

Hawking Radiation as Seen by Observers

Bachelor thesis

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Massless scalar field in curved spacetime

- Spacetime metric: $g_{\mu\nu}$
- Lagrangian: $\mathcal{L} = -\frac{1}{2}\sqrt{|g|}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$
- Klein-Gordon equation: $\nabla_\mu\nabla^\mu\phi = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\partial_\nu\phi\right) = 0$
- Scalar product: $(\phi|\psi) := i\int_\Sigma dS^\mu\phi^*\nabla_\mu\psi - \psi\nabla_\mu\phi^*$
- Orthonormal basis: $(u_i|u_j) = \delta_{ij}$, $(u_i|u_j^*) = 0$, $(u_i^*|u_j^*) = -\delta_{ij}$
- Quantisation: $\phi(\mathbf{x}) = \sum_i u_i a_i + u_i^* a_i^\dagger$

State of the QFT

- Vacuum: $a_i |0\rangle = 0$
- Problem: $u_i \rightarrow v_j, a_i \rightarrow b_j: b_i |0\rangle \neq 0$
→ Need to guess state!
- Static spacetime: choose vacuum w.r.t. positive frequency modes:
 $u_i \sim e^{-i\omega t}$
- What does an observer see?

Greens functions

- Vacuum:

- Wightman function $D^+(\mathbf{x}, \mathbf{x}') := \langle 0 | \phi(\mathbf{x}) \phi(\mathbf{x}') | 0 \rangle$
- $iD(\mathbf{x}, \mathbf{x}') := [\phi(\mathbf{x}), \phi(\mathbf{x}')] = 2i \operatorname{Im} D^+(\mathbf{x}, \mathbf{x}')$
- $D^{(1)}(\mathbf{x}, \mathbf{x}') := \langle 0 | \{ \phi(\mathbf{x}), \phi(\mathbf{x}') \} | 0 \rangle = 2 \operatorname{Re} D^+(\mathbf{x}, \mathbf{x}')$

- Thermal:

- replace $\langle 0 | \dots | 0 \rangle$ by $\frac{1}{Z} \operatorname{Tr} e^{-\beta H} \dots$
- D is c-number: $D_\beta = D$
- $D_\beta^{(1)}(t, \vec{x}; t', \vec{x}') = \sum_n D^{(1)}(t - i\beta n, \vec{x}; t', \vec{x}')$

Unruh detector

- Detector model: $c \cdot m(\tau) \phi(\mathbf{x}(\tau))$, $c \ll 1$
- Transition amplitude: $Q_{|0,0\rangle \rightarrow |E,\psi\rangle}(\tau) \sim \int_{-\infty}^{\tau} e^{iE\tau'} \langle \psi | \phi(\mathbf{x}(\tau')) | 0 \rangle d\tau'$
- Transition rate: $\frac{dF_E}{d\tau} = 2\text{Re} \int_{-\infty}^0 d\tau' e^{-iE\tau'} D^+(\mathbf{x}(\tau + \tau'), \mathbf{x}(\tau))$
- For constant rate: $\frac{dF_E}{d\tau} = \int_{-\infty}^{\infty} d\tau e^{-iE\tau} D^+(\mathbf{x}(\tau), \mathbf{x}(0))$
- Interpretation: F_E is particle population for observer

Minkowski space

- Wightman function: $D^+(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\pi^2} \frac{1}{(t-t'-i\varepsilon)^2 - |\vec{x}-\vec{x}'|^2}$
 - Static observer: $t(\tau) = \tau, \vec{x}(\tau) = \text{const}$
 - $D^+(\mathbf{x}(\tau), \mathbf{x}(0)) = -\frac{1}{4\pi^2} \frac{1}{(\tau-i\varepsilon)^2}$
 - Fourier transform: $\frac{dF_E}{d\tau} = 0$
- Inertial observer: vacuum contains no particles

image

Unruh effect

- Accelerating observer: $t(\tau) = 1/\alpha \sinh \alpha\tau$, $x(\tau) = 1/\alpha \cosh \alpha\tau$
 - $D^+(\mathbf{x}(\tau), \mathbf{x}(\tau')) = -\frac{\alpha^2}{16\pi^2} \frac{1}{\sinh^2 \frac{\alpha(\tau-\tau')}{2}}$
- Thermal state: $D_\beta^+(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\beta^2} \frac{1}{\sinh^2 \left(\frac{\pi}{\beta} \sqrt{(t-t'-i\epsilon)^2 - |\vec{x}-\vec{x}'|^2} \right)}$
 - Static observer: $D_\beta^+(\mathbf{x}(\tau), \mathbf{x}(\tau')) = -\frac{1}{4\beta^2} \frac{1}{\sinh^2 \left(\frac{\pi}{\beta} (\tau-\tau') \right)}$
- Set $\beta = 2\pi/\alpha$
- Accelerating observer: vacuum is a thermal state

Static spacetimes

- Metric: $ds^2 = -\beta(\vec{x}) dt^2 + g_{ij}(\vec{x}) dx^i dx^j$
- Positive frequency solutions: $u_i(t, \vec{x}) = \frac{1}{\sqrt{2\omega_i}} e^{-i\omega_i t} A_i(\vec{x})$
- State of QFT – vacuum: $a_i |0\rangle = 0$
- Normal coordinates ($a = \beta(0)$):

$$D^+(\mathbf{x}, 0) = -\frac{1}{4\pi^2} \frac{1}{a(t - i\epsilon)^2 - |\vec{x}|^2} + \mathcal{O}(x^2)$$

Pole at $x = x'$

- Consider trajectory $x(\tau)$, $x(0) = 0$
- D^+ has second order pole at $\tau = 0$: $\frac{1}{a(t(\tau) - i\varepsilon)^2 - |\vec{x}(\tau)|^2}$
- ε shift pole to upper half:
 - $a(t(\tau_\varepsilon) - i\varepsilon)^2 - |\vec{x}(\tau_\varepsilon)|^2 = 0$
 - $\delta\tau = \left. \frac{d\tau_\varepsilon}{d\varepsilon} \right|_{\varepsilon=0} = iat(0) \pm \sqrt{-a^2\dot{t}(0)^2 + 1}$
 $\rightarrow \delta\tau$ has positive imaginary part
- Only poles in the lower half contribute
 \rightarrow drop pole at $\tau = 0$

Other singularities of D^+

- $D^+(\mathbf{x}, \mathbf{x}') = \langle 0 | \phi(\mathbf{x}) \phi(\mathbf{x}') | 0 \rangle$ satisfies $\nabla_\mu \nabla^\mu D^+(\mathbf{x}, \mathbf{x}') = 0$
- Define $A = 1/D^+ \Rightarrow A \nabla_\mu \nabla^\mu A = 2 \nabla_\mu A \nabla^\mu A$
- $D^+ = \infty \Rightarrow A = 0 \Rightarrow \nabla_\mu A \nabla^\mu A = 0$
 $\rightarrow D^+ = \infty$ is a lightlike hypersurface
- D^+ is singular on the lightcone
- Assume no more singularities

image

- \rightarrow no singularities on timelike trajectories
- $D = [\phi(\mathbf{x}), \phi(\mathbf{x}')] is only non zero on lightcone$
 $\rightarrow D^+$ is real for all trajectories

Static observers

Lemma:

box

In a static spacetime a static observer does not observe any particles

- $t(\tau) = \frac{\tau}{\sqrt{a}}$
- $u_i = \frac{1}{\sqrt{2\omega_i}} e^{-i\omega_i t} A_i(\vec{x})$
- $D^+(\mathbf{x}(\tau), \mathbf{x}(0)) = \sum_i \frac{1}{2\omega_i} e^{-i\omega_i \tau / \sqrt{a}} A_i(\vec{x}_0) A_i^*(\vec{x}_0)$

$$\begin{aligned} \frac{dF_E}{d\tau} &= \sum_i \frac{1}{2\omega_i} A_i(\vec{x}_0) A_i^*(\vec{x}_0) \left(\int_{-\infty}^{\infty} d\tau e^{-iE\tau} e^{-i\omega\tau/\sqrt{a}} \right) \\ &= \sum_i \frac{1}{2\omega_i} A_i(\vec{x}_0) A_i^*(\vec{x}_0) \delta(E + \omega/\sqrt{a}) = 0 \end{aligned}$$

Observers along Killing vectors

Lemma:

box

An observer moving along a Killing vector will see excitations if and only if there exists at least one solution u : $\frac{A}{|B|} < \frac{|m|}{\omega_m}$

- $\dot{\mathbf{x}} = A\partial_t + B\mathbf{k}$
- $\mathbf{k}u = imu$
- $u_{m,i} = \tilde{A}_i(y_1, y_2)e^{-i\omega_{m,i}t}e^{im\xi}$
- Schwarzschild/Minkowski circular orbit: $\mathbf{k} = \partial_\phi$, $\frac{|m|}{\omega_m}$ unbound
- Minkowski inertial: $\mathbf{k} = \partial_x$, $\frac{A}{|B|} > 1 = \frac{|m|}{\omega}$

Lemma:

box

An observer moving along a Killing vector will see excitations if and only if there exists at least one solution u : $\frac{g_{\xi\xi}(\mathbf{x})}{|g_{tt}(\mathbf{x})|} < \frac{m^2}{\omega_m^2}$

General observers

- Transition rate not constant:

$$\frac{dF_E}{d\tau} = 2\text{Re} \int_{-\infty}^0 d\tau' e^{-iE\tau'} D^+(\mathbf{x}(\tau + \tau'), \mathbf{x}(\tau))$$

→ can not use residuum theorem

- How to handle pole at $\tau' = 0$?
- Expansion: $D^+(\mathbf{x}(\tau'), 0) = -\frac{1}{4\pi^2\tau'^2} + W(\tau')$
- $\frac{1}{\tau'^2}$ term does not contribute
- $W(\tau')$ is non singular

Hawking radiation

- Consider star collapsing to black hole
- $ds^2 = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega$ where $f(r) = 1 - 2M/r$
- Before collapse: vacuum state
- After collapse: Hawking
 - cite
 - thermal state with temperature $\beta_H = 8\pi M$
- What spectrum will observers see before and after the collapse?

Solutions of Klein-Gordon equation

- $u_{\omega lm} = A e^{-i\omega t} \frac{R_{\omega l}}{r} Y_l^m(\theta, \varphi)$
- $\frac{d^2 R_{\omega l}}{dr_*^2} + \omega^2 R_{\omega l} - \left(\frac{l(l+1)}{r^2} + \frac{f'(r)}{r} \right) f(r) R_{\omega l} = 0$
- Asymptotic modes: $u_{\omega lm} = \frac{1}{\sqrt{\pi\omega}} e^{-i\omega t} \frac{\sin(\omega r_* - l\frac{\pi}{2})}{r} Y_l^m(\theta, \varphi)$
- Wightman function: $D^+(\mathbf{x}, \mathbf{x}') = \int_0^\infty \frac{d\omega}{\pi\omega} \sum_{l,m} e^{-i\omega(t-t')} \frac{\sin(\omega r_* - l\frac{\pi}{2})}{r} \frac{\sin(\omega r'_* - l\frac{\pi}{2})}{r'} Y_l^m(\theta, \varphi) Y_l^{m*}(\theta', \varphi')$
- Problems:
 - IR divergence: $\int_0^\infty \frac{d\omega}{\pi\omega} e^{i\omega \dots}$
 - Angular dependence: $\sum_{l,m} Y_l^m(\theta, \varphi) Y_l^{m*}(\theta', \varphi') \sim \delta(\theta - \theta') \delta(\varphi - \varphi')$

Intermezzo: Minkowski space spherical modes

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- $u_{\omega,l,m}^M = \frac{\sqrt{\omega}}{\sqrt{\pi}} e^{-i\omega t} j_l(\omega r) Y_l^m(\theta, \varphi)$
- Asymptotic: $u_{\omega,l,m}^M \rightarrow \frac{1}{\sqrt{\pi\omega}} e^{-i\omega t} \frac{\sin(\omega r - l\frac{\pi}{2})}{r} Y_l^m(\theta, \varphi)$
- Wightman function: $D^+(\mathbf{x}, \mathbf{x}')$

$$= \int_0^\infty \frac{\omega d\omega}{\pi} \sum_{l,m} e^{-i\omega(t-t')} j_l(\omega r) j_l(\omega r') Y_l^m(\theta, \varphi) Y_l^{m*}(\theta', \varphi')$$

$$\rightarrow \int_0^\infty \frac{d\omega}{\pi\omega} \sum_{l,m} e^{-i\omega(t-t')} \frac{\sin(\omega r - l\frac{\pi}{2})}{r} \frac{\sin(\omega r' - l\frac{\pi}{2})}{r'} Y_l^m(\theta, \varphi) Y_l^{m*}(\theta', \varphi')$$

- Replace $\frac{\sin(\omega r_* - l \frac{\pi}{2})}{\omega r} \approx F(r) j_l(\omega r_*)$
- Fix $F(r)$ for limit $\mathbf{x} \rightarrow \mathbf{x}'$

Wightman function

$$D^+(\mathbf{x}, \mathbf{x}') \approx -\frac{1}{4\pi^2 \sqrt{f(r)f(r')}} \frac{1}{(t-t'-i\epsilon)^2 - r_*^2 - r'^2 + 2r_* r'_* \cos \alpha}$$

- Static observer: $r = \text{const}, \alpha = 0$
 - $D^+(\mathbf{x}(\tau), \mathbf{x}(\tau')) = -\frac{1}{4\pi^2} \frac{1}{(\tau - \tau' - i\epsilon)^2}$
 - No particles

Circular Observer before collapse

- Circular observer: $r = \text{const}$, $t = A\tau$, $\phi = B\tau$

- $D^+(\mathbf{x}(\tau), \mathbf{x}(\tau')) = -\frac{1}{4\pi^2 f(r)} \frac{1}{(A(\tau-\tau') - i\epsilon)^2 - 2r_*^2 (1 - \cos B(\tau-\tau'))}$

- Small velocities $\frac{A}{Br_*} > 1$: $\frac{dF_E}{d\tau} \sim e^{-E/Bx_0}$

- High velocities $\frac{A}{Br_*} < 1$: Poles on real axis \rightarrow not possible

- Problem:

$$D_\beta^+(\mathbf{x}(\tau), \mathbf{x}(0)) = -\frac{1}{4\pi^2 \tau^2} \frac{1}{f(r)A^2 - f(r)r_*^2 B^2} + \mathcal{O}(\tau^0) \neq -\frac{1}{4\pi^2 \tau^2} + \mathcal{O}(\tau^0)$$

- Approximation only valid for $r^2 \approx f(r)r_*^2$ ($r > 200M$)

Thermal Wightman function

- Thermal Wightman function:

$$\begin{aligned}
 D_{\beta}^{+}(t(\tau'), \vec{x}(\tau'); 0) &= \sum_{n=-\infty}^{\infty} D^{+}(t(\tau') - i\beta n, \vec{x}(\tau'); 0) \\
 &= \sum_{n=-\infty}^{\infty} -\frac{1}{4\pi^2(\tau' - i\beta\sqrt{a}n)^2} + \sum_{n=-\infty}^{\infty} W(\tau'(t - i\beta n)) \\
 &= -\frac{1}{4\beta^2 a} \frac{1}{\sinh^2\left(\frac{\pi}{\beta\sqrt{a}}\tau'\right)} + W_{\beta}(\tau')
 \end{aligned}$$

- Static observers: $T = \frac{T_0}{\sqrt{a}}$ (Tolman effect)
- General: Corrections from $W_{\beta}(\tau')$
 - temperature shift?
 - non thermal?

Determining the temperature

image

- Expect thermal spectrum + non thermal spectrum
- Fit thermal spectrum
- Minimize $\int_{-\infty}^0 d\tau' \left(D^+(\mathbf{x}(\tau + \tau'), \mathbf{x}(\tau)) - D_{\beta}^M(\tau') \right)^2$
- Expect small shift: $D_{\beta}^M(\tau') \approx D_{\beta_H}^M(\tau') + \alpha g(\tau')$ where $\alpha = \frac{\Delta\beta}{\beta_H} = -\frac{\Delta T}{T_H}$
- $\alpha = \frac{\int_{-\infty}^0 d\tau' h(\tau') \cdot g(\tau')}{\int_{-\infty}^0 d\tau' g(\tau')^2}$ where $h(\tau') = D^+(\mathbf{x}(\tau + \tau'), \mathbf{x}(\tau)) - D_{\beta_H}^M(\tau')$

Determining the temperature

image

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Quellen

- C. S. Wu u. a. 'Experimental Test of Parity Conservation in Beta Decay'. In: Phys. Rev. 105 (4 1957), S. 1413-1415.
doi:10.1103/PhysRev.105.1413.
url:<https://link.aps.org/doi/10.1103/PhysRev.105.1413>.
- ccweb.org/documents/parity/parity.html
- www.spektrum.de/lexikon/physik/paritaet/10923
- www.physi.uni-heidelberg.de/~menzemer/KeyexperimentsWS1415/lamparth.pdf
- www.weltderphysik.de/gebiet/teilchen/news/2012/verletzung-der-zeitsymmetrie-beobachtet/

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