Lie-Hamilton systems and their role in the current Covid pandemic

Cristina Sardón
UPM-ICMAT, Spain

Online Friday Fish Seminar
August 2020

Contents

- Why are we interested in Lie systems?
- How are Lie systems related to the current pandemic?
- Brief recall of geometric fundamentals for Lie systems
- What is a Lie system? Their solutions as nonlinear superposition rules
- Which Lie systems are Hamiltonian?
- Remarkable geometric properties of Lie–Hamilton systems. Retrieving a solution as a (nonlinear) superposition rule through the coalgebra method.
- Can we write down a Hamiltonian formulation for a pandemic model using the Lie–Hamilton theory?

Motivation

Properties and applications

Why Lie systems?

... because **Lie systems** are first-order ODEs that admit general solutions in form of (generally nonlinear) **superposition rules** or functions $\Phi : N^m \times N \to N$ of the form $x = \Phi(x_{(1)}, \dots, x_{(m)}; k)$ allowing us to write the general solution as

$$x(t) = \Phi(x_{(1)}(t), \dots, x_{(m)}(t); k),$$

where $x_{(1)}(t), \dots, x_{(m)}(t)$ is a generic family of particular solutions and $k \in N$.

They enjoy a plethora of geometric properties:

- Finite dimensional Lie algebras
- Lie group actions
- The Poisson coalgebras
- They are compatible with multiple geometric structures (Dirac, Jacobi, contact...)
- Superposition rules can be interpreted as zero-curvature connections

Why Lie systems? II

Lie systems appear in the study of:

- Relevant physical models
- Mathematics
- Control theory
- Quantum Mechanics
- Biology and ecology. Predator-prey systems, viral dynamics...

Are there any epidemic models that are Lie systems?

Lie systems and superposition rules have been extrapolated to *higher-order systems of ODEs*.

- Higher-order Riccati equation,
- The second- and third-order Kummer–Schwarz,
- Milne–Pinney and dissipative Milne–Pinney equations.

Lie systems have also been extended to the realm of PDEs, the so-called PDE-Lie systems.

Lie systems

Geometric Fundamentals

Definitions

Lie algebra is a pair $(V, [\cdot,\cdot])$, where V is a linear space equipped with Lie bracket $[\cdot,\cdot] : V \times V \to V$. We define by $Lie(\mathfrak{B}, V, [\cdot,\cdot])$, $\mathfrak{B} \subset V$, the smallest Lie subalgebra of $(V, [\cdot,\cdot])$ containing $\mathfrak{B}$, namely the linear space generated by $\mathfrak{B}$ and

$$[\mathfrak{B},\mathfrak{B}], [\mathfrak{B},[\mathfrak{B},\mathfrak{B}]], [\mathfrak{B},[\mathfrak{B},[\mathfrak{B},\mathfrak{B}]]], [[\mathfrak{B},\mathfrak{B}],[\mathfrak{B},\mathfrak{B}]], \ldots.$$

A **t-dependent vector field** is the map $X : \mathbb{R} \times N \to TN$ such that the following diagram is commutative

$$\begin{tikzcd} & TN \arrow[d, "\pi"] \\ \mathbb{R} \times N \arrow[ur, bend left, "X"] \arrow[r, "\pi_2"] & N \end{tikzcd} \qquad X(t,x) \in \pi^{-1}(x) = T_xN$$

Thus, the maps $X_t : x \in N \to X(t,x) \in TN$ are $\{X_t\}_{t\in\mathbb{R}}$.

Definitions

We call **integral curve of X**, the integral curve of its **suspension**.

For every $\gamma : t \in \mathbb{R} \mapsto (t, x(t)) \in \mathbb{R} \times N$ we have an **associated system**

$$\frac{d(\pi_2 \circ \gamma)}{dt}(t) = (X \circ \gamma)(t)$$

So, X determines a single first-order ODE. Conversely, given such system, there exists a unique X whose integral curves $(t, x(t))$ are its particular solutions.

Minimal Lie algebra of a t-dependent vect. field X is the smallest real Lie algebra of vector fields V^X containing $\{X_t\}_{t\in\mathbb{R}}$, namely,

$$V^X = Lie(\{X_t\}_{t\in\mathbb{R}}, [\cdot,\cdot]).$$

Lie–Scheffers theorem

Theorem

(The Lie–Scheffers Theorem) *A first-order system*

$$\frac{dx}{dt} = F(t, x), \qquad x \in N,$$

admits a superposition rule if and only if X can be written as

$$X_t = \sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha$$

for a certain family $b_1(t), \ldots, b_r(t)$ of t-dependent functions and a family $X_1, \ldots, X_r$ of vector fields on N spanning an r-dimensional real Lie algebra of vector fields V^X. A **Vessiot–Guldberg Lie algebra** *(VG henceforth).*

Theorem

(The abbreviated Lie–Scheffers Theorem) *A system X admits a superposition rule if and only if V^X is finite-dimensional.*

Examples of Lie systems

The Riccati equation on the real line

$$\frac{dx}{dt} = a_0(t) + a_1(t)x + a_2(t)x^2,$$

where $a_0(t), a_1(t), a_2(t)$ are arbitrary t-dependent functions, admits the *superposition rule* $\Phi : (x_{(1)}, x_{(2)}, x_{(3)}; k) \in \mathbb{R}^3 \times \mathbb{R} \mapsto x \in \mathbb{R}$ given by

$$x(t) = \frac{x_{(1)}(t)(x_{(3)}(t) - x_{(2)}(t)) + kx_{(2)}(t)(x_{(1)}(t) - x_{(3)}(t))}{(x_{(3)}(t) - x_{(2)}(t)) + k(x_{(1)}(t) - x_{(3)}(t))}.$$

The first-order Riccati equation $X = a_0(t)X_1 + a_1(t)X_2 + a_2(t)X_3$, where

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = x\frac{\partial}{\partial x}, \qquad X_3 = x^2\frac{\partial}{\partial x}$$

span a VG isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Then, $\Phi : (x_{(1)}, x_{(2)}, x_{(3)}; k) \in \mathbb{R}^3 \times \mathbb{R}$ such that it provides us with a solution $x \in \mathbb{R}$

Lie systems on Lie groups

Every Lie system X associated with a VG gives rise by integrating V^X to a (generally local) Lie group action $\varphi : G \times N \to N$ whose fundamental vector fields are the elements of V and such that $T_e G \simeq V$ with e being the neutral element of G.

$$x(t) = \varphi(g_1(t), x_0), \qquad x_0 \in \mathbb{R}^n,$$

with $g_1(t)$ being a particular solution of

$$\frac{dg}{dt} = -\sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha^R(g),$$

where $X_1^R, \ldots, X_r^R$ is a certain basis of right-invariant vector fields on G such that $X_\alpha^R(e) = \mathrm{a}_\alpha \in T_e G$, with $\alpha = 1, \ldots, r$, and each a_α is the element of $T_e G$ associated with the fundamental vector field X_α.

Since $X_1^R, \ldots, X_r^R$ span a finite-dimensional real Lie algebra, the Lie–Scheffers Theorem guarantees there exists a superposition rule and becomes a Lie system.

A contemporary application

Lie systems in viral infection dynamics

A primitive viral infection

Finally, let us consider a simple viral infection model given by

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = (\alpha(t) - g(y))x, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = \beta(t)xy - \gamma(t)y, \end{cases}$$

where $g(y)$ is a real positive function taking into account the power of the infection. Note that if a particular solution satisfies $x(t_0) = 0$ or $y(t_0) = 0$ for a $t_0 \in \mathbb{R}$, then $x(t) = 0$ or $y(t) = 0$, respectively, for all $t \in \mathbb{R}$. As these cases are trivial, we restrict ourselves to studying particular solutions within $\mathbb{R}^2_{x,y\neq 0} = \{(x, y) \in \mathbb{R}^2 \mid x \neq 0, y \neq 0\}$.
The simplest possibility consists in setting $g(y) = \delta$, where δ is a constant. Then, (21) describes the integral curves of the t-dependent vector field $X_t = (\alpha(t) - \delta)X_1 + \gamma(t)X_2 + \beta(t)X_3$, on $\mathbb{R}^2_{x,y\neq 0}$, where the vector fields

$$X_1 = x\frac{\partial}{\partial x}, \qquad X_2 = -y\frac{\partial}{\partial y}, \qquad X_3 = xy\frac{\partial}{\partial y},$$

close a finite-dimensional Lie algebra. So, X is a Lie system related to a Vessiot–Guldberg Lie algebra $V \simeq \mathbb{R} \ltimes \mathbb{R}^2$ where $\langle X_1\rangle \simeq \mathbb{R}$ and $\langle X_2, X_3\rangle \simeq \mathbb{R}^2$.

The SIS model with fluctuations

This is the susceptible-infectious-susceptible (SIS) epidemic model. There are only two states: infected or susceptible (no immunization).
The instantaneous density of infected individuals $\rho(t)$ taking values in $[0, 1]$ and the fluctuations have been neglected. The density of infected individuals decreases with rate $\gamma\rho$, where γ is the recovery rate, and the rate of growth of new infections is proportional to $\alpha\rho(1-\rho)$, where the intensity of contagion is given by the transmission rate α.

$$\frac{d\rho}{dt} = \alpha\rho(1-\rho) - \gamma\rho. \tag{1}$$

One can redefine the timescale as $\tau \equiv \alpha t$ and $\rho_0 \equiv 1 - \gamma/\alpha$, so we can rewrite (1) as

$$\frac{d\rho}{d\tau} = \rho(\rho_0 - \rho) \tag{2}$$

The equilibrium density is reached if $\rho = 0$ or $\rho = \rho_0$. This model involves random mixing and large population assumptions. To add fluctuations, we need to introduce stochastic mechanics, since temporal fluctuations can drastically alter the prevalence of pathogens and spatial heterogeneity.

"Stochastic" considerations

In *Hamiltonian dynamics of the SIS epidemic model with stochastic fluctuations*, (G.M. Nakamura, A.S. Martinez), the spreading of the disease is assumed as a Markov chain in discrete time δt in which at most one single recovery or transmission occurs in the duration of this infinitesimal interval.

$$\frac{dP_\mu}{dt} = -\sum_{\nu=0}^{2^N-1} H_{\mu\nu} P_\nu \tag{3}$$

The instantaneous probability of finding the system in the μ-configuration is expressed by $P_\mu(t)$ and the configuration label follows the rule

$$\mu = n_0 \cdot 2^0 + n_1 \cdot 2^1 + \cdots + n_{N-1} \cdot 2^{N-1},$$

where $n_k = 1$ if the k-th agent is infected and $n_k = 0$ if it is not. The matrix elements $H_{\mu\nu}$ express the transition rates from configuration ν to configuration μ.
From these considerations, to evaluate the time evolution of relevant statistical moments of $\rho(t)$, we take the average density of infected agents

$$\langle \rho(t) \rangle = \frac{1}{N} \sum_{\mu=0}^{2^N-1} \sum_{k=0}^{N-1} \langle \mu | n_k | \mu \rangle P_\mu(t) \tag{4}$$

The first two equations for instantaneous mean dentisity of infected people $\langle \rho \rangle$ and variance $\sigma^2 = \langle \rho^2 \rangle - \langle \rho \rangle^2$ are

$$\begin{aligned} \frac{d\langle \rho \rangle}{d\tau} &= \langle \rho \rangle (\rho_0 - \langle \rho \rangle) - \sigma^2(\tau), \\ \frac{d\sigma^2}{d\tau} &= 2\sigma^2 (\rho_0 - \langle \rho \rangle) - \Delta_3(\tau) - \frac{1}{N}\langle \rho(1-\rho) \rangle + \frac{\gamma}{N\alpha}\langle \rho \rangle \end{aligned} \tag{5}$$

where $\Delta_3(\tau) = \langle \rho^3(\tau) \rangle - \langle \rho(\tau) \rangle^3$.

Hamiltonization of SIS w/ fluctuations

We can "Hamiltonize" the previous eq. when $\sigma(\tau)$ becomes irrelevant compared to $\langle\rho\rangle$. We only require mean and variance, neglecting higher statistical moments and($N \gg 1$).

$$
\begin{aligned}
\frac{d\ln\langle\rho\rangle}{d\tau} &= \rho_0 - \langle\rho\rangle - \frac{\sigma^2}{\langle\rho\rangle}, \\
\frac{1}{2}\frac{d\ln\sigma^2}{d\tau} &= \rho_0 - 2\langle\rho\rangle.
\end{aligned} \tag{6}
$$

We can define dynamical variables that describe a Hamiltonian system when $q = \langle\rho\rangle$ and $p = 1/\sigma$ while preserving the independent variable τ. So,

$$
\begin{aligned}
\frac{dq}{d\tau} &= q\rho_0 - q^2 - \frac{1}{p^2}, \\
\frac{dp}{d\tau} &= -p\rho_0 + 2pq.
\end{aligned} \tag{7}
$$

System (8) is a Hamiltonian system coming from the Hamiltonian function $H = qp\,(\rho_0 - q) + \frac{1}{p}$.

Solutions SISf

We employ the abbreviation SISf for system (8) to differentiate it from the classical SIS model in (2). The letter "f" accounts for "fluctuations".

$$\begin{aligned} \frac{dq}{dt} &= q\rho_0 - q^2 - \frac{1}{p^2}, \\ \frac{dp}{dt} &= -p\rho_0 + 2pq. \end{aligned} \tag{8}$$

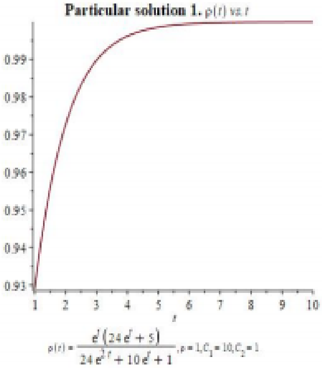
We have computed the general solution to this system as

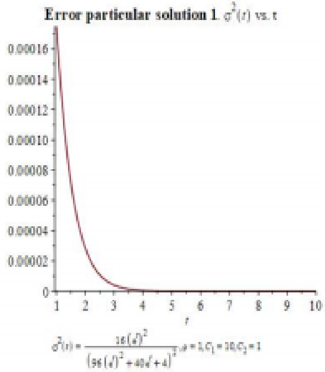
$$q(t) = \frac{\rho e^{\rho t}(C_1\rho^2 - 4)e^{\rho t} + 2C_1C_2\rho^2}{(C_1^2\rho^2 - 4)e^{2\rho t} + 4C_2\rho^2(C_1e^{\rho t} + C_2)}, \tag{9}$$

$$p(t) = C_1 + \frac{C_1^2\rho^2 - 4}{4\rho^2 - C_2} + C_2e^{-\rho t} \tag{10}$$

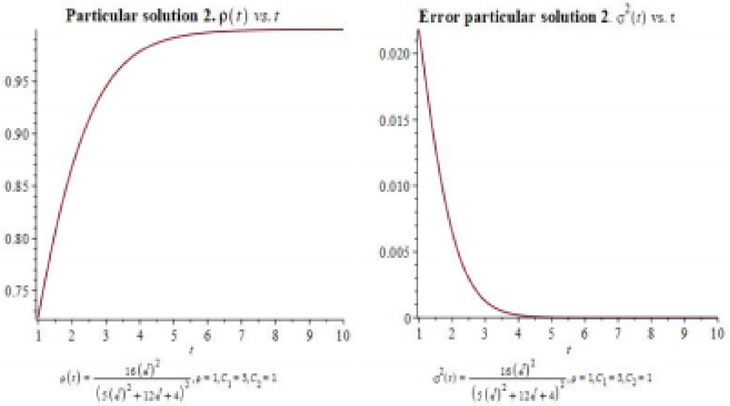
We present three different choices of three particular solutions and their corresponding graphs according to the change of variables $q =< \rho >$ and $p = 1/\sigma$.

Particular solution 1

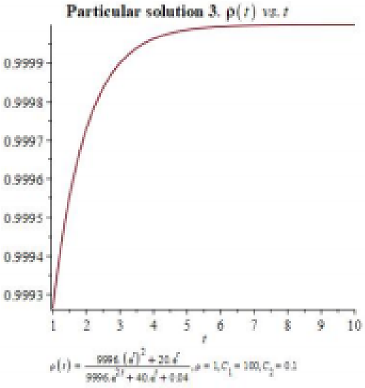


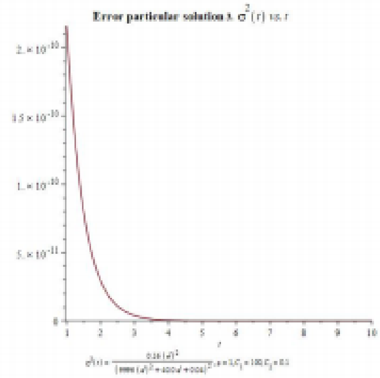


Particular solution 2



Particular solution 3





SISf Lie system

The previous equations can be generalised to a model represented by a time-dependent vector field

$$X_t = \rho_0(t)X_1 + X_2, \qquad X_1 = q\frac{\partial}{\partial q} - p\frac{\partial}{\partial p}, \quad X_2 = \left(-q^2 - \frac{1}{p^2}\right)\frac{\partial}{\partial q} + 2qp\frac{\partial}{\partial p}.$$

The generalisation comes from the fact that $\rho_0(t)$ is no longer a constant, but it can evolve in time. A direct calculation shows that $[X_1, X_2] = X_2$, which means that the Lie algebra spanned by X_1, X_2 is a finite dimensional Lie algebra.

This means that the pandemic SIS model admits a solution in terms of a superposition rule.

In the next slides we will derive its solution in terms of a superposition principle using the geometric properties of Lie systems. Let us start with it in the next section.

Lie systems

and superposition rules

Derivation of superposition rules

Definition

Given a t-dependent vector field X on N, its **diagonal prolongation** $\widetilde{X}$ to $N^{(m+1)}$ is the unique t-dependent vector field on $N^{(m+1)}$ such that

- Given $\mathrm{pr} : (x_{(0)}, \ldots, x_{(m)}) \in N^{(m+1)} \mapsto x_{(0)} \in N$, we have that $\mathrm{pr}_* \widetilde{X}_t = X_t$ $\forall t \in \mathbb{R}$.
- $\widetilde{X}$ is invariant under the permutations $x_{(i)} \leftrightarrow x_{(j)}$, with $i, j = 0, \ldots, m$.

In coordinates, we have that

$$X_j = \sum_{i=1}^{n} X^i(t,x)\frac{\partial}{\partial x_i} \Rightarrow \widetilde{X} = \sum_{j=0}^{m+1} X_j = \sum_{j=0}^{m+1}\sum_{i=1}^{n} X^i(t,x)\frac{\partial}{\partial x_i}.$$

Derivation of superposition rules

- Take a basis $X_1, \ldots, X_r$ of a Vessiot–Guldberg Lie algebra V associated with the Lie system.
- Choose the minimum integer m so that the diagonal prolongations to N^m of $X_1, \ldots, X_r$ are linearly independent at a generic point.
- Obtain n functionally independent first-integrals $F_1, \ldots, F_n$ common to all the diagonal prolongations, $\widetilde{X}_1, \ldots, \widetilde{X}_r$, to $N^{(m+1)}$, for instance, by the *method of characteristics*. We require such functions to hold that

$$\frac{\partial(F_1, \ldots, F_n)}{\partial\left((x_1)_{(0)}, \ldots, (x_n)_{(0)}\right)} \neq 0.$$

- Assume that these integrals take certain constant values, i.e., $F_i = k_i$ with $i = 1, \ldots, n$, and employ these equalities to express the variables $(x_1)_{(0)}, \ldots, (x_n)_{(0)}$ in terms of the variables of the other copies of N within $N^{(m+1)}$ and the constants $k_1, \ldots, k_n$. The obtained expressions constitute a superposition rule in terms of any generic family of m particular solutions and n constants.

Applications of superposition rules in:

SISf pandemic systems

Superposition rule for SISf model

The model (8) can be generalized to a model represented by a time-dependent vector field

$$X_t = \rho_0(t) X_1 + X_2 \tag{11}$$

where the constitutive vector fields are computed to be

$$X_1 = q\frac{\partial}{\partial q} - p\frac{\partial}{\partial p}, \quad X_2 = \left(-q^2 - \frac{1}{p^2}\right)\frac{\partial}{\partial q} + 2qp\frac{\partial}{\partial p}. \tag{12}$$

Let us apply the steps introduced in the previous section one by one to arrive at the general solution.

Step 1. For the vector fields in (12), a direct calculation shows that the Lie bracket

$$[X_1, X_2] = X_2 \tag{13}$$

is closed within the Lie algebra. This implies that the SISf model (8) is a Lie system. The Vessiot-Guldberg algebra spanned by X_1, X_2 is an imprimitive Lie algebra of type I_{14}

Step 2. If we copy the configuration space twice, we will have four degrees of freedom (q_1, p_1, q_2, p_2) and we will archieve precisely two first-integrals in vinicity of the Fröbenius theorem. A first-integral for X_t has to be a first-integral for X_1 and X_2 simultaneously. We define the diagonal prolongation $\widetilde{X}_1$ of the vector field X_1 and we look for a first integral F_1 such that $\widetilde{X}_1[F_1]$ vanishes identically. Notice that if F_1 is a first-integral of $\widetilde{X}_1$ then it is of $\widetilde{X}_2$ due to the commutation relation.

$$\widetilde{X}_1 = q_1\frac{\partial}{\partial q_1} + q_2\frac{\partial}{\partial q_2} - p_1\frac{\partial}{\partial p_1} - p_2\frac{\partial}{\partial p_2} \tag{14}$$

through the following characteristic system

$$\frac{dq_1}{q_1} = \frac{dq_2}{q_2} = \frac{dp_1}{-p_1} = \frac{dp_2}{-p_2}. \tag{15}$$

Fix the dependent variable q_1 and obtain a new set of dependent variables, say (K_1, K_2, K_3), which are computed to be

$$K_1 = \frac{q_1}{q_2}, \qquad K_2 = q_1 p_1, \qquad K_3 = q_1 p_2. \tag{16}$$

Step 3. This induces the following basis in the tangent space

$$\frac{\partial}{\partial K_1}=q_2\frac{\partial}{\partial q_1}-\frac{q_2p_1}{q_1}\frac{\partial}{\partial p_1}-\frac{q_2p_2}{q_1}\frac{\partial}{\partial p_1},\qquad \frac{\partial}{\partial K_2}=\frac{1}{q_1}\frac{\partial}{\partial p_1},\qquad \frac{\partial}{\partial K_3}=\frac{1}{q_1}\frac{\partial}{\partial p_2}. \tag{17}$$

provided that q_1 is not zero. Introducing the coordinate changes exhibited in (16) into the diagonal projection $\widetilde{X}_2$ of the vector field X_2, we arrive at the following expression

$$\widetilde{X}_2=\left(2K_1-\left(1+\frac{1}{K_1^2}\right)\right)\frac{\partial}{\partial K_1}+\left(\left(\frac{1}{K_2^2}+\frac{1}{K_3^2}\right)K_2^2-\left(1+\frac{1}{K_1^2}\right)K_2\right)\frac{\partial}{\partial K_2}$$
$$+\left(2\frac{K_3}{K_2}-\left(1+\frac{1}{K_1^2}\right)K_3\right)\frac{\partial}{\partial K_3}.$$

To integrate the system once more, we use the method of characteristics again and obtain

$$\frac{d\ln|K_1|}{1-\frac{1}{K_1^2}}=\frac{d\ln|K_2|}{\frac{1}{K_2}+\frac{K_2}{K_3}-1-\frac{1}{K_1^2}}=\frac{d\ln|K_3|}{\frac{2}{K_2}-\left(1+\frac{1}{K_1^2}\right)}. \tag{18}$$

Superposition principle SISf pandemic model

Exact solution. We obtain two first integrals by integrating in pairs (K_1, K_2) and (K_1, K_3), where we have fixed K_1. After some cumbersome calculations we obtain

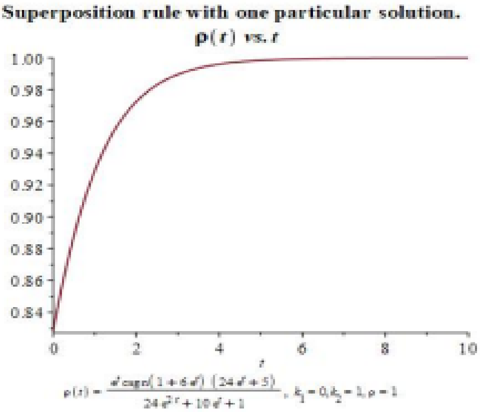
$$K_2 = \frac{K_1\left(4k_2^2K_1^2 + 4k_1k_2K_1 + k_1^2 - 4\right)}{2(K_1+1)(K_1-1)k_2(2k_2K_1+k_1)}, \qquad K_3 = \frac{K_1\left(k_2K_1^2 + k_1K_1 + \frac{k_1^2-4}{4k_2}\right)}{(K_1+1)(K_1-1)}. \tag{19}$$

By substituting back the coordinate transformation (16) into the solution (19) (please notice the difference between capitalized constants (K_1, K_2, K_3) and lower case constants (k_1, k_2), we arrive at the following implicit equations

$$\begin{aligned} q_1 &= -\frac{q_2\left(k_1k_2 \pm \sqrt{4k_2^2p_2^2q_2^2 + k_1^2k_2p_2q_2 - 4k_2^3p_2q_2 - 4k_2p_2q_2 + 4k_2^2}\right)}{2k_2(-p_2q_2+k_2)} \\ p_1 &= \frac{4q_1^2k_2^2 + 4q_1q_2k_1k_2 + q_2^2k_1^2 - 4q_2^2}{2k_2(2q_1^3k_2 + q_1^2k_1q_2 - 2q_1k_2q_2^2 - k_1q_2^3)}. \end{aligned} \tag{20}$$

Let us notice that the equations (20) depend on a particular solution (q_2, p_2) and two constants of integration (k_1, k_2) which are related to initial conditions.

Let us show now the graphs and values of the initial conditions for which the solution reminds us of sigmoid behavior, which is the expected growth of $\rho(t)$. As particular solution for (q_2, p_2), we have made use of particular solution 2 given in some previous slide through its corresponding values of q, p through the change of variables $q = < \rho >$ and $p = 1/\sigma$.



The error graph since it gives a constant zero graph because $k_1 \to 0$

Lie–Hamilton systems

Superposition rules with the coalgebra method

Lie–Hamilton systems

Many instances of relevant Lie systems possess VG of Hamiltonian vector fields with respect to a Poisson structure. Such Lie systems are hereafter called **Lie–Hamilton systems**.

Every Lie–Hamilton system can be interpreted as a curve in finite-dimensional Lie algebra of functions (with respect to a certain Poisson structure).

Additionally, Lie–Hamilton systems appear in the analysis of relevant physical and mathematical problems,

- second-order Kummer–Schwarz equations
- Riccati equations
- Some viral infections
- Lotka-Volterra, Predator-Prey models and many others...

A geometric reminder

A **Poisson algebra** is $(A, \star, \{\cdot,\cdot\})$, A linear space, $\star : A \times A \to A$, $(A, \star)$ is an associative $\mathbb{R}$-algebra and $\{\cdot,\cdot\}$ a Lie bracket on A, the so-called *Poisson bracket* with $(A, \{\cdot,\cdot\})$ a Lie algebra and also the *Leibnitz rule*.

$$\{b \star c, a\} = b \star \{c, a\} + \{b, a\} \star c, \forall a, b, c \in A.$$

Given a manifold N, the pair $(N, \{\cdot,\cdot\})$ is a **Poisson manifold** such that $(C^\infty(N), \cdot, \{\cdot,\cdot\})$ is a Poisson algebra.

As $\{\cdot, f\}$ with $f \in C^\infty(N)$ is a derivation on $(C^\infty(N), \cdot)$, there exists an unique vector field X_f on N **the hamiltonian vector field** assoc. with f such that

$$X_f g = \{g, f\}, \forall g \in C^\infty(N).$$

The Jacobi identity for the Poisson structure entails $X_{\{f,g\}} = -[X_f, X_g]$. This is a *Lie algebra antihomorphism* between $(C^\infty(N), \{\cdot,\cdot\})$ and $(\Gamma(TN), [\cdot,\cdot])$.

As every Poisson structure is a derivation in each entry, it determines an unique $\Lambda \in \Gamma(\Lambda^2 TN)$

$$\{f, g\} = \Lambda(df, dg) \quad \forall f, g \in C^\infty(N)$$

We call Λ the **Poisson bivector** st. $[\cdot,\cdot]_{SN} = 0$ being the **Schouten-Nijenhuis Lie bracket.**

An example of Lie–Hamilton system

Consider the system of differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a_0(t) + a_1(t)x + a_2(t)(x^2 - y^2), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = a_1(t)y + a_2(t)2xy,$$

with $a_0(t), a_1(t), a_2(t)$ being arbitrary t-dependent real functions. By writing $z = x + \mathrm{i}y$, we find that it is equivalent to

$$\frac{\mathrm{d}z}{\mathrm{d}t} = a_0(t) + a_1(t)z + a_2(t)z^2, \qquad z \in \mathbb{C},$$

which is a particular type of complex Riccati equations. Let us show that it is a Lie system on $\mathbb{R}^2_{y\neq 0}$. This is related to the t-dependent vector field

$$X_t = a_0(t)X_1 + a_1(t)X_2 + a_2(t)X_3,$$

where

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad X_3 = (x^2 - y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}$$

span a Vessiot–Guldberg real Lie algebra $V \simeq \mathfrak{sl}(2)$ with commutation relations

$$[X_1, X_2] = X_1, \qquad [X_1, X_3] = 2X_2, \qquad [X_2, X_3] = X_3.$$

Hence, $\{X_t\}_{t\in\mathbb{R}} \subset V^X \subset V$ is finite-dimensional and X is a Lie system.

We search for a symplectic form, $\omega = f(x,y)\mathrm{d}x \wedge dy$, turning $V = \langle X_1, X_2, X_3 \rangle$ into a Lie algebra of Hamiltonian vector fields with respect to it.
We impose $\mathcal{L}_{X_i}\omega = 0$ $(i = 1,2,3)$. In coordinates, these conditions read

$$\frac{\partial f}{\partial x} = 0, \qquad x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + 2f = 0, \qquad (x^2 - y^2)\frac{\partial f}{\partial x} + 2xy\frac{\partial f}{\partial y} + 4xf = 0.$$

From the first equation, we obtain $f = f(y)$. Using the second, $f = y^{-2}$ is a particular solution of both equations.
This leads to a closed and non-degenerate two-form on $\mathbb{R}^2_{y\neq 0}$, namely

$$\omega = \frac{\mathrm{d}x \wedge \mathrm{d}y}{y^2}.$$

Using the relation $\iota_X\omega = \mathrm{d}h$ between a Hamiltonian vector field X and one of its corresponding Hamiltonian functions h, we observe that X_1, X_2 and X_3 are Hamiltonian vector fields with Hamiltonian functions

$$h_1 = -\frac{1}{y}, \qquad h_2 = -\frac{x}{y}, \qquad h_3 = -\frac{x^2 + y^2}{y},$$

respectively. Obviously, the remaining vector fields of V become also Hamiltonian. Thus, planar Riccati equations admit a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields relative to $\omega = \frac{\mathrm{d}x \wedge \mathrm{d}y}{y^2}$.

The Poisson structure

Definition

A **Lie–Hamiltonian structure** is a triple (N, Λ, h), where (N, Λ) stands for a Poisson manifold and h represents a t-parametrized family of functions $h_t : N \to \mathbb{R}$ such that $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}}, \{\cdot,\cdot\}_\Lambda)$ is a finite-dimensional real Lie algebra.

Definition

A t-dependent vector field X is said to admit a Lie–Hamiltonian structure (N, Λ, h) if X_t is the Hamiltonian vector field corresponding to h_t for each $t \in \mathbb{R}$. $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}}, \{\cdot,\cdot\}_\Lambda)$ is called a **Lie–Hamilton algebra** for X.

Example of Lie–Hamiltonian

Reconsider the Planar Riccati equations,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a_0(t) + a_1(t)x + a_2(t)(x^2 - y^2), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = a_1(t)y + a_2(t)2xy,$$

whose vector fields are Hamiltonian w.r.t the symplectic form $\omega = \frac{\mathrm{d}x \wedge \mathrm{d}y}{y^2}$ and form a basis for a VG isomorphic to $V \simeq \mathfrak{sl}(2, \mathbb{R})$.

If $\{\cdot, \cdot\}_\omega : C^\infty(\mathbb{R}^2_{y\neq 0}) \times C^\infty(\mathbb{R}^2_{y\neq 0}) \to C^\infty(\mathbb{R}^2_{y\neq 0})$ stands for the Poisson bracket induced by ω, then

$$\{h_1, h_2\}_\omega = -h_1, \qquad \{h_1, h_3\}_\omega = -2h_2, \qquad \{h_2, h_3\}_\omega = -h_3.$$

Hence, the planar Riccati equation X possesses a Lie–Hamiltonian structure of the form $\left(\mathbb{R}^2_{y\neq 0}, \omega, h = a_0(t)h_1 + a_1(t)h_2 + a_2(t)h_3\right)$.

We have that $(\mathcal{H}_\Lambda, \{\cdot, \cdot\}_\omega) \equiv (\langle h_1, h_2, h_3 \rangle, \{\cdot, \cdot\}_\omega)$ is a Lie–Hamilton algebra for X isomorphic to $\mathfrak{sl}(2)$.

Lie–Hamiltonian viral infection

Recall the simple viral infection model

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (\alpha(t) - g(y))x, \tag{21}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \beta(t)xy - \gamma(t)y, \tag{22}$$

restricting ourselves to studying particular solutions within $\mathbb{R}^2_{x,y\neq 0} = \{(x,y) \in \mathbb{R}^2 \mid x \neq 0, y \neq 0\}$ and $g(y) = \delta$, and with δ a constant. The vector fields

$$X_1 = x\frac{\partial}{\partial x}, \qquad X_2 = -y\frac{\partial}{\partial y}, \qquad X_3 = xy\frac{\partial}{\partial y},$$

are Hamiltonian with respect to $\omega = \frac{dx \wedge dy}{xy}$.
Then, the vector fields X_1, X_2 and X_3 have Hamiltonian functions $h_1 = \ln y$, $h_2 = \ln x$, $h_3 = -x$, which along $h_0 = 1$ define a Lie–Hamilton algebra $\overline{(\mathbb{R} \ltimes \mathbb{R}^2)}$ with the associated t-dependent Hamiltonian

$$h_t = (\alpha(t) - \delta)h_1 + \gamma(t)h_2 + \beta(t)h_3$$

giving rise to a Lie–Hamiltonian structure $(\mathbb{R}^2_{x,y\neq 0}, \omega, h)$ for X.

The Lie–Hamiltonian SISf model

The Lie–Hamiltonian SISf model

Consider now the canonical symplectic form $\omega = dq \wedge dp$. It is easy to check that the vector fields X_1 and X_2

$$X_t = \rho_0(t)X_1 + X_2, \qquad X_1 = q\frac{\partial}{\partial q} - p\frac{\partial}{\partial p}, \quad X_2 = \left(-q^2 - \frac{1}{p^2}\right)\frac{\partial}{\partial q} + 2qp\frac{\partial}{\partial p}.$$

are Hamiltonian with respect to the canonical symplectic form and Hamiltonian functions

$$h_1 = -qp, \qquad h_2 = -q^2 p + \frac{1}{p}, \tag{23}$$

respectively. It is easy to see that the Poisson bracket of these two functions reads $\{h_1, h_2\} = h_2$. It means that the Hamiltonian functions form a finite dimensional Lie algebra and it is isomorphic to the one defined by vector fields X_1, X_2. The time-dependent vector field of the Lie system X_t retrieves the Hamiltonian function as $X_t = -\widehat{\Lambda}(dh_t)$. Therefore, the Hamiltonian of the system is

$$h = \rho_0(t)h_1 + h_2 = -q^2 p + \frac{1}{p} - \rho_0(t)qp \tag{24}$$

and it is exactly the Hamiltonian proposed in *Hamiltonian dynamics of the SIS epidemic model with stochastic fluctuations*, (G.M. Nakamura, A.S. Martinez),

A different derivation of superposition rules

Using the coalgebra method

Geometry for the coalgebra method

Two examples of Poisson algebras are the **symmetric Lie algebra** $S_{\mathfrak{g}}$ and the **universal Lie algebra** $U_{\mathfrak{g}}$.

If we consider a finite-dim. real Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$, since $\mathfrak{g} \simeq (\mathfrak{g}^*)^*$, we can consider $\mathfrak{g}$ as linear functions on $\mathfrak{g}^*$.

We define $S_{\mathfrak{g}}$ as the quotient of set of polynomials $T_{\mathfrak{g}}$ on $\mathfrak{g}$ by the ideal generated by $u \otimes w - w \otimes u$ with $u, w \in \mathfrak{g}$, we refer to it as $(S_{\mathfrak{g}}, \cdot, \{\cdot, \cdot\}_{S_{\mathfrak{g}}})$. And $U_{\mathfrak{g}}$ is obtained from the quotient $T_{\mathfrak{g}}/\mathcal{R}$ of the tensor algebra $(T_{\mathfrak{g}}, \otimes)$ of $\mathfrak{g}$ by the bilateral ideal $\mathcal{R}$ spanned by $v \otimes \omega - \omega \otimes v - [v, \omega]$ with $v, \omega \in \mathfrak{g}$.

Geometry for the coalgebra method

If we have two Poisson algebras $(A, \star_A, \{\cdot,\cdot\}_A)$ and $(B, \star_B, \{\cdot,\cdot\}_B)$, the space $A \otimes B$ becomes a Poisson algebra $(A \otimes B, \star_{A\otimes B}, \{\cdot,\cdot\}_{A\otimes B})$ by defining

$$(a \otimes b) \star_{A\otimes B} (c \otimes d) = (a \star_A c) \otimes (b \star_B d) \qquad a, c \in A \quad b, d \in B$$
$$\{a \otimes b, c \otimes d\} = \{a, c\}_A \otimes b \star_B d + a \star_A c \otimes \{b, d\}_B$$

We say that $(A, \star_A, \{\cdot,\cdot\}_A)$ is a **Poisson coalgebra** if the so-called **coproduct** $\Delta : (A, \star_A, \{\cdot,\cdot\}_A) \to (A \otimes A, \star_{A\otimes A}, \{\cdot,\cdot\}_{A\otimes A})$ is an coassociative Poisson algebra homomorphism,

$$(\Delta \circ Id) \circ \Delta = (Id \circ \Delta) \circ \Delta.$$

Similarly, we can have Poisson structures on the m-th tensor product $A^{(m)} \equiv A \otimes \cdots \otimes A$. The m-coproduct map $\Delta^m : A \to A \otimes \cdots \otimes A$ can be defined as

$$\Delta^m(v) = v \otimes Id \otimes \cdots \otimes Id + \cdots + Id \otimes \cdots \otimes v^{(k)} \otimes \cdots \otimes Id + Id \otimes \cdots \otimes v \quad (25)$$

which is the primitive coproduct.

The coalgebra method for s.r.

Lemma

Given a Lie-Hamilton system X with Lie–Hamiltonian structure (N, Λ, h), the space $(S_{\mathfrak{g}}, \cdot, \{\cdot,\cdot\}_{\mathfrak{g}}, \Delta)$ with $\mathfrak{g} \simeq (\mathcal{H}_\Lambda, \{\cdot,\cdot\}_\Lambda)$ is a Poisson coalgebra with a coproduct $\Delta : S_{\mathfrak{g}} \to S_{\mathfrak{g}} \otimes S_{\mathfrak{g}}$ satisfying $\Delta(v) = v \otimes Id + Id \otimes v$.

If X is a Lie–Hamilton system with a Lie–Hamiltonian structure (N, Λ, h), then the diagonal prolongation $\tilde{X}$ to each N^{m+1} is also a Lie–Hamilton system endowed with a Lie–Hamilton structure $(N^{m+1}, \Lambda^{(m+1)}, \tilde{h})$ given by

$$\Lambda^{(m+1)}(x_{(0)}, \ldots, x_{(m)}) = \sum_{a=0}^{m} \Lambda(x_{(a)})$$

using the vector bundle isomorphism $\Lambda^2 TN^{m+1} \simeq \Lambda^2 TN \oplus \cdots \oplus \Lambda^2 TN$ and $\tilde{h}_t = D^{(m)}(\Delta^{(m)}(h_t))$ where $D^{(m)}$ the representation morphism.

Coalgebra method

- The map $\Delta^{(m)} : S_{\mathfrak{g}} \to S_{\mathfrak{g}}^{(m)} \equiv S_{\mathfrak{g}} \otimes \cdots \otimes S_{\mathfrak{g}}$ defined by recursion as

$$\Delta^{(m+1)} = (Id \otimes \cdots \otimes Id \otimes \Delta^{(2)}) \circ \Delta^{(m)}, \quad m > 2$$

  is a P.A. morphism from $(S_{\mathfrak{g}}, \cdot, \{\cdot,\cdot\}_{S_{\mathfrak{g}}})$ to $(S_{\mathfrak{g}}^{(m)}, \cdot, \{\cdot,\cdot\}_{S_{\mathfrak{g}^{(m)}}})$

$$\{P_1 \otimes \cdots \otimes P_m, Q_1 \otimes \cdots \otimes Q_m\}_{S_{\mathfrak{g}}^{(m)}} = \sum_{i=1}^{m} P_1 Q_1 \otimes \cdots \otimes \{P_i, Q_i\} \otimes \cdots \otimes P_m Q_m$$

- The Lie algebra morphism $\mathfrak{g} \hookrightarrow C^\infty(N)$ gives rise to a family of P.A. morphisms

$$D^{(m)} : S_{\mathfrak{g}} \otimes \cdots \otimes S_{\mathfrak{g}} \hookrightarrow C^\infty(N) \otimes \cdots \otimes C^\infty(N) \subset C^\infty(N \times \cdots \times N)$$

  satisfying

$$\left[D^{(m)}(v_1 \otimes \cdots \otimes v_m)\right](x_{(1)}, \ldots, x_{(m)}) = [D(v_1)](x_{(1)}) \ldots [D(v_m)](x_{(m)})$$

Coalgebra method

Theorem

If X is a Lie–Hamilton system with a Lie–Hamiltonian structure (N, Λ, h) and C is a Casimir element of the Poisson algebra $(S_{\mathfrak{g}}, \cdot, \{\cdot, \cdot\}_{S_{\mathfrak{g}}})$ then,

1. *The functions defined as*

$$F^{(k)} = D^{(k)}(\Delta^{(k)}(C)), \quad k = 2, \ldots, m$$

are t-indp. constants of motion for diagonal prolon. $\tilde{X}$ to N^m. Furthermore, $m-1$ functionally functions in involution.

2. *The functions given by*

$$F_{ij}^{(k)} = S_{ij}(F^{(k)}), \quad 1 \leq i < j \leq k, \quad k = 2, \ldots, m$$

where S_{ij} is the permutation of variables $x_{(i)} \leftrightarrow x_{(j)}$, are t-indp. const. of motion for the diagonal prolong. $\tilde{X}$ to N^m.

Example: coalgebra method for $\mathfrak{iso}(2)$

Over the plane, consider the following vector fields

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \tag{26}$$

with commutation relations

$$[X_1, X_2] = 0, \qquad [X_1, X_3] = -X_2, \qquad [X_2, X_3] = X_1.$$

Step 1. With respect to the canonical symplectic structure $\omega = \mathrm{d}x \wedge \mathrm{d}y$, this corresponds Lie algebra, denoted by $\overline{\mathfrak{iso}}(2)$, determined by a basis

$$h_1 = y, \qquad h_2 = -x, \qquad h_3 = \frac{1}{2}(x^2 + y^2), \qquad h_0 = 1 \tag{27}$$

satisfying commutation relations

$$\{h_1, h_2\}_\omega = h_0, \qquad \{h_1, h_3\}_\omega = h_2, \qquad \{h_2, h_3\}_\omega = -h_1, \qquad \{h_0, \cdot\}_\omega = 0, \tag{28}$$

Example: coalgebra method for $\mathfrak{iso}(2)$

The symmetric Poisson algebra $S\left(\overline{\mathfrak{iso}}(2)\right)$ has a non-trivial Casimir invariant given by

$$C = v_3 v_0 - \tfrac{1}{2}(v_1^2 + v_2^2).$$

Choosing the representation given in (27), we obtain a trivial constant of motion on $(x, y) \equiv (x_1, y_1)$

$$\begin{aligned} F = D(C) &= \phi(v_3)\phi(v_0) - \tfrac{1}{2}\left(\phi^2(v_1) + \phi^2(v_2)\right) \\ &= h_3(x_1, y_1)h_0(x_1, y_1) - \tfrac{1}{2}\left(h_1^2(x_1, y_1) + h_2^2(x_1, y_1)\right) \\ &= \tfrac{1}{2}(x_1^2 + y_1^2) \times 1 - \tfrac{1}{2}(y_1^2 + x_1^2) = 0. \end{aligned}$$

Introducing the coalgebra structure in $S\left(\overline{\mathfrak{iso}}(2)\right)$ through the coproduct, we obtain nontrivial first integrals.

$$
\begin{aligned}
F^{(2)} &= D^{(2)}(\Delta(C)) \\
&= \left(h_3(x_1,y_1)+h_3(x_2,y_2)\right)\left(h_0(x_1,y_1)+h_0(x_2,y_2)\right) \\
&\qquad\qquad\qquad\qquad - \tfrac{1}{2}\Big(\left((h_1(x_1,y_1)+h_1(x_2,y_2)\right)^2+\left(h_2(x_1,y_1)+h_2(x_2,y_2)\right)^2\Big) \\
&= \tfrac{1}{2}(x_1-x_2)^2+\tfrac{1}{2}(y_1-y_2)^2, \\
F^{(3)} &= D^{(3)}(\Delta(C)) \\
&= \sum_{i=1}^{3} h_3(x_i,y_i)\sum_{j=1}^{3} h_0(x_j,y_j) - \tfrac{1}{2}\Big(\big(\sum_{i=1}^{3} h_1(x_i,y_i)\big)^2+\big(\sum_{i=1}^{3} h_2(x_i,y_i)\big)^2\Big) \\
&= \tfrac{1}{2}\sum_{1\le i<j}^{3}\left((x_i-x_j)^2+(y_i-y_j)^2\right).
\end{aligned}
\tag{29}
$$

Furthermore, using the property of permutating subindices (**??**), we find more first integrals

$$
\begin{aligned}
&F_{12}^{(2)} = S_{12}(F^{(2)}) \equiv F^{(2)}, \qquad F_{13}^{(2)} = S_{13}(F^{(2)}) = \tfrac{1}{2}(x_3-x_2)^2+\tfrac{1}{2}(y_3-y_2)^2, \\
&F_{23}^{(2)} = S_{23}(F^{(2)}) = \tfrac{1}{2}(x_1-x_3)^2+\tfrac{1}{2}(y_1-y_3)^2
\end{aligned}
\tag{30}
$$

Coalgebra superposition rule

for SISf Pandemic model

Coalgebra method for SIS pandemic model

We need to find a Casimir function for the Poisson algebra $I_{14A}^{r=1} \simeq \mathbb{R} \ltimes \mathbb{R}$, but unfortunately, there exists no nontrivial Casimir.

We can circumvent this problem by considering an inclusion of $I_{14A}^{r=1}$ as a Lie subalgebra of a Lie algebra of another class admitting a Lie–Hamiltonian algebra with a nontrivial Casimir. For example, consider the $I_8 \simeq \mathfrak{iso}(1,1)$ due to the simple form of its Casimir. If we obtain the superposition rule for I_8, we simultaneously obtain the superposition for $I_{14A}^{r=1}$ as a byproduct.

The Lie–Hamilton algebra $\mathfrak{iso}(1,1)$ has the commutation relations

$$\{h_1, h_2\} = h_0, \quad \{h_1, h_3\} = -h_1, \quad \{h_2, h_3\} = h_2, \quad \{h_0, \cdot\} = 0, \tag{31}$$

with respect to $\omega = dx \wedge dy$ in the basis $\{h_1 = y, h_2 = -x, h_3 = xy, h_0 = 1\}$. Applying the coalgebra method to the Casimir associated to this Lie–hamilton algebra $\mathcal{C} = h_1 h_2 + h_3 h_0$ and mapping the representation without coproduct, the first iteration is trivial, i.e., $F = 0$, but

$$F^{(2)} = (x_1 - x_2)(y_1 - y_2) = k_1,$$

$$F_{23}^{(2)} = (x_1 - x_3)(y_1 - y_3) = k_2,$$

$$F_{13}^{(2)} = (x_3 - x_2)(y_3 - y_2) = k_3.$$

From them, we can choose two functionally independent constants of motion. In this case, $F^{(2)} = k_1, F^{(2)}_{23} = k_2$ can be understood as the equations on $\mathbb{R}^2$. The introduction of k_3 again simplifies the final result which reads

$$
\begin{aligned}
x_1(x_2, y_2, x_3, y_3, k_1, k_2, k_3) &= \frac{1}{2}(x_2 + x_3) + \frac{k_2 - k_1 \pm B}{2(y_2 - y_3)},\\
y_1(x_2, y_2, x_3, y_3, k_1, k_2, k_3) &= \frac{1}{2}(y_2 + y_3) + \frac{k_2 - k_1 \mp B}{2(x_2 - x_3)},
\end{aligned}
\tag{32}
$$

where

$$B = \sqrt{k_1^2 + k_2^2 + k_3^2 - 2(k_1 k_2 + k_1 k_3 + k_2 k_3)}.$$

In the case that matters to us, $I^{r=1}_{14A}$, the third constant k_3 is a function $k_3 = k_3(x_2, y_2, x_3, y_3)$ and $B \geq 0$. We see there is a change of coordinates between this system and SIS,

$$x = -qp, \qquad y = q - \frac{1}{qp^2} \tag{33}$$

So, the superposition principle for SISf would read

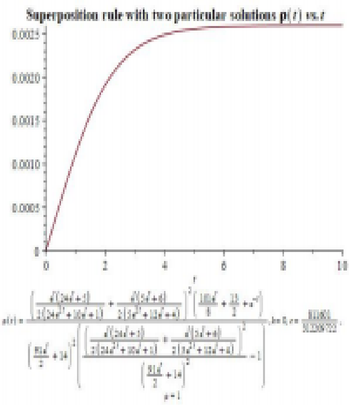
$$q = \frac{\left(\frac{\frac{1}{2}q_2+\frac{1}{2}q_3+(k_2-k_1\pm B)}{(2p_2-2p_3)}\right)^2\left(\frac{1}{2}p_2+\frac{1}{2}p_3+\frac{(k_2-k_1\mp B)}{(2q_2-2q_3)}\right)}{\left(\frac{\frac{1}{2}q_2+\frac{1}{2}q_3+(k_2-k_1\pm B)}{(2p_2-2p_3)}\right)^2-1} \tag{34}$$

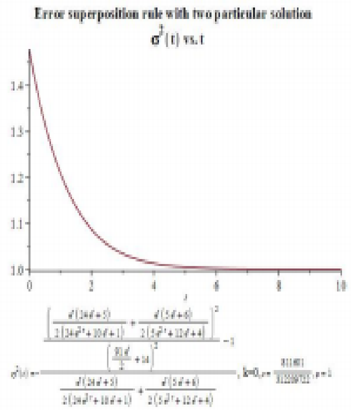
$$p = -\frac{\left(\frac{\frac{1}{2}q_2+\frac{1}{2}q_3+(k_2-k_1\pm B)}{(2p_2-2p_3)}\right)^2-1}{\frac{1}{2}q_2+\frac{1}{2}q_3+\frac{(k_2-k_1\pm B)}{(2p_2-2p_3)}\left(\frac{1}{2}p_2+\frac{1}{2}p_3+\frac{(k_2-k_1\mp B)}{(2q_2-2q_3)}\right)} \tag{35}$$

Here, (q_2, p_2) and (q_3, p_3) are two particular solutions and k_1, k_2, k_3 are constants of integration.

Here, (q_2, p_2) and (q_3, p_3) are two particular solutions and k_1, k_2, k_3 are constants of integration.

Now, we show the graphics for $<\rho>= q(t)$ and $\sigma^2 = 1/p^2$ using the two particular solutions provided in the introduction. Notice that we have renamed $c = (k_2 - k_1 \pm B)$ and $k = (k_2 - k_1 \mp B)$.





Conclusions about Lie systems and Covid

One may wonder how the current pandemic of COVID19 could be related to a SISf-pandemic model. Let us state clear some points.

- The SISf model is a very first approximation for a trivial infection process, in which there is only two possible states: infected or susceptible. Hence, this model does not provide the possibility of adquiring immunity. It seems that COVID19 provides some certain type of immunity, but only to a thirty percent of the infected individuals.
- Hence, a SIR model that considers “R” for recuperated individuals (not susceptible anymore, i.e., immune) is not a proper model for the current situation. One should have a model contemplating immune and nonimmunized individuals.
- Unfortunately, we are still in search of a Lie system including potential immunity and nonimmunity.
- There exists a stochastic theory of Lie systems. In the present work we were lucky to find a theory with fluctuations that happened to match a stochastic expansion, but this is rather more of an exception than a rule. Maybe the stochastic Lie system approach could lead us to Hamiltonian Lie systems?

Thanks for the attention!

A Guide to Lie Systems with Compatible Geometric Structures.
J. de Lucas, C. Sardón. World Scientific (2020).

Part of the content free at: https://arxiv.org/abs/1508.00726

Preprint for SISf : arXiv:2008.02484