An Introduction to String Theory

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Not necessary?

- Metric:
  - $\eta_{\mu\nu}$ = diag(-1,+1, $\ldots$, +1)
- Products
  - $X^2 = X^\mu X^\nu \eta_{\mu\nu}$

The Relativistic Point Particle – The Action

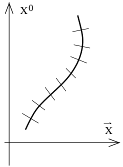
- Consider the action of a point particle (with fixed coordinates $X_\mu = (t, \vec{x})$ in a given frame):

$$S = -m \int dt \sqrt{1 - \dot{\vec{x}}\dot{\vec{x}}}$$

$\rightarrow$ not Lorentz-invariant, due to mixture of spacial and temporal coordinates under a Lorentz-transformation Λ.

- Consider instead for a generalized coordinate τ along the line element:

$$S = -m \int d\tau \sqrt{-\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\eta_{\mu\nu}}$$



Ref: [1]

Remark: S is proportional to the integral over the worldline of the particle

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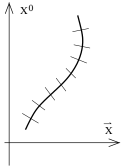

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The Relativistic Point Particle – Action Symmetries

What are the symmetries of

$$S = -m \int d\tau \sqrt{-\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\eta_{\mu\nu}}$$

- Reparametrization invariance:

  Let $\tilde{\tau} = \tilde{\tau}(\tau)$. Then:

$$S' = -m \int d\tilde{\tau} \sqrt{-\frac{dX^\mu}{d\tilde{\tau}}\frac{dX^\nu}{d\tilde{\tau}}\eta_{\mu\nu}} = S$$

  $\rightarrow$ gauge symmetry of the action $\rightarrow$ still D-1 dof!
- Poincaré invariance:

  Let:

$$X'^\mu = \Lambda^\mu{}_\nu X^\nu + c^\mu$$

  Then:

$$S' = S, \qquad \text{as} \qquad \Lambda^\mu{}_\rho \, \eta_{\mu\nu} \, \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$$

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Dynamics of a Relativistic String – The Nambu-Goto Action

Action of the Relativistic String?

$$\text{particle} \quad \Leftrightarrow \quad \text{worldline} \quad \Leftrightarrow \quad S = -m \int \underbrace{d\tau \sqrt{-\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\eta_{\mu\nu}}}_{\text{line element } ds}$$

$$\text{closed string} \quad \Leftrightarrow \quad ? \quad \Leftrightarrow \quad ?$$

Boundary condition of closed strings living in D dimensions:

$$X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi) \quad \text{for } \mu = 0, 1, ..., D-1; \ \sigma \in [0; 2\pi)$$

Shorthand notation: $\sigma^\alpha = (\tau, \sigma)$ for $\alpha \in \{0; 1\}$

image of a string

Action of the Relativistic String?

particle $\Leftrightarrow$ worldline $\Leftrightarrow$ $S = -m \int d\tau \sqrt{-\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\eta_{\mu\nu}}$

closed string $\Leftrightarrow$ worldsheet $\Leftrightarrow$?

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Dynamics of a Relativistic String – The Nambu-Goto Action

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closed string $\Leftrightarrow$ worldsheet $\Leftrightarrow$ Nambu-Goto/Dirac Action

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image of a string

Dynamics of a Relativistic String – The Nambu-Goto Action

How to describe the surface area to construct the action?

$$S \propto \int\limits_{\partial V} d^2x = \int\limits_{\partial V} d^2\sigma \, |\det J|$$

For this, first remember that any metric is given by:

$$g_{\alpha\beta} = g(e_\alpha, e_\beta)$$

So the metric on the worldsheet (the pullback metric) can be written as:

$$\gamma_{\alpha\beta} = \underbrace{\frac{\partial X^\mu}{\partial \sigma^\alpha}\frac{\partial X^\nu}{\partial \sigma^\beta}\eta_{\mu\nu}}_{(J^T \eta J)_{\alpha\beta}}$$

Thus,

$$\det\gamma = \det\eta \, \det\{^2\}J = -\det\{^2\}J$$

$$|\det J| = \sqrt{-\det\gamma} = \sqrt{-\gamma}$$

Dynamics of a Relativistic String – The Nambu-Goto Action

So write the action as:

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int d^2\sigma \, \sqrt{-\gamma}$$

This action in invariant under,

- Poincaré transformations
- Reparametrization

and the eqations of motion (EoM) are given by:

$$\partial_\alpha \left(\sqrt{-\gamma} \, \gamma^{\alpha\beta} \partial_\beta X^\mu \right) = 0$$

$\Rightarrow$ rather hard to quantize in this form! Alternatives?

Dynamics of a Relativistic String – The Polyakov Action

Way out: **The Polyakov Action**:

$$S_P = -\frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{-g}\, g^{\alpha\beta}\,\partial_\alpha X^\mu \partial_\beta X^\nu\,\eta_{\mu\nu}$$

The EoMs:

- for X^μ:
  - same as for the Nambu-Goto action!

$$\partial_\alpha\left(\sqrt{-g}\, g^{\alpha\beta}\partial_\beta X^\mu\right) = 0$$

- for $g_{\alpha\beta}$:

$$g_{\alpha\beta} = 2\frac{\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}}{g^{\rho\sigma}\partial_\rho X^\mu \partial_\sigma X^\nu \eta_{\mu\nu}} \equiv 2\frac{\partial_\alpha X \partial_\beta X}{g^{\rho\sigma}\partial_\rho X \partial_\sigma X}$$

Gains of the Polyakov action?

Dynamics of a Relativistic String – The Polyakov Action

Symmetries of the Polyakov action

$$S_P = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-g}\,g^{\alpha\beta}\,\partial_\alpha X^\mu \partial_\beta X^\nu\,\eta_{\mu\nu}$$

- Poincaré invariance
- Reparametrization invariance
- Invariance under:

$$g'_{\alpha\beta} = \Omega^2(\sigma) g_{\alpha\beta}$$

$\Rightarrow$ **Weyl invariance**

However, using the conformal gauge and writing g as

$$g_{\alpha\beta} = e^{2\phi}\eta_{\alpha\beta}$$

can be undone by a Weyl transformation ($\phi = 0$). Thus,

$$g_{\alpha\beta} = \eta_{\alpha\beta}$$

Plot of the transformation in action

Equation of Motion of the Polyakov Action

With $g_{\alpha\beta} = \eta_{\alpha\beta}$ the EoMs

$$\begin{cases} \partial_\alpha \left(\sqrt{-g}\, g^{\alpha\beta} \partial_\beta X^\mu \right) = 0 \\ g_{\alpha\beta} = 2 \frac{\partial_\alpha X \partial_\beta X}{g^{\rho\sigma} \partial_\rho X \partial_\sigma X} \end{cases}$$

are reduced to:

$$\begin{cases} \partial_\alpha \partial^\alpha X^\mu & = 0 \\ T_{\alpha\beta} & = \partial_\alpha X \partial_\beta X - \frac{1}{2} \eta_{\alpha\beta} \eta^{\mu\nu} \partial_\mu X \partial_\nu X = 0 \end{cases}$$

With the second constraints explicitly as:

$$\begin{cases} T_{01} & = \dot{X} \cdot X' = 0 \\ T_{00} & = T_{11} = \frac{1}{2} \left(\dot{X}^2 + X'^2 \right) = 0 \end{cases}$$

Solution to the EoMs

To find a solution, define the lightcone coordinates as

$$\sigma^{\pm} = \tau \pm \sigma$$

Then, the EoMs are given by

$$\begin{cases} \partial_+\partial_- X^\mu & = 0 \\ (\partial_+ X)^2 & = 0 \\ (\partial_- X)^2 & = 0 \end{cases} \quad \text{, with } X^\mu(\tau,\sigma) = X^\mu(\tau, \sigma + 2\pi)$$

$$\Rightarrow X^\mu(\tau,\sigma) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-)$$

$$X^\mu_L(\sigma^+) = \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^\mu e^{-in\sigma^+}$$

$$X^\mu_R(\sigma^-) = \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu e^{-in\sigma^-}$$

Solution to the EoMs

The constraints $(\partial_\pm X)^2 = 0$:

- First:

$$\partial_- X^\mu = \frac{\alpha'}{2} p^\mu + \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \alpha_n^\mu e^{-in\sigma^-} \equiv \sum_n \alpha_n^\mu e^{-in\sigma^-}$$

$$\text{with} \quad \alpha_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu$$

- Then:

$$(\partial_- X)^2 = \frac{\alpha'}{2} \sum_{n,p} \alpha_m \alpha_n e^{-i(n+m)\sigma^-}$$

$$= \alpha' \sum_n \underbrace{\frac{1}{2} \sum_m \alpha_m \alpha_{n-m}}_{L_n} e^{-in\sigma^-} \stackrel{!}{=} 0$$

Solution to the EoMs

$\Rightarrow \infty$ number of constraints on L_n:

$$L_n = \tilde{L}_n = 0$$

However, L_0 contains the mass $M = -p^2$ as:

$$L_0 = \frac{\alpha'}{4}p^2 + \frac{1}{2}\sum_m \alpha_m \alpha_{-m}$$

$$\Rightarrow M^2 = \frac{4}{\alpha'}\sum_{m>0} \alpha_m \alpha_{-m} = \frac{4}{\alpha'}\sum_{m>0} \tilde{\alpha}_m \tilde{\alpha}_{-m}$$

$\Rightarrow$ **Level matching**

A brief summary before moving to the covariant quantization

Covariant Quantization

First, promote X^μ and $\Pi^\mu = 1/2\pi\alpha' \dot{X}^\mu$ to operators with the usual commutators:

$$\left[X^\mu(\tau,\sigma), \Pi^\nu(\tau,\sigma')\right] = i\delta(\sigma' - \sigma)\delta^{\mu\nu}$$

Then:

$$[x^\mu, p_\nu] = i\delta^\mu_\nu$$
$$[\alpha^\mu_n, \alpha^\nu_m] = [\tilde{\alpha}^\mu_n, \tilde{\alpha}^\nu_m] = n\eta^{\mu\nu}\delta_{n,-m}$$

Introduce the creation operators as

$$a_n = \frac{\alpha_n}{\sqrt{n}}, \quad a^\dagger_n = \frac{\alpha_{-n}}{\sqrt{n}}, \quad n > 0$$

which gives the usual

$$\left[a_n, a^\dagger_m\right] = \delta_{nm}$$

$\Rightarrow$ construction of the Fock space!

Covariant Quantization

1^{st} problem with this approach: appearance of ghosts, due to the relation:

$$[\alpha_n^\mu, \alpha_m^\nu] = n\eta^{\mu\nu}\delta_{n,-m} \quad \rightarrow \quad \text{negative norms for } \mu = \nu = 0 \quad \rightarrow \quad \text{ghosts (unphysical)}$$

2^{nd} problem: What about the constraints $L_n = \tilde{L}_n = 0$? How to order the operators in L_n?

- one could require for any physical state $|\varphi\rangle$:

$$\langle\varphi| L_n |\varphi\rangle = \langle\phi| \tilde{L}_n |\varphi\rangle = 0$$

$$\Rightarrow L_n |\varphi\rangle = 0,\ n > 0 \quad (L_n^\dagger = L_{-n})$$

- For L_0:

$$L_0 = \sum_{m=1}^{\infty} \alpha_{-m}\alpha_m + \frac{1}{2}\alpha_0^2$$

$$\Rightarrow (L_0 - a)|\varphi\rangle = 0 \quad \rightarrow \quad M^2 = \frac{4}{\alpha'}\left(-a + \sum_{m=1}^{\infty} \alpha_{-m}\alpha_m\right)$$

Quantization in the Lightcone Gauge

To resolve these problems, introduce lightcone coordinates:

$$X^{\pm} = \sqrt{\frac{1}{2}}\left(X^0 \pm X^{D-1}\right)$$

and solve the EoMs $\left(\partial_+\partial_- X^\mu = 0, \quad (\partial_+ X)^2 = 0, \quad (\partial_- X)^2 = 0\right)$ in the lightcone gauge:

$$X^+ = x^+ + \alpha' p^+ \tau = \underbrace{\frac{1}{2}x^+ + \frac{1}{2}\alpha' p^+ \sigma^+}_{X_L^+(\sigma^+)} + \underbrace{\frac{1}{2}x^+ + \frac{1}{2}\alpha' p^+ \sigma^-}_{X_L^+(\sigma^-)}$$

Solution for X^-:

$$X_L^-(\sigma^+) = \frac{1}{2}x^- + \frac{1}{2}\alpha' p^- \sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^- e^{-in\sigma^+}$$

$$X_R^-(\sigma^-) = \frac{1}{2}x^- + \frac{1}{2}\alpha' p^- \sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^- e^{-in\sigma^-}$$

Quantization in the Lightcone Gauge

Now, the oscillator modes α_n^-:

$$\alpha_n^- = \sqrt{\frac{1}{2\alpha'}} \frac{1}{p^+} \sum_{m=-\infty}^{m=+\infty} \sum_{i=1}^{D-2} \alpha_{n-m}^i \alpha_m^i$$

And, the level matching condition becomes:

$$M^2 = \frac{4}{\alpha'} \sum_{i=1}^{D-2} \sum_{n>0} \alpha_{-n}^i \alpha_n^i = \frac{4}{\alpha'} \sum_{i=1}^{D-2} \sum_{n>0} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i$$

Gain: we have 2(D-2) oscillator modes with spacial indices only $\Rightarrow$ after quantization we only have positive norms, due to

$$\left[\alpha_n^i, \alpha_m^j\right] = \left[\tilde{\alpha}_n^i, \tilde{\alpha}_m^j\right] = n\delta^{ij}\delta_{n,-m}$$

$\Rightarrow$ 1st problem solved.

What about the mass spectrum and the ordering ambiguity?

Take the classical result and swap the operators using the commutator $\left[\alpha_n^i, \alpha_{-m}^j\right] = n\delta^{ij}\delta_{n,m}$

$$M^2 = \frac{4}{\alpha'}\left(\frac{1}{2}\sum_{i=1}^{D-2}\sum_{n\neq 0}\alpha_{-n}^i\alpha_n^i - a\right) = \frac{4}{\alpha'}\left(\sum_{i=1}^{D-2}\sum_{n>0}\alpha_{-n}^i\alpha_n^i + \frac{D-2}{2}\sum_{n>0} n\right)$$

Interpreting the term $\sum_{n>0} n$:

$$\sum_{n>0} n = \lim_{\epsilon\to 0}\sum_{n>0} ne^{-\epsilon n} = -\frac{\partial}{\partial\epsilon}\sum_{n>0} e^{-\epsilon n} =$$
$$= -\frac{\partial}{\partial\epsilon}(1-e^{-\epsilon})^{\ 1} = \frac{1}{\epsilon^2} - \frac{1}{12} + O(\epsilon)$$

After renormalization: $\sum n = -\frac{1}{12} \quad \Rightarrow \quad a = -(D-2)/24$

The Critical Dimension

What is the value D?

The First Excited States and the Emergence of the Graviton

The First Excited States and the Emergence of the Graviton

The Polyakov Action and Boundary Conditions

The EoMs and its Solutions & Quantization

Bosonic States