

Hawking Radiation as Seen by Observers

Bachelor thesis

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07.09.2018

Introduction

- Hawking 1974 [1]: black holes radiate at $T_H = \frac{1}{8\pi M}$
- Minkowski space:
 - Inertial observer: no particles
 - Accelerating observer: heat bath (Unruh effect)
- Observers around the black hole:
 - Freely falling (e.g. orbiting): no particles?
 - Static observer: heat bath?

- 1 QFT in curved spacetime
- 2 Static spacetimes and equivalence principle
- 3 Black holes
- 4 Conclusion

Massless scalar field in curved spacetime[2]

- Spacetime metric: $g_{\mu\nu}$
- Lagrangian: $\mathcal{L} = -\frac{1}{2}\sqrt{|g|}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$
- Klein-Gordon equation: $\nabla_\mu\nabla^\mu\phi = 0$
- Scalar product: $(\phi|\psi) := i\int_\Sigma dS^\mu\phi^*\nabla_\mu\psi - \psi\nabla_\mu\phi^*$
- Orthonormal basis: $(u_i|u_j) = \delta_{ij}$, $(u_i|u_j^*) = 0$, $(u_i^*|u_j^*) = -\delta_{ij}$
- Quantisation: $\phi(\mathbf{x}) = \sum_i u_i a_i + u_i^* a_i^\dagger$

State of the QFT[2]

- Vacuum: $a_i |0\rangle = 0$
- Problem: $u_i \rightarrow v_j, a_i \rightarrow b_j: b_i |0\rangle \neq 0$
→ Need to guess state!
- Static spacetime: choose vacuum w.r.t. positive frequency modes:
 $u_i \sim e^{-i\omega t}$
- What does an observer see?

Wightman function[2]

- Vacuum Wightman function:

- $D^+(\mathbf{x}, \mathbf{x}') := \langle 0 | \phi(\mathbf{x}) \phi(\mathbf{x}') | 0 \rangle$
- $D^+(\mathbf{x}, \mathbf{x}') = \sum_i u_i(\mathbf{x}) u_i^*(\mathbf{x}')$
- $\nabla_\mu \nabla^\mu D^+(\mathbf{x}, \mathbf{x}') = 0$

- Thermal Wightman function:

- Replace $\langle 0 | \dots | 0 \rangle$ by $\frac{1}{Z} \text{Tr} e^{-\beta H} \dots$
- If D^+ real: $D_\beta^+(t, \vec{x}; t', \vec{x}') = \sum_n D^+(t - i\beta n, \vec{x}; t', \vec{x}')$

Unruh detector[2]

- Detector model: $\mathcal{H}_{\text{detector}} = c \cdot m(\tau) \cdot \phi(\mathbf{x}(\tau))$, $c \ll 1$

Transition rate

$$\frac{dF_E}{d\tau} = 2\text{Re} \int_{-\infty}^0 d\tau' e^{-iE\tau'} D^+(\mathbf{x}(\tau + \tau'), \mathbf{x}(\tau))$$

Constant rate

$$\frac{dF_E}{d\tau} = \int_{-\infty}^{\infty} d\tau e^{-iE\tau} D^+(\mathbf{x}(\tau), \mathbf{x}(0))$$

- Interpretation: F_E is particle population for observer

Minkowski space[2]

- Wightman function: $D^+(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\pi^2} \frac{1}{(t-t'-i\epsilon)^2 - |\vec{x}-\vec{x}'|^2}$
 - Static observer: $t(\tau) = \tau, \vec{x}(\tau) = \text{const}$
 - $D^+(\mathbf{x}(\tau), \mathbf{x}(0)) = -\frac{1}{4\pi^2} \frac{1}{(\tau-i\epsilon)^2}$
 - Fourier transform: $\frac{dF_E}{d\tau} = 0$
- Inertial observer: vacuum contains no particles

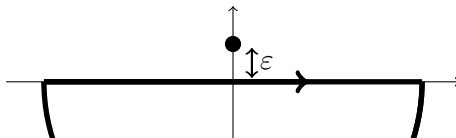


Figure: Pole structure

Unruh effect[2]

- Accelerating observer: $t(\tau) = 1/\alpha \sinh \alpha\tau$, $x(\tau) = 1/\alpha \cosh \alpha\tau$
 - $D^+(\mathbf{x}(\tau), \mathbf{x}(\tau')) = -\frac{\alpha^2}{16\pi^2} \frac{1}{\sinh^2 \frac{\alpha(\tau-\tau')}{2}}$
- Thermal state: $D_\beta^+(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\beta^2} \frac{1}{\sinh^2 \left(\frac{\pi}{\beta} \sqrt{(t-t'-i\varepsilon)^2 - |\vec{x}-\vec{x}'|^2} \right)}$
 - Static observer: $D_\beta^+(\mathbf{x}(\tau), \mathbf{x}(\tau')) = -\frac{1}{4\beta^2} \frac{1}{\sinh^2 \left(\frac{\pi}{\beta} (\tau-\tau') \right)}$
- Set $\beta = 2\pi/\alpha$
- Accelerating observer: vacuum is a thermal state

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Static spacetimes

- Metric: $ds^2 = -\beta(\vec{x}) dt^2 + g_{ij}(\vec{x}) dx^i dx^j$
- Positive frequency solutions: $u_i(t, \vec{x}) = \frac{1}{\sqrt{2\omega_i}} e^{-i\omega_i t} A_i(\vec{x})$
- State of QFT – vacuum: $a_i |0\rangle = 0$

Properties of Wightman function

Wightman function in normal coordinates

$$D^+(\mathbf{x}, 0) = -\frac{1}{4\pi^2} \frac{1}{a(t-i\varepsilon)^2 - |\vec{x}|^2} + \mathcal{O}(x^2) \quad a = \beta(0)$$

- D^+ has second order pole at $\mathbf{x} = 0$
- ε shifts pole to upper half
→ drop pole at $\mathbf{x} = 0$
- No more singularities inside lightcone
- D^+ is real inside lightcone

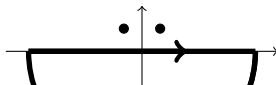


Figure: Pole shift

Static observers

Lemma 1

In a static spacetime a static observer does not observe any particles.

- $t(\tau) = \frac{\tau}{\sqrt{a}}$
- $u_i = \frac{1}{\sqrt{2\omega_i}} e^{-i\omega_i t} A_i(\vec{x})$
- $D^+(\mathbf{x}(\tau), \mathbf{x}(0)) = \sum_i \frac{1}{2\omega_i} e^{-i\omega_i \tau / \sqrt{a}} A_i(\vec{x}_0) A_i^*(\vec{x}_0)$

$$\begin{aligned} \frac{dF_E}{d\tau} &= \sum_i \frac{1}{2\omega_i} A_i(\vec{x}_0) A_i^*(\vec{x}_0) \left(\int_{-\infty}^{\infty} d\tau e^{-iE\tau} e^{-i\omega\tau/\sqrt{a}} \right) \\ &= \sum_i \frac{1}{2\omega_i} A_i(\vec{x}_0) A_i^*(\vec{x}_0) \delta(E + \omega/\sqrt{a}) = 0 \end{aligned}$$

Observers along Killing vectors

Lemma 2

An observer moving along a spatial Killing vector \mathbf{k} will see excitations if and only if there exists at least one solution u : $\frac{A}{|B|} < \frac{|m|}{\omega}$.

- $\dot{\mathbf{x}} = A\partial_t + B\mathbf{k}$
- $\mathbf{k}u = im u, \partial_t u = -i\omega u$
- $u_{m,i} = \tilde{A}_i(y_1, y_2)e^{-i\omega t}e^{im\xi}$
- Schwarzschild/Minkowski circular orbit: $\mathbf{k} = \partial_\varphi$,
 - $u \sim e^{-i\omega t}e^{im\varphi}$
 - $\frac{|m|}{\omega}$ unbounded \rightarrow sees particles
- Geodesic observers see particles!

Equivalence principle?

- Equivalence principle: local effects
- Unruh detector: global effect

$$\frac{dF_E}{d\tau} = \int_{-\infty}^{\infty} d\tau e^{-iE\tau} D^+(\mathbf{x}(\tau), \mathbf{x}(0))$$

- Depends on whole history of observer

General observers

- Transition rate not constant:

$$\frac{dF_E}{d\tau} = 2\text{Re} \int_{-\infty}^0 d\tau' e^{-iE\tau'} D^+(\mathbf{x}(\tau + \tau'), \mathbf{x}(\tau))$$

→ can not use residuum theorem

- How to handle pole at $\tau' = 0$?
- Expansion: $D^+(\mathbf{x}(\tau'), 0) = -\frac{1}{4\pi^2\tau'^2} + W(\tau')$
- $\frac{1}{\tau'^2}$ term does not contribute
- $W(\tau')$ is non-singular

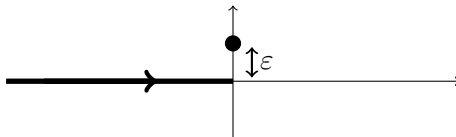


Figure: Pole structure

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Black holes

- Consider star
- $ds^2 = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega$ where $f(r) = 1 - 2M/r$
- QFT in vacuum state
- Collapse to black hole: thermal state with $\beta_H = 8\pi M[1]$
- What spectrum will observers see before and after the collapse?

Vacuum Wightman function

$$D^+(\mathbf{x}, \mathbf{x}') \approx -\frac{1}{4\pi^2 \sqrt{f(r)f(r')}} \frac{1}{(t-t'-i\epsilon)^2 - r_*^2 - r'^2 + 2r_* r'_* \cos \alpha} \quad r > 200M$$

- Tortoise coordinate: $r_* = r + 2M \ln \frac{r-2M}{2M}$

Circular Observer before collapse

- Circular observer: $r = \text{const}, t = A\tau, \phi = B\tau$
- $D^+(\mathbf{x}(\tau), \mathbf{x}(\tau')) = -\frac{1}{4\pi^2 f(r)} \frac{1}{(A(\tau-\tau')-i\epsilon)^2 - 2r_*^2(1-\cos B(\tau-\tau'))}$
- Fourier transform: $\frac{dF_E}{d\tau} \sim e^{-E/Bx_0}$

After collapse

1. Find thermal Wightman function
2. How to determine observed temperature?
3. Apply to different observers:
 - Static
 - Circular geodesic
 - Infalling geodesic (dropped at infinity, no initial velocity)

Thermal Wightman function

- Thermal Wightman function:

$$\begin{aligned}
 D_{\beta}^{+}(t(\tau')) &= \sum_{n=-\infty}^{\infty} D^{+}(t(\tau') - i\beta n) \\
 &= \sum_{n=-\infty}^{\infty} -\frac{1}{4\pi^2(\tau' - i\beta\sqrt{a}n)^2} + W(\tau'(t - i\beta n)) \\
 &= -\frac{1}{4\beta^2 a} \frac{1}{\sinh^2\left(\frac{\pi}{\beta\sqrt{a}}\tau'\right)} + W_{\beta}(\tau')
 \end{aligned}$$

- Static observers: $T = \frac{T_H}{\sqrt{a}}$ (Tolman relation)
- General: Corrections from $W_{\beta}(\tau')$
 - Temperature shift?
 - Non thermal?

Determining the temperature

- Expect thermal spectrum + non thermal spectrum
- Fit temperature: Minimize $\int_0^\infty dE \left| \frac{dF_E}{d\tau} - \left(\frac{dF_E}{d\tau} \right)_\beta \right|^2$

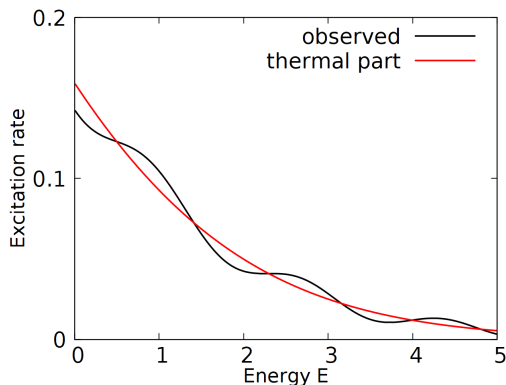


Figure: Observed and thermal spectrum

Determining the temperature

- Minimize instead $\int_{-\infty}^0 d\tau' \left| D^+(\mathbf{x}(\tau + \tau'), \mathbf{x}(\tau)) - D_{\beta}^M(\tau') \right|^2$

- Expect small shift:

$$D_{\beta}^M(\tau') \approx D_{\beta_H}^M(\tau') + \alpha g(\tau') \text{ where } \alpha = \frac{\Delta\beta}{\beta_H} = -\frac{\Delta T}{T_H}$$

- Minimize $\int_{-\infty}^0 d\tau' |h(\tau') - \alpha g(\tau')|^2$
where $h(\tau') = D^+(\mathbf{x}(\tau + \tau'), \mathbf{x}(\tau)) - D_{\beta_H}^M(\tau')$

$$\alpha = \frac{\int_{-\infty}^0 d\tau' h(\tau') \cdot g(\tau')}{\int_{-\infty}^0 d\tau' g(\tau')^2}$$

Static observer

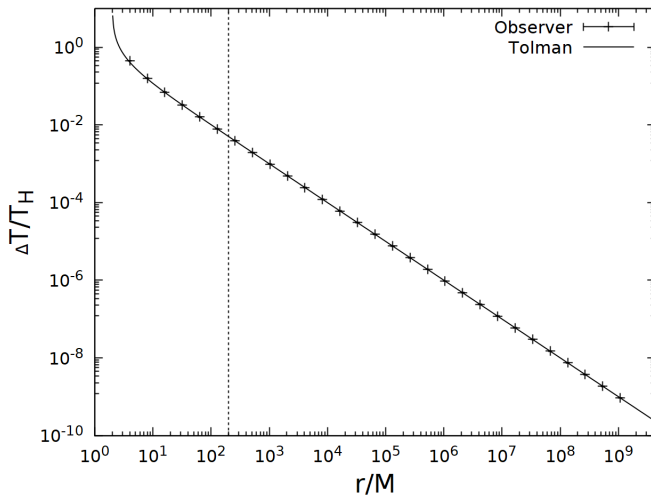


Figure: Relative temperature shift

Static observer

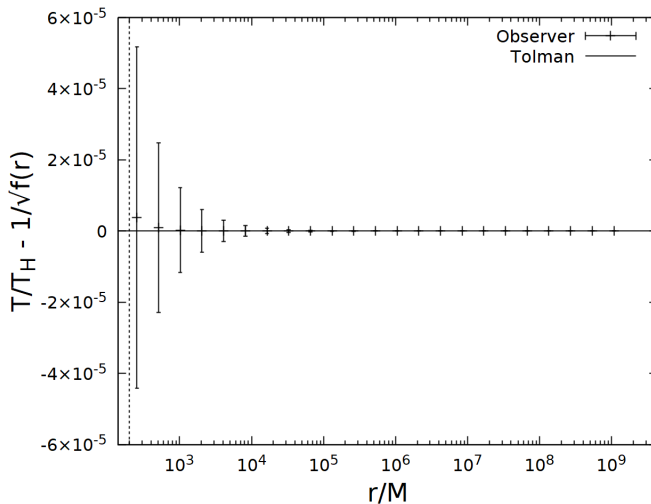


Figure: Difference to Tolman relation

Circular observer

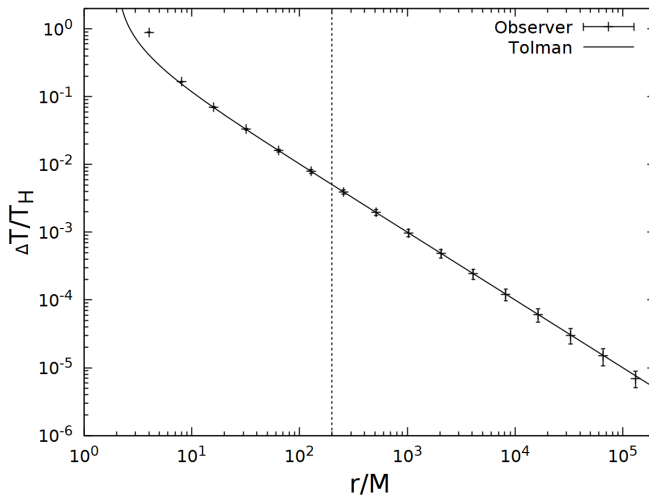


Figure: Relative temperature shift

Circular observer

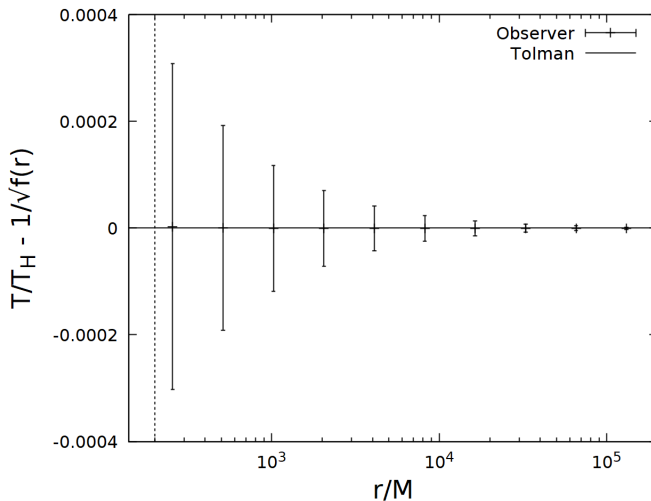


Figure: Difference to Tolman relation

Infalling radial observer

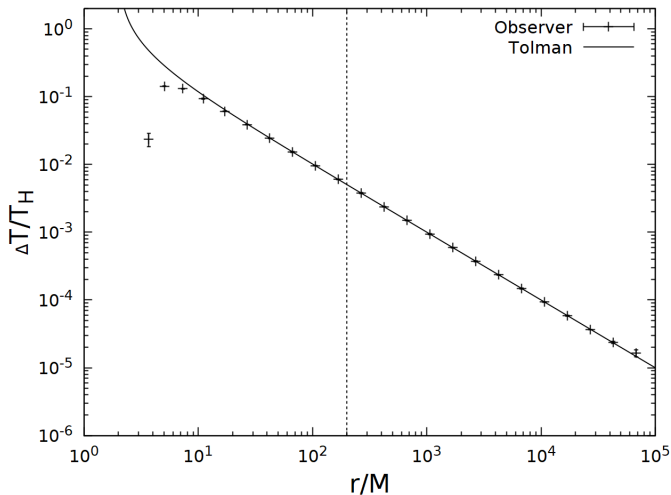


Figure: Relative temperature shift

Infalling radial observer

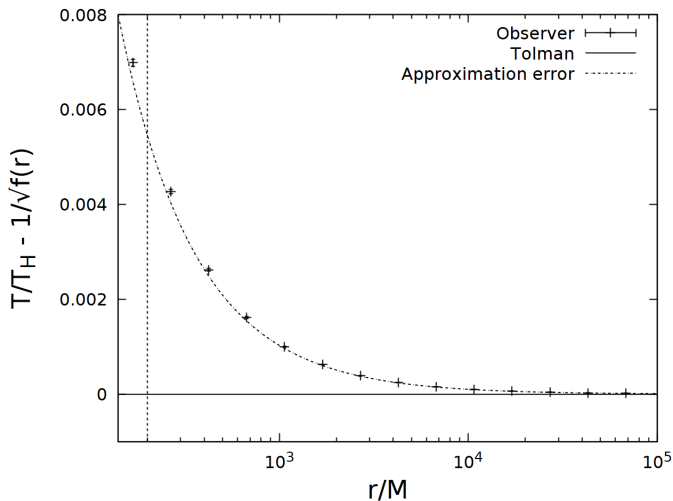


Figure: Difference to Tolman relation

Conclusion

- Equivalence principle not applicable
 - Approximated D^+ for $r > 200M$
 - Determined the temperature out of a spectrum
 - For all observers temperature follows Tolman relation
 - Deviations not significant enough
- Need better approximation

Sources



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Solutions of Klein-Gordon equation [2]

- $u_{\omega lm} = A e^{-i\omega t} \frac{R_{\omega l}}{r} Y_l^m(\theta, \varphi)$
- $\frac{d^2 R_{\omega l}}{dr_*^2} + \omega^2 R_{\omega l} - \left(\frac{l(l+1)}{r^2} + \frac{f'(r)}{r} \right) f(r) R_{\omega l} = 0$
- Asymptotic: $R_{\omega l} = e^{\pm i\omega r_*}$ $r_* = r + 2M \ln \frac{r-2M}{2M}$
- $u_{\omega lm} \approx \frac{1}{\sqrt{\pi\omega}} e^{-i\omega t} \frac{\sin(\omega r_* - l\frac{\pi}{2})}{r} Y_l^m(\theta, \varphi)$

Wightman function

- Wightman function: $D^+(\mathbf{x}, \mathbf{x}') =$

$$\int_0^\infty \frac{d\omega}{\pi\omega} \sum_{l,m} e^{-i\omega(t-t')} \frac{\sin(\omega r_* - l\frac{\pi}{2})}{r} \frac{\sin(\omega r'_* - l\frac{\pi}{2})}{r'} Y_l^m(\theta, \varphi) Y_l^{m*}(\theta', \varphi')$$
- Problems:
 - IR divergence: $\int_0^\infty \frac{d\omega}{\pi\omega} e^{i\omega\ldots}$
 - Angular dependence: $\sum_{l,m} Y_l^m(\theta, \varphi) Y_l^{m*}(\theta', \varphi') \sim \delta(\theta - \theta')\delta(\varphi - \varphi')$

Intermezzo: Minkowski space spherical modes

- $u_{\omega,l,m}^M = \frac{\sqrt{\omega}}{\sqrt{\pi}} e^{-i\omega t} j_l(\omega r) Y_l^m(\theta, \varphi)$
- Asymptotic: $u_{\omega,l,m}^M \rightarrow \frac{1}{\sqrt{\pi\omega}} e^{-i\omega t} \frac{\sin(\omega r - l\frac{\pi}{2})}{r} Y_l^m(\theta, \varphi)$
- Wightman function: $D^+(\mathbf{x}, \mathbf{x}')$
 $\rightarrow \int_0^\infty \frac{d\omega}{\pi\omega} \sum_{l,m} e^{-i\omega(t-t')} \frac{\sin(\omega r - l\frac{\pi}{2})}{r} \frac{\sin(\omega r' - l\frac{\pi}{2})}{r'} Y_l^m(\theta, \varphi) Y_l^{m*}(\theta', \varphi')$

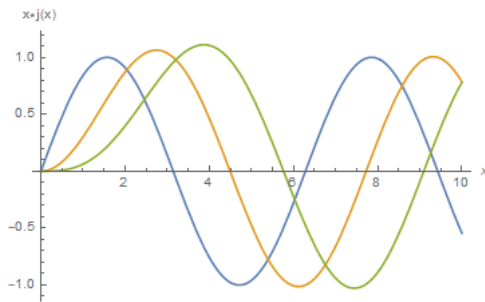


Figure: Spherical Bessels: $x \cdot j_l(x)$

- Replace $\frac{\sin(\omega r_* - l \frac{\pi}{2})}{\omega r} \approx F(r) j_l(\omega r_*)$
- Fix $F(r)$ for limit $\mathbf{x} \rightarrow \mathbf{x}'$

Wightman function

$$D^+(\mathbf{x}, \mathbf{x}') \approx -\frac{1}{4\pi^2 \sqrt{f(r)f(r')}} \frac{1}{(t-t'-i\epsilon)^2 - r_*^2 - r'^2 + 2r_* r'_* \cos \alpha}$$

- Static observer: $r = \text{const}, \alpha = 0$
 - $D^+(\mathbf{x}(\tau), \mathbf{x}(\tau')) = -\frac{1}{4\pi^2} \frac{1}{(\tau - \tau' - i\epsilon)^2}$
 - No particles

Pole at $x = 0$

- Consider trajectory $\mathbf{x}(\tau)$, $\mathbf{x}(0) = 0$
- D^+ has second order pole at $\tau = 0$: $\frac{1}{a(t(\tau) - i\varepsilon)^2 - |\vec{x}(\tau)|^2}$
- ε shift pole to upper half:
 - $a(t(\tau_\varepsilon) - i\varepsilon)^2 - |\vec{x}(\tau_\varepsilon)|^2 = 0$
 - $\delta\tau = \left. \frac{d\tau_\varepsilon}{d\varepsilon} \right|_{\varepsilon=0} = iat(0) \pm \sqrt{-a^2\dot{t}(0)^2 + 1}$
 $\rightarrow \delta\tau$ has positive imaginary part
- Only poles in the lower half contribute
 \rightarrow drop pole at $\tau = 0$

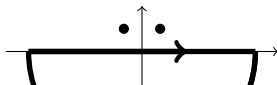


Figure: Pole shift

Other singularities of D^+

- $D^+(\mathbf{x}, \mathbf{x}') = \langle 0 | \phi(\mathbf{x}) \phi(\mathbf{x}') | 0 \rangle$ satisfies $\nabla_\mu \nabla^\mu D^+(\mathbf{x}, \mathbf{x}') = 0$
- Define $A = 1/D^+ \Rightarrow A \nabla_\mu \nabla^\mu A = 2 \nabla_\mu A \nabla^\mu A$
- $D^+ = \infty \Rightarrow A = 0 \Rightarrow \nabla_\mu A \nabla^\mu A = 0$
 $\rightarrow D^+ = \infty$ is a lightlike hypersurface
- D^+ is singular on lightcone

- Other singularities appear at $t = 0$
→ no singularities on timelike trajectories
- $D = [\phi(\mathbf{x}), \phi(\mathbf{x}')]]$ is only non zero on lightcone
→ D^+ is real for all trajectories

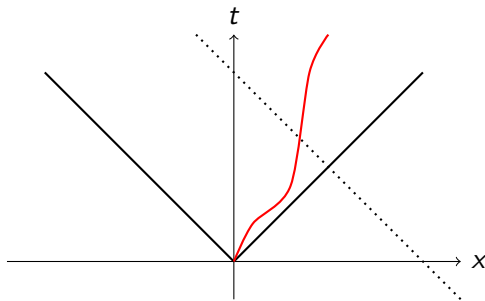


Figure: Singularities on trajectories