Hawking Radiation as Seen by Observers

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Introduction

CHAPTER 2

Quantum field theory in spacetimes

Introducing a QFT in a spacetime is done in a lot of literature. We will mainly follow the treatment from Birell and Davies [1] together with some aspects from [2] and [3].

klingt echt kacke

2.1 Klein-Gordon-Field

Consider a massless real Klein-Gordon-field in a curved spacetime with metric $g_{\mu\nu}$ given by the Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\sqrt{|g|}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi \tag{2.1}$$

The equation of motion is given by the Klein-Gordon-equation:

$$\sqrt{|g|}\nabla_\mu\nabla^\mu\phi = \partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\partial_\nu\phi\right) = 0 \tag{2.2}$$

For solutions we also require to drop to zero at the boundary. Define a scalar product of two such solutions ϕ, ψ over a Cauchy surface Σ via:

$$(\phi|\psi) := i\int_\Sigma \mathrm{d}S^\mu\ \phi^*\nabla_\mu\psi - \psi\nabla_\mu\phi^* = i\int_\Sigma \mathrm{d}S^\mu\ \phi^*\overleftrightarrow{\nabla}_\mu\psi \tag{2.3}$$

The scalar product is independent of the choice of Σ[3]. Note $(\phi^*|\psi^*) = -(\phi|\psi)^*$ and $(\phi|\psi)^* = (\psi|\phi)$. Now choose a complete set of solutions $\{u_i\}$ with

$$(u_i|u_j) = \delta_{ij},\ (u_i^*|u_j^*) = -\delta_{ij} \text{ and } (u_i^*|u_j) = 0 \tag{2.4}$$

The completeness of the modes implies $(\phi|\psi) = \sum_i(\phi|u_i)(u_i|\psi) - (\phi|u_i^*)(u_i^*|\psi)$.

2.2 Quantisation, Bogolyubov Transformations and Vacua

We can quantize the field by introducing the canonical commutation relations CCR on a cauchysurface Σ with (future directed) normal vector S^μ[2]:

$$[\phi(\mathbf{x}), \phi(\mathbf{x}')]_\Sigma = 0 \tag{2.5}$$
$$[\phi(\mathbf{x}), \nabla_S \phi(\mathbf{x}')]_\Sigma = i\delta(\mathbf{x} - \mathbf{x}') \tag{2.6}$$
$$[\nabla_S \phi(\mathbf{x}), \nabla_S \phi(\mathbf{x}')]_\Sigma = 0 \tag{2.7}$$

One can show that if they hold on one cauchysurface they hold on every cauchysurface[2]. Given a complete set of modes this leads to $\phi = \sum_i u_i a_i + u_i^* a_i^\dagger$, with a_i bosonic annihilation operators satisfying $[a_i, a_j^\dagger] = \delta_{ij}$[2].

zu viele quellen

Of course there are many different complete sets. One could also expand it in a different set $\{v_j\}$: $\phi = \sum_j v_j b_j + v_j^* b_j^\dagger$. The b's are then given by

$$b_j = \sum_i (v_j|u_i) a_i + (v_j|u_i*) a_i^\dagger \tag{2.8}$$

This is called a Bogolyubov transformation[3].

So far everything was in complete analogy to quantisation in Minkowski space. However problems arise when one tries to define the ground state of the system which is defined as the state with the lowest energy. The notion of energy (and thus the hamiltonian) depends on the notion of time. Therefore different coordinate systems will have different hamiltonians and thus different ground states. Since on a manifold there is no preferred coordinate system as in flat space we will have to guess the state of the field. This state may appear as the vacuum to some observers but will appear as an excited state to others (this is for example the reason why an eternal black hole seems to be thermal for an observer outside[1].)[1].

In a static spacetime one usually chooses the state given by $a_i |0\rangle = 0$, where a_i are annihilation operators for positive frequency modes (i.e. $i\partial_t u_i = \omega_i u_i, \omega > 0$). For the collapsing star we will choose the groundstate of the (static) spacetime before the collapse (which will then eventually convert into an excited state)[1].

2.3 Greens functions

After defining the groundstate of the QFT on can define several Greens functions (there are many more, but we will only need those)[1].

2.3.1 Vacuum Greens function

- The Wightman function $D^+(\mathbf{x}, \mathbf{x}') := \langle 0| \phi(\mathbf{x})\phi(\mathbf{x}') |0\rangle$
- Expectation value of the commutator: $iD(\mathbf{x}, \mathbf{x}') := [\phi(\mathbf{x}), \phi(\mathbf{x}')] = 2i \operatorname{Im} D^+(\mathbf{x}, \mathbf{x}')$
- Expectation value of the anticommutator $D^{(1)}(\mathbf{x}, \mathbf{x}') := \langle 0| \{\phi(\mathbf{x}), \phi(\mathbf{x}')\} |0\rangle = 2 \operatorname{Re} D^+(\mathbf{x}, \mathbf{x}')$

One does not need to take the expectation value of the commutator since (using the commutation relations) it is a c-number.

$$iD(\mathbf{x},\mathbf{x}') = \sum_{i,j} [u_i(\mathbf{x})a_i + u_i^*(\mathbf{x})a_i^\dagger, u_j(\mathbf{x}')a_j + u_j^*(\mathbf{x}')a_j^\dagger] \tag{2.9}$$

$$= \sum_i u_i(\mathbf{x})u_i^*(\mathbf{x}') - u_i^*(\mathbf{x})u_i(\mathbf{x}') \tag{2.10}$$

Since $\nabla_\mu \nabla^\mu \phi(\mathbf{x}) = 0$ this also holds for all Greens functions, i.e $\nabla_\mu \nabla^\mu D^+(\mathbf{x},\mathbf{x}') = 0$.

If the ground state is defined as $a_i |0\rangle = 0$ for a complete set of modes u_i (as for example for positiv frequency modes in a static spacetime) we can calculate $D^+(\mathbf{x},\mathbf{x}')$ by summing over all modes:

$$D^+(\mathbf{x},\mathbf{x}') = \langle 0| \phi(\mathbf{x})\phi(\mathbf{x}') |0\rangle = \sum_i u_i(\mathbf{x})u_i^*(\mathbf{x}') \tag{2.11}$$

2.3.2 Thermal Greens function

Later we will also need thermal greens function. These are given by replacing the vacuum expectation value $\langle 0| \dots |0\rangle$ by the thermal expectation value $\langle \dots \rangle_\beta = \frac{1}{Z} \mathrm{Tr}\, e^{-\beta H} \dots$ with $\beta = \frac{1}{k_B T}$, the hamiltonian H and $Z = \mathrm{Tr}\, e^{-\beta H}$.

It can be shown [1] that $D_\beta^{(1)}$ is given by shifting the time by $i\beta n$ and then summing over n

$$D_\beta^{(1)}(t,\vec{\mathbf{x}};t',\vec{\mathbf{x}}') = \sum_n D^{(1)}(t - i\beta n, \vec{\mathbf{x}}; t', \vec{\mathbf{x}}') \tag{2.12}$$

To find D_β^+ we can use that D (which is the imaginary part of D^+ and D_β^+) is just a c-number and therefore independent of the state of the field. If one is only interested in points where D^+ is real (as we will) one can replace $D^{(1)}$ by D^+ in the above formula since both greens functions are then proportional. [1]

2.4 Particle Detectors

We have already seen that there is no suitable definition of vacuum in a spacetime. This implies that in the rest frame of an observer the vacuum state could differ from the vacuum state we defined. Therefore also the notion of what a particle will be different for different observers. To analyse what particles a specific observer sees, Unruh and DeWitt invented a model for a particle detector which measures the energy excitations (particles) of the field along a specific trajectory.

The calculations are done in the appendix A.3. The important result is that one can split the result in a contribution from the detector and one from the field given by an excitation rate at energy E:

$$\frac{\mathrm{d}F_E}{\mathrm{d}\tau} = 2\mathrm{Re} \int_{-\infty}^0 \mathrm{d}\tau'\, e^{-iE\tau'} D^+(\mathbf{x}(\tau+\tau'), \mathbf{x}(\tau)) \tag{2.13}$$

This excitation rate is considered as energy distribution of particles an observer E will measure or see. In case the Wightmanfunction only depends on the difference $\tau - \tau'$ one can simplify this further to achieve:

$$\frac{\mathrm{d}F_E}{\mathrm{d}\tau} = \int_{-\infty}^{\infty} \mathrm{d}\tau' \, e^{-iE\tau'} D^+(\mathbf{x}(\tau'), \mathbf{x}(0)) \tag{2.14}$$

The excitation rate is constant and given by the Fourier transform of the Wightman function evaluated along the curve.

2.5 QFT in Minkowski space

Before considering more general spacetimes it is useful to have a look on how the formalism of introducing a QFT and the Unruh detector works in the well known Minkowski space.

2.5.1 Solutions of the Klein-Gordon-equation

The Klein-Gordon-equation in Minkowskispace is the normal wave equation:

$$\partial_\mu \partial^\mu \phi = 0. \tag{2.15}$$

The solutions are given by plane waves:

$$u_{\vec{k}}(x) = \frac{1}{\sqrt{2|k|}} \frac{e^{i\mathbf{kx}}}{\sqrt{2\pi}^3}, \text{ with } k^0 = |k| \tag{2.16}$$

The prefactor $\frac{1}{\sqrt{2|k|}}$ is required for normalisation $(u_{\vec{\mathbf{k}}}|u_{\vec{\mathbf{k}}'}) = \delta^3(\vec{\mathbf{k}} - \vec{\mathbf{k}}')$. The vacuum in Minkowskispace is given by $a_{\vec{\mathbf{k}}} |0_M\rangle = 0$. Throughout this thesis we will always exclude the mode $\omega = |\vec{\mathbf{k}}| = 0$[1].[1]

Since the Minkowskispace is also spherical symmetric one could also choose spherical modes

$$u^{\mathrm{M}}_{\omega,l,m} = \frac{\sqrt{\omega}}{\sqrt{\pi}} e^{-i\omega t} j_l(\omega r) Y_l^m(\theta, \varphi), \tag{2.17}$$

where j_l is a spherical Bessel function. The prefactor is again due to normalisation and can be achieved using the completeness relation for spherical Bessels $\int_0^\infty r^2 \, \mathrm{d}r \, j_l(wr) j_l(w'r) = \frac{\pi}{2\omega^2} \delta(\omega - \omega')$[4]. For great distances from the origin one can approximate the Bessels by their asymptotic behaviour $j_l(x) \overset{x \gg 1}{\rightarrow} \frac{\sin\left(x - l\frac{\pi}{2}\right)}{x}$ [4] and achieves:

$$u^{\mathrm{M}}_{\omega,l,m} \approx \frac{1}{\sqrt{\pi\omega}} e^{-i\omega t} \frac{\sin\left(\omega r - l\frac{\pi}{2}\right)}{r} Y_l^m(\theta, \varphi) \tag{2.18}$$

It is important to note that for this approximation it is necessary to have $r \gg 1/\omega$. So if one fixes r than the approximation will break down for small ω.

[1] This might seem a bit ad hoc first but it is mainly to exclude some $\delta(\omega)$ terms which lead to clearly unphysical behaviour, e.g. an infinite transition rate to the groundstate of our detector.

2.5.2 The Wightman function

The Wightmanfunction is calculated in the appendix A.1:

$$D^+(\mathbf{x},\mathbf{x}') = -\frac{1}{4\pi^2}\frac{1}{(t-t'-i\varepsilon)^2 - |\vec{\mathbf{x}}-\vec{\mathbf{x}}'|^2} \tag{2.19}$$

Up to the small imaginary number $i\varepsilon$ this is a real function. This means that the imaginary part (or $D(x,x')$) can only be non-vanishing if the denominator goes to 0 for $\varepsilon \to 0$. This is only the case for lightlike seperated x and x' or equivalently if x is on the lightcone of x'. This means when computing $D^+(\mathbf{x}(\tau),\mathbf{x}(\tau'))$ on a trajectory (like for a detector in eq. 2.14) the imaginary part will always vanish, since our detector stays strictly inside the lightcone[2].

Since we will later use spherical modes in order to calculate D^+ in the Schwarzschildmetric it is useful to have an expression for the Wightmanfunction in terms of the spherical modes. This is given by:

$$D^+(\mathbf{x},\mathbf{x}') = \int_0^\infty \frac{\omega\,\mathrm{d}\omega}{\pi}\sum_{l,m} e^{-i\omega(t-t')} j_l(\omega r) j_l(\omega r') Y_l^m(\theta,\varphi) Y_l^{m*}(\theta',\varphi') \tag{2.20}$$

or when using the approximate forms for great r

$$D^+(\mathbf{x},\mathbf{x}') = \int_0^\infty \frac{\mathrm{d}\omega}{\pi\omega}\sum_{l,m} e^{-i\omega(t-t')} \frac{\sin\left(\omega r - l\frac{\pi}{2}\right)}{r}\frac{\sin\left(\omega r' - l\frac{\pi}{2}\right)}{r'} Y_l^m(\theta,\varphi) Y_l^{m*}(\theta',\varphi') \tag{2.21}$$

When using the approximate form two main problems occur. They can be seen by expanding $\sin\left(\omega r - l\frac{\pi}{2}\right) = \frac{e^{i\omega r} i^{-l} - e^{-i\omega r} i^{l}}{2i}$.

1. After expanding one has to integrate $\int_0^\infty \frac{\mathrm{d}\omega}{\omega} e^{i\omega\dots}$. This integral does not converge since the real part has a (logarithmic) divergence. This IR-divergence is due to the fact that the asymptotic approximation is only true for $r \gg 1/\omega$ and therefore fails for small ω. The spherical Bessel functions remain finite when approaching $\omega \to 0$, while in the asymptotic form $\frac{\cos\omega\dots}{\omega}$ diverges.

2. Instead of integrating over ω one could also do the summation over l,m first. Taking care of the $i^{\pm l}$ one either has to sum $\sum_{l,m} Y_l^m(\theta,\varphi)Y_l^{m*}(\theta',\varphi') \sim \delta(\varphi-\varphi')\delta(\theta-\theta')$ or $\sum_{l,m}(-1)^l Y_l^m(\theta,\varphi)Y_l^{m*}(\theta',\varphi') = \sum_{l,m} Y_l^m(\pi-\theta,\pi+\varphi)Y_l^{m*}(\theta',\varphi') \sim \delta(\varphi-\varphi'+\pi)\delta(\theta-\pi+\theta')$ which means that there's only a contribution in two directions namely the direction of the detector and the opposite direction. This is again an artefact of the asymptotic form but in this case it is due to the assumption that r is big. Since $D^+(\mathbf{x},\mathbf{x}')$ has the same 'spatial size' independent of $\mathbf{x}$, moving it far away from the origin will shrink the corresponding solid angle. For r quite big all contribution will appear only in one direction.

In asymptotic flat spacetimes (as the Schwarzschild metric) one often cannot find exact solutions but rather asymptotic forms of them. Therefore these two effects can (and will in our case) occur when calculating D^+ in such spacetimes.

[2] Of course there's also the case $\mathbf{x} = \mathbf{x}'$ which we have to treat separately. But from the CCR we can conclude $2i\,\mathrm{Im}\,D^+(x,x) = iD(x,x) = [\phi(x),\phi(x)] = 0$.

2.5.3 Inertial Observer

We can now calculate the excitation rate for observers in Minkowskispace. Let's start with an steady observer $t(\tau) = \tau, \vec{\mathbf{x}}(\tau) = 0$:

$$D^+(\mathbf{x}(\tau), \mathbf{x}(\tau')) = -\frac{1}{4\pi^2}\frac{1}{(\tau - \tau' - i\varepsilon)^2} \tag{2.22}$$

Clearly $D^+(\mathbf{x}(\tau), \mathbf{x}(\tau'))$ only depends on $\Delta\tau = \tau - \tau'$. So we can use eq. (2.14) to obtain:

$$\frac{\mathrm{d}F_E(\tau)}{\mathrm{d}\tau} = \int_{-\infty}^{\infty} \mathrm{d}\tau\, e^{-iE\tau} D^+(\mathbf{x}(\tau), \mathbf{x}(0)) \tag{2.23}$$

$$= -\frac{1}{4\pi^2}\int_{-\infty}^{\infty} \mathrm{d}\tau\, e^{-iE\tau}\frac{1}{(\tau - i\varepsilon)^2} = 0 \tag{2.24}$$

For the last step use contour integration and close the contour in the lower half plane. Since $e^{-iE\tau}$ drops to 0 for large τ with negative imaginary part the integral is given by the sum over all residuals in the lower half plane. Because the ε moves the pole at $\tau = 0$ into the upper half plane there are no poles in the lower half plane and therefore the integral vanishes. So there are no particle excitations for an observer on a inertial worldline[3] which simply means that hetreats the Minkowski vacuum as a state with no particles. [1]

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2.5.4 The Unruh effect

Another interesting observer is an observer which is accelerating with an constant proper acceleration $\alpha > 0$, i.e. $t(\tau) = 1/\alpha \sinh \alpha\tau$, $x(\tau) = 1/\alpha \cosh \alpha\tau$, $y(\tau) = z(\tau) = 0$. Define $\lambda = \alpha\tau$. For simplicity calculate D^+ with $\varepsilon = 0$[4]

$$-\frac{\alpha^2}{4\pi^2 D^+(\mathbf{x}(\lambda/\alpha), \mathbf{x}(\lambda'/\alpha))} = (\sinh\lambda - \sinh\lambda')^2 - (\cosh\lambda - \cosh\lambda')^2 \tag{2.25}$$

$$= 4\sinh^2\frac{\lambda - \lambda'}{2} \tag{2.26}$$

$$D^+(x(\tau), x(\tau')) = -\frac{\alpha^2}{16\pi^2}\frac{1}{\sinh^2\frac{\alpha(\tau-\tau')}{2}} \tag{2.27}$$

Computing the thermal Wightman function (see appendix A.2)

$$D^+_\beta(x, x') = -\frac{1}{4\beta^2}\frac{1}{\sinh^2\left(\frac{\pi}{\beta}\sqrt{(t - t' - i\varepsilon)^2 - |\vec{\mathbf{x}} - \vec{\mathbf{x}}'|^2}\right)} \tag{2.28}$$

[3] Note that by poincare invariance of $D^+(x, x')$ one can always choose a frame in which an inertial observer does not move.

[4] The result with ε is analogue, see for example [1]

and evaluating it on a inertial trajectory, i.e. $t = \tau, \vec{x} = 0$ leads to the same formula as in eq. (2.27)

$$D^+_\beta(\mathbf{x}(\tau), \mathbf{x}(\tau')) = -\frac{1}{4\beta^2}\frac{1}{\sinh^2\left(\frac{\pi}{\beta}(\tau - \tau')\right)} \tag{2.29}$$

for $\beta = 2\pi/\alpha$. So an accelerating observer actually sees the Minkowski vacuum as a heat bath. So indeed the notion of a particle is observer dependent.

Using eq. (2.14) one can then easily evaluate the excitation spectrum for such an observer [1]:

$$\frac{\mathrm{d}F_E}{\mathrm{d}\tau} = \frac{1}{2\pi}\frac{E}{e^{\beta E} - 1}.[1] \tag{2.30}$$

CHAPTER 3

Unruh-Detector in static Spacetimes

Static spacetimes have a metric that looks like

$$\mathrm{d}s^2 = -\beta(\vec{\mathbf{x}})\,\mathrm{d}t^2 + g_{ij}(\vec{\mathbf{x}})\,\mathrm{d}x^i\,\mathrm{d}x^j \tag{3.1}$$

The metric only depends on the spatial coordinates and therefore ∂_t is a global timelike killing vector. For simplicity we will denote $g = \det\left(g_{ij}\right)$ instead of the four dimensional determinant.

3.1 Positive frequency modes

A solution u is called positive frequency if

$$i\partial_t u = \omega u,\ \omega > 0 \tag{3.2}$$

In case of a static metric it is possible to find a complete set of positive frequency solutions [3]

$$u(t, x^i) \sim e^{-i\omega t} A(x^i) \tag{3.3}$$

Using the normalisation condition on a cauchysurface $t = \text{const.}$ one finds that

$$u_i(t, \vec{\mathbf{x}}) = \frac{1}{\sqrt{2\omega_i}} e^{-i\omega_i t} A_i(\vec{\mathbf{x}}) \tag{3.4}$$

$$\delta_{ij} = \int_\Sigma \mathrm{d}^3x\, \frac{\sqrt{g}}{\sqrt{\beta}} A_i^*(\vec{\mathbf{x}}) A_j(\vec{\mathbf{x}}) \tag{3.5}$$

$$\sum_k A_k(\vec{\mathbf{x}}) A_k^*(\vec{\mathbf{x}}') = \frac{\sqrt{\beta}}{\sqrt{g}} \delta^3(\vec{\mathbf{x}} - \vec{\mathbf{x}}') \tag{3.6}$$

As mentioned before in section 2.2 there is a natural choice for the groundstate, namely the vacuum with $a_i\,|0\rangle = 0$.

3.2 Properties of the Wightman function

The Wightman function is given by

$$D^{+}(\mathbf{x},\mathbf{x}') = \sum_i \frac{1}{2\omega_i} e^{-i\omega_i(t-t')} A_i(\vec{\mathbf{x}}) A_i(\vec{\mathbf{x}}') \tag{3.7}$$

A first property of D^{+} is obtained by derivating w.r.t t and then setting $t = 0$

$$i\partial_t\, D^{+}(\mathbf{x},0)\Big|_{t=0} = \frac{1}{2}\sum_i A_i(\vec{\mathbf{x}}) A_i(\vec{\mathbf{x}}') \tag{3.8}$$

$$= \frac{1}{2}\frac{\sqrt{\beta}}{\sqrt{g}}\delta^3(\vec{\mathbf{x}}) \tag{3.9}$$

3.2.1 Wightman function in normal coordinates

We can choose any coordinate system we like for $g_{ij}(\vec{\mathbf{x}})$. It will be useful to have the Wightman function in normal coordinates around a point $\mathbf{x}' = 0$ (see appendix A.4):

$$D^{+}(\mathbf{x},0) = -\frac{1}{4\pi^2}\frac{1}{a(t-i\varepsilon)^2 - |\vec{\mathbf{x}}|^2} + O\left(x^2\right) \tag{3.10}$$

which is basically the same as in Minkowski space up to a prefactor $a = \beta(0)$.

3.2.2 The pole at the origin

When calculating the excitation rate we will first evaluate D^{+} on a timelike trajectory $\mathbf{x}(\tau)$ with $\mathbf{x}(0) = 0$ and $\dot{\mathbf{x}}^2 = -\beta\dot{t}^2 + g_{ij}\dot{x}^i\dot{x}^j = -1$ and then integrate over τ. Thereby we encounter a pole on the real axis at $\tau = 0$. Due to the ε this (second order) pole will move either in upper or in the lower half or could even split into two poles. To examine the behaviour of this pole define τ_ε as the position of the pole at $\tau = 0$ for a non vanishing ε, i.e. τ_ε satisfies

$$a(t(\tau_\varepsilon) - i\varepsilon)^2 - |\vec{\mathbf{x}}(\tau_\varepsilon)|^2 = 0 \tag{3.11}$$

Differentiate this twice with respect to ε and then setting $\varepsilon \longrightarrow 0$ yields (Note that $\tau_0 = 0$)

$$a(\dot{t}(0)\delta\tau - i)^2 - |\dot{\vec{\mathbf{x}}}(0)|^2\delta\tau^2 = 0 \tag{3.12}$$

where we defined $\delta\tau = \frac{\mathrm{d}\tau_\varepsilon}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}$. Noting that $a\dot{t}(0)^2 - |\dot{\vec{\mathbf{x}}}(0)|^2 = 1$ one finds

$$0 = \delta\tau^2 - 2ia\dot{t}(0)\delta\tau - 1 \tag{3.13}$$

$$\delta\tau = ia\dot{t}(0) \pm \sqrt{-a^2\dot{t}(0)^2 + 1} \tag{3.14}$$

$\delta\tau$ has two solutions and therefore the pole will split into two poles. But we know that $a\dot{t}(0) > 0$ and so will both values of $\delta\tau$ have positive imaginary part. Recall that $\delta\tau = \frac{\mathrm{d}\tau_\varepsilon}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}$ and so

$\tau_\varepsilon = \delta\tau\varepsilon + O\left(\varepsilon^2\right)$ which means that for a sufficient small value of ε both poles will lie in the upper half of the complex plane.

3.2.3 Singularities of the Wightman function

We know that the Wightman function solves the Klein-Gordon-Equation, i.e. words missing

$$\nabla_\mu \nabla^\mu D^+(\mathbf{x}, \mathbf{x}') = 0 \tag{3.15}$$

Now again fix $\mathbf{x}'$ and define $A(\mathbf{x}) = \frac{1}{D^+(\mathbf{x},\mathbf{x}')}$.

$$0 = \nabla_\mu \nabla^\mu \frac{1}{A} \tag{3.16}$$

$$= -\frac{A\nabla_\mu \nabla^\mu A - 2\nabla_\mu A \nabla^\mu A}{A^3} \tag{3.17}$$

$$0 = A\nabla_\mu \nabla^\mu A - 2\nabla_\mu A \nabla^\mu A \tag{3.18}$$

$$\nabla_\mu A \nabla^\mu A = \frac{A}{2} \nabla_\mu \nabla^\mu A \tag{3.19}$$

This must be also the case for points where the Wightman function has a pole, i.e. $A = 0$. Then we conclude $\nabla_\mu A \nabla^\mu A = 0$ which means that ∇A is a lightlike vector. Poles can now have two different behaviours: either they are an isolated singularity or they are part of a hypersurface on which $D^+ = \infty$. We will exclude the first type by the following handwaving argument: since D^+ solves the Klein-Gordon-equation with well we don't expect such solutions to create isolated poles. The second type appears for example in Minkowski space on the light cone. We know that such a hypersurface is given by $A = 0$ and since ∇A is a lightlike vector it is a null hypersurface. Since the $t = t'$ plane is Cauchy this hypersurfaces will cross it[1]. So if D^+ stays finite on $t = t'$ except for $\mathbf{x} = \mathbf{x}'$ (which will be the case for our examples) we can assume no singular behaviour of D^+ apart from the lightcone of $\mathbf{x}'$. Since observers always stay inside the lightcone this implies that it will not encounter any pole on its trajectory except for $\mathbf{x}'$.

Note that we can repeat the same argumentation for $D = 2\mathrm{Im}D^+$. Around the origin D vanishes except for being singular on the lightcone. It will therefore remain singular there. Apart from that we know that by causality outside the lightcone $iD = [\phi(\mathbf{x}), \phi(\mathbf{x}')] = 0$ which implies that there are no more hypersurfaces with $D = \infty$. So inside the lightcone D is nonsingular. If we assume that D is analytically in a region without singularities we can conclude $D = 0$ inside the lightcone (since this is true in a small region around the origin). So D^+ is a real function inside the lightcone.

With this argumentation (although it is not a proof) we will assume from now on that D^+ is real and finite on all trajectories (except for $\mathbf{x} = \mathbf{x}'$).

3.3 Observers on Trajectories

Before considering observers in the Schwarzschild metric it is useful to analyse the behaviour of the Unruh detector in a static spacetime in general.

[1] We will exclude spacetimes with closed null curves

3.3.1 Static observers

We will start by showing the following important lemma:

Lemma 1 *In a static spacetime a static observer does not observe any particles.*

beta(x) -> a

Proof: Recall that the modes in a static spacetime (see eq. 3.5) can be written as $u_i = \frac{1}{\sqrt{2\omega_i}} e^{-i\omega_i t} A_i(\vec{\mathbf{x}})$. Since the observer moves only along ∂_t he will have four-velocity $\dot{\mathbf{x}} = \frac{1}{\sqrt{a}}\partial_t$. Note that all components of the metric are independent of τ because the spatial coordinates stay constant $\vec{\mathbf{x}}(\tau) = \vec{\mathbf{x}}_0$. Integrating yields $t(\tau) = \frac{1}{\sqrt{a}}\tau + t_0$. Now we can evaluate $D^+(\mathbf{x}(\tau), \mathbf{x}(\tau'))$:

$$D^+(\mathbf{x}(\tau), \mathbf{x}(\tau')) = \langle 0|\, \phi(\mathbf{x}(\tau))\phi(\mathbf{x}(\tau'))\, |0\rangle = \sum_i u_i(\mathbf{x}(\tau)) u_i^*(\mathbf{x}(\tau')) \tag{3.20}$$

$$= \sum_i \frac{1}{2\omega_i} e^{-i\omega_i (t(\tau)-t(\tau'))} A_i(\vec{\mathbf{x}}_0) A_i^*(\vec{\mathbf{x}}_0) \tag{3.21}$$

$$= \sum_i \frac{1}{2\omega_i} e^{-i\omega_i (\tau-\tau')/\sqrt{a}} A_i(\vec{\mathbf{x}}_0) A_i^*(\vec{\mathbf{x}}_0) \tag{3.22}$$

So D^+ only depends on the difference $\tau - \tau'$. Therefore we can apply eq. 2.14:

$$\frac{\mathrm{d}F_E}{\mathrm{d}\tau} = \int_{-\infty}^{\infty} \mathrm{d}\tau\, e^{-iE\tau} D^+(\mathbf{x}(\tau), \mathbf{x}(0)) \tag{3.23}$$

$$= \sum_i \frac{1}{2\omega_i} A_i(\vec{\mathbf{x}}_0) A_i^*(\vec{\mathbf{x}}_0) \left(\int_{-\infty}^{\infty} \mathrm{d}\tau\, e^{-iE\tau} e^{-i\omega\tau/\sqrt{a}} \right) \tag{3.24}$$

$$= \sum_i \frac{1}{2\omega_i} A_i(\vec{\mathbf{x}}_0) A_i^*(\vec{\mathbf{x}}_0) \delta\left(E + \omega/\sqrt{a}\right) = 0 \tag{3.25}$$

The deltafunction is always zero because $E > 0$, $\omega > 0$, and $a > 0$[2]. So the observer does not detect any particles. QED.

This exact result will be later used to show whether the approximate form of the Wightmanfunction in the Schwarzschild metric is applicable.

rewrite

3.3.2 Detector on general Trajectories

In this section we will use the information we gathered about the Wightman function to tackle the problem how to calculate the detector response function on a general trajectory. In particular we have the problem that in general we cannot integrate from $-\infty$ to ∞ and use the residue theorem but we would rather have to integrate from $-\infty$ to 0. Since there's a singularity at 0 we will have in all cases a diverging integral.

[2] Here one can see why it is sensible to exclude $\omega = 0$ because it would lead to an infinite transition rate to the groundstate $E = 0$ which is clearly not physical (This is due to the first order perturbation theory).

Vacuum case

The last sentence is not really true. There is the small ε which removes the singularity from the real axis. We will assume that the integral over the trajectory will remain finite for $\varepsilon \to 0$[3].

Recall that by eq. 2.13 the excitation rate is given by

$$\frac{\mathrm{d}F_E}{\mathrm{d}\tau} = 2\mathrm{Re}\int_{-\infty}^{0} \mathrm{d}\tau'\, e^{-iE\tau'} D^+(\mathbf{x}(\tau+\tau'), \mathbf{x}(\tau)) \tag{3.26}$$

Without loss of generality we can consider the current proper time $\tau = 0$ and set $\mathbf{x}(\tau) = 0$. We will now do a series expansion of the Wightmanfunction around $\tau' = 0$, i.e.

$$D^+(\mathbf{x}(\tau'), 0) = \frac{a_{-2}}{\tau'^2} + W(\tau') \tag{3.27}$$

where $W(\tau')$ is finite at $\tau' = 0$. We can find a_{-2} by the following limit

$$a_{-2} = \lim_{\tau'\to 0} \tau'^2 \cdot D^+(\mathbf{x}(\tau'), 0) \tag{3.28}$$

$$= -\frac{1}{4\pi^2} \lim_{\tau'\to 0} \frac{\tau'^2}{at(\tau')^2 - |\vec{\mathbf{x}}(\tau)|^2} + O\left(x^2\right) \tag{3.29}$$

$$= -\frac{1}{4\pi^2} \lim_{\tau'\to 0} \frac{\tau'}{at\dot{t} - \vec{\mathbf{x}}\dot{\vec{\mathbf{x}}}} \tag{3.30}$$

$$= -\frac{1}{4\pi^2} \frac{1}{a\dot{t}^2 - \dot{\vec{\mathbf{x}}}^2} \tag{3.31}$$

$$= -\frac{1}{4\pi^2} \tag{3.32}$$

Where we have used $t(0) = \vec{\mathbf{x}}(0) = 0$ and $a\dot{t}^2 - \dot{\vec{\mathbf{x}}}^2 = 1$. Redoing the calculation with the ε shifts the pole to the upper half: $-\frac{1}{4\pi^2(\tau'-i\varepsilon)^2}$. So the singular part of the Wightmanfunction does neither depend on the specific trajectory nor on the geometry of the spacetime at all. So to calculate this we can take any trajectory we like, for example a static trajectory for which the remaining part vanishes $W(\tau') = 0$. Since the rate for this trajectory is zero the $\frac{1}{\tau'^2}$ term will not contribute in general.

This means that instead of integrating over $D^+(\mathbf{x}(\tau'), 0)$ we can equivalently integrate over $W(\tau') = D^+(\mathbf{x}(\tau'), 0) + \frac{1}{4\pi^2\tau'^2}$ which is well defined. Since we don't expect any singular behaviour of W we can do the $\varepsilon \to 0$ limit and therefore drop the ε.

Thermal case

In a thermal field we can also extract the contribution from an inertial observer to be left with a non singular function. To do this plug the expansion of $D^+(\mathbf{x}(\tau'), 0) = -\frac{1}{4\pi^2\tau'^2} + W(\tau')$ into formula (2.12)

[3] One can for example proof this by an explicit calculation for an inertial trajectory.

for D_β^+

$$D_\beta^+(t(\tau),\vec{\mathbf{x}}(\tau);0) = \sum_{n=-\infty}^{\infty} D^+(t(\tau)-i\beta n,\vec{\mathbf{x}}(\tau);0) \tag{3.33}$$
$$= \sum_{n=-\infty}^{\infty} -\frac{1}{4\pi^2(\tau'-i\beta\sqrt{a}n)^2} + \sum_{n=-\infty}^{\infty} W(\tau(t-i\beta n)) \tag{3.34}$$
$$= -\frac{1}{4\beta^2 a}\frac{1}{\sinh^2\left(\frac{\pi}{\beta\sqrt{a}}\tau\right)} + W_\beta(\tau) \tag{3.35}$$

The last equation is obtained by comparing to an inertial observer in Minkowski space (also see A.2). So the observer will see a thermal spectrum of temperature $T_{\text{static}} = \frac{T}{\sqrt{a}}$ (compare with eq. 2.29) plus some corrections coming from $W_\beta(\tau)$. These corrections will vanish for a static observer, be small for slow observers and might become the dominating spectrum for fast observers. The relation $T_{\text{static}} = \frac{T}{\sqrt{a}}$ is also known from other analysis of thermal systems in general relativity and is called Tolman relation [5].

Another property of the thermal Wightman function will be important later: Going through similar steps as in the vacuum case one finds that around the origin it looks like $-\frac{1}{4\pi^2\tau'^2} + O(\tau'^0)$. So again the singularity at $\tau' = 0$ is trajectory and geometry invariant.

3.3.3 Equivalence principle?

The statement of lemma 1 that a static observer does not recognize particles might seem surprising as in many spacetimes a static observer needs to accelerate in order to stay at his position (take the schwarzschild metric for example). As a first guess on could think that by equivalence principle an proper accelerating observer would see a heat bath as given by the unruh effect. This would also imply that a freely falling observer does not detect any particles. In order to show that this assumption is misleading we will first analyse the properties of trajectories on which no particles will be detected in general. To conclude the discussion we will show that on circular geodesics in the Schwarzschildmetric one actually detects something.

Recall that transition probability (not the rate) for a detector proportional to the square norm of the following state (see A.3)

$$|\psi\rangle = \int_{-\infty}^{\tau} \mathrm{d}\tau'\, e^{iE\tau'}\phi(\mathbf{x}(\tau'))\,|0\rangle \tag{3.36}$$
$$= \sum_i \frac{1}{\sqrt{2\omega_i}}\int_{-\infty}^{\tau} \mathrm{d}\tau'\, e^{iE\tau'} e^{+i\omega_i t(\tau')} A_i(\vec{\mathbf{x}}(\tau'))^*\,|\mathbf{1}_i\rangle \tag{3.37}$$

Since we would like to have no transitions at all the transition probability has to be zero and (note that the scalarproduct of states is positive definite) therefore the state $|\psi\rangle = 0$. But this implies since the one particle states $|\mathbf{1}_i\rangle$ are linear independent that all

$$Q_i := \int_{-\infty}^{\tau} \mathrm{d}\tau'\, e^{iE\tau'} e^{+i\omega_i t(\tau')} A_i(\vec{\mathbf{x}}(\tau'))^* \stackrel{!}{=} 0 \tag{3.38}$$

have to vanish. If we were not dealing with distributions but rather with functions this would imply (since we need this for all τ):

$$\forall\tau : e^{iE\tau}e^{+i\omega_i t(\tau)}A_i(\vec{\mathbf{x}}(\tau))^* \stackrel{!}{=} 0 \tag{3.39}$$

$$\Rightarrow A_i(\vec{\mathbf{x}}(\tau)) \stackrel{!}{=} 0 \tag{3.40}$$

However this is impossible since the A_i are supposed to form a complete basis over the full space.

We cannot apply this argument directly since D^+ is a distribution. But apart from the $-\frac{1}{4\pi^2\tau^2}$ term the Wightman function behaves like a function and therefore $W(\tau)$ has to vanish in order to see no excitations. Therefore D^+ evaluated on the trajectory can only be given by $-\frac{1}{4\pi^2\tau^2}$ [4]. It is clear that it will be quite hard to figure out a trajectory that satisfies that apart from static trajectories. This argument shows that it is very unlikely that all free falling observers will not recognize any particles.

We will conclude the argumentation with giving an explicit geodesic on which the detector response function is non zero. To do this we will need the following lemma for observers moving along killing vectors:

Lemma 2 *In a static spacetime an observer moving with constant velocity* $\dot{\mathbf{x}} = A\partial_t + B\mathbf{k}$ *along a spatial killingvector* $\mathbf{k}$ *will see excitations if and only if there exists at least one eigenfunction* u [5] *to* $\mathbf{k}$ *with eigenvalue 'im' such that* $\frac{A}{|B|} < \frac{|m|}{\omega_m}$.

Proof: Choose a coordinate system (ξ, y_1, y_2) for the spatial metric such that it has a coordinate ξ with $\partial_\xi = \mathbf{k}$. Since ∂_ξ is killing the metric will not depend on ξ:

$$[\partial_t, \partial_\xi] = [\partial_t, \nabla_\mu\nabla^\mu] = [\partial_\xi, \nabla_\mu\nabla^\mu] = 0 \tag{3.41}$$

We can therefore simultaneously diagonalize the operators which implies that we can find a complete set of solutions such that

$$u_{m,i} = \tilde{A}_i(y_1, y_2)e^{-i\omega_m t} \cdot e^{im\xi} \tag{3.42}$$

Note that if $u_{m,i}$ solves the Klein-Gordon-equation then also $u_{-m,i}$ is a solution with the same frequency $\omega_{-m} = \omega_m$.

The observer will only see nothing if he will see nothing for $\tau \to \infty$ [6]:

$$Q_m \sim \int_{-\infty}^{\infty} \mathrm{d}\tau'\, e^{iE\tau'}e^{+i\omega_m t(\tau')}e^{-im\xi(\tau')} = \int_{-\infty}^{\infty} \mathrm{d}\tau'\, e^{iE\tau'}e^{+i\omega_m A\tau'}e^{-imB\tau'} \tag{3.43}$$

$$= \delta(E + \omega_m A - mB) \tag{3.44}$$

This will be non zero at least for one energy only if $\omega_m A - mB < 0$. For $mB > 0$ we directly find $\frac{A}{|B|} < \frac{|m|}{\omega_m}$. For $mB < 0$ we can take the solution with $-m$ which will give a contribution. QED.

[4] This is for example the case for inertial trajectories in Minkowski space

[5] It is implied that u is a solution to the Klein-Gordon-equation with $\omega = \omega_m$

[6] This is because the transition rate is constant

We can now apply this to a circular geodesic in the Schwarzschild metric. Since the metric is spherically symmetric the trajectory is along the killing vector ∂_φ. Also the values of ω are continuous from 0 to ∞ and are especially independent of m (this will be derived in the next chapter). This means no matter how big $\frac{A}{B}$ is we will always find a combination of m and ω that fulfils the second condition. So we have found one explicit example for a geodesic on which particle excitations occur.

But how does this work with the equivalence principle? This simply resolved by recalling that the equivalence principle only states that it is impossible to distinguish between flat space and curved space using only local measurements. But the calculations above required integration over the whole worldline of the observer. Therefore this is a non local effect and we cannot apply the equivalence principle here.

We can remove the detailed trajectory dependence in lemma 2 by showing a further lemma (for simplicity we will work again in a coordinate system where ξ is a coordinate.)

Lemma 3 *In a static spacetime there exists an observer moving with constant velocity along a spatial killingvector $\mathbf{k} = \partial_\xi$ who will see excitations if and only if there exists a spacetime point $\mathbf{x}$ and at least one eigenfunction u[7] to $\mathbf{k}$ with eigenvalue 'im' such that $\frac{g_{\xi\xi}(\mathbf{x})}{|g_{tt}(\mathbf{x})|} < \frac{m^2}{\omega_m^2}$.*

Proof: We are considering trajectories $\dot{\mathbf{x}} = A\partial_t + B\partial_\xi$. The metric in the chosen coordinates will not change during the movement and so $\dot{\mathbf{x}}^2 = -1$ implies

$$|g_{tt}|A^2 - g_{\xi\xi}B^2 = 1 \tag{3.45}$$

$$\frac{A^2}{B^2} = \frac{1}{B^2 g_{tt}} + \frac{g_{\xi\xi}}{g_{tt}} \tag{3.46}$$

Since there are no restriction to B we conclude that $\frac{g_{\xi\xi}}{g_{tt}} < \frac{A^2}{B^2} < \infty$. The value $\frac{g_{\xi\xi}}{g_{tt}}$ cannot be achieved, however we can get infinitesimal close. Assuming $\frac{g_{\xi\xi}(\mathbf{x})}{-g_{tt}(\mathbf{x})} < \frac{m^2}{\omega_m^2}$ we will therefore always find a trajectory such that $\frac{A}{B} < \frac{m}{\omega_m}$ and so the conditions for lemma 2 are satisfied. If on the other side $\frac{g_{\xi\xi}(\mathbf{x})}{-g_{tt}(\mathbf{x})} \geq \frac{m^2}{\omega_m^2}$ there is no trajectory for lemma 2 and no excitations will be detected. QED.

This lemma implies that there are circular trajectories on which one will encounter particles as well as all inertial observers in Minkowskispace will see no excitations. There all inertial trajectories are along killing vectors (e.g. take $\xi = x$) and $\frac{g_{xx}}{|g_{tt}|} = 1$, but $\frac{k_x^2}{\omega^2} \leq 1$. So lemma 3 implies no excitations.

[7] Again u is a solution to the Klein-Gordon-equation with $\omega = \omega_m$

CHAPTER 4

Hawking radiation as seen by Observers

4.1 The metric

In this thesis we will only use the Schwarzschild-metric to describe stars and black holes (which means that they have no charge and no angular momentum). The metric for an spherical symmetric object with mass M is given by

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + \frac{1}{f(r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega \qquad\qquad \mathrm{d}\Omega = \mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\varphi^2 \tag{4.1}$$

where $f(r) = 1 - \frac{2M}{r}$. The metric is only valid outside the boundary of the star or for $r > 2M$. The two vector fields ∂_t and ∂_φ are killing.

4.2 The Klein-Gordon-Equation in the Schwarzschild metric

Consider a static spherical star where the outer metric is the Schwarzschild metric (i.e. non rotating and uncharged). It's important that we are considering a star because a black hole does not provide a global timelike killing vector field (analytic extension of ∂_t leads to a spacelike vectorfield inside the black hole). From now on we will only consider the outer region. To achieve a global solution one need to match inner with outer solutions.

The Klein-Gordon-Equation $\nabla_\mu \nabla^\mu \phi = 0$ can be written as:

$$-\frac{r^2}{f(r)}\partial_t^2\phi + \left(\partial_r r^2 f(r)\partial_r\right)\phi - L^2\phi = 0 \tag{4.2}$$

where L^2 is the usual angular momentum operator.

4.2.1 Spherical Modes

Since the spacetime is spherical symmetric and has the killing vector field ∂_t we can do the following ansatz for the modes

$$u_{\omega lm} = Ae^{-i\omega t}\frac{R_{\omega l}}{r}Y_l^m(\theta,\varphi) \tag{4.3}$$

Plugging this into the Klein-Gordon-Equation $\nabla_\mu\nabla^\mu u_{\omega lm} = 0$ yields to the following equation for $R_{\omega l}$ (see for example [1]):

$$\frac{\mathrm{d}^2R_{\omega l}}{\mathrm{d}r_*^2} + \omega^2R_{\omega l} - \left(\frac{l(l+1)}{r^2} + \frac{f'(r)}{r}\right)f(r)R_{\omega l} = 0 \tag{4.4}$$

$$\frac{\mathrm{d}^2R_{\omega l}}{\mathrm{d}r_*^2} + \omega^2R_{\omega l} - O\left(r^{-2}\right)R_{\omega l} = 0 \tag{4.5}$$

So for $\omega r \gg l$ one can neglect the r dependent part. In this case we find the asymptotic solutions

$$R_{\omega l} = e^{\pm i\omega r_*} \tag{4.6}$$

$$u_{\omega lm} = \frac{A_{\omega lm}}{r}e^{-i\omega t+i\omega r_*}Y_l^m(\theta,\varphi) + \frac{B_{\omega lm}}{r}e^{-i\omega t-i\omega r_*}Y_l^m(\theta,\varphi) \tag{4.7}$$

$$= \frac{A_{\omega lm}}{r}e^{-i\omega u}Y_l^m(\theta,\varphi) + \frac{B_{\omega lm}}{r}e^{-i\omega v}Y_l^m(\theta,\varphi) \tag{4.8}$$

Unfortunately we either cannot determine the (quite important) phase between $A_{\omega lm}$ and $B_{\omega lm}$ nor can we normalise the modes by integrating over all space. Instead I will impose that very far away from the star (where $r_* \approx r$) the field behaves as in Minkowskispace (which means that e.g. experiments give the same results). This implies that $D^+(\mathbf{x},\mathbf{x}')$ is the same as in Minkowskispace. Comparing with the asymptotic spherical modes in Minkowskispace (see eq. (2.18)) yields to:

$$u_{\omega lm} = \frac{1}{\sqrt{\pi\omega}}e^{-i\omega t}\frac{\sin\left(\omega r_* - l\frac{\pi}{2}\right)}{r}Y_l^m(\theta,\varphi) = \frac{r_*}{r}u^{\mathrm{M}}_{\omega lm}(r_*,\theta,\varphi) \tag{4.9}$$

$$= \frac{i^{-l}}{2i\sqrt{\pi\omega}r}e^{-i\omega u}Y_l^m(\theta,\varphi) - \frac{i^l}{2i\sqrt{\pi\omega}r}e^{-i\omega v}Y_l^m(\theta,\varphi) \tag{4.10}$$

4.2.2 The Wightman function

The next step would be to calculate the Wightman function $D^+(\mathbf{x},\mathbf{x}')$. Note that since ∂_t is a timelike killing vector we define the ground state $|0\rangle$ by $a_{\omega lm}|0\rangle = 0$ and use eq. 2.11.

$$D^+(\mathbf{x},\mathbf{x}') = \int_0^\infty \frac{\mathrm{d}\omega}{\pi\omega}\sum_{l,m} e^{-i\omega(t-t')}\frac{\sin\left(\omega r_* - l\frac{\pi}{2}\right)}{r}\frac{\sin\left(\omega r'_* - l\frac{\pi}{2}\right)}{r'}Y_l^m(\theta,\varphi)Y_l^{m*}(\theta',\varphi') \tag{4.11}$$

Since this integral is nearly the same as in Minkowskispace (see eq. 2.21) we also encounter the same problems, namely the IR divergence and the fact that it is non-zero only in two directions. This is due to the fact that the approximation $\omega r \gg l$ breaks down for small ω and for large l. Recall that the problems in the non approximate calculation in Minkowskispace didn't occur because the $j_l(\omega r)$ remain finite at $\omega \to 0$. In other words the essential feature of the $j_l(\omega r)$ is that they let all the $\cos(\omega \ldots)$ terms in the integral drop to zero for $\omega \to 0$ instead of $\cos 0 = 1$ in the approximate case. Therefore we can assume that the same happens for exact solutions around the star.

Unfortunately the exact behaviour for small ω is the same as for small r which will depend on the specific geometry of the star. But the metric of a star is almost flat (since the radius of a star R_0 is much bigger than its Schwarzschildradius R_S). Therefore we will approximate the global mode by replacing the sine with the spherical Bessel function:

$$u_{\omega lm} = \frac{\sqrt{\omega}}{\sqrt{\pi}} e^{-i\omega t} \frac{r_*}{r} F(r) j_l(\omega r_*) Y_l^m(\theta, \phi) = \frac{r_*}{r} F(r) u^{\mathrm{M}}_{\omega lm}(r_*, \theta, \phi) \tag{4.12}$$

We introduced an extra factor $F(r)$ to correct the r dependence again. This factor must approach 1 at infinity. Because the modes now are the same (up to a prefactor and replacing $r \to r_*$) as in Minkowski space we can find the Wightman function by adjusting the Wightman function of Minkowski space (see eq. (2.22))

$$D^+(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\pi^2} \frac{r_* r'_* F(r) F(r')}{r r'} \frac{1}{(t - t' - i\varepsilon)^2 - |\vec{\mathbf{x}}_* - \vec{\mathbf{x}}'_*|^2} \tag{4.13}$$

$$= -\frac{1}{4\pi^2} \frac{r_* r'_* F(r) F(r')}{r r'} \frac{1}{(t - t' - i\varepsilon)^2 - r_*^2 - r'^2_* + 2 r_* r'_* \cos\alpha} \tag{4.14}$$

where $\vec{\mathbf{x}}_*$ is obtained by replacing $r \to r_*$ in $\vec{\mathbf{x}}$ and α is the angle between the two vectors. We still have the factor $F(r)$. This can be fixed because we know the behaviour around $\mathbf{x} \approx \mathbf{x}'$ by eq. (3.10). First we know exact that for $\vec{\mathbf{x}} = \vec{\mathbf{x}}'$ it should look like $-\frac{1}{4\pi^2}\frac{1}{f(r)(t-t')^2}$. This implies $F(r) = \frac{r}{r_* \sqrt{f(r)}}$. Second for $t = t'$ the denominator is given by the spatial distance $|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|^2$. The distance between two infinitesimal separated r and $r + \mathrm{d}r$ is given by $|\vec{\mathbf{x}}(r + \mathrm{d}r) - \vec{\mathbf{x}}(r)| = \sqrt{g_{rr}}\,\mathrm{d}r = \frac{\mathrm{d}r}{\sqrt{f(r)}}$. But $|r_*(r + \mathrm{d}r) - r_*(r)| = \frac{\mathrm{d}r_*}{\mathrm{d}r}\mathrm{d}r = \frac{\mathrm{d}r}{f(r)}$. Again we can correct this by setting $F(r) = \frac{r}{r_*\sqrt{f(r)}}$. Since both independent arguments lead to the same result we conclude that the asymptotic form of the Wightmanfunction is given by:

$$D^+(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\pi^2\sqrt{f(r) f(r')}} \frac{1}{(t - t' - i\varepsilon)^2 - |\vec{\mathbf{x}}_* - \vec{\mathbf{x}}'_*|^2} \tag{4.15}$$

$$= -\frac{1}{4\pi^2\sqrt{f(r) f(r')}} \frac{1}{(t - t' - i\varepsilon)^2 - r_*^2 - r'^2_* + 2 r_* r'_* \cos\alpha} \tag{4.16}$$

4.2.3 The Wightman function after the collapse

To calculate the Wightman function after the collapse we will apply the result of Hawking to find that the expectation value of two operators thereafter is given by the thermal expectation value with $\beta = 8\pi M$ in the spacetime before the collapse. The corresponding temperature $T_H = \frac{1}{8\pi M k_{\mathrm{B}}}$ is called Hawking temperature. [6]

Therefore the (vacuum) Wightman function at late times is given by the thermal Wightman function at early times (it can be computed analogously to the one in Minkowski space)

$$D^+_\beta(\mathbf{x},\mathbf{x}') = -\frac{1}{4\beta^2\sqrt{f(r)f(r')}}\frac{1}{\sinh^2\left(\frac{\pi}{\beta}\sqrt{(t-t')^2-|\vec{\mathbf{x}}_*-\vec{\mathbf{x}}'_*|^2}\right)} \tag{4.17}$$

4.3 Observers in the Schwarzschildmetric

In this section we will finally calculate what an observer will see when moving on static, circular and radial trajectories using the thermal Wightman function. We are not interested in the concrete spectrum but rather in the observed temperature. Apart from the static trajectories we cannot solve this analytically, so we need to use numerical methods (see section A.5). For a static and circular observer we will also calculate the observed spectrum in the spacetime before the collapse (using the vacuum Wightman function) because we dealt with that before in section 3.3.

4.3.1 Static observer

Before the collapse

We already know by lemma 1 that a static observer will not see any excitations. We will now show that this is also true when using our approximate form of D^+. A static observer is given by $t = \frac{\tau}{\sqrt{f(r)}}$ and all other coordinates constant. Inserting this into the Wightman function yields

$$D^+(\mathbf{x}(\tau),\mathbf{x}(\tau')) = -\frac{1}{4\pi^2}\frac{1}{(\tau-\tau'-i\varepsilon)^2} \tag{4.18}$$

This is the same as for an inertial observer in Minkowski space (see eq. 2.22). So a static observer indeed does not recognize any particles.

After the collapse

The thermal Wightman function is given by

$$D^+_\beta(\mathbf{x}(\tau),0) = -\frac{1}{4\beta^2 f(r)}\frac{1}{\sinh^2\left(\frac{\pi}{\beta\sqrt{f(r)}}\tau\right)} \tag{4.19}$$

which results in an observed temperature of $T = f(r)^{-1/2}T_{\mathrm{H}}$ which agrees with the Tolman effect found in section 3.3.2. So a static observer will see a slightly higher temperature than the Hawking temperature.

Since we know the exact result we can use this as a benchmark for our method to approximate the temperature. Running this on a static observer gives the result depicted in fig. 4.1. It fits with the Tolman relation over all orders of magnitude. The small difference lies within the numerical errors[1]. We can therefore conclude that our method is suitable for determining the temperature.

ok?

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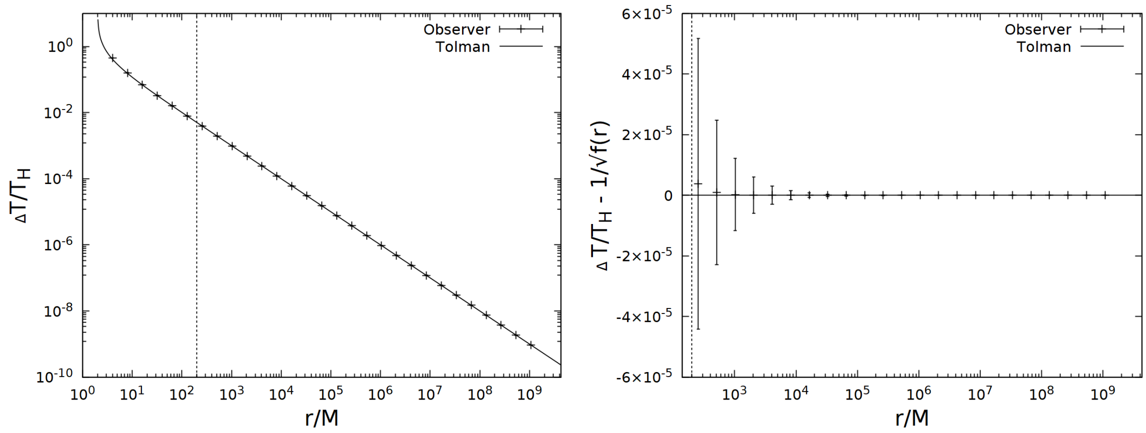
[1] Note that all error depicted in the errorbars is the estimated numerical error, not errors induced by the approximative Wightman function



(a) Relative temperature shift

(b) Difference to Tolman relation

Figure 4.1: Static observer – The relative temperature fits with Tolman-relation as shown in a). In b) the Tolman relation was subtracted. It matches with the Tolman relation within the numerical errors. Below the dotted line at $r = 200M$ the approximation is not suitable.

4.3.2 Circular observer

Before the collapse

By Lemma 2 we know that a circular observer should see some excitations. Using our approximate form we can now calculate what he will actually measure. A circular observer is given by $t = a\tau$, $\varphi = B\tau$, $r = \text{const.}$, $\theta = \frac{\pi}{2}$[2]. The geodesic equation and $\dot{\mathbf{x}}^2 = -1$ give constrains on the constants (see for example [7])

$$A^2 = \frac{r}{r - 3M} \tag{4.20}$$

$$B^2 = \frac{1}{r^2}\frac{M}{r - 3M} \tag{4.21}$$

The Wightman function evaluated on the curve is

$$D^+(\mathbf{x}(\tau), \mathbf{x}(\tau')) = -\frac{1}{4\pi^2 f(r)} \frac{1}{(A(\tau - \tau') - i\varepsilon)^2 - 2r_*^2(1 - \cos B(\tau - \tau'))} \tag{4.22}$$

We see that D^+ only depends on $\tau - \tau'$. So we can use the simplified formular (2.14). For this we need to find the poles in the lower half of $D^+(\mathbf{x}(\tau), 0)$ which means finding the roots of

$$0 = A^2\tau^2 - 2r_*^2(1 - \cos B\tau) \tag{4.23}$$

$$0 = \xi^2 x^2 - 2(1 - \cos x) \tag{4.24}$$

where $x = B\tau$ and $\xi = \frac{A}{Br_*}$. Clearly $\tau = 0$ is a root (which will lie in the upper half). Apart from that

[2] Since the spacetime is spherically symmetric we can choose a coordinate system such that $\theta = \frac{\pi}{2}$ and $B > 0$

we have to differentiate between two cases:

- $\xi < 1$: In this case there are another two roots on the real axis. The reason is that for $\xi < 1$ (which is a very fast circular motion) the trajectory hits the Minkowski-lightcone. But we know that this is not possible in the Schwarzschild-spacetime by the argumentation in section 3.2.3. So we assume that this behaviour is due to our approximation and therefore exclude this case[3].
- $\xi > 1$: This case represents slower motions. There are two more (first order) poles at $\pm i x_0$ of whom one is in the lower half.

The rate is given by eq. (2.14)

$$\frac{\mathrm{d}F_E}{\mathrm{d}\tau} = \int_{-\infty}^{\infty} \mathrm{d}\tau\, e^{-iE\tau} D^+(\mathbf{x}(\tau), \mathbf{x}(0)) \tag{4.25}$$

$$= -\frac{1}{4\pi^2 f(r)} \int_{-\infty}^{\infty} \mathrm{d}\tau\, e^{-iE\tau} \frac{1}{A^2\tau^2 - 2r_*^2(1-\cos B(\tau))} \tag{4.26}$$

$$= -\frac{1}{4\pi^2 r_*^2 f(r) B} \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-iEx/B} \frac{1}{\xi^2 x^2 - 2(1-\cos x)} \tag{4.27}$$

$$= \frac{i}{2\pi r_*^2 f(r) B} \mathrm{Res}\left(e^{-iEx/B} \frac{1}{\xi^2 x^2 - 2(1-\cos x)}, -ix_0 \right) \tag{4.28}$$

$$= \frac{i}{2\pi r_*^2 f(r) B} e^{-E/Bx_0} \lim_{x\to -ix_0} \frac{x + ix_0}{\xi^2 x^2 - 2(1-\cos x)} \tag{4.29}$$

$$= \frac{1}{2\pi r_*^2 f(r) B} e^{-E/Bx_0} \frac{1}{-2\xi^2 x_0 + 2\sinh x_0} \tag{4.30}$$

Erklärung nicht thermisch

So basically a circular observer sees a exponentially falling energy distribution.

After the collapse

An observer on a circular orbit $t = A\tau$ and $\varphi = B\tau$ has the following thermal Wightman function:

$$D^+_\beta(\mathbf{x}(\tau), \mathbf{x}(0)) = -\frac{1}{4\beta^2 f(r)} \frac{1}{\sinh^2\left(\frac{\pi}{\beta}\sqrt{A^2\tau^2 - 2r_*^2(1-\cos B\tau)}\right)} \tag{4.31}$$

This function is problematic because expanding around zero yields:

$$D^+_\beta(\mathbf{x}(\tau), \mathbf{x}(0)) = -\frac{1}{4\pi^2\tau^2} \frac{1}{f(r)A^2 - f(r) r_*^2 B^2} + O(\tau^0) \neq -\frac{1}{4\pi^2\tau^2} + O(\tau^0) \tag{4.32}$$

Unfortunately the τ^{-2} term has a different prefactor since we are only given $\dot{\mathbf{x}}^2 = -f(r)A^2 + r^2B^2 = -1$. However we know by 3.3.2 that for every non approximate Wightman function on a trajectory this prefactor looks like $-1/4\pi^2$. So this deviation is due to our approximation which is therefore only applicable if $r^2 \approx f(r) r_*^2$. For $r > 200M$ the deviation is smaller than 10%. We will ignore smaller values of r for the interpretation of the results.

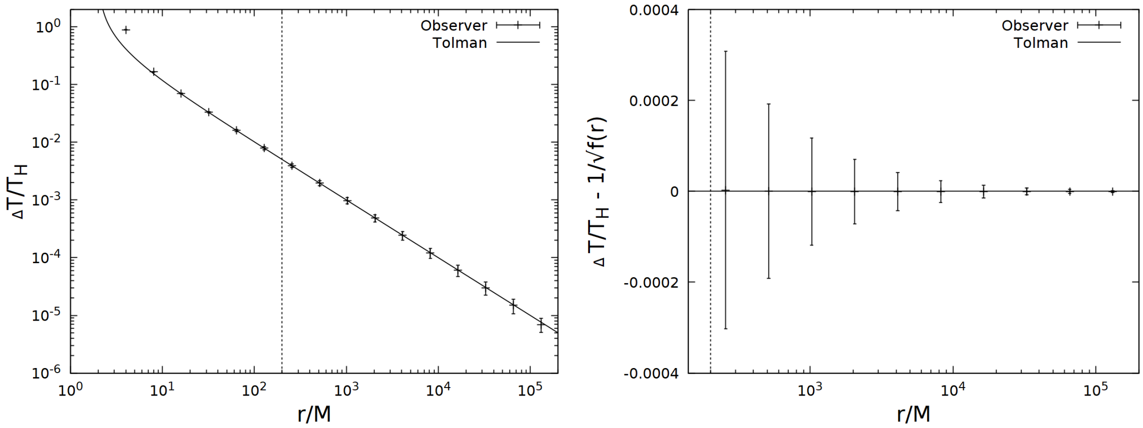
plot

[3] Note that $\xi < 1$ for a geodesic can only happen for $r \lesssim 1.1 R_S$ which is definitely not far away from the black hole

The main problem with the different prefactor is that the difference with another thermal Wightman function will remain divergent at $\tau = 0$[4]. We can deal with this problem by replacing r_*^2 by $r^2/f(r)$ in the Wightman function

$$D_\beta^+(\mathbf{x}(\tau),\mathbf{x}(0)) = -\frac{1}{4\beta^2 f(r)} \frac{1}{\sinh^2\left(\frac{\pi}{\beta}\sqrt{A^2\tau^2 - 2\frac{r^2}{f(r)}(1-\cos B\tau)}\right)} \tag{4.33}$$

This is possible because the approximation is only valid when the difference is neglectable. Now we can determine the temperature shift according to section A.5. The result is given in fig. 4.2. One can see as long as our approximation is suitable (above the dotted line) the result fits to the Tolman relation. Again the difference between both lies inside the numerical errors.



(a) Relative temperature shift

(b) Difference to Tolman relation

Figure 4.2: Circular observer – The relative temperature fits with Tolman-relation as shown in a). In b) the Tolman relation was subtracted. It matches with the Tolman relation within the numerical errors. Below the dotted line at $r = 200M$ the approximation is not suitable.

4.3.3 Radial observer after the collapse

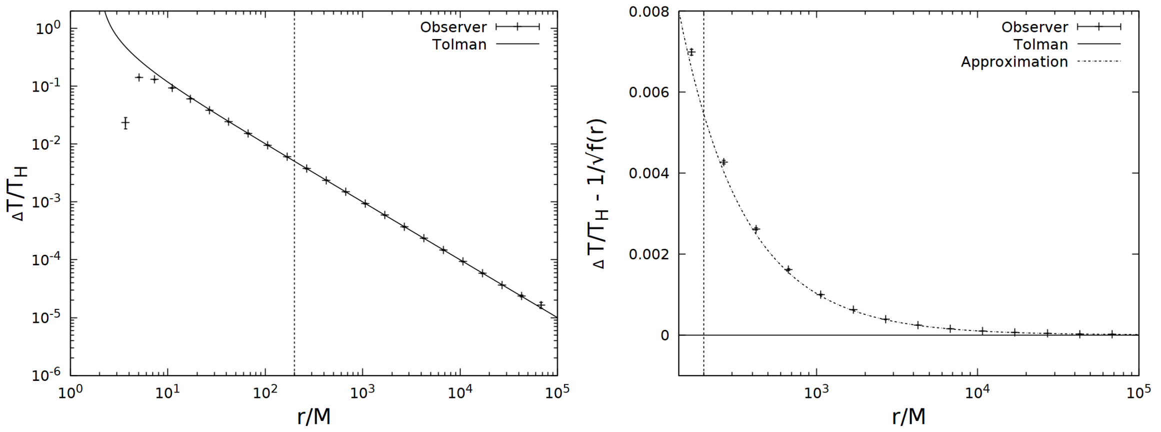
We will restrict ourselves to the case of a freely falling observer that is dropped at infinity with zero velocity. This means $E = f(r)\dot{t} = 1$. The geodesic equation and $\dot{\mathbf{x}}^2 = -1$ result in the following trajectory [8]:

$$r(\tau) = \left(-\frac{3}{2}\sqrt{2M}(\tau - \tau_0)\right)^{\frac{2}{3}} \tag{4.34}$$

$$t(\tau) = \tau - \tau_0 - 3\tau_0\left(\frac{1}{x(\tau)} + \frac{1}{2}\ln\frac{x(\tau)-1}{x(\tau)+1}\right) \tag{4.35}$$

[4] This difference must not be singular since we would like to integrate it numerically, see eq. (A.65)

where $x(\tau) = -\dot{r} = \left(1 - \frac{\tau}{\tau_0}\right)^{-\frac{1}{3}}$ and $\tau_0 = \frac{4M}{3}$[5]. This is inserted in the thermal Wightman function (4.17) and the measured temperature is determined (see fig. 4.3).



(a) Relative temperature shift

(b) Difference to Tolman relation

Figure 4.3: Radial infalling observer – The relative temperature fits with Tolman-relation as shown in a). In b) the Tolman relation was subtracted. The difference is outside of the numerical errors even for radii $r > 200M$ (dotted vertical line). However the difference has the same order of magnitude of the estimated error caused by approximating the Wightman function (dotted curve).

The result again fits with the Tolman relation. After subtracting the Tolman relation we find a difference much larger than the numerical errors. Note that the error bars only depict the numerical error, not the error done by approximating the Wightman function. In the last section we found that we can only use our Wightman function when $\frac{r_*^2}{r^2} f(r) \approx 1$. We need to estimate the order of magnitude of the error caused by this approximation in order to show whether or not the difference in fig. 4.3 b) is a significant deviation from the Tolman relation. We can give a (rough) estimate by applying the relative error done by the approximation on the Tolman relation, i.e. $\frac{r_*^2}{r^2} f(r) \cdot \left(\frac{1}{\sqrt{f(r)}} - 1\right)$. This is given as a dotted curve in fig. 4.3 b). The calculated temperature deviation has about the same order of magnitude. This means that the difference could be completely due to our approximation. We will therefore be conservative and conclude that up to the errors of our method there is no temperature shift for an infalling observer.

Fehler berechnen?

[5] τ_0 is chosen such that at $\tau = 0$ the observer hits the event horizon.

CHAPTER 5

Conclusion

We showed properties of the Wightman function and the spectrum of an Unruh detector in a static spacetime. Using normal coordinates we were able to find that the singularity at $\mathbf{x} = \mathbf{x}'$ will – when evaluated on a curve – give a second order pole $-\frac{1}{4\pi^2\tau^2}$ which is shifted to the upper half by a small regularisation ε. This term does not contribute to the spectrum. We argued that apart from this singularity the Wightmanfunction on any timelike curve will remain finite.

Furthermore we found that in general in a static spacetime in the vacuum state a static observer does not encounter any particles. If the spacetime is in a thermal state with temperature T_0 the observer sees a thermal spectrum with a different temperature according to the Tolmanrelation: $T = \frac{T_0}{\sqrt{g_{tt}}}$. For an observer moving along a spatial killing vectorfield we found a condition when he will encounter particles in the vacuum state which is independent of whether the trajectory is a geodesic or not. This means that the particle spectrum is a global effect and cannot be treated in a local manner.

We were able to apply this knowledgeto find an approximate form of the Wightman function in the outer Schwarzschild metric. This approximation is valid for $r > 200M$. Using it we showed the general results from before explicitly, namely a static observer does not encounter any particles and a circular geodesic observer sees a non vanishing spectrum. Then we applied the result of Hawking that after the collapse to a black hole the field is in a thermal state with $\beta = 8\pi M$. A method to determine the temperature measured by different observers moving in this spacetime was developed and applied to static, circular and radial infalling observers. In all cases the temperature followed the Tolman relation. For static and circular observers there was no difference up to numerical errors. For radial observers we found a slight deviation outside the numerical errors. However this deviation has the same order of magnitude as expected by our approximation. Therefore a better approximation would be necessary to decide whether this is significant or not. For the time being we can only conclude that for all considered observers the temperature is given by the Tolman relation.

better

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APPENDIX A

Appendix

A.1 Wightmanfunction in Minkowskispace

Using eq. (2.11) we can calculate the Wightman function:

$$D^+(\mathbf{x},\mathbf{x}') = \int \mathrm{d}^3k\, u_{\vec{\mathbf{k}}}(x) u^*_{\vec{\mathbf{k}}}(x') \tag{A.1}$$

$$= \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{1}{2|\vec{\mathbf{k}}|} e^{-i|\vec{\mathbf{k}}|(t-t')+i\vec{\mathbf{k}}(\vec{\mathbf{x}}-\vec{\mathbf{x}}')} \tag{A.2}$$

$$\overset{\omega=|\vec{\mathbf{k}}|}{=} \int_0^\infty \int_{-1}^1 \frac{\omega^2\,\mathrm{d}\omega\,\mathrm{d}\cos\theta}{(2\pi)^2} \frac{1}{2\omega} e^{-i\omega(t-t')+i\omega|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|\cos\theta} \tag{A.3}$$

$$= \frac{1}{2i|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|} \int_0^\infty \frac{\mathrm{d}\omega}{(2\pi)^2}\, e^{-i\omega(t-t')} \left(e^{i\omega|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|} - e^{-i\omega|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|} \right) \tag{A.4}$$

This oscillating integral does not converge. Therefore we will first calculate $D^+(\mathbf{x},\mathbf{x}')$ for complex times by setting $t \longrightarrow t - i\varepsilon, \varepsilon > 0$ and then treating $D^+(\mathbf{x},\mathbf{x}')$ as a distribution when setting $\varepsilon \longrightarrow 0$.

$$D^+(\mathbf{x},\mathbf{x}') = \frac{1}{2i|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|} \int_0^\infty \frac{\mathrm{d}\omega}{(2\pi)^2}\, e^{-i\omega(t-t'-i\varepsilon-|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|)} - e^{-i\omega(t-t'-i\varepsilon+|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|)} \tag{A.5}$$

$$= -\frac{1}{2i|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|} \frac{1}{(2\pi)^2} \left(\frac{i}{t-t'-i\varepsilon-|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|} - \frac{i}{t-t'-i\varepsilon+|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|} \right) \tag{A.6}$$

$$= -\frac{1}{4\pi^2} \frac{1}{(t-t'-i\varepsilon)^2 - |\vec{\mathbf{x}}-\vec{\mathbf{x}}'|^2} \tag{A.7}$$

Alternatively one can achieve the form by first choosing a coordinate system such that $\mathbf{x} = \tilde{t}\partial_0$ (as we will do for the thermal case) and then transforming it back (one has to be very careful with the ε here).

A.2 Thermal Wigthman function in Minkowski space

To calculate the thermal Wightman function we will restrict to the interior of the light cone (we will only need that). At a specific point choose the following coordinate system: First we will set $\mathbf{x}' = 0$ and then choose a coordinate system such that $\mathbf{x} = \tilde{t}\partial_0$. Here $\tilde{t} = \pm\sqrt{-\mathbf{x}^2}$ with the upper sign for $t > 0$ and the lower for $t < 0$. In this coordinate system we can redo the calculation of the Wightman function and achieve

$$D^+(\mathbf{x}, 0) = -\frac{1}{4\pi^2}\frac{1}{(\tilde{t} - i\varepsilon)^2} \tag{A.8}$$

To find the thermal Wightmanfunction we to replace $\tilde{t} \to \tilde{t} - i\beta n$ (see eq. (2.12)) and add all the contributions (note that inside the lightcone $D^+ = D^{(1)}$)

$$\begin{aligned}
D^+_\beta(\mathbf{x}, 0) &= -\frac{1}{4\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{(\tilde{t} - i\beta n - i\varepsilon)^2} && \text{(A.9)}\\
&= \frac{1}{4\pi^2\beta^2}\sum_{n=-\infty}^{\infty}\frac{1}{(-i\frac{\tilde{t}}{\beta} - n - \varepsilon)^2} && \text{(A.10)}\\
&= \frac{1}{4\beta^2}\frac{1}{\sin^2\left(-i\pi\frac{\tilde{t}}{\beta} - \varepsilon\right)} && \text{(A.11)}\\
&= -\frac{1}{4\beta^2}\frac{1}{\sinh^2\left(\frac{\pi}{\beta}(\tilde{t} - i\varepsilon)\right)} && \text{(A.12)}
\end{aligned}$$

Here we used $\sin^{-2}(\pi x) = \pi^{-2}\sum_n (x-n)^{-2}$ [**davies**]. Now go back to the old frame and replace $\tilde{t} = \pm\sqrt{-\mathbf{x}^2}$

$$\begin{aligned}
D^+_\beta(\mathbf{x}, \mathbf{x}') &= -\frac{1}{4\beta^2}\frac{1}{\sinh^2\left(\frac{\pi}{\beta}\left(\pm\sqrt{(t-t')^2 - |\vec{\mathbf{x}} - \vec{\mathbf{x}}'|^2} - i\varepsilon\right)\right)} && \text{(A.13)}\\
&= -\frac{1}{4\beta^2}\frac{1}{\sinh^2\left(\frac{\pi}{\beta}\sqrt{(t-t'-i\varepsilon)^2 - |\vec{\mathbf{x}} - \vec{\mathbf{x}}'|^2}\right)} && \text{(A.14)}
\end{aligned}$$

A.3 The Unruh-Detector

shorten?

Our treatment of such a detector will follow Birrell and Davies [**davies**]. One describes a detector by a operator $m(\tau)$ which couples to the field via a interaction term $c \cdot m(\tau)\phi(\mathbf{x}(\tau))$, where c is small and $\mathbf{x}(\tau)$ is the trajectory of the detector. For $\tau \to -\infty$ the detector is in the groundstate $|E_0\rangle$ and the field is in the vacuum state $|0\rangle$. The detector develops with time according to $m(\tau) = e^{iH_0\tau}m(0)e^{-iH_0\tau}$ with $H_0|E\rangle = E|E\rangle$.

We would like to calculate the probability that the detector detects a particle with energy E. Since c is small one can use first order perturbation theory where the transition amplitude to another state

$|E,\psi\rangle$ at time τ is given by

$$Q_{|E_0,0\rangle\to|E,\psi\rangle}(\tau) = ic\,\langle E,\psi|\int_{-\infty}^{\tau} m(\tau')\phi(\mathbf{x}(\tau'))\,\mathrm{d}\tau'\;|E_0,0\rangle \tag{A.15}$$

$$= ic\,\langle E,\psi|\int_{-\infty}^{\tau} e^{iH_0\tau'}m(0)e^{-iH_0\tau'}\phi(\mathbf{x}(\tau'))\,\mathrm{d}\tau'\;|E_0,0\rangle \tag{A.16}$$

$$= ic\,\langle\psi|\int_{-\infty}^{\tau} e^{iE\tau'}\,\langle E|\,m(0)\,\big|E_0\big\rangle\,e^{-iE_0\tau'}\phi(\mathbf{x}(\tau'))\,\mathrm{d}\tau'\;|0\rangle \tag{A.17}$$

$$= ic\,\langle E|\,m(0)\,\big|E_0\big\rangle\int_{-\infty}^{\tau} e^{i(E-E_0)\tau'}\,\langle\psi|\,\phi(\mathbf{x}(\tau'))\,|0\rangle\,\mathrm{d}\tau' \tag{A.18}$$

The transition probability is $P_{|E_0,0\rangle\to|E,\psi\rangle}(\tau) = |Q_{|E_0,0\rangle\to|E,\psi\rangle}(\tau)|^2$. But since we are only interested in the state of the detector we sum over all field configurations:

$$P_E(\tau) := \sum_i P_{|E_0,0\rangle\to|E,\psi_i\rangle}(\tau) = \sum_i |Q_{|E_0,0\rangle\to|E,\psi\rangle}(\tau)|^2 \tag{A.19}$$

$$= c^2|\,\langle E|\,m(0)\,|E_0\rangle\,|^2 F_{E-E_0}(\tau) \tag{A.20}$$

$$\text{with } F_E(\tau) = \sum_i\left|\int_{-\infty}^{\tau} e^{iE\tau}\,\langle\psi_i|\,\phi(\mathbf{x}(\tau'))\,|0\rangle\,\mathrm{d}\tau'\right|^2 \tag{A.21}$$

$$= \sum_i\int_{-\infty}^{\tau} e^{-iE\tau''}\,\langle 0|\,\phi(\mathbf{x}(\tau''))\,\mathrm{d}\tau''\;|\psi_i\rangle\;\langle\psi_i|\int_{-\infty}^{\tau} e^{iE\tau'}\phi(\mathbf{x}(\tau'))\,|0\rangle\,\mathrm{d}\tau' \tag{A.22}$$

$$= \int_{-\infty}^{\tau}\mathrm{d}\tau'\int_{-\infty}^{\tau}\mathrm{d}\tau''\,e^{-iE(\tau''-\tau')}\,\langle 0|\,\phi(\mathbf{x}(\tau''))\phi(\mathbf{x}(\tau'))\,|0\rangle \tag{A.23}$$

$$= \int_{-\infty}^{\tau}\mathrm{d}\tau'\int_{-\infty}^{\tau}\mathrm{d}\tau''\,e^{-iE(\tau''-\tau')}D^+(\mathbf{x}(\tau''),\mathbf{x}(\tau')) \tag{A.24}$$

Here we introduced the Wightman function $D^+(\mathbf{x},\mathbf{x}') = \langle 0|\,\phi(\mathbf{x})\phi(\mathbf{x}')\,|0\rangle$. The probability splits in a product of two parts. The first one only depends on the model of the detector while the second part only depends on the trajectory. We will therefore interpret the (so called detector response) function $F_E(\tau)$ as the distribution of energy excitations (or particles) as 'seen' by an observer on the trajectory $\mathbf{x}(\tau)$.

The transition rate is then given by:

$$\frac{\mathrm{d}F_E}{\mathrm{d}\tau} = \int_{-\infty}^{\tau}\mathrm{d}\tau''\,e^{-iE(\tau''-\tau)}D^+(\mathbf{x}(\tau''),\mathbf{x}(\tau)) + \int_{-\infty}^{\tau}\mathrm{d}\tau'\,e^{-iE(\tau-\tau')}D^+(\mathbf{x}(\tau),\mathbf{x}(\tau')) \tag{A.25}$$

$$= 2\mathrm{Re}\int_{-\infty}^{0}\mathrm{d}\tilde{\tau}\,e^{-iE\tilde{\tau}}D^+(\mathbf{x}(\tilde{\tau}+\tau),\mathbf{x}(\tau)) \tag{A.26}$$

since $D^+(\mathbf{x},\mathbf{x}')^* = D^+(\mathbf{x}',\mathbf{x})$. For the special case that the Wightman function does only depend on the difference of the τ's, i.e. $D^+(\mathbf{x}(\tau_1+\tau'),\mathbf{x}(\tau_2+\tau')) = D^+(\mathbf{x}(\tau_1),\mathbf{x}(\tau_2))$ one can simplify this further:

$$
\begin{aligned}
\frac{\mathrm{d}F_E}{\mathrm{d}\tau} &= \int_{-\infty}^{0} \mathrm{d}\tilde{\tau}\, e^{-iE\tilde{\tau}} D^+(\mathbf{x}(\tilde{\tau}+\tau), \mathbf{x}(\tau)) + \int_{0}^{\infty} \mathrm{d}\tilde{\tau}\, e^{-iE\tilde{\tau}} D^+(\mathbf{x}(\tau), \mathbf{x}(\tau-\tilde{\tau})) && \text{(A.27)}\\
&= \int_{-\infty}^{0} \mathrm{d}\tilde{\tau}\, e^{-iE\tilde{\tau}} D^+(\mathbf{x}(\tilde{\tau}+\tau), \mathbf{x}(\tau)) + \int_{0}^{\infty} \mathrm{d}\tilde{\tau}\, e^{-iE\tilde{\tau}} D^+(\mathbf{x}(\tau+\tilde{\tau}), \mathbf{x}(\tau)) && \text{(A.28)}\\
&= \int_{-\infty}^{\infty} \mathrm{d}\tilde{\tau}\, e^{-iE\tilde{\tau}} D^+(\mathbf{x}(\tilde{\tau}+\tau), \mathbf{x}(\tau)) = \int_{-\infty}^{\infty} \mathrm{d}\tilde{\tau}\, e^{-iE\tilde{\tau}} D^+(\mathbf{x}(\tilde{\tau}), \mathbf{x}(0)) && \text{(A.29)}
\end{aligned}
$$

The rate is the Fourier transform of the Wightman function and is independent of τ.

A.4 Wightman function in normal coordinates

Fix $\mathbf{x}' = 0$. The metric in normal coordinates then looks like [**davies**]:

$$
g_{ij} = \delta_{ij} - \frac{1}{3} R_{iajb} x^a x^b + O\left(x^3\right) \tag{A.30}
$$

Since the metric is given to the second order we will also expand other quantities[1]

$$
\begin{aligned}
g^{ij} &= \delta^{ij} + \frac{1}{3} R^{i\ j}_{\ a\ b} x^a x^b + O\left(x^3\right) && \text{(A.31)}\\
\partial_i g^{ij} &= -\frac{1}{3} R^j_{\ i} x^i + O\left(x^2\right) = -\frac{1}{3} R_{ji} x^i + O\left(x^2\right) && \text{(A.32)}\\
g = \det g_{ij} &= 1 - \frac{1}{3} R_{ij} x^i x^j + O\left(x^3\right) && \text{(A.33)}\\
\frac{1}{g}\partial_i g &= -\frac{2}{3} R_{ij} x^j + O\left(x^2\right) && \text{(A.34)}\\
\beta &= a + b_i x^i + \frac{1}{2} c_{ij} x^i x^j + O\left(x^3\right) && \text{(A.35)}
\end{aligned}
$$

[1] Note that one can raise and lower indices with δ_{ij} if one is neglecting $O(x^2)$.

A.4.1 Solutions of the Klein-Gordon-equation

The Klein-Gordon-equation is given by

$$
\begin{aligned}
0 = \nabla_\mu \nabla^\mu \phi &= -\frac{1}{\sqrt{\beta g}}\partial_t\left(\sqrt{\beta g}\frac{1}{\beta}\partial_t\phi\right) + \frac{1}{\sqrt{\beta g}}\partial_i\left(\sqrt{\beta g}g^{ij}\partial_j\phi\right) && \text{(A.36)}\\
&= -\frac{1}{\beta}\partial_t^2\phi + \frac{1}{\sqrt{\beta g}}\partial_i\left(\sqrt{\beta g}\right)g^{ij}\partial_j\phi + \left(\partial_i g^{ij}\right)\partial_j\phi + g^{ij}\partial_i\partial_j\phi && \text{(A.37)}\\
&= -\frac{1}{\beta}\partial_t^2\phi + \frac{1}{2\beta g}\partial_i(\beta g)g^{ij}\partial_j\phi + \left(\partial_i g^{ij}\right)\partial_j\phi + g^{ij}\partial_i\partial_j\phi && \text{(A.38)}\\
\partial_t^2\phi &= \frac{\partial_i\beta}{2}g^{ij}\partial_j\phi + \frac{\beta\partial_i g}{2g}g^{ij}\partial_j\phi + \beta\left(\partial_i g^{ij}\right)\partial_j\phi + \beta g^{ij}\partial_i\partial_j\phi && \text{(A.39)}\\
&= \frac{1}{2}\left(b_i + c_{ik}x^k\right)\partial_i\phi - \frac{a}{3}R_{ik}x^k\partial_i\phi - \frac{a}{3}R_{ik}x^k\partial_i\phi + \left(a + b_k x^k\right)\partial_i\partial_i\phi + O\left(x^2\right) && \text{(A.40)}\\
&= \frac{1}{2}\left(b_i + c_{ik}x^k\right)\partial_i\phi - \frac{2a}{3}R_{ik}x^k\partial_i\phi + \left(a + b_k x^k\right)\partial_i\partial_i\phi + O\left(x^2\right) && \text{(A.41)}
\end{aligned}
$$

To solve these equations make the ansatz $\tilde{u}_{\vec{\mathbf{k}}}(\mathbf{x}) = \exp\left(-i\omega t + ik_i x^i + i\frac{1}{2}k_a B^a_{ij}(\vec{\mathbf{k}})x^i x^j + iO(x^3)\right)$ and separate the different orders:

$$
\begin{aligned}
-\omega^2 &= \frac{1}{2}\left(b_i + c_{ik}x^k\right)i\left(k_i + k_a B^a_{ik}x^k\right) - \frac{2a}{3}R_{ik}x^k ik_i\\
&\quad + \left(a + b_k x^k\right)\left(ik_a B^a_{ii} - \left(k_i + k_a B^a_{ik}x^k\right)\left(k_i + k_b B^b_{il}x^l\right)\right) + O\left(x^2\right) && \text{(A.42)}\\
-\omega^2 &= \frac{1}{2}ib_i k_i + aik_a B^a_{ii} - ak_i k_i && \text{(A.43)}\\
0 &= ik_a\left(\frac{1}{2}b_i B^a_{ik} + \frac{1}{2}c_{ak} - \frac{2a}{3}R_{ak} + b_k B^a_{ii}\right) - 2ak_i k_a B^a_{ik} - b_k k_i k_i && \text{(A.44)}
\end{aligned}
$$

If we demand that all parameters should be real then we have 8 equations for 18 free parameters of B^a_{ij}[2]. We could now fix some more properties of B but it is not necessary for our argumentation. Note that the dispersion relation now reads $\omega = \sqrt{a}\left|\vec{\mathbf{k}}\right|$.

A.4.2 Normalising the modes

Next we need to find the right normalisation of the modes. Since we can't integrate our modes over the whole spacetime we will use the CCR to find the right normalisation[3]. So expand $\phi(\mathbf{x}) = \int \frac{\mathrm{d}^3k}{\sqrt{2\pi}^3}\frac{1}{\sqrt{2\omega N_{\vec{\mathbf{k}}}}}\tilde{u}_{\vec{\mathbf{k}}}(\mathbf{x})a_{\vec{\mathbf{k}}} + \frac{1}{\sqrt{2\omega N_{\vec{\mathbf{k}}}}}\tilde{u}_{\vec{\mathbf{k}}}(\mathbf{x})^* a^\dagger_{\vec{\mathbf{k}}}$ and calculate the CCR for a surface $t = \text{const.}$:

[2] Note that B^a_{ij} is symmetric in i, j.

[3] This works because the CCR are only valid in the right normalisation

$$[\phi(\mathbf{x}), \phi(0)] = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{2\omega N_{\vec{\mathbf{k}}}} \tilde{u}_{\vec{\mathbf{k}}}(\mathbf{x}) \tilde{u}_{\vec{\mathbf{k}}}(0)^* - \frac{1}{2\omega N_{\vec{\mathbf{k}}}} \tilde{u}_{\vec{\mathbf{k}}}(\mathbf{x})^* \tilde{u}_{\vec{\mathbf{k}}}(0) \tag{A.45}$$

$$= i \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{2\omega N_{\vec{\mathbf{k}}}} e^{i\vec{\mathbf{k}}\vec{\mathbf{x}} + O(x^2)} - \frac{1}{2\omega N_{\vec{\mathbf{k}}}} e^{-i\vec{\mathbf{k}}\vec{\mathbf{x}} + O(x^2)} \stackrel{!}{=} 0 \tag{A.46}$$

$$[\phi(\mathbf{x}), \sqrt{g}\partial_0 \phi(0)] = i \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{2N_{\vec{\mathbf{k}}}} \tilde{u}_{\vec{\mathbf{k}}}(\mathbf{x}) \tilde{u}_{\vec{\mathbf{k}}}(0)^* + \frac{1}{2N_{\vec{\mathbf{k}}}} \tilde{u}_{\vec{\mathbf{k}}}(\mathbf{x})^* \tilde{u}_{\vec{\mathbf{k}}}(0) + O\left(x^2\right) \tag{A.47}$$

$$= i \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{2N_{\vec{\mathbf{k}}}} e^{i\vec{\mathbf{k}}\vec{\mathbf{x}} + O(x^2)} + \frac{1}{2N_{\vec{\mathbf{k}}}} e^{-i\vec{\mathbf{k}}\vec{\mathbf{x}} + O(x^2)} + O\left(x^2\right) \tag{A.48}$$

$$\stackrel{!}{=} i\delta^3(\vec{\mathbf{x}}) = i \int \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{i\vec{\mathbf{k}}\vec{\mathbf{x}}} \tag{A.49}$$

$$[\sqrt{g}\partial_0 \phi(\mathbf{x}), \sqrt{g}\partial_0 \phi(0)] = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{\omega}{2N_{\vec{\mathbf{k}}}} \tilde{u}_{\vec{\mathbf{k}}}(\mathbf{x}) \tilde{u}_{\vec{\mathbf{k}}}(0)^* - \frac{\omega}{2N_{\vec{\mathbf{k}}}} \tilde{u}_{\vec{\mathbf{k}}}(\mathbf{x})^* \tilde{u}_{\vec{\mathbf{k}}}(0) + O\left(x^2\right) \tag{A.50}$$

$$= i \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{\omega}{2N_{\vec{\mathbf{k}}}} e^{i\vec{\mathbf{k}}\vec{\mathbf{x}} + O(x^2)} - \frac{\omega}{2N_{\vec{\mathbf{k}}}} e^{-i\vec{\mathbf{k}}\vec{\mathbf{x}} + O(x^2)} + O\left(x^2\right) \stackrel{!}{=} 0 \tag{A.51}$$

Since $e^{i\vec{\mathbf{k}}\vec{\mathbf{x}}}$ is a basis we find:

$$\frac{1}{2\omega N_{\vec{\mathbf{k}}}} - \frac{1}{2\omega N_{-\vec{\mathbf{k}}}} = 0 \tag{A.52}$$

$$\frac{1}{2N_{\vec{\mathbf{k}}}} + \frac{1}{2N_{-\vec{\mathbf{k}}}} = 1 \tag{A.53}$$

$$\frac{\omega}{2N_{\vec{\mathbf{k}}}} - \frac{\omega}{2N_{-\vec{\mathbf{k}}}} = 0 \tag{A.54}$$

This system of equations is only solved for $N_{\vec{\mathbf{k}}} = 1$ and so the normalised modes are

$$u_{\vec{\mathbf{k}}} = \frac{1}{\sqrt{2\pi}^3 \sqrt{2\omega}} e^{-i\omega t + i\vec{\mathbf{k}}\vec{\mathbf{x}} + O(x^2)} \tag{A.55}$$

$$= \frac{1}{\sqrt{2\pi}^3 \sqrt{2\omega}} e^{-i\omega t + i\vec{\mathbf{k}}\vec{\mathbf{x}}} \left(1 + O\left(x^2\right)\right) \tag{A.56}$$

So up to linear order we achieve the plane wave modes as in Minkowski space. Therefore the Wightman function is also the equivalent and given by

$$D^+(\mathbf{x}, 0) = -\frac{1}{4\pi^2} \frac{1}{a(t - i\varepsilon)^2 - |\vec{\mathbf{x}}|^2} + O\left(x^2\right) \tag{A.57}$$

A.5 Determination of the temperature

In chapter 4 we need to find the temperature that an observer will see on the basis of the energy spectrum. The energy spectrum is basically given by a Fourier-like transform of D^+ evaluated along the curve (see eq. (2.13) and (2.14)).

A.5.1 Idea

Imaging an observer equipped with an Unruh-detector on a trajectory in the Schwarzschild metric. By eq. (3.35) he expects some thermal spectrum together with some corrections. These corrections could either shift the observed temperature or could be part of the non thermal spectrum. However he won't be able to distinguish which part of the spectrum is thermal or not. So to determine the temperature he will fit a thermal spectrum into the observed spectrum, i.e. find a value β such that the following minimizes:

$$\int_0^\infty \mathrm{d}E \left(\frac{\mathrm{d}F_E}{\mathrm{d}\tau} - \frac{1}{2\pi}\frac{E}{e^{\beta E}-1}\right)^2 \to \min \tag{A.58}$$

We will use this measurement process to define the temperature observed on a trajectory. To minimize this we don't need to calculate the spectrum first, we can equivalently directly minimize the difference of D^+ and the thermal Wightman function in Minkowski space

understandable

$$D^{\mathrm{M}}_\beta(\tau') = -\frac{1}{4\beta^2}\frac{1}{\sinh^2\left(\frac{\pi}{\beta}\tau'\right)} \tag{A.59}$$

corresponding to β[4]:

$$\int_{-\infty}^0 \mathrm{d}\tau' \left(D^+(\mathbf{x}(\tau+\tau'),\mathbf{x}(\tau)) - D^{\mathrm{M}}_\beta(\tau')\right)^2 \to \min \tag{A.60}$$

We expect the temperature shift to be small compared to the Hawking temperature $\beta_{\mathrm{H}} = 8\pi M$ (and indeed this will be the case). So we can do a Taylor expansion of the thermal Wightman function around β_{H}[5] This yields to

$$\begin{aligned} D^{\mathrm{M}}_\beta(\tau') &\approx D^{\mathrm{M}}_{\beta_{\mathrm{H}}}(\tau') + \frac{\Delta\beta}{\beta_{\mathrm{H}}}\frac{\sinh\left(\frac{\pi}{\beta_{\mathrm{H}}}\tau'\right) - \frac{\pi}{\beta_{\mathrm{H}}}\tau'\cosh\left(\frac{\pi}{\beta_{\mathrm{H}}}\tau'\right)}{2\beta_{\mathrm{H}}^2\sinh^3\left(\frac{\pi}{\beta_{\mathrm{H}}}\tau'\right)} && \text{(A.61)} \\ &=: D^{\mathrm{M}}_{\beta_{\mathrm{H}}}(\tau') + \alpha g(\tau') && \text{(A.62)} \end{aligned}$$

[4] This is due to the Plancherel theorem which states that the integral over the square of a function equals the integral over the square of its Fourier transform.

[5] Note that we are expanding around the fixed Hawking temperature and not around the position dependent Tolman temperature $\beta_{\mathrm{static}} = \sqrt{f(r)}\beta_{\mathrm{H}}$. This is useful to compare the observed temperatures.

We would like to compute $\alpha = \frac{\Delta\beta}{\beta_{\mathrm{H}}} = \frac{\beta-\beta_{\mathrm{H}}}{\beta_{\mathrm{H}}}$. This is easily done because we can find the minimum by differentiating w.r.t. α:

$$\int_{-\infty}^{0} \mathrm{d}\tau' \left(D^{+}(\mathbf{x}(\tau+\tau'),\mathbf{x}(\tau)) - D^{\mathrm{M}}_{\beta_{\mathrm{H}}}(\tau') - \alpha g(\tau')\right)^2 \to \min \tag{A.63}$$

$$\alpha = \frac{\int_{-\infty}^{0} \mathrm{d}\tau' \left(D^{+}(\mathbf{x}(\tau+\tau'),\mathbf{x}(\tau)) - D^{\mathrm{M}}_{\beta_{\mathrm{H}}}(\tau')\right) \cdot g(\tau')}{\int_{-\infty}^{0} \mathrm{d}\tau'\, g(\tau')^2} \tag{A.64}$$

$$=: \frac{\int_{-\infty}^{0} \mathrm{d}\tau'\, h(\tau') \cdot g(\tau')}{\int_{-\infty}^{0} \mathrm{d}\tau'\, g(\tau')^2} \tag{A.65}$$

The integral over g^2 can be calculated ones and yields $I_g = \int_{-\infty}^{0} \mathrm{d}\tau'\, g(\tau')^2 = \frac{15-\pi^2}{92160\pi^4} M^{-3} \approx$
explain? $5.7149 \cdot 10^{-7} M^{-3}$. In the diagrams we will not plot α but rather $-\alpha = \Delta T / T_{\mathrm{H}}$.

A.5.2 Error estimation

The other integral $I = \int_{-\infty}^{0} \mathrm{d}\tau'\, h(\tau') g(\tau')$ is evaluated numerically using the trapezoidal rule (see for example [**ron**]). Since we cannot integrate the infinite range $(-\infty, 0)$ with this method we will integrate over a finite range (τ_1, τ_2)[6]. In order to get a meaningful result we need to estimate the errors. There are basically 3 sources of errors: the lower and upper integration limit and the error by the method itself.

For big absolute values of τ' we can replace the sinh and cosh in g by an exponential function:

$$g(\tau') \approx \frac{1 - \frac{\pi}{\beta_{\mathrm{H}}}\tau'}{2\beta_{\mathrm{H}}^2} \exp\left(2\frac{\pi}{\beta_{\mathrm{H}}}\tau'\right) \tag{A.66}$$

$$\approx -\frac{\pi}{2\beta_{\mathrm{H}}^3}\tau' \exp\left(2\frac{\pi}{\beta_{\mathrm{H}}}\tau'\right) \tag{A.67}$$

check Integrating this from $-\infty$ to τ_1 yields:

$$\int_{-\infty}^{\tau_1} g(\tau') = \frac{1}{4\beta_{\mathrm{H}}}\left(\frac{-\tau_1}{\beta_H} + \frac{1}{2\pi}\right) \exp\left(2\frac{\pi}{\beta_{\mathrm{H}}}\tau_1\right) \tag{A.68}$$

$$\approx \frac{\beta_{\mathrm{H}}}{2\pi} g(\tau_1) \tag{A.69}$$

Since h will drop to zero we can assume an upper bound of the error for sufficiently small τ_1 by

$$\Delta I_1 \approx \frac{\beta_{\mathrm{H}}}{2\pi} |g(\tau_1) h(\tau_1)| \tag{A.70}$$

anders? Another error arises from the factthat around zero h is the (finite) difference of two divergent functions. Evaluating the function is therefore afflicted with increasing sampling errors when

[6] Note that all τ are negative.

approaching zero. Since we cannot control those errors we will stop integrating at $\tau_2 \lesssim 0$. When we assume that the function is nearly constant on the small interval we can add the missing contribution $g(\tau_2)h(\tau_2)\cdot|\tau_2|$. However this will lead to an error which we estimate generously by the expected amount of the contribution:

$$\Delta I_2 \approx |g(\tau_2)h(\tau_2)\tau_2| \approx |g(0)h(0)\tau_2| \tag{A.71}$$

The last contribution comes from the error of the method used. For the trapezoidal method this error is given by ($\Delta\tau$ is the stepwidth)[**ron**]

$$\Delta I_{\text{trapez}} = \frac{\Delta\tau^2}{12}(\tau_2-\tau_1)|(h\cdot g)''(\chi)| \tag{A.72}$$

where χ is some value in the interval (τ_1, τ_2). Since the $h\cdot g$ will asymptotically drop to zero at least exponentially the second derivative will also drop. We are therefore interested in the curvature nearby $\chi \approx \tau_2 \approx 0$. In fact we will use the second derivative at zero to estimate the error:

$$\Delta I_{\text{trapez}} \approx \frac{\Delta\tau^2}{12}(\tau_2-\tau_1)|(h\cdot g)''(0)| \tag{A.73}$$

Recall that we minimized $\int(h-\alpha g)^2$ and therefore expect $h \approx \alpha g$. So $(h\cdot g)''(0) \approx (\alpha g^2)''(0)$ that inserting the definition yields to check

$$\Delta I_{\text{trapez}} \approx \alpha\frac{\Delta\tau^2}{12}(\tau_2-\tau_1)\frac{2\pi^2}{45\beta^6} \tag{A.74}$$

To compare the order of magnitude of the different error contributions we will replace $h \approx \alpha g$, use the asymptotic form of g in (A.67) and express all quantities in units of the mass of the star (i.e. set $M = 1$)

$$\Delta I_1[M^{-3}] = 4|g(\tau_1)h(\tau_1)| \approx \alpha\frac{1}{524288\pi^4}\tau_1[M]^2\exp\left(\frac{1}{2}\tau_1[M]\right) \tag{A.75}$$

$$\Delta I_2[M^{-3}] = |g(0)h(0)\tau_2| \approx \alpha\frac{1}{36\beta^4}|\tau_2| = \alpha\frac{1}{147456\pi^4}|\tau_2[M]| \tag{A.76}$$

$$\Delta I_{\text{trapez}}[M^{-3}] = \alpha\frac{\Delta\tau^2}{12}(\tau_2-\tau_1)\frac{2\pi^2}{45\beta^6} \approx \alpha\Delta\tau[M]^2(\tau_2[M]-\tau_1[M])\frac{1}{70778880\pi^4} \tag{A.77}$$

Apart from the τ_2 we can minimize the errors by choosing sufficient values for τ_1 and $\Delta\tau$. If we set

$$\Delta\tau[M] \ll 1/(\tau_2[M]-\tau_1[M]) \tag{A.78}$$

$$\Delta\tau[M] \ll |\tau_2[M]| \tag{A.79}$$

the contribution of ΔI_{trapez} is neglectable compared to ΔI_2. If we further set τ_1 such that

$$\tau_1[M]^2\exp\left(\frac{1}{2}\tau_1[M]\right) \ll |\tau_2[M]| \tag{A.80}$$

Table A.1: Parameters used for numerical integration: integration from τ_1 to τ_2 using $\Delta\tau$ as stepwidth.

Observer	$\tau_1[M]$	$\tau_2[M]$	$\Delta\tau[M]$
static	-40	-0.1	0.0001
circular	-40	$-(\log_2(r)/10)^2$	0.0001
radial	-40	-0.1	0.0001

we can also neglect ΔI_1. So the dominant error contribution will come from $\Delta I_2 = |g(0)h(0)\tau_2|$. This error is – as the value – divided by I_g to achieve the error on α which is then given in the diagrams as errorbars.

For computing those diagrams the parameters in tab. A.1 were used. One can easily check that they fulfil relations (A.78) - (A.80). It was necessary to adjust τ_2 for circular observers because the numerical errors increased with increasing r. For the other observers $g \cdot h$ became thinner with increasing r till the sampling errors dominated. Therefore it was not possible to use this algorithm for arbitrary high radii.

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