An Introduction to String Theory

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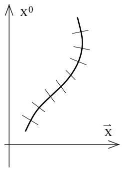
$$S = -m \int dt \sqrt{1 - \dot{\vec{x}}\dot{\vec{x}}}$$

  $\rightarrow$ not Lorentz-invariant, due to mixture of spacial and temporal coordinates under a Lorentz-transformation Λ.

- Consider instead for a generalized coordinate τ along the line element:

$$S = -m \int d\tau \sqrt{-\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\eta_{\mu\nu}}$$



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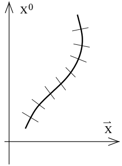

Remark: S is proportional to the integral over the worldline of the particle

The Relativistic Point Particle – Action Symmetries

What are the symmetries of

$$S = -m \int d\tau \sqrt{-\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\eta_{\mu\nu}}$$

- Reparametrization invariance:

  Let $\tilde{\tau} = \tilde{\tau}(\tau)$. Then:

  $$S' = -m \int d\tilde{\tau} \sqrt{-\frac{dX^\mu}{d\tilde{\tau}}\frac{dX^\nu}{d\tilde{\tau}}\eta_{\mu\nu}} = S$$

  $\rightarrow$ gauge symmetry of the action $\rightarrow$ still D-1 dof!
- Poincaré invariance:

  Let:

  $$X'^\mu = \Lambda^\mu{}_\nu X^\nu + c^\mu$$

  Then:

  $$S' = S, \qquad \text{as} \qquad \Lambda^\mu{}_\rho\, \eta_{\mu\nu}\, \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$$

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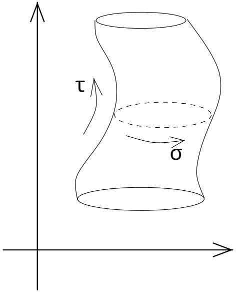
Dynamics of a Relativistic String – The Nambu-Goto Action

Action of the Relativistic String?

particle $\Leftrightarrow$ worldline $\Leftrightarrow$ $S = -m \int \underbrace{d\tau \sqrt{-\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\eta_{\mu\nu}}}_{\text{line element } ds}$

closed string $\Leftrightarrow$? $\Leftrightarrow$?



Boundary condition of closed strings living in D dimensions:

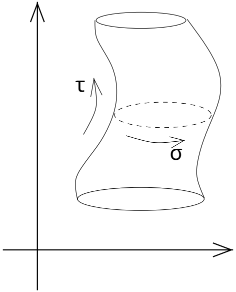
$$X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+2\pi) \quad \text{for } \mu = 0,1,...,D-1;\ \sigma \in [0;2\pi)$$

Shorthand notation: $\sigma^\alpha = (\tau,\sigma)$ for $\alpha \in \{0;1\}$

Action of the Relativistic String?

particle $\Leftrightarrow$ worldline $\Leftrightarrow$ $S = -m \int d\tau \sqrt{-\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\eta_{\mu\nu}}$

closed string $\Leftrightarrow$ worldsheet $\Leftrightarrow$?



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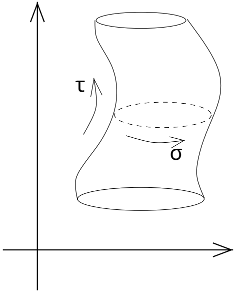
closed string $\Leftrightarrow$ worldsheet $\Leftrightarrow$ Nambu-Goto/Dirac Action



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Shorthand notation: $\sigma^\alpha = (\tau, \sigma)$ for $\alpha \in \{0; 1\}$

Dynamics of a Relativistic String – The Nambu-Goto Action

How to describe the surface area to construct the action?

$$S \propto \int_{\partial V} d^2x = \int_{\partial V} d^2\sigma \, |\det J|$$

For this, first remember that any metric is given by:

$$g_{\alpha\beta} = g(e_\alpha, e_\beta)$$

So the metric on the worldsheet (the pullback metric) can be written as:

$$\gamma_{\alpha\beta} = \underbrace{\frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} \eta_{\mu\nu}}_{(J^T \eta J)_{\alpha\beta}}$$

Thus,

$$\det \gamma = \det \eta \, (\det J)^2 = -(\det J)^2$$
$$|\det J| = \sqrt{-\det \gamma} = \sqrt{-\gamma}$$

Dynamics of a Relativistic String – The Nambu-Goto Action

So write the action as:

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{-\gamma}$$

This action in invariant under,

- Poincaré transformations
- Reparametrization

and the eqations of motion (EoM) are given by:

$$\partial_\alpha \left(\sqrt{-\gamma}\, \gamma^{\alpha\beta} \partial_\beta X^\mu \right) = 0$$

$\Rightarrow$ rather hard to quantize in this form! Alternatives?

Dynamics of a Relativistic String – The Polyakov Action

Way out: **The Polyakov Action**:

$$S_P = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-g}\, g^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu\, \eta_{\mu\nu}$$

The EoMs:

- for X^μ:
  - same as for the Nambu-Goto action!

$$\partial_\alpha \left(\sqrt{-g}\, g^{\alpha\beta} \partial_\beta X^\mu \right) = 0$$

- for $g_{\alpha\beta}$:

$$g_{\alpha\beta} = 2 \frac{\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}}{g^{\rho\sigma} \partial_\rho X^\mu \partial_\sigma X^\nu \eta_{\mu\nu}} \equiv 2 \frac{\partial_\alpha X \partial_\beta X}{g^{\rho\sigma} \partial_\rho X \partial_\sigma X}$$

Gains of the Polyakov action?

Dynamics of a Relativistic String – The Polyakov Action

Symmetries of the Polyakov action

$$S_P = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-g}\, g^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu\, \eta_{\mu\nu}$$

- Poincaré invariance
- Reparametrization invariance
- Invariance under:

$$g'_{\alpha\beta} = \Omega^2(\sigma) g_{\alpha\beta}$$

$\Rightarrow$ **Weyl invariance**

However, using the conformal gauge and writing g as

$$g_{\alpha\beta} = e^{2\phi}\eta_{\alpha\beta}$$

can be undone by a Weyl transformation ($\phi = 0$). Thus,

$$g_{\alpha\beta} = \eta_{\alpha\beta}$$

The Polyakov Action – Equation of Motion

With $g_{\alpha\beta} = \eta_{\alpha\beta}$ the EoMs

$$\begin{cases} \partial_\alpha \left(\sqrt{-g}\, g^{\alpha\beta} \partial_\beta X^\mu\right) = 0 \\ g_{\alpha\beta} = 2\frac{\partial_\alpha X \partial_\beta X}{g^{\rho\sigma}\partial_\rho X \partial_\sigma X} \end{cases}$$

are reduced to:

$$\begin{cases} \partial_\alpha \partial^\alpha X^\mu & = 0 \\ T_{\alpha\beta} & = \partial_\alpha X \partial_\beta X - \frac{1}{2}\eta_{\alpha\beta}\eta^{\mu\nu}\partial_\mu X \partial_\nu X = 0 \end{cases}$$

With the second constraints explicitly as:

$$\begin{cases} T_{01} & = \dot{X} \cdot X' = 0 \\ T_{00} & = T_{11} = \frac{1}{2}\left(\dot{X}^2 + X'^2\right) = 0 \end{cases}$$

The Polyakov Action – Solution to the EoMs

To find a solution, define the lightcone coordinates as

$$\sigma^{\pm} = \tau \pm \sigma$$

Then, the EoMs are given by

$$\begin{cases} \partial_+ \partial_- X^\mu & = 0 \\ (\partial_+ X)^2 & = 0 \\ (\partial_- X)^2 & = 0 \end{cases} \quad \text{, with } X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi)$$

$$\Rightarrow X^\mu(\tau, \sigma) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-)$$

$$X_L^\mu(\sigma^+) = \frac{1}{2} x^\mu + \frac{1}{2} \alpha' p^\mu \sigma^+ + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-in\sigma^+}$$

$$X_R^\mu(\sigma^-) = \frac{1}{2} x^\mu + \frac{1}{2} \alpha' p^\mu \sigma^- + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\sigma^-}$$

The constraints $(\partial_\pm X)^2 = 0$:

- First:

$$\partial_- X^\mu = \frac{\alpha'}{2} p^\mu + \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \alpha_n^\mu e^{-in\sigma^-} \equiv \sqrt{\frac{\alpha'}{2}} \sum_n \alpha_n^\mu e^{-in\sigma^-}$$

$$\text{with} \quad \alpha_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu$$

- Then:

$$(\partial_- X)^2 = \frac{\alpha'}{2} \sum_{n,p} \alpha_m \alpha_n e^{-i(n+m)\sigma^-}$$

$$= \alpha' \sum_n \underbrace{\frac{1}{2} \sum_m \alpha_m \alpha_{n-m}}_{L_n} e^{-in\sigma^-} \overset{!}{=} 0$$

The Polyakov Action – Solution to the EoMs

$\Rightarrow \infty$ number of constraints on L_n:

$$L_n = \tilde{L}_n = 0$$

However, L_0 contains the mass $M^2 = -p^2$ as:

$$L_0 = \frac{\alpha'}{4}p^2 + \frac{1}{2}\sum_m \alpha_m \alpha_{-m}$$

$$\Rightarrow M^2 = \frac{4}{\alpha'}\sum_{m>0} \alpha_m \alpha_{-m} = \frac{4}{\alpha'}\sum_{m>0} \tilde{\alpha}_m \tilde{\alpha}_{-m}$$

$\Rightarrow$ **Level matching**

Polyakov Action – Summary

- The Polyakov action

$$S_P = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-g}\, g^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu\, \eta_{\mu\nu}$$

gives rise to solutions in lightcone coordinates with left and right-moving modes:

$$X^\mu_{L;R}(\sigma^\pm) = \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^\pm + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \overset{(\sim)}{\alpha}_n{}^\mu e^{-in\sigma^\pm}$$

- These modes have to fulfil the constraints of the action $\to$ level matching

$$L_n = \tilde{L}_n = 0$$

- They give the mass of the particle through:

$$M^2 = \frac{4}{\alpha'} \sum_{m>0} \alpha_m \alpha_{-m} = \frac{4}{\alpha'} \sum_{m>0} \tilde{\alpha}_m \tilde{\alpha}_{-m}$$

Polyakov Action – Summary

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The Quantum String – Covariant Quantization

First, promote X^μ and $\Pi^\mu = 1/2\pi\alpha' \dot{X}^\mu$ to operators with the usual commutators:

$$\left[X^\mu(\tau,\sigma), \Pi^\nu(\tau,\sigma')\right] = i\delta(\sigma' - \sigma)\delta^{\mu\nu}$$

Then:

$$[x^\mu, p_\nu] = i\delta^\mu_\nu$$
$$[\alpha^\mu_n, \alpha^\nu_m] = [\tilde{\alpha}^\mu_n, \tilde{\alpha}^\nu_m] = n\eta^{\mu\nu}\delta_{n,-m}$$

Introduce the creation operators as

$$a_n = \frac{\alpha_n}{\sqrt{n}}, \quad a_n^\dagger = \frac{\alpha_{-n}}{\sqrt{n}}, \quad n > 0$$

which gives the usual

$$\left[a_n, a_m^\dagger\right] = \delta_{nm}$$

$\Rightarrow$ construction of the Fock space!

1^{st} problem with this approach: appearance of ghosts, due to the relation:

$$[\alpha_n^\mu, \alpha_m^\nu] = n\eta^{\mu\nu}\delta_{n,-m} \quad \rightarrow \quad \text{negative norms for } \mu = \nu = 0 \quad \rightarrow \quad \text{ghosts (unphysical)}$$

2^{nd} problem: What about the constraints $L_n = \tilde{L}_n = 0$? How to order the operators in L_n?

- one could require for any physical state $|\varphi\rangle$:

$$\langle\phi| L_n |\varphi\rangle = \langle\phi| \tilde{L}_n |\varphi\rangle = 0$$

$$\Rightarrow L_n |\varphi\rangle = 0,\ n > 0 \quad (L_n^\dagger = L_{-n})$$

- For L_0:

$$L_0 = \sum_{m=1}^{\infty} \alpha_{-m}\alpha_m + \frac{1}{2}\alpha_0^2$$

$$\Rightarrow (L_0 - a)|\varphi\rangle = 0 \quad \rightarrow \quad M^2 = \frac{4}{\alpha'}\left(-a + \sum_{m=1}^{\infty} \alpha_{-m}\alpha_m\right)$$

The Quantum String – Quantization in the Lightcone Gauge

To resolve these problems, introduce lightcone coordinates:

$$X^{\pm} = \sqrt{\frac{1}{2}}\left(X^0 \pm X^{D-1}\right)$$

and solve the EoMs $\left(\partial_+\partial_- X^\mu = 0, \quad (\partial_+ X)^2 = 0, \quad (\partial_- X)^2 = 0\right)$ in the lightcone gauge:

$$X^+ = x^+ + \alpha' p^+ \tau = \underbrace{\frac{1}{2}x^+ + \frac{1}{2}\alpha' p^+ \sigma^+}_{X_L^+(\sigma^+)} + \underbrace{\frac{1}{2}x^+ + \frac{1}{2}\alpha' p^+ \sigma^-}_{X_R^+(\sigma^-)}$$

Solution for X^-:

$$X_L^-(\sigma^+) = \frac{1}{2}x^- + \frac{1}{2}\alpha' p^- \sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^- e^{-in\sigma^+}$$

$$X_R^-(\sigma^-) = \frac{1}{2}x^- + \frac{1}{2}\alpha' p^- \sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^- e^{-in\sigma^-}$$

The Quantum String – Quantization in the Lightcone Gauge

Now, the oscillator modes α_n^-:

$$\alpha_n^- = \sqrt{\frac{1}{2\alpha'}}\frac{1}{p^+}\sum_{m=-\infty}^{m=+\infty}\sum_{i=1}^{D-2}\alpha_{n-m}^i\alpha_m^i$$

And, the level matching condition becomes:

$$M^2 = \frac{4}{\alpha'}\sum_{i=1}^{D-2}\sum_{n>0}\alpha_{-n}^i\alpha_n^i = \frac{4}{\alpha'}\sum_{i=1}^{D-2}\sum_{n>0}\tilde{\alpha}_{-n}^i\tilde{\alpha}_n^i$$

Gain: we have 2(D-2) oscillator modes with spacial indices only

$\Rightarrow$ after quantization we only have positive norms, due to

$$\left[\alpha_n^i, \alpha_m^j\right] = \left[\tilde{\alpha}_n^i, \tilde{\alpha}_m^j\right] = n\delta^{ij}\delta_{n,-m}$$

$\Rightarrow$ 1^{st} problem solved.

What about the mass spectrum and the ordering ambiguity?

Take the classical result and swap the operators using the commutator $\left[\alpha_n^i, \alpha_{-m}^j\right] = n\delta^{ij}\delta_{n,m}$

$$M^2 = \frac{4}{\alpha'}\left(\frac{1}{2}\sum_{i=1}^{D-2}\sum_{n\neq 0}\alpha_{-n}^i\alpha_n^i - a\right) = \frac{4}{\alpha'}\left(\sum_{i=1}^{D-2}\sum_{n>0}\alpha_{-n}^i\alpha_n^i + \frac{D-2}{2}\sum_{n>0} n\right)$$

Interpreting the term $\sum n$:

$$\sum_{n>0} n = \lim_{\epsilon\to 0}\sum_{n>0} n e^{-\epsilon n} = -\frac{\partial}{\partial\epsilon}\sum_{n>0} e^{-\epsilon n} =$$

$$= -\frac{\partial}{\partial\epsilon}(1 - e^{-\epsilon})^{\ 1} = \frac{1}{\epsilon^2} - \frac{1}{12} + O(\epsilon)$$

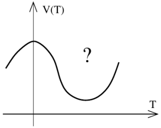
After renormalization: $\sum n = -\frac{1}{12} \quad \Rightarrow \quad a = -(D-2)/24$

The Quantum String – The Critical Dimension

What is the value D?

- No excitations are present: ground state with

$$M^2 = -\frac{1}{\alpha'}\frac{D-2}{6} \quad < 0$$



$\rightarrow$ Tachyons: the string sits at an unstable point in the tachyon field; not well understood

- First excitations $\tilde{\alpha}^i_{-1}\alpha^j_{-1}\,|0;p\rangle$ yield:

$$M^2 = \frac{4}{\alpha'}\left(1 - \frac{D-2}{24}\right)$$

with $(D-2)^2$ degrees of freedom (dof). However, in the rest frame of the particle there are $(D-1)^2$ dof $\rightarrow$ no rest frame $\rightarrow M^2 = 0$

$$\rightarrow D = 26$$

The Graviton

The states $\tilde{\alpha}^i_{-1}\alpha^j_{-1}\,|0;p\rangle$ transform in the 24 $\otimes$ 24 representation of SO(24), which decomposes into:

$$24 \otimes 24 = \underbrace{\text{traceless symmetric}}_{\text{massless spin 2-particle}} \oplus \underbrace{\text{anti-symmetric}}_{\text{Kalb-Ramond field } B_{\mu\nu}} \oplus \underbrace{\text{singlet}}_{\text{dilaton}}$$

Feynman and Weinberg:

Given a massless, spin 2 particle. Then, it must be invariant at linearized level under

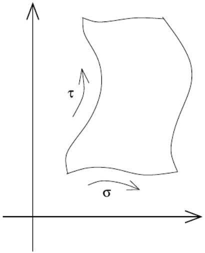
$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

to avoid negative norm states, and it should be present in interactions. This is only possible if the theory obeys diffeomorphism invariance $\to$ General Relativity

Open Strings – Action and Boundary Conditions

Open strings are described by the Polyakov action:

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \, \partial_\alpha X \partial^\alpha X \qquad \sigma \in [0, \pi]$$



which under variation yields

$$\delta S = \frac{1}{2\pi\alpha'} \cdot \int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma (\partial_\alpha \partial^\alpha X) \delta X + \underbrace{\int_0^\pi d\sigma \dot{X} \delta X \Big|_{\tau_i}^{\tau_f}}_{=0} - \underbrace{\int_{\tau_i}^{\tau_f} d\tau \, X' \delta X \Big|_0^\pi}_{\overset{!}{=}0}$$

- Neumann conditions: moving ends ($\delta X \neq 0$)

$$\partial_\sigma X = 0 \quad \text{for } \sigma = 0, \pi$$

- Dirichlet conditions: fixed ends

$$\delta X = 0 \quad \text{for } \sigma = 0, \pi$$

Open Strings – Dp-Branes

- Quantization: regular mode expansion + lightcone gauge

$$X_L^\mu(\sigma^+) = \frac{1}{2}x^\mu + \alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-in\sigma^+}$$

$$X_R^\mu(\sigma^-) = \frac{1}{2}x^\mu + \alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\sigma^-}$$

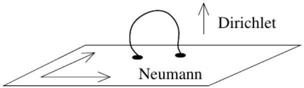
$$X^\pm = \sqrt{\frac{1}{2}}\left(X^0 \pm X^p\right)$$

- Dirichlet boundary conditions fix ends, i.e. for

$$\partial_\sigma X^a = 0 \qquad \text{for } a = 0, ..., p$$

$$X^I = c^I \qquad \text{for } I = p+1, ..., D-1$$

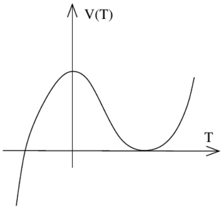
we have 2 endpoints moving freely on a (p+1)-dimensional hypersurface = **Dp-brane**



Bosonic States – Mass Spectrum

- Mass spectrum:

$$M^2 = \frac{1}{\alpha'} \left(\underbrace{\sum_{i=1}^{p-1} \sum_{n>0} \alpha^i_{-n} \alpha^i_n}_{\text{longitudinal oscillations}} + \underbrace{\sum_{i=p+1}^{D-1} \sum_{n>0} \alpha^i_{-n} \alpha^i_n}_{\text{transversal oscillations}} + \frac{2-D}{24} \right)$$



- The ground state: $M^2 = -\frac{1}{\alpha'}$
  - tachyonic
  - interpretation: unstable brane decaying into closed strings

Bosonic States – Mass Spectrum

- The first excited states:
  - $M^2 = 0 \rightarrow$ massless
  - Longitudinal oscillations:

    $$\alpha^a_{-1} \left|0;p\right\rangle \qquad a = 1, ..., p-1$$

    Spin 1 particles living in the brane $\rightarrow$ photon
  - Transverse oscillations:

    $$\alpha^I_{-1} \left|0;p\right\rangle \qquad I = p+1, ..., D-1$$

    hints the dynamics of the brane
      $\rightarrow$ string theory $=$ string $+$ brane dynamics!

Conclusion

- Strings are described by the Polyakov action
- Both theories require $D = 26$ to sustain Lorentz-invariance
  - Closed Strings:
    - Constraints imply the level-matching conditions $L_n = \tilde{L_n} = 0$
    - Solutions can be quantized in the lightcone gauge $\rightarrow \alpha_n$
    - Imply a massless spin 2 particle
  - Open Strings:
    - Dynamics constrained by Dirichlet and Neumann boundary conditions
    - Imply massless spin 1 particles

Further Reading

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