathrm{d}u\,e^{i\omega Be^{-\frac{u}{4M}}}e^{i\omega' u} + 2\omega' A^*A'^*e^{-i\omega v}\delta(\omega') \tag{A.136}$$

$$= 2\omega' AA'^*\int_{-\infty}^{\infty} \mathrm{d}u\,e^{i\omega Be^{-\frac{u}{4M}}}e^{i\omega' u} \tag{A.137}$$

Next substitute $x = e^{-\frac{u}{4M}}$ and then use contour integration to integrate over the positive imaginary axis ($x = iy$):

$$(u_{\omega'}|\psi_\omega) = 8M\omega' A^*A'^*\int_0^{\infty} \frac{\mathrm{d}x}{x}e^{i\omega Bx}e^{-4Mi\omega'\ln x} \tag{A.138}$$

$$= 8M\omega' AA'^*\int_0^{\infty} \mathrm{d}x\,e^{i\omega Bx}x^{-4Mi\omega'-1} \tag{A.139}$$

$$= i8M\omega' AA'^*\int_0^{\infty} \mathrm{d}y\,e^{-\omega By}(iy)^{-4Mi\omega'-1} \tag{A.140}$$

$$= i^{-4Mi\omega'}8M\omega' AA'^*\int_0^{\infty} \mathrm{d}y\,e^{-\omega By}y^{-4Mi\omega'-1} \tag{A.141}$$

$$\overset{z=\omega By}{=} i^{-4Mi\omega'}8M\omega' AA'^*\int_0^{\infty} \frac{\mathrm{d}z}{B\omega}e^{-z}z^{-4Mi\omega'-1}(\omega B)^{4Mi\omega'+1} \tag{A.142}$$

$$= i^{-4Mi\omega'}(\omega B)^{4Mi\omega'}8M\omega' AA'^*\Gamma(-4Mi\omega') \tag{A.143}$$

To calculate $(u_{\omega'}|\psi_\omega*)$ one can redo the same calculation but choose a contour over the negative imaginary axis ($x = -iy$)

$$
\begin{aligned}
(u_{\omega'}|\psi^*_\omega) &= -2i\int_{-\infty}^{\infty}\mathrm{d}u\,\tilde{\psi}^*_\omega\partial_u\tilde{u}^*_{\omega'} && \text{(A.144)}\\
&= 2\omega' A^* A'^* \int_{-\infty}^{\infty}\mathrm{d}u\, e^{-i\omega B e^{-\frac{u}{4M}}}e^{i\omega' u} && \text{(A.145)}\\
&= 8M\omega' A^* A'^* \int_0^{\infty}\mathrm{d}x\, e^{-i\omega B x}x^{-4Mi\omega'-1} && \text{(A.146)}\\
&= -i8M\omega' A^* A'^* \int_0^{\infty}\mathrm{d}y\, e^{-\omega B y}(-iy)^{-4Mi\omega'-1} && \text{(A.147)}\\
&= i^{4Mi\omega'}8M\omega' A^* A'^* \int_0^{\infty}\mathrm{d}y\, e^{-\omega B y}y^{-4Mi\omega'-1} && \text{(A.148)}\\
&\overset{z=\omega By}{=} i^{4Mi\omega'}8M\omega' A^* A'^* \int_0^{\infty}\frac{\mathrm{d}z}{\omega B}e^{-z}z^{-4Mi\omega'-1}(\omega B)^{4Mi\omega'+1} && \text{(A.149)}\\
&= i^{4Mi\omega'}(\omega B)^{4Mi\omega'}8M\omega' A^* A'^*\Gamma(-4Mi\omega') && \text{(A.150)}\\
&= (-1)^{l+1}i^{8Mi\omega'}(u_{\omega'}|\psi_\omega) = (-1)^{l+1}e^{-4\pi M\omega'}(u_{\omega'}|\psi_\omega) && \text{(A.151)}
\end{aligned}
$$

So both scalar products lead (up to a prefactor) to the same result.

Expectation values

For simplicity I will drop the angular momentum indices l, m since they will be equal for all modes. Since $\{\psi\}$ is a complete set of modes for the whole spacetime we can write[11]

$$
\begin{aligned}
\delta_{\tilde{\omega}\omega'} = (u_{\tilde{\omega}}|u_{\omega'}) &= \sum_\omega (u_{\tilde{\omega}}|\psi_\omega)(\psi_\omega|u_{\omega'}) - (u_{\tilde{\omega}}|\psi^*_\omega)(\psi^*_\omega|u_{\omega'}) && \text{(A.152)}\\
&= \sum_\omega (u_{\tilde{\omega}}|\psi_\omega)(\psi_\omega|u_{\omega'}) - e^{-4M\pi(\tilde{\omega}+\omega')}(u_{\tilde{\omega}}|\psi_\omega)(\psi_\omega|u_{\omega'}) && \text{(A.153)}\\
&= \left(1-e^{-4M\pi(\tilde{\omega}+\omega')}\right)\sum_\omega (u_{\tilde{\omega}}|\psi_\omega)(\psi_\omega|u_{\omega'}) && \text{(A.154)}
\end{aligned}
$$

$$
\sum_\omega (u_{\tilde{\omega}}|\psi_\omega)(\psi_\omega|u_{\omega'}) = \frac{\delta_{\tilde{\omega}\omega'}}{1-e^{-4M\pi(\tilde{\omega}+\omega')}} \tag{A.155}
$$

$$
\sum_\omega (u_{\tilde{\omega}}|\psi^*_\omega)(\psi^*_\omega|u_{\omega'}) = \frac{e^{-4M\pi(\tilde{\omega}+\omega')}}{1-e^{-4M\pi(\tilde{\omega}+\omega')}}\delta_{\tilde{\omega}\omega'} = \frac{1}{e^{8M\pi\omega'}-1}\delta_{\tilde{\omega}\omega'} \tag{A.156}
$$

To calculate D^+ now we need the vacuum expectation values like $\langle 0|\, b_{\tilde{\omega}}b_{\omega'}\,|0\rangle$ and $\langle 0|\, b^\dagger_{\tilde{\omega}}b_{\omega'}\,|0\rangle$.

[11] Actually the sum over ω is an integral.

$$\langle 0| b^\dagger_{\tilde{\omega}} b_{\omega'} |0\rangle = \sum_\omega (u_{\tilde{\omega}}|\psi^*_\omega)^*(u_{\omega'}|\psi^*_\omega) \tag{A.157}$$

$$= \sum_\omega (\psi^*_\omega|u_{\tilde{\omega}})(u_{\omega'}|\psi^*_\omega) \tag{A.158}$$

$$= \frac{1}{e^{8M\pi\omega'} - 1}\delta_{\tilde{\omega}\omega'} \tag{A.159}$$

To calculate $\langle 0| b_{\tilde{\omega}} b_{\omega'} |0\rangle$ make use of $[b_{\tilde{\omega}}, b_{\omega'}] = 0$

$$\langle 0| b_{\tilde{\omega}} b_{\omega'} |0\rangle = \langle 0| b_{\omega'} b_{\tilde{\omega}} |0\rangle \tag{A.160}$$

$$\sum_\omega (u_{\tilde{\omega}}|\psi_\omega)(u_{\omega'}|\psi^*_\omega) = \sum_\omega (u_{\omega'}|\psi_\omega)(u_{\tilde{\omega}}|\psi^*_\omega) \tag{A.161}$$

$$(-1)^{l+1}e^{-4M\pi\omega'}\sum_\omega (u_{\tilde{\omega}}|\psi_\omega)(u_{\omega'}|\psi_\omega) = (-1)^{l+1}e^{-4M\pi\tilde{\omega}}\sum_\omega (u_{\omega'}|\psi_\omega)(u_{\tilde{\omega}}|\psi_\omega) \tag{A.162}$$

So unless $\tilde{\omega} = \omega'$ the expectation value has to vanish. To show that is also vanishes for $\tilde{\omega} = \omega'$ one has to calculate this directly:

$$\langle 0| b_{\tilde{\omega}} b_{\omega'} |0\rangle = (-1)^{l+1}e^{-4M\pi\omega'}\int_0^\infty \mathrm{d}\omega\, (u_{\omega'}|\psi_\omega)(u_{\omega'}|\psi_\omega) \tag{A.163}$$

$$= (-1)^{l+1}e^{-4M\pi\omega'}\int_0^\infty \mathrm{d}\omega\, i^{-8Mi\omega'}(\omega B)^{8Mi\omega'}(8M)^2\omega'^2 A^2 A'^{2*}\Gamma(-4Mi\omega')^2 \tag{A.164}$$

$$= B^{8Mi\omega'}(8M)^2\omega'^2\frac{1}{4\pi}A'^{2*}\Gamma(-4Mi\omega')^2\int_0^\infty \mathrm{d}\omega\,\omega^{8Mi\omega'}\frac{1}{\omega} \tag{A.165}$$

$$= (8M)^2 B^{8Mi\omega'}\omega'^2\frac{1}{4\pi}A'^{2*}\Gamma(-4Mi\omega')^2\int_0^\infty \mathrm{d}\omega\, e^{(8Mi\omega'-1)\ln\omega} \tag{A.166}$$

$$\overset{x=\ln\omega}{=} B^{8Mi\omega'}(8M)^2\omega'^2\frac{1}{4\pi}A'^{2*}\Gamma(-4Mi\omega')^2\int_{-\infty}^\infty \mathrm{d}x\, e^x e^{(8Mi\omega'-1)x} \tag{A.167}$$

$$= B^{8Mi\omega'}(8M)^2\omega'^2\frac{1}{4\pi}A'^{2*}\Gamma(-4Mi\omega')^2\delta(8M\omega') \tag{A.168}$$

$$= B^{8Mi\omega'}8M\omega'^2\frac{1}{4\pi}A'^{2*}\Gamma(-4Mi\omega')^2\delta(\omega') \tag{A.169}$$

There is only a contribution for $\omega' = 0$ which we excluded from our analysis. The other expectation values $\langle 0| b_{\tilde{\omega}} b^\dagger_{\omega'} |0\rangle$ and $\langle 0| b^\dagger_{\tilde{\omega}} b^\dagger_{\omega'} |0\rangle$ can easily be achieved by complex conjugating the other other two results.

The expectation values in the later spacetime coincide with the expectation values for a thermal state with $\beta = 8M\pi$ in the early spacetime, i.e.

$$\langle 0|\, b^{\dagger}_{\tilde{\omega}} b_{\omega'} \,|0\rangle = \frac{1}{e^{8M\pi\omega'} - 1}\delta_{\tilde{\omega}\omega'} \qquad = \langle a^{\dagger}_{\tilde{\omega}} a_{\omega'}\rangle_{\beta} \tag{A.170}$$

$$\langle 0|\, b_{\tilde{\omega}} b_{\omega'} \,|0\rangle = 0 \qquad = \langle a_{\tilde{\omega}} a_{\omega'}\rangle_{\beta} \tag{A.171}$$

It can easily be shown by expanding the definition that this also happens for D^+:

$$D^+(\mathbf{x}, \mathbf{x}') = \langle 0|\, \phi(\mathbf{x})\phi(\mathbf{x}') \,|0\rangle \tag{A.172}$$

$$= \sum_{ij} \langle 0|\, \left(u_i(\mathbf{x})b_i + u_i(\mathbf{x})^* b_i^{\dagger}\right)\left(u_j(\mathbf{x})b_j + u_j(\mathbf{x})^* b_j^{\dagger}\right) |0\rangle \tag{A.173}$$

$$= \sum_{ij} \langle \left(u_i(\mathbf{x})b_i + u_i(\mathbf{x})^* b_i^{\dagger}\right)\left(u_j(\mathbf{x})b_j + u_j(\mathbf{x})^* b_j^{\dagger}\right)\rangle_{\beta} \tag{A.174}$$

$$= \langle \phi(\mathbf{x})\phi(\mathbf{x}')\rangle_{\beta} \tag{A.175}$$

So as long as we are only interested in two point correlations in the later spacetime we can ignore the gravitational collapse by putting the quantum system not in the ground state but in a thermal state with temperature $T_{\mathrm{H}} = \frac{1}{k_{\mathrm{B}}\beta} = \frac{1}{8\pi k_{\mathrm{B}} M}$.

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