Weighted normal bundles

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$$M \supseteq N$$

$$\nu(M,N) = TM|_N / TN$$

We are studying a "weighted" version of the normal bundle & related geometry.

$\mathbb{R}^n$ coords $x_1, \ldots, x_n$, $w_1, \ldots, w_n$, $w_i \in \{0,1,2,\ldots\}$

weighted scalar mult. is $x_i \mapsto t^{w_i} x_i$, $t \in \mathbb{R}$.

There is a weighted version of the normal bundle $\nu_W(M,N)$

- fibre bundle over N
- diffeo to $\nu(M,N)$, but not canonically
- not a vector bundle
- it has a "scalar mult." — it has an action of the monoid $(\mathbb{R}, \cdot)$ — so it is an example of a "graded bundle"

- algebraic geometry
- R. Melrose "quasi-homogeneous structure"
- hypoelliptic operators

② Weightings

whole numbers

weight sequence $0 \leq w_1 \leq w_2 \leq \cdots \leq w_n = r$

$w = (w_1, \ldots, w_n)$

if $U \subseteq \mathbb{R}^n$, define ideals

$$C^\infty(U)_{(i)} = \langle x^s = x_1^{s_1} \cdots x_n^{s_n} \mid s \cdot w \geq i \rangle$$

ideal generated by, $s_j \in \{0, 1, \ldots\}$

Ex (1) $w = (1, \ldots, 1)$

$C^\infty(U)_{(i)}$ = ideal of smooth fcns vanishing at origin to degree i in the ordinary sense

$= \mathcal{I}_0^i$ $\quad \mathcal{I}_0$ = vanishing ideal of the origin.

(2) $w = (0, 1, \ldots, 1)$

$C^\infty(U)_{(i)}$ = ideal of fcns vanishing on the x_1 coord axis to degree i in the ordinary sense.

(3) $w = (1, 2)$ $\quad (\mathbb{R}^2)$

$C^\infty(\mathbb{R}^2)_{(1)} = \langle x_1, x_2 \rangle$ ideal of fcns vanishing at the origin

$C^\infty(\mathbb{R}^2)_{(2)} = \langle x_1^2, \underline{\underline{x_2}} \rangle$ $\quad N = \{(0,0)\} \subseteq \mathbb{R}^2$ not the standard example

$C^\infty(\mathbb{R}^2)_{(3)} = \langle x_1^3, x_1 x_2, x_2^2 \rangle$

$C^\infty(\mathbb{R}^2)_{(4)} = \langle x_1^4, x_1^2 x_2, x_2^2 \rangle$

<u>Def</u> An order r weighting on a manifold M is a descending sequence of ideal sheaves

$$C^\infty_M = C^\infty_{M,(0)} \supseteq C^\infty_{M,(1)} \supseteq C^\infty_{M,(2)} \supseteq \cdots$$

sheaf of smooth fcns

such that around any point there is a chart U such that the restricted sequence of ideals $C^\infty_M(U)_{(i)}$, $i=0,1,\dots$ is of the above form,

<u>Rmk</u>: $C^\infty_{M,(1)} = I_N$ is the sheaf of fcn vanishing on a submanifold $N \subseteq M$.

<u>Ex</u>: $C^\infty_{M,(i)} = I_N^i$ ← standard/trivial weighting

By looking which coord variables survive we get a filtration of $\nu(M,N)$: let $i \geq 1$

$$\underbrace{C^\infty_{M,(i)} \Big/ C^\infty_{M,(i)} \cap I_N^2} \subseteq I_N / I_N^2 \cong \Gamma_{\nu(M,N)^*}$$

this is the sheaf of sections of a subbundle of $\nu(M,N)^*$

call $\operatorname{ann}(F_{-i+1})$ ← this defines a subbundle

annihilator

$$F_{-i+1} \subseteq \nu(M,N)$$

Basically think of F_{-i} = directions with weight $\leq i$

$$\nu(M,N) = F_{-r} \supseteq F_{-r+1} \supseteq \cdots \supseteq F_{-1} \supseteq F_0 = N$$

x_j has wt w_j, $\partial/\partial x_j$ has wt $-w_j$

Def $\nu_W(M,N) = \mathrm{Hom}_{alg}(\mathrm{gr}\, C^\infty(M), \mathbb{R})$

with respect to filtration
$C^\infty(M)_{(0)} \supset C^\infty(M)_{(1)} \supset \cdots$

Rmk For the std weighting this is the ordinary normal bundle $TM|_N/TN$. $C^\infty(M)_{(i)}/C^\infty(M)_{(i+1)}$

Is there a description like in the general case ??

Yes – using jet bundles.

③ **Jets** $T_n M = \mathcal{J}^n_0(\mathbb{R}, M)$ jet bundle

$$T_1 M = TM$$

$$A_r = \mathbb{R}[\epsilon]/\epsilon^{r+1}$$

$$T_r M \cong \mathrm{Hom}_{alg}(C^\infty(M), A_r),$$

$$[\gamma] \in \mathcal{J}^r_0(\mathbb{R}, M) \longmapsto \left(u_\gamma : f \mapsto \sum_{i=0}^{r} \frac{1}{i!} \frac{d^i}{dt^i}\Big|_{t=0} f(\gamma(t))\epsilon^i\right)$$

Structures:

① Action of $\mathrm{Aut}_{alg}(A_r)$ — reparametrizations of curves.

$$\mathrm{Aut}_{alg}(A_r) \subseteq \mathrm{End}_{alg}(A_r) = \Lambda_r \quad \text{monoid acts.}$$

$\Lambda_r \supseteq \Lambda_1 = (\mathbb{R}, \cdot) \rightsquigarrow T_r M$ is a graded bundle.

② Tautological map $\quad \boxed{T_r M = \mathrm{Hom}_{alg}(C^\infty(M), A_r)}$

$$C^\infty(M) \longrightarrow C^\infty(T_r M) \otimes A_r$$

$c, T_r M$

$$f \longmapsto \left(u \longmapsto u(f) \in A_r \right)$$

③ $$T_u(T_r M) = \mathrm{Der}_u\left(C^\infty(M), A_r\right)$$

$$D : C^\infty(M) \to A_r$$

$$D(fg) = (Df)\,u(g) + u(f)\,Dg.$$

$\Rightarrow$ tgt spaces have A_r-module strs.

<u>Back to weightings</u>

order r weighting $C^\infty(M)_{(i)}$

subbundle $Q \subseteq T_r M|_N$

$$Q = \left\{ u \,\middle|\, u(f) \in \epsilon^i A_r \ \forall f \in C^\infty(M)_{(i)} \text{ for all } i \right\}$$

$$T_r : C^\infty(M) \to C^\infty(T_r M) \otimes A_r$$

$$C^\infty(M)_{(i)} = \left\{ f \,\middle|\, T_r f|_Q \in C^\infty(Q) \otimes \epsilon^i A_r \right\}$$

<u>Weighted normal bundle</u>

$$Q \subset T_r M$$

$$TQ \subset T(T_r M)$$

it's invariant under A_r-mod structure

$$\epsilon \cdot TQ \subseteq TQ$$

it's integrable $\rightsquigarrow$ foliation $\mathcal{F}_{\epsilon TQ}$ of Q.

$$\nu_W(M, N) \cong Q / \mathcal{F}_{\epsilon TQ}.$$

Thank you!

Iakovos: $M \times \mathbb{R}$, $t^0\mathcal{H}_0 + t^1\mathcal{H}_{-1} + \cdots + t^r\mathcal{H}_{-r}$ sing. fol. of $M \times \mathbb{R}$.

Lie filtration

$$H_0 \subseteq H_{-1} \subseteq \cdots H_{-r} = TM$$

subbundles such that

$$[\Gamma(H_i), \Gamma(H_j)] \subset \Gamma(H_{i+j})$$

H_0 is involutive. $\rightsquigarrow$ foliation

Leaves of $\mathcal{H}_0$ have canonical weightings st $\Gamma(H_i)$ are homogeneous of degree i in the weighted sense

L = leaf of $\mathcal{H}_0$

$$\mathcal{H} = \bigoplus_{i=0}^{r} \Gamma(H_{-i})\,\epsilon^i \subseteq \mathcal{X}(M) \otimes A_r \xrightarrow{T_r} \mathcal{X}_0(T_rM)$$

$$\mathcal{X}(M) \otimes \epsilon^0\mathbb{R} \subseteq \mathcal{X}(M) \otimes A_r$$

$$T_r(\mathcal{H}) \cdot L = Q$$