The singular Weinstein conjecture

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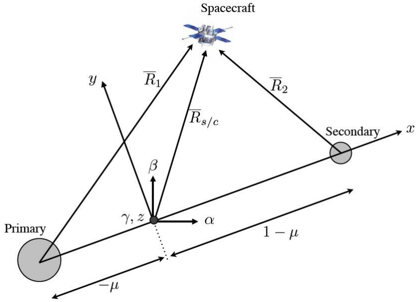
Motivating examples from celestial mechanics

Restricted planar circular 3-body problem I

Simplified version of the general 3-body problem:

- One of the bodies has negligible mass.
- The other two bodies move in circles following Kepler's laws for the 2-body problem.
- The motion of the small body is in the same plane.



Restricted planar circular 3-body problem II

- Time-dependent potential: $U(q,t) = \frac{1-\mu}{|q-q_E(t)|} + \frac{\mu}{|q-q_M(t)|}$
- Time-dependent Hamiltonian:
  $H(q,p,t) = \frac{|p|^2}{2} - U(q,t), \quad (q,p) \in \mathbb{R}^2 \setminus \{q_E, q_M\} \times \mathbb{R}^2$
- Rotating coordinates: Time independent Hamiltonian
  $H(q,p) = \frac{|p|^2}{2} - \frac{1-\mu}{|q-q_E|} + \frac{\mu}{|q-q_M|} + p_1 q_2 - p_2 q_1$
- H has 5 critical points: L_i Lagrange points $(H(L_1) \leq \cdots \leq H(L_5))$
- Periodic orbits of X_H?
- Perturbative methods (dynamical systems) or.... contact topology!

Level-sets of Hamiltonians

Let (W, ω) be a symplectic manifold and $\Sigma \subset W$ hypersurface.

Definition

A Liouville vector field is a v.f. $X \in \mathfrak{X}(W)$ such that $\mathcal{L}_X \omega = \omega$.

Proposition

Let X be a Liouville vector field transverse to Σ. Then $(\Sigma, \alpha = \iota_X \omega)$ is a contact manifold. If $\Sigma = H^{-1}(c)$, then $R_\alpha \cong X_H|_{H=c}$.

Conjecture (Weinstein conjecture)

Let (M, α) closed contact manifold. Then R_α admits periodic orbits.

Contact Geometry of the RPC3BP

- For $c < H(L_1)$, $\Sigma_c = H^{-1}(c)$ has 3 connected components: Σ_c^E (the satellite stays close to the earth), Σ_c^M (to the moon), or it is far away.

Proposition (Albers–Frauenfelder–Koert–Paternain)

For $c < H(L_1)$, $X = (q - q_E)\frac{\partial}{\partial q}$ is transverse to Σ_c^E.

Hence $(\Sigma_c^E, \iota_X \omega)$ is contact.
But Weinstein conjecture does not apply because of non-compactness (collision!)

☹

Moser regularization of the restricted 3-body problem

Via Moser's regularization Σ_c^E can be compactified to $\overline{\Sigma}_c^E \cong \mathbb{R}P(3)$.

Theorem (Albers–Frauenfelder–Koert–Paternain)

For any value $c < H(L_1)$, the regularized RPC3BP has a closed orbit with energy c.

☺

But...

- Where are those periodic orbits?
- Maybe on the collision set?
- Keep track of the singularities in the geometric structure?
- ...b^m-symplectic and b^m-contact geometry!

Or manifold at infinity?

- Consider the canonical change of coordinates to polar coordinates:

$$(q, p) \mapsto (r, \alpha, P_r, P_\alpha)$$

- McGehee change of coordinates: $r = \frac{2}{x^2}$, where $x \in \mathbb{R}^+$
- Non-canonical, the symplectic form becomes singular:

$$\omega = -\frac{4}{x^3} dx \wedge d\alpha + dP_r \wedge dP_\alpha$$

- This is a b^3-symplectic form.

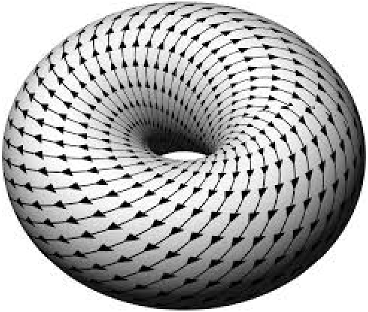
Dynamics of X_H?

b^m-symplectic and b^m-contact geometry

Introducing b-symplectic

- b-symplectic structures can be seen as symplectic structures modeled over a Lie algebroid (the b-cotangent bundle).
- A vector field v is a b-vector field if $v_p \in T_pZ$ for all $p \in Z$. The b-**tangent bundle** ${}^bTM$ is defined by

$$\Gamma(U, {}^bTM) = \left\{ \begin{array}{c} \text{b-vector fields} \\ \text{on } (U, U \cap Z) \end{array} \right\}$$



b-cotangent bundle

Consider a hypersurface $Z = f^{-1}(0)$ of M, the **critical set**

$${}^b\mathfrak{X}(M) = \{\text{v.f. tangent to } Z\} = \langle f\frac{\partial}{\partial f}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}\rangle$$

<u>Serre–Swan</u>: There exists a bundle ${}^bTM$ such that $\Gamma({}^bTM) = {}^b\mathfrak{X}(M)$.

The dual: ${}^bT^*M$ and forms: ${}^b\Omega^k(M) = \Gamma(\Lambda^k({}^bT^*M))$.

Extending differential calculus

$\omega \in {}^b\Omega^k(M)$ can be decomposed

$$\omega = \alpha \wedge \frac{df}{f} + \beta \text{ where } \alpha \in \Omega^{k-1}(M), \beta \in \Omega^k(M).$$

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Extension of the exterior derivative by defining

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b-symplectic and b-contact manifolds

Definition ([GMP])

A b-symplectic form on W^{2n} is $\omega \in {}^b\Omega^2(W)$ such that

- $d\omega = 0$,
- ω is non-degenerate.

Definition

A manifold (M^{2n+1}, α) where $\alpha \in {}^b\Omega^1(M)$ is b-contact if $\alpha \wedge (d\alpha)^n \neq 0$.

b-symplectic and b-contact manifolds

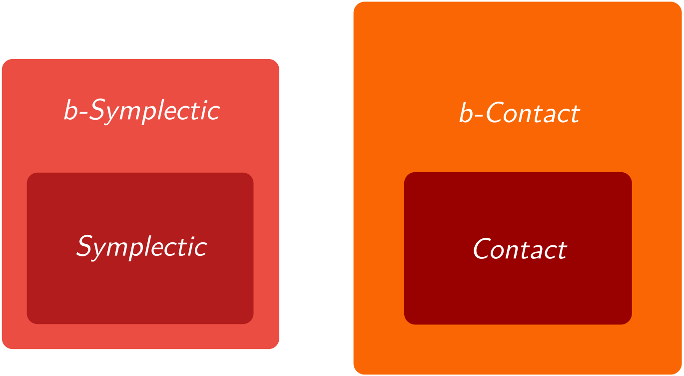
Definition ([GMP])

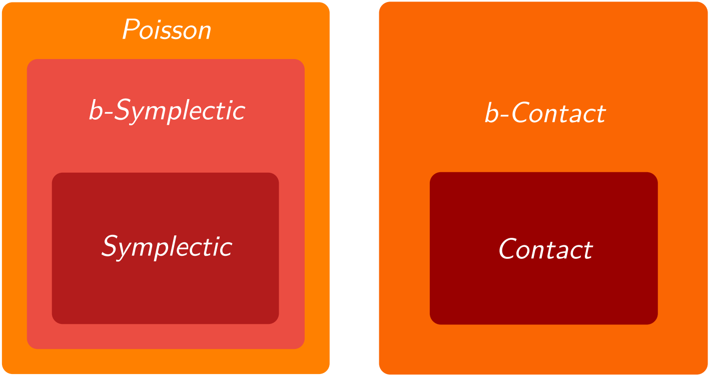
A b^m-symplectic form on W^{2n} is $\omega \in {}^{b^m}\Omega^2(W)$ such that

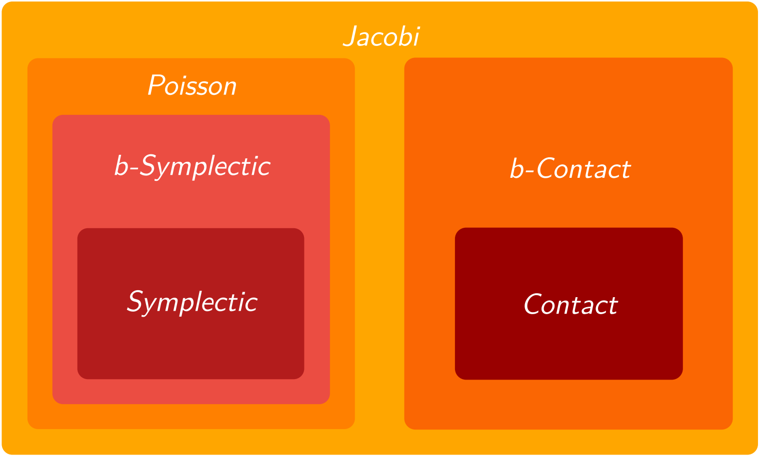
- $d\omega = 0$,
- ω is non-degenerate.

Definition

A manifold (M^{2n+1}, α) where $\alpha \in {}^{b^m}\Omega^1(M)$ is b^m-contact if $\alpha \wedge (d\alpha)^n \neq 0$.


b-Symplectic
Symplectic
b-Contact
Contact


Poisson
b-Symplectic
Symplectic
b-Contact
Contact


Jacobi
Poisson
b-Symplectic
Symplectic
b-Contact
Contact

Local study of b^m-contact manifolds I

Example

- $(\mathbb{R}^3, \frac{dz}{z} + xdy), \; R_\alpha = z\frac{\partial}{\partial z}$
- $(\mathbb{R}^3, dx + y\frac{dz}{z}), \; R_\alpha = \frac{\partial}{\partial x}$

The Reeb vector field R_α is defined by the equations

$$\begin{cases} \iota_{R_\alpha} d\alpha = 0 \\ \iota_{R_\alpha} \alpha = 1 \end{cases}$$

Local study of b^m-contact and b^m-symplectic manifolds II

One can prove Darboux theorem, analyze the induced structure on the critical set...see [MO1].

Proposition

Let (W, Z, ω) be a b^m-symplectic manifold and $X \in {}^{b^m}\mathfrak{X}(W)$ such that $\mathcal{L}_X\omega = \omega$ and $X \pitchfork \Sigma$. Then $(\Sigma, \iota_X\omega)$ is b^m-contact with critical set $\widetilde{Z} = Z \cap \Sigma$.

Dynamics on b^m-contact manifolds

The Reeb vector field R_α is defined by the equations

$$\begin{cases} \iota_{R_\alpha} d\alpha = 0 \\ \iota_{R_\alpha} \alpha = 1. \end{cases}$$

The Reeb vector field can vanish!

Do there exists plugs?

A *trap* is a smooth vector field on the manifold $D^{n-1} \times [0,1]$ such that

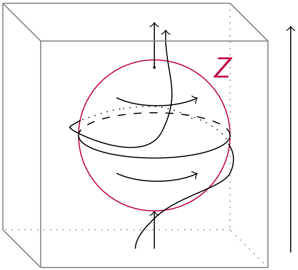
1. the flow of the vector field is given by $\frac{\partial}{\partial t}$ near the boundary of $\partial D \times [0,1]$, where t is the coordinate on $[0,1]$;
2. there are no periodic orbits contained in $D \times [0,1]$;
3. the orbit entering at the origin of the disk $D \times \{0\}$ does not leave $D \times [0,1]$ again.

If the vector field additionally satisfies *entry-exit matching condition*, that is that the orbit entering at $(x,0)$ leaves at $(x,1)$ for all $x \in D \setminus \{0\}$, then the trap is called a *plug*.

- Weinstein conjecture: There are no plugs.
- Eliashberg–Hofer: non-existence of traps for dim=3.
- Geiges–Roettgen–Zehmisch: existence in higher dimension.
- Traps and plugs for b^m-contact?

Theorem

There exists traps for the b^m-Reeb flow.



Theorem

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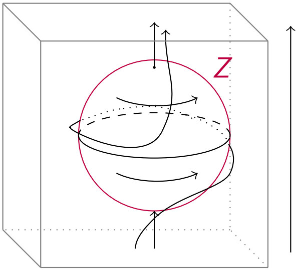


Question: Existence/Non-existence of periodic Reeb orbits away and on Z?

Proposition

Let $(M, \alpha = u\frac{dz}{z} + \beta)$ be a b^m-contact manifold of dimension 3. Then the restriction on Z of the 2-form $\Theta = ud\beta + \beta \wedge du$ is symplectic and the Reeb vector field is Hamiltonian with respect to Θ with Hamiltonian function u, i.e. $\iota_R \Theta = du$.

This is highly 3-dimensional!

Infinitely many periodic orbits on Z

Proposition

Let (M, α) be a 3-dimensional b^m-contact manifold and assume the critical hypersurface Z to be closed. Then there exists infinitely many periodic Reeb orbits on Z.

Proof.

1. $\alpha = u\frac{dz}{z} + \beta$
2. u is non-constant on Z
3. R_α is Hamiltonian on Z for $-u$,
4. $u^{-1}(p)$ where p regular is a circle,
5. R_α periodic on $u^{-1}(p)$.

$\square$

No periodic orbits away from Z

There are compact b^m-contact manifolds (M, Z) of any dimension for all $m \in \mathbb{N}$ without periodic Reeb orbits on $M \setminus Z$.

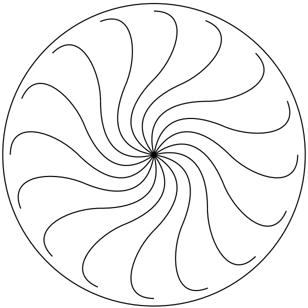
Example

- $S^3 \subset (\mathbb{R}^4, \omega = \frac{dx_1}{x_1} \wedge dy_1 + dx_2 \wedge dy_2)$
- $X = \frac{1}{2} x_1 \frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial y_1} + \frac{1}{2}(x_2 \frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial y_2})$ Liouville v.f.
- $R_\alpha = 2x_1^2 \frac{\partial}{\partial x_1} - x_1 y_1 \frac{\partial}{\partial y_1} + 2x_2 \frac{\partial}{\partial y_2} - 2y_2 \frac{\partial}{\partial x_2}$
- On $Z = S^2$: rotation,
- Away from Z, no periodic orbits.

Periodic orbits away from Z?

Definition

$(M^3, \xi = \ker \alpha)$ is *overtwisted* if there exists D^2 s.t. $TD \cap \xi$ defines a 1-dimensional foliation given by



A contact manifold that is not overtwisted is called *tight*.

Theorem (Hofer '93)

Let (M^3, α) a closed OT contact manifold. Then there exists a periodic orbit.

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Definition

A b^m-contact manifold is overtwisted if there exists an overtwisted disk away from the critical hypersurface Z.

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Let (M^3, α) a closed OT contact manifold. Then there exists a periodic orbit.

Not true for open OT manifolds!

Definition

A b^m-contact manifold is overtwisted if there exists an overtwisted disk away from the critical hypersurface Z.

Definition

A contact form α is $\mathbb{R}^+$-invariant around the critical set if there exists a contact vector field that $\alpha = u\frac{dz}{z^m} + \beta$, where $u \in C^\infty(Z)$ and $\beta \in \Omega^1(Z)$

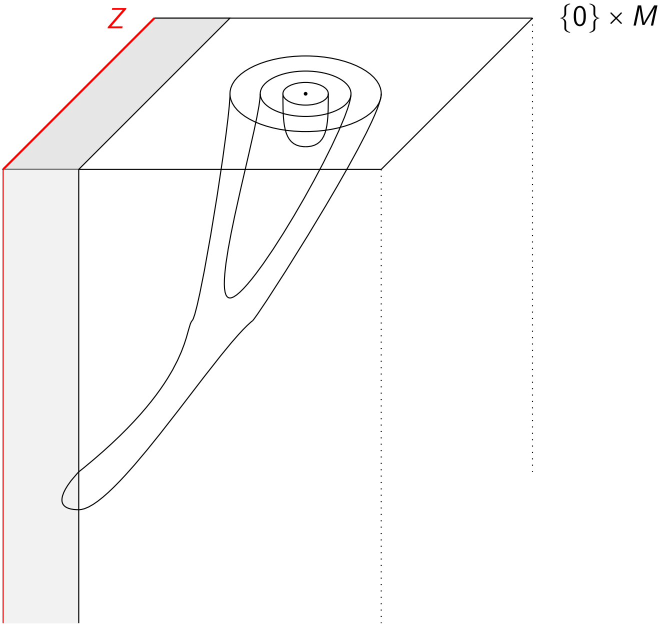
Theorem

Let (M, α) be a closed b^m-contact manifold with critical set Z. Assume there exists an overtwisted disk in $M \setminus Z$ and assume that α is $\mathbb{R}^+$-invariant in a tubular neighbourhood around Z. Then there exists

1. *a periodic Reeb orbit in $M \setminus Z$ or*
2. *a family of periodic Reeb orbits approaching the critical set Z.*

The proof is an adaptation of Hofer's technique.

Question: Other applications of this theorem?


Z
$\{0\} \times M$

Back to the motivating example

Contact geometry of RPC3BP revisited

In rotating coordinates: $H(q,p) = \frac{|p|^2}{2} - \frac{1-\mu}{|q-q_E|} - \frac{\mu}{|q-q_M|} + p_1 q_2 - p_2 q_1$

Lemma

The vector field $Y = p\frac{\partial}{\partial p}$ is a Liouville vector field and is transverse to Σ_c for $c > 0$.

- Symplectic polar coordinates: $(r, \alpha, P_r, P_\alpha)$.
- McGehee change of coordinates: $r = \frac{2}{x^2}$.

b^3-symplectic form: $-4\frac{dx}{x^3} \wedge dP_r + d\alpha \wedge dP_\alpha$.
Is Σ_c b^3-contact after McGehee? Can we apply the results on periodic orbits?

b^3-contact form in the RPC3BP

Theorem

After the McGehee change, the Liouville vector field $Y = p\frac{\partial}{\partial p}$ is a b^3-vector field that is everywhere transverse to Σ_c for $c > 0$ and the level-sets $(\Sigma_c, \iota_Y\omega)$ for $c > 0$ are b^3-contact manifolds. Topologically, the critical set is a cylinder and the Reeb vector field admits infinitely many non-trivial periodic orbits on the critical set.

Proof.

- Y transverse at the critical set?
- On critical set, Hamiltonian $H = \frac{1}{2}P_r^2 - P_\alpha$, so that $Y(H)|_{H=c} = P_r^2 - P_\alpha = \frac{1}{2}P_r^2 + c > 0$;
- b^3-contact form $\alpha = (P_r \frac{dx}{x^3} + P_\alpha d\alpha)|_{H=c}$ with $Z = \{(x, \alpha, P_r, P_\alpha) | x = 0, \frac{1}{2}P_r^2 - P_\alpha = c\}$;
- $R_\alpha|_Z = X_{P_r}$;
- Cylinder is foliated by periodic orbits.

$\square$

Open questions and future work

Can those periodic orbits be continued away from the critical set?

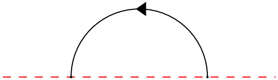


Figure: A Singular periodic orbit a.k.a. heteroclinic

Conjecture (Singular Weinstein conjecture)

Let (M, α) be a compact b^m-contact manifold. Then there exists always a singular periodic Reeb orbit.

Recent work (joint with Miranda and Peralta-Salas: "Generically", the conjecture is satisfied.

Thanks for listening

References

- Albers, Peter, Urs Frauenfelder, Otto Van Koert, and Gabriel P. Paternain. "Contact geometry of the restricted three-body problem." Communications on pure and applied mathematics 65, no. 2 (2012): 229-263.
- Guillemin, Victor, Eva Miranda, and Ana Rita Pires. "Symplectic and Poisson geometry on b-manifolds." Advances in mathematics 264 (2014): 864-896.
- Miranda, Eva, and Cédric Oms. "The geometry and topology of contact structures with singularities." arXiv preprint arXiv:1806.05638 (2018).
- Miranda, Eva, and Cédric Oms. "The singular Weinstein conjecture." arXiv preprint arXiv:2005.09568 (2020).