An Introduction to String Theory

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Introduction – Notation

- Metric:
  - $\eta_{\mu\nu} = \text{diag(-1,+1, } \ldots, +1)$

The Relativistic Point Particle – The Action

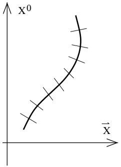
- Consider the action of a point particle (with fixed coordinates $X_\mu = (t, \vec{x})$ in a given frame):

$$S = -m \int dt \sqrt{1 - \dot{\vec{x}}\dot{\vec{x}}}$$

$\rightarrow$ not Lorentz-invariant, due to mixture of spacial and temporal coordinates under a Lorentz-transformation Λ.

- Consider instead for a generalized coordinate τ along the line element:

$$S = -m \int d\tau \sqrt{-\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\eta_{\mu\nu}}$$



Ref: [1]

Remark: S is proportional to the integral over the worldline of the particle

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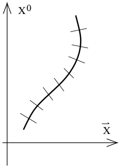

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The Relativistic Point Particle – Action Symmetries

What are the symmetries of

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- Reparametrization invariance:

  Let $\tilde{\tau} = \tilde{\tau}(\tau)$. Then:

$$S' = -m \int d\tilde{\tau} \sqrt{-\frac{dX^\mu}{d\tilde{\tau}}\frac{dX^\nu}{d\tilde{\tau}}\eta_{\mu\nu}} = S$$

  $\rightarrow$ gauge symmetry of the action $\rightarrow$ still D-1 dof!

- Poincaré invariance:

  Let:

$$X'^\mu = \Lambda^\mu{}_\nu X^\nu + c^\mu$$

  Then:

$$S' = S, \qquad \text{as} \qquad \Lambda^\mu{}_\rho\, \eta_{\mu\nu}\, \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$$

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Action of the Relativistic String?

particle $\Leftrightarrow$ worldline $\Leftrightarrow$ $S = -m \int \underbrace{d\tau \sqrt{-\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\eta_{\mu\nu}}}_{\text{line element } ds}$

closed string $\Leftrightarrow$? $\Leftrightarrow$?

Boundary condition of closed strings living in D dimensions:

$$X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+2\pi) \quad \text{for } \mu = 0, 1, ..., D-1;\ \sigma \in [0; 2\pi)$$

Shorthand notation: $\sigma^\alpha = (\tau, \sigma)$ for $\alpha \in \{0; 1\}$

image of a string

Dynamics of a Relativistic String – The Nambu-Goto Action

Action of the Relativistic String?

particle $\Leftrightarrow$ worldline $\Leftrightarrow$ $S = -m \int d\tau \sqrt{-\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\eta_{\mu\nu}}$

closed string $\Leftrightarrow$ worldsheet $\Leftrightarrow$?

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closed string $\Leftrightarrow$ worldsheet $\Leftrightarrow$ Nambu-Goto/Dirac Action

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Dynamics of a Relativistic String – The Nambu-Goto Action

How to describe the surface area to construct the action?

$$S \propto \int_{\partial V} d^2x = \int_{\partial V} d^2\sigma \, |\det J|$$

For this, first remember that any metric is given by:

$$g_{\alpha\beta} = g(e_\alpha, e_\beta)$$

So the metric on the worldsheet (the pullback metric) can be written as:

$$\gamma_{\alpha\beta} = \underbrace{\frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} \eta_{\mu\nu}}_{(J^T \eta J)_{\alpha\beta}}$$

Thus,

$$\det \gamma = \det \eta \, \det{}^2 J = -\det{}^2 J$$
$$|\det J| = \sqrt{-\det \gamma} = \sqrt{-\gamma}$$

Dynamics of a Relativistic String – The Nambu-Goto Action

So write the action as:

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{-\gamma}$$

This action in invariant under,

- Poincaré transformations
- Reparametrization

and the eqations of motion (EoM) are given by:

$$\partial_\alpha \left(\sqrt{-\gamma}\, \gamma^{\alpha\beta} \partial_\beta X^\mu \right) = 0$$

$\Rightarrow$ rather hard to quantize in this form! Alternatives?

Dynamics of a Relativistic String – The Polyakov Action

Way out: **The Polyakov Action**:

$$S_P = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-g}\, g^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu\, \eta_{\mu\nu}$$

The EoMs:

- for X^μ:
  - same as for the Nambu-Goto action!

$$\partial_\alpha \left(\sqrt{-g}\, g^{\alpha\beta} \partial_\beta X^\mu \right) = 0$$

- for $g_{\alpha\beta}$:

$$g_{\alpha\beta} = 2 \frac{\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}}{g^{\rho\sigma} \partial_\rho X^\mu \partial_\sigma X^\nu \eta_{\mu\nu}} \equiv 2 \frac{\partial_\alpha X \partial_\beta X}{g^{\rho\sigma} \partial_\rho X \partial_\sigma X}$$

Gains of the Polyakov action?

Dynamics of a Relativistic String – The Polyakov Action

Symmetries of the Polyakov action

$$S_P = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-g}\,g^{\alpha\beta}\,\partial_\alpha X^\mu\partial_\beta X^\nu\,\eta_{\mu\nu}$$

- Poincaré invariance
- Reparametrization invariance
- Invariance under:

$$g'_{\alpha\beta} = \Omega^2(\sigma)g_{\alpha\beta}$$

$\Rightarrow$ **Weyl invariance**

However, using the conformal gauge and writing g as

$$g_{\alpha\beta} = e^{2\phi}\eta_{\alpha\beta}$$

can be undone by a Weyl transformation ($\phi = 0$). Thus,

$$g_{\alpha\beta} = \eta_{\alpha\beta}$$

Plot of the transformation in action

Equation of Motion of the Polyakov Action

With $g_{\alpha\beta} = \eta_{\alpha\beta}$ the EoMs

$$\begin{cases} \partial_\alpha \left(\sqrt{-g}\, g^{\alpha\beta} \partial_\beta X^\mu \right) = 0 \\ g_{\alpha\beta} = 2 \frac{\partial_\alpha X \partial_\beta X}{g^{\rho\sigma} \partial_\rho X \partial_\sigma X} \end{cases}$$

are reduced to:

$$\begin{cases} \partial_\alpha \partial^\alpha X^\mu & = 0 \\ T_{\alpha\beta} & = \partial_\alpha X \partial_\beta X - \frac{1}{2} \eta_{\alpha\beta} \eta^{\mu\nu} \partial_\mu X \partial_\nu X = 0 \end{cases}$$

With the second constraints explicitly as:

$$\begin{cases} T_{01} & = \dot{X} \cdot X' = 0 \\ T_{00} & = T_{11} = \frac{1}{2} \left(\dot{X}^2 + X'^2 \right) = 0 \end{cases}$$

Solution to the EoMs

To find a solution, define the lightcone coordinates as

$$\sigma^{\pm} = \tau \pm \sigma$$

Then, the EoMs are given by

$$\begin{cases} \partial_+ \partial_- X^\mu & = 0 \\ (\partial_+ X)^2 & = 0 \\ (\partial_- X)^2 & = 0 \end{cases} \quad \text{, with } X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi)$$

$$\Rightarrow X^\mu(\tau, \sigma) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-)$$

$$X_L^\mu(\sigma^+) = \frac{1}{2} x^\mu + \frac{1}{2} \alpha' p^\mu \sigma^+ + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}^\mu e^{-in\sigma^+}$$

$$X_R^\mu(\sigma^-) = \frac{1}{2} x^\mu + \frac{1}{2} \alpha' p^\mu \sigma^- + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha^\mu e^{-in\sigma^-}$$

Solution to the EoMs

The constraints $(\partial_\pm X)^2 = 0$:

- First:

$$\partial_- X^\mu = \frac{\alpha'}{2} p^\mu + \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \alpha_n^\mu e^{-in\sigma^-} \equiv \sum_n \alpha_n^\mu e^{-in\sigma^-}$$

$$\text{with} \quad \alpha_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu$$

- Then:

$$(\partial_- X)^2 = \frac{\alpha'}{2} \sum_{n,p} \alpha_m \alpha_n e^{-i(n+m)\sigma^-}$$

$$= \alpha' \sum_n \underbrace{\frac{1}{2} \sum_m \alpha_m \alpha_{n-m} e^{-in\sigma^-}}_{L_n} \stackrel{!}{=} 0$$

Solution to the EoMs

$\Rightarrow \infty$ number of constraints on L_n:

$$L_n = \tilde{L}_n = 0$$

However, L_0 contains the mass $M = -p^2$ as:

$$L_0 = \frac{\alpha'}{2}p^2 + \frac{1}{2}\sum_{m>0} \alpha_m \alpha_{-m} e^{-in\sigma^-}$$

$$\Rightarrow M^2 = \frac{4}{\alpha'}\sum_{m>0} \alpha_m \alpha_{-m} = \frac{4}{\alpha'}\sum_{m>0} \tilde{\alpha}_m \tilde{\alpha}_{-m}$$

$\Rightarrow$ **Level matching**

A brief summary before moving to the covariant quantization

Quantization of X^μ in the Lightcone Gauge

References

[1] David Tong. “Lectures on String Theory”. In: *arXiv:0908.0333 [hep-th]* (Feb. 2012). arXiv: 0908.0333. URL: `http://arxiv.org/abs/0908.0333` (visited on 07/13/2020).