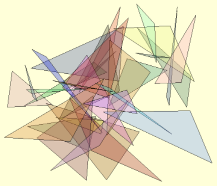
Singular chains on Lie groups and the Cartan relations

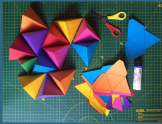
Camilo Arias Abad



Universidad Nacional de Colombia
(joint with Alexander Quintero)

Friday Fish
Utrecht
2020

Outline

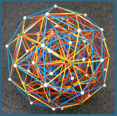


Chains on Lie groups and the Cartan relations

The compact case and Chern-Weil theory

Local systems on classifying spaces

Lie theory



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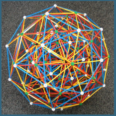
Lie theory



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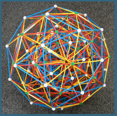

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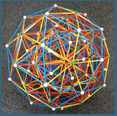

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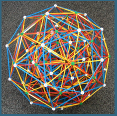

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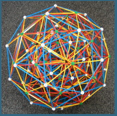

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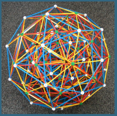

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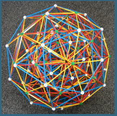

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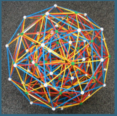

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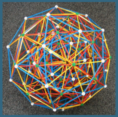
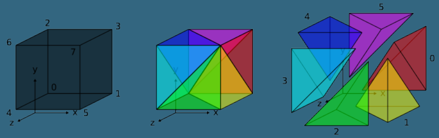
$$\mathsf{EZ}(\sigma \otimes \nu) \quad := \quad \sum_{\chi \in \Sigma_{(r,s)}} (-1)^{|\chi|} (\sigma \times \nu) \circ \overline{\chi},$$

Eilenberg-Zilber





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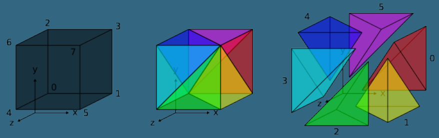

where $\overline{\chi}$ is the map:

$$\overline{\chi} : \Delta_{r+s} \to \Delta_r \times \Delta_s$$
$$\overline{\chi}(t_1, \ldots, t_{r+s}) = ((t_{\chi(1)}, \ldots, t_{\chi(r)}), (t_{\chi(r+1)}, \ldots, t_{\chi(r+s)})).$$

The group ring and singular chains

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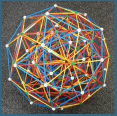

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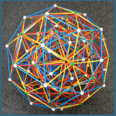

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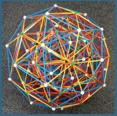

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Question: Describe the modules over $C(G)$ infinitesimally.

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$$\begin{aligned} [L_v, L_w] &= L_{[v,w]} \\ [L_v, \iota_w] &= \iota_{[v,w]} \\ [\iota_v, \iota_w] &= 0 \\ d\iota_v &= L_v \end{aligned}$$

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The relations above are called the Cartan relations.

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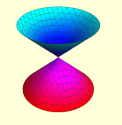

Theorem [C.A]

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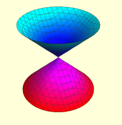

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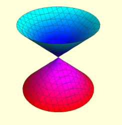

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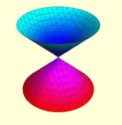

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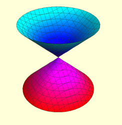

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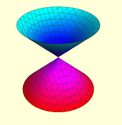

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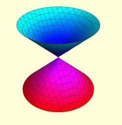

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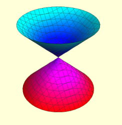

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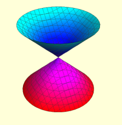

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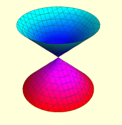

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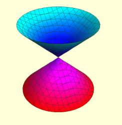

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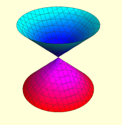

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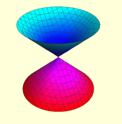

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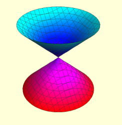

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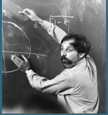
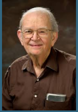
Bott-Shulman-Stasheff algebra





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\Omega^2(G_0) \arrow[r, "\partial"] \arrow[u, "\bar{d}"] & \Omega^2(G_1) \arrow[r, "\partial"] \arrow[u, "\bar{d}"] & \Omega^2(G_2) \arrow[r, "\partial"] \arrow[u, "\bar{d}"] & \cdots \\
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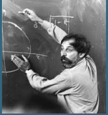

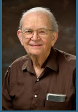

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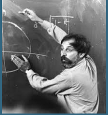

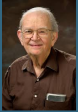

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$\alpha \in \Omega(BG) \otimes \mathrm{End}(V)$ satisfies the Maurer-Cartan equation.

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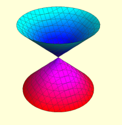

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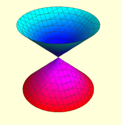

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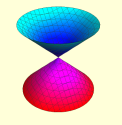

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is a homomorphism of differential graded algebras.

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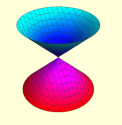

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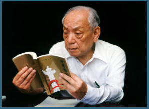
$$\rho(\sigma) := \int_{\Delta_k} \sigma^*(\alpha_k)$$

is a homomorphism of differential graded algebras.

Summary

$$\begin{tikzcd}
\Big\{\text{Representations of } T\mathfrak{g}\Big\} \arrow[r] & \Big\{\text{Marurer-Cartan in } \Omega(BG)\otimes \mathrm{End}(V)\Big\} \arrow[d, "\text{integration}"] \\
& \Big\{\text{Modules over } C(G)\Big\}
\end{tikzcd}$$

Chern-Weil theory



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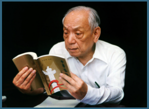

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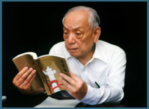

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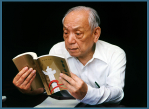

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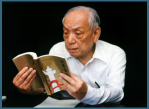

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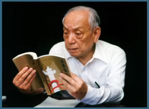

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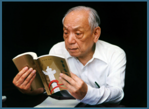

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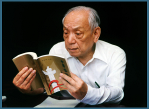

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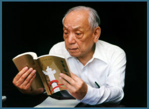

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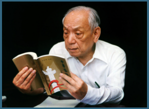

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A dg-category of representations of $T\mathfrak{g}$

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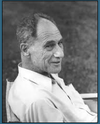

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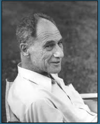

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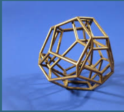
A_∞-equivalence



Theorem [C. A., A. Quintero]

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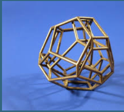

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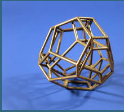

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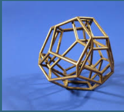

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$$\begin{tikzcd}
\overline{\mathsf{Rep}}(T\mathfrak{g}) & & \mathsf{BSS}(G) \arrow[dr] & \\
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Chen's iterated integrals

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$$\Omega(M)^{\otimes k} \longrightarrow \Omega(M^k) \xrightarrow{\text{ev}^*} \Delta_k \times PM \xrightarrow{\int_{\Delta_k}} PM$$

Gugenheim's A_∞ de Rham theorem

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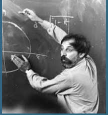

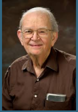

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & \\
\Omega^2(G_0) \arrow[r, "\partial"] \arrow[u, "\bar{d}"] & \Omega^2(G_1) \arrow[r, "\partial"] \arrow[u, "\bar{d}"] & \Omega^2(G_2) \arrow[r, "\partial"] \arrow[u, "\bar{d}"] & \cdots \\
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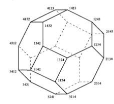
$$

A_∞ de Rham theorem for classifying spaces



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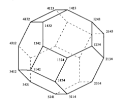

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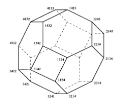

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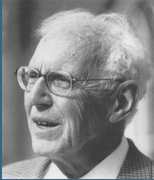
Applications to Lie theory and Chern-Weil theory.

The Van Est map



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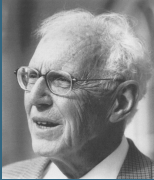

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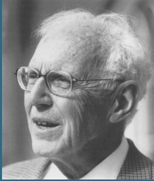

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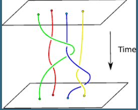
Higher local systems



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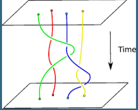

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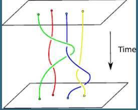

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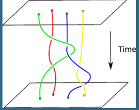

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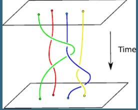

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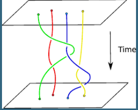

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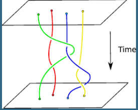

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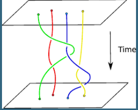

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For each of the points of view above, the corresponding notions of local system can be organized into a dg-category, and the resulting dg-categories are quasi-equivalent.

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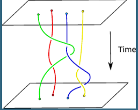

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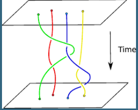

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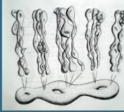
We will abuse the notation and write $\mathsf{Loc}(X)$ for any of these dg-categories.

Higher local systems on classifying spaces



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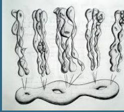

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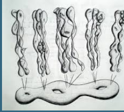

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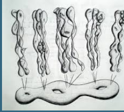

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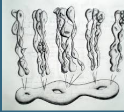

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$$H(BG) \simeq \mathsf{End}_{\overline{\mathsf{Mod}}(C_\bullet(G))}(\mathbb{R}) \simeq \overline{\mathsf{Rep}}(\mathrm{T}\mathfrak{g})_{\overline{\mathsf{Rep}}(T\mathfrak{g})}(\mathbb{R})$$

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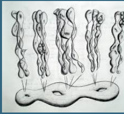

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$$H(BG) \simeq \mathsf{End}_{\overline{\mathsf{Mod}}(C_\bullet(G))}(\mathbb{R}) \simeq \overline{\mathsf{Rep}}(\mathsf{T}\mathfrak{g})_{\overline{\mathsf{Rep}}(T\mathfrak{g})}(\mathbb{R})$$

And one recovers the usual fact that:

$$H(BG) \simeq HC^\bullet(C_\bullet(G)) \simeq W(\mathfrak{g})^{\mathsf{basic}} \simeq S(\mathfrak{g}^*)^G.$$

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The loop space fibration $\pi : L(BG) \to BG$

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There is additional structure, that of a Batalin-Vilkovisky algebra...

Thank you for your attention!