Hawking Radiation as Seen by Observers

Bachelor thesis

Friedrich Hübner
Universität Bonn

6. September 2018

Introduction

- Consider star collapsing to black hole
- Hawking 1974 [1]: black hole radiate at $T_{\mathrm{H}} = \frac{1}{8\pi M}$
- Minkowski space:
  - Inertial observer: no particles
  - Accelerating observer: heat bath (Unruh effect)
- Observers around the black hole:
  - Freely falling (e.g. orbiting): no particles?
  - Static observer: heat bath?

Massless scalar field in curved spacetime

- Spacetime metric: $g_{\mu\nu}$
- Lagrangian: $\mathcal{L} = -\frac{1}{2}\sqrt{|g|}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi$
- Klein-Gordon equation: $\nabla_\mu\nabla^\mu\phi = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\partial_\nu\phi\right) = 0$
- Scalar product: $(\phi|\psi) := i\int_\Sigma \mathrm{d}S^\mu\,\phi^*\nabla_\mu\psi - \psi\nabla_\mu\phi^*$
- Orthonormal basis: $(u_i|u_j) = \delta_{ij},\ (u_i|u_j^*) = 0,\ (u_i^*|u_j^*) = -\delta_{ij}$
- Quantisation: $\phi(\mathbf{x}) = \sum_i u_i a_i + u_i^* a_i^\dagger$

State of the QFT

- Vacuum: $a_i \left|0\right\rangle = 0$
- Problem: $u_i \to v_j, a_i \to b_j$: $b_i \left|0\right\rangle \neq 0$

  $\to$ Need to guess state!
- Static spacetime: choose vacuum w.r.t. positive frequency modes: $u_i \sim e^{-i\omega t}$
- What does an observer see?

Greens functions[2]

- Vacuum:
  - Wightman function $D^+(\mathbf{x},\mathbf{x}') := \langle 0|\,\phi(\mathbf{x})\phi(\mathbf{x}')\,|0\rangle = \sum_i u_i(\mathbf{x})u_i^*(\mathbf{x}')$
  - $iD(\mathbf{x},\mathbf{x}') := [\phi(\mathbf{x}),\phi(\mathbf{x}')] = 2i\,\mathrm{Im}\,D^+(\mathbf{x},\mathbf{x}')$
  - $D^{(1)}(\mathbf{x},\mathbf{x}') := \langle 0|\,\{\phi(\mathbf{x}),\phi(\mathbf{x}')\}\,|0\rangle = 2\,\mathrm{Re}\,D^+(\mathbf{x},\mathbf{x}')$
- Thermal:
  - replace $\langle 0|\ldots|0\rangle$ by $\frac{1}{Z}\mathrm{Tr}\,e^{-\beta H}\ldots$
  - D is c-number: $D_\beta = D$
  - $D_\beta^{(1)}(t,\vec{x};t',\vec{x}') = \sum_n D^{(1)}(t-i\beta n,\vec{x};t',\vec{x}')$

Unruh detector

- Detector model: $\mathcal{H}_{\mathrm{detector}} = c \cdot m(\tau) \cdot \phi(\mathbf{x}(\tau)),\ c \ll 1$

Transition rate

$$\frac{\mathrm{d}F_E}{\mathrm{d}\tau} = 2\mathrm{Re} \int_{-\infty}^{0} \mathrm{d}\tau' \, e^{-iE\tau'} D^+(\mathbf{x}(\tau+\tau'), \mathbf{x}(\tau))$$

Constant rate

$$\frac{\mathrm{d}F_E}{\mathrm{d}\tau} = \int_{-\infty}^{\infty} \mathrm{d}\tau \, e^{-iE\tau} D^+(\mathbf{x}(\tau), \mathbf{x}(0))$$

- Interpretation: F_E is particle population for observer

Minkowski space

- Wightman function: $D^+(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\pi^2} \frac{1}{(t - t' - i\varepsilon)^2 - |\vec{x} - \vec{x}'|^2}$
- Static observer: $t(\tau) = \tau, \vec{x}(\tau) = \mathrm{const}$
  - $D^+(\mathbf{x}(\tau), \mathbf{x}(0)) = -\frac{1}{4\pi^2} \frac{1}{(\tau - i\varepsilon)^2}$
  - Fourier transform: $\frac{\mathrm{d}F_E}{\mathrm{d}\tau} = 0$

$\rightarrow$ Inertial observer: vacuum contains no particles

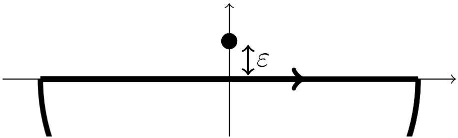

Abbildung: Pole structure

Unruh effect

- Accelerating observer: $t(\tau) = 1/\alpha \sinh \alpha\tau$, $x(\tau) = 1/\alpha \cosh \alpha\tau$
  - $D^+(\mathbf{x}(\tau), \mathbf{x}(\tau')) = -\frac{\alpha^2}{16\pi^2} \frac{1}{\sinh^2 \frac{\alpha(\tau-\tau')}{2}}$
- Thermal state: $D^+_\beta(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\beta^2} \frac{1}{\sinh^2\left(\frac{\pi}{\beta}\sqrt{(t-t'-i\varepsilon)^2 - |\vec{x}-\vec{x}'|^2}\right)}$
  - Static observer: $D^+_\beta(\mathbf{x}(\tau), \mathbf{x}(\tau')) = -\frac{1}{4\beta^2} \frac{1}{\sinh^2\left(\frac{\pi}{\beta}(\tau-\tau')\right)}$
- Set $\beta = 2\pi/\alpha$
- Accelerating observer: vacuum is a thermal state

Static spactimes

- Metric: $\mathrm{d}s^2 = -\beta(\vec{x})\,\mathrm{d}t^2 + g_{ij}(\vec{x})\,\mathrm{d}x^i\,\mathrm{d}x^j$
- Positive frequency solutions: $u_i(t,\vec{x}) = \frac{1}{\sqrt{2\omega_i}} e^{-i\omega_i t} A_i(\vec{x})$
- State of QFT – vacuum: $a_i \left|0\right\rangle = 0$
- Normal coordinates ($a = \beta(0)$):

$$D^+(\mathbf{x},0) = -\frac{1}{4\pi^2}\frac{1}{a(t-i\varepsilon)^2 - |\vec{x}|^2} + \mathcal{O}\left(x^2\right)$$

Pole at $\mathbf{x} = 0$

- Consider trajectory $\mathbf{x}(\tau)$, $\mathbf{x}(0) = 0$
- D^+ has second order pole at $\tau = 0$: $\frac{1}{a(t(\tau)-i\varepsilon)^2-|\vec{x}(\tau)|^2}$
- ε shift pole to upper half:
  - $a(t(\tau_\varepsilon) - i\varepsilon)^2 - |\vec{x}(\tau_\varepsilon)|^2 = 0$
  - $\delta\tau = \frac{\mathrm{d}\tau_\varepsilon}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} = ia\dot{t}(0) \pm \sqrt{-a^2\dot{t}(0)^2 + 1}$
    $\rightarrow \delta\tau$ has positive imaginary part
- Only poles in the lower half contribute

  $\rightarrow$ drop pole at $\tau = 0$

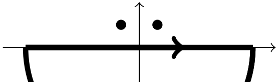

Abbildung: Pole shift

Other singularities of D^+

- $D^+(\mathbf{x}, \mathbf{x}') = \langle 0 | \phi(\mathbf{x})\phi(\mathbf{x}') | 0 \rangle$ satisfies $\nabla_\mu \nabla^\mu D^+(\mathbf{x}, \mathbf{x}') = 0$
- Define $A = 1/D^+ \Rightarrow A\nabla_\mu \nabla^\mu A = 2\nabla_\mu A \nabla^\mu A$
- $D^+ = \infty \Rightarrow A = 0 \Rightarrow \nabla_\mu A \, \nabla^\mu A = 0$
  $\rightarrow D^+ = \infty$ is a lightlike hypersurface
- D^+ is singular on lightcone

- Other singularities appear at $t = \mathrm{const}$

  $\rightarrow$ no singularities on timelike trajectories
- $D = [\phi(\mathbf{x}), \phi(\mathbf{x}')]$ is only non zero on lightcone

  $\rightarrow$ D^+ is real for all trajectories

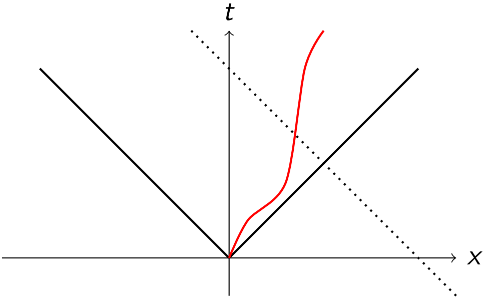

Abbildung: Singularities on trajectories

Static observers

Lemma 1

In a static spacetime a static observer does not observe any particles

- $t(\tau) = \frac{\tau}{\sqrt{a}}$
- $u_i \;=\; \frac{1}{\sqrt{2\omega_i}} e^{-i\omega_i t} A_i(\vec{x})$
- $D^+(\mathbf{x}(\tau), \mathbf{x}(0)) = \sum_i \frac{1}{2\omega_i} e^{-i\omega_i \tau/\sqrt{a}} A_i(\vec{x}_0) A_i^*(\vec{x}_0)$

$$\begin{aligned}\frac{\mathrm{d}F_E}{\mathrm{d}\tau} &= \sum_i \frac{1}{2\omega_i} A_i(\vec{x}_0) A_i^*(\vec{x}_0) \left(\int_{-\infty}^{\infty} \mathrm{d}\tau \, e^{-iE\tau} e^{-i\omega\tau/\sqrt{a}} \right) \\ &= \sum_i \frac{1}{2\omega_i} A_i(\vec{x}_0) A_i^*(\vec{x}_0) \delta\big(E + \omega/\sqrt{a}\big) = 0\end{aligned}$$

Observers along Killing vectors

Lemma 2

An observer moving along a Killing vector **k** will see excitations if and only if there exists at least one solution u: $\frac{A}{|B|} < \frac{|m|}{\omega_m}$

- $\dot{\mathbf{x}} = A\partial_t + B\mathbf{k}$
- $\mathbf{k}u = im\,u$

- $u_{m,i} = \tilde{A}_i(y_1, y_2)e^{-i\omega_{m,i}t}e^{im\xi}$
- Schwarzschild/Minkowski circular orbit: $\mathbf{k} = \partial_\phi$, $\frac{|m|}{\omega_m}$ unbound
- Geodesic observers see particles!

Observers along Killing vectors

Lemma 3

There is an observer moving along a Killing vector $\mathbf{k}$ who will see excitations if and only if there exists at least one solution u: $\frac{g_{\xi\xi}(\mathbf{x})}{|g_{tt}(\mathbf{x})|} < \frac{m^2}{\omega_m^2}$

- Minkowski inertial: $\mathbf{k} = \partial_x$, $\frac{g_{xx}}{|g_{tt}|} = 1 \geqslant \frac{|m|}{\omega}$

$\rightarrow$ Minkowski space: All inertial observers do not encounter particles.

- Global effect: equivalence principle does not apply
- Depends on whole history of observer

General observers

- Transition rate not constant:
  $\frac{\mathrm{d}F_E}{\mathrm{d}\tau} = 2\mathrm{Re}\int_{-\infty}^{0} \mathrm{d}\tau'\, e^{-iE\tau'} D^+(\mathbf{x}(\tau+\tau'), \mathbf{x}(\tau))$
  $\rightarrow$ can not use residuum theorem
- How to handle pole at $\tau' = 0$?

- Expansion: $D^+(\mathbf{x}(\tau'), 0) = -\frac{1}{4\pi^2\tau'^2} + W(\tau')$
- $\frac{1}{\tau'^2}$ term does not contribute
- $W(\tau')$ is non-singular

Black holes

- Consider star
- $\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + \frac{1}{f(r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega$ where $f(r) = 1 - 2M/r$
- QFT in vacuum state
- Collapse to black hole: thermal state with temperature $\beta_{\mathrm{H}} = 8\pi M$
- What spectrum will observers see before and after the collapse?

Wightman function

$$D^+(\mathbf{x},\mathbf{x}') \approx -\frac{1}{4\pi^2\sqrt{f(r)f(r')}}\frac{1}{(t-t'-i\varepsilon)^2-r_*^2-r_*'^2+2r_*r_*'\cos\alpha}$$

Circular Observer before collapse

- Circular observer: $r = \mathrm{const}, t = A\tau, \phi = B\tau$
  - $D^+(\mathbf{x}(\tau), \mathbf{x}(\tau')) = -\frac{1}{4\pi^2 f(r)} \frac{1}{(A(\tau-\tau')-i\varepsilon)^2 - 2r_*^2(1-\cos B(\tau-\tau'))}$
  - Small velocities $\frac{A}{Br_*} > 1$: $\frac{\mathrm{d}F_E}{\mathrm{d}\tau} \sim e^{-E/Bx_0}$
  - High velocities $\frac{A}{Br_*} < 1$: Poles on real axis $\rightarrow$ not possible
- Problem:
  $D^+_\beta(\mathbf{x}(\tau), \mathbf{x}(0)) = -\frac{1}{4\pi^2\tau^2} \frac{1}{f(r)A^2 - f(r)r_*^2 B^2} + \mathcal{O}(\tau^0) \neq -\frac{1}{4\pi^2\tau^2} + \mathcal{O}(\tau^0)$
- Approximation only valid for $r^2 \approx f(r) r_*^2 \qquad (r > 200M)$

After collapse

1. Find thermal Wightman function

2. How to determine observed temperature?

3. Apply on different observers:
    - static
    - circular geodesic
    - infalling geodesic ($E = m$)

Thermal Wightman function

- Thermal Wightman function:

$$
\begin{aligned}
D^+_\beta(t(\tau')) &= \sum_{n=-\infty}^{\infty} D^+(t(\tau') - i\beta n) \\
&= \sum_{n=-\infty}^{\infty} -\frac{1}{4\pi^2(\tau' - i\beta\sqrt{a}n)^2} + W(\tau'(t - i\beta n)) \\
&= -\frac{1}{4\beta^2 a}\frac{1}{\sinh^2\left(\frac{\pi}{\beta\sqrt{a}}\tau'\right)} + W_\beta(\tau')
\end{aligned}
$$

- Static observers: $T = \frac{T_0}{\sqrt{a}}$ (Tolman relation)
- General: Corrections from $W_\beta(\tau')$
  - temperature shift?
  - non thermal?

Determining the temperature

- Expect thermal spectrum + non thermal spectrum
- Fit temperature: Minimize $\int_0^\infty \mathrm{d}E \left| \frac{\mathrm{d}F_E}{\mathrm{d}\tau} - \left(\frac{\mathrm{d}F_E}{\mathrm{d}\tau}\right)_\beta \right|^2$

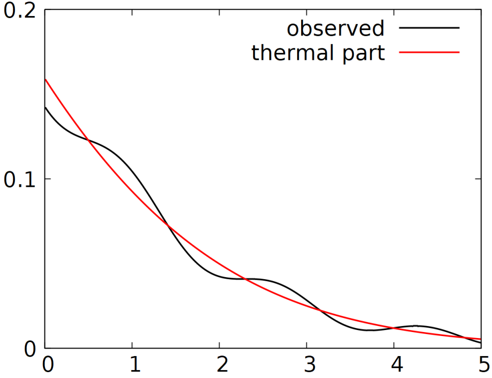

Abbildung: Observed and thermal spectrum

Determining the temperature

- Minimize instead $\int_{-\infty}^{0} \mathrm{d}\tau' \left| D^{+}(\mathbf{x}(\tau+\tau'), \mathbf{x}(\tau)) - D_{\beta}^{\mathrm{M}}(\tau') \right|^{2}$
- Expect small shift:
  $D_{\beta}^{\mathrm{M}}(\tau') \approx D_{\beta_{\mathrm{H}}}^{\mathrm{M}}(\tau') + \alpha g(\tau')$ where $\alpha = \frac{\Delta\beta}{\beta_{\mathrm{H}}} = -\frac{\Delta T}{T_{\mathrm{H}}}$
- Minimize $\int_{-\infty}^{0} \mathrm{d}\tau' \left| h(\tau') - \alpha g(\tau') \right|^{2}$
  where $h(\tau') = D^{+}(\mathbf{x}(\tau+\tau'), \mathbf{x}(\tau)) - D_{\beta_{\mathrm{H}}}^{\mathrm{M}}(\tau')$

$$\alpha = \frac{\int_{-\infty}^{0} \mathrm{d}\tau'\, h(\tau') \cdot g(\tau')}{\int_{-\infty}^{0} \mathrm{d}\tau'\, g(\tau')^{2}}$$

Static observer

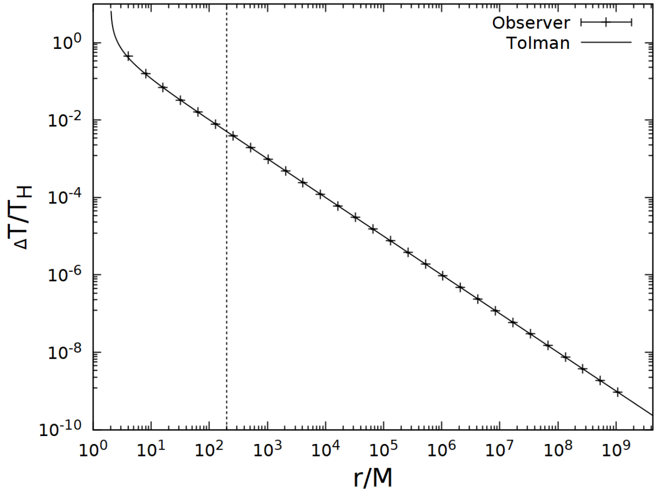

Abbildung: Relative temperature shift

Static observer

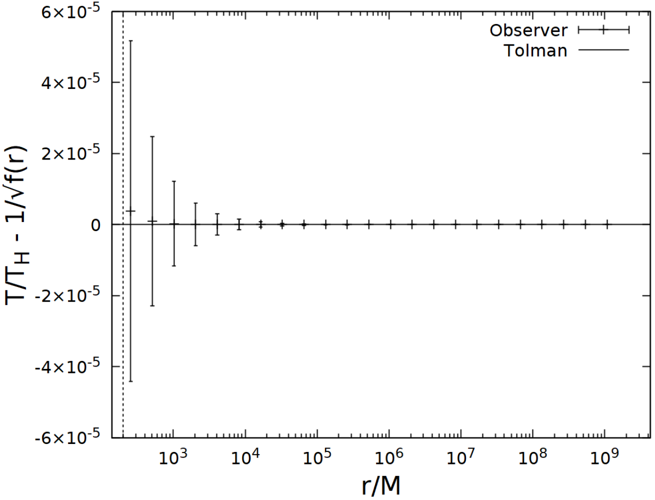

Abbildung: Difference to Tolman relation

Circular observer

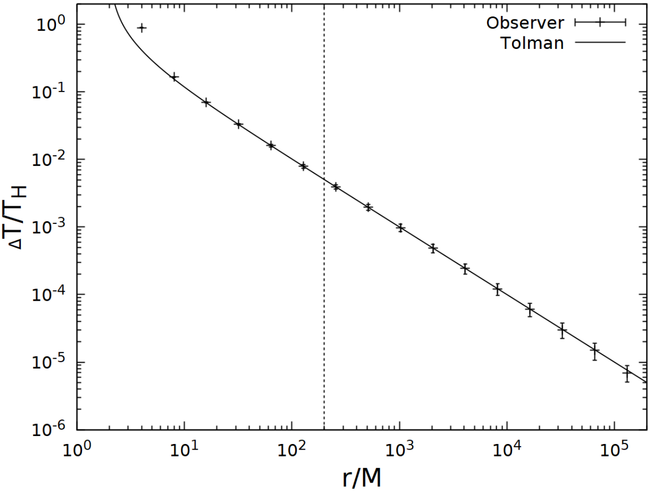

Abbildung: Relative temperature shift

Circular observer

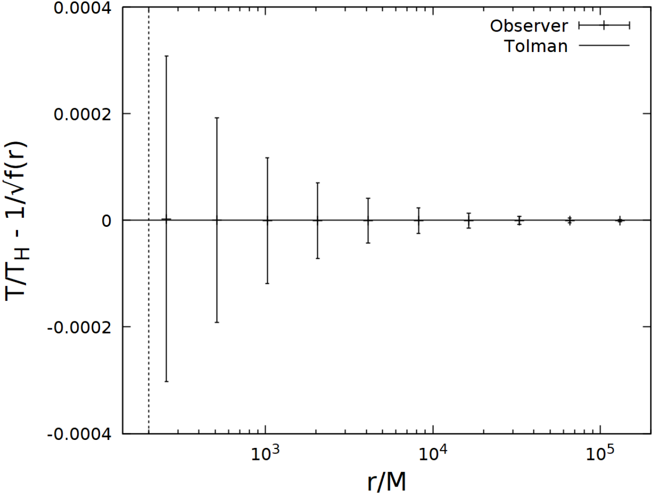

Abbildung: Difference to Tolman relation

Infalling radial observer

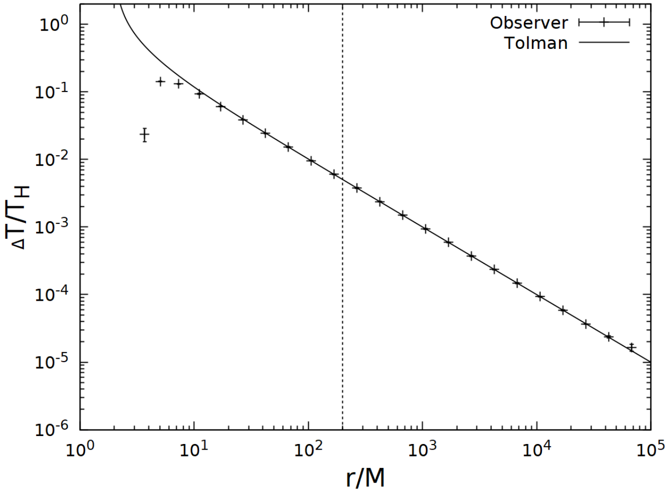

Abbildung: Relative temperature shift

Infalling radial observer

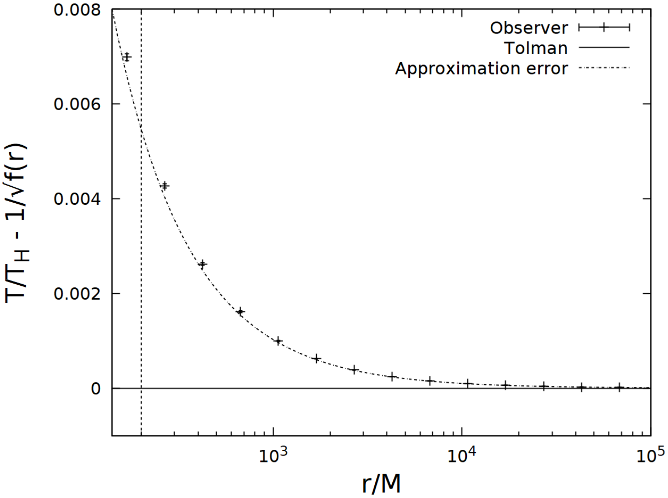

Abbildung: Difference to Tolman relation

Conclusion

- Equivalence principle not applicable
- Approximated D^+ for $r > 200M$
- Determined the temperature out of a spectrum
- For all observers temperature follows Tolman relation
- Deviations not significant enough

Sources

- S. W. Hawking. “Particle creation by black holes”. In: *Communications in Mathematical Physics* 43.3 (1975), S. 199–220. ISSN: 1432-0916. DOI: 10.1007/BF02345020. URL: https://doi.org/10.1007/BF02345020.
- Davies Birrell. *Quantum fields in curved space*. Camebridge University Press, 1982.

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Solutions of Klein-Gordon equation [2]

- $u_{\omega lm} = Ae^{-i\omega t}\frac{R_{\omega l}}{r}Y_l^m(\theta,\varphi)$
- $\frac{\mathrm{d}^2 R_{\omega l}}{\mathrm{d}r_*^2} + \omega^2 R_{\omega l} - \left(\frac{l(l+1)}{r^2} + \frac{f'(r)}{r}\right)f(r)R_{\omega l} = 0$
- Asymptotic: $R_{\omega l} = e^{\pm i\omega r_*}$ $\qquad r_* = r + 2M\ln\frac{r-2M}{2M}$
- $u_{\omega lm} \approx \frac{1}{\sqrt{\pi\omega}}e^{-i\omega t}\frac{\sin\left(\omega r_* - l\frac{\pi}{2}\right)}{r}Y_l^m(\theta,\varphi)$

Wightman function

- Wightman function: $D^+(\mathbf{x}, \mathbf{x}') =$
$\int_0^\infty \frac{\mathrm{d}\omega}{\pi\omega} \sum_{l,m} e^{-i\omega(t-t')} \frac{\sin\left(\omega r_* - l\frac{\pi}{2}\right)}{r} \frac{\sin\left(\omega r'_* - l\frac{\pi}{2}\right)}{r'} Y_l^m(\theta,\varphi) Y_l^{m*}(\theta',\varphi')$
- Problems:
  - IR divergence: $\int_0^\infty \frac{\mathrm{d}\omega}{\pi\omega} e^{i\omega\dots}$
  - Angular dependence: $\sum_{l,m} Y_l^m(\theta,\varphi) Y_l^{m*}(\theta',\varphi') \sim \delta(\theta-\theta')\delta(\varphi-\varphi')$

Intermezzo: Minkowski space spherical modes

- $u^{\mathrm{M}}_{\omega,l,m} = \frac{\sqrt{\omega}}{\sqrt{\pi}} e^{-i\omega t} j_l(\omega r) Y_l^m(\theta,\varphi)$
- Asymptotic: $u^{\mathrm{M}}_{\omega,l,m} \to \frac{1}{\sqrt{\pi\omega}} e^{-i\omega t} \frac{\sin\left(\omega r - l\frac{\pi}{2}\right)}{r} Y_l^m(\theta,\varphi)$
- Wightman function: $D^+(\mathbf{x},\mathbf{x}')$
  $\to \int_0^\infty \frac{\mathrm{d}\omega}{\pi\omega} \sum_{l,m} e^{-i\omega(t-t')} \frac{\sin\left(\omega r - l\frac{\pi}{2}\right)}{r} \frac{\sin\left(\omega r' - l\frac{\pi}{2}\right)}{r'} Y_l^m(\theta,\varphi) Y_l^{m*}(\theta',\varphi')$

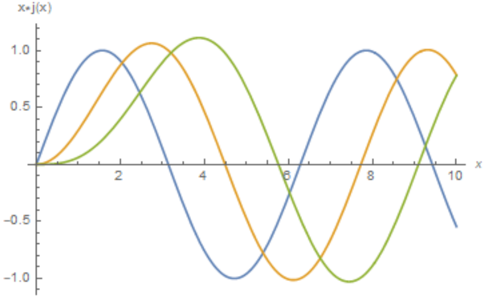

Abbildung: Spherical Bessels: $x \cdot j_l(x)$

- Replace $\frac{\sin\left(\omega r_* - l\frac{\pi}{2}\right)}{\omega r} \approx F(r) j_l(\omega r_*)$
- Fix $F(r)$ for limit $\mathbf{x} \to \mathbf{x}'$

Wightman function

$$D^+(\mathbf{x}, \mathbf{x}') \approx -\frac{1}{4\pi^2\sqrt{f(r)f(r')}} \frac{1}{(t-t'-i\varepsilon)^2 - r_*^2 - r_*'^2 + 2r_* r_*' \cos\alpha}$$

- Static observer: $r = \mathrm{const}, \alpha = 0$
  - $D^+(\mathbf{x}(\tau), \mathbf{x}(\tau')) = -\frac{1}{4\pi^2}\frac{1}{(\tau-\tau'-i\varepsilon)^2}$

  $\rightarrow$ No particles