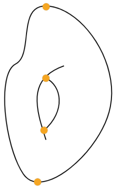
Topology, symplectic topology, . . . What's next?

Ronen Brilleslijper & Oliver Fabert

0-dimensional objects

Morse theory on smooth manifolds



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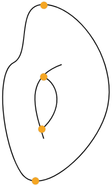
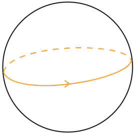


1-dimensional objects

Floer theory on symplectic manifolds



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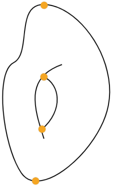

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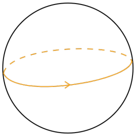
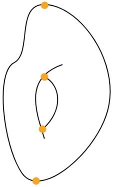


2-dimensional objects

???

0-dimensional objects

Morse theory on smooth manifolds



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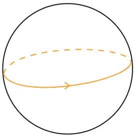

2-dimensional objects

Holomorphic symplectic? Polysymplectic?

Physics motivation

Symplectic geometry

Mechanics

Physics motivation

Symplectic geometry

Mechanics

Polysymplectic geometry

Field theory

Physics motivation

Symplectic geometry

- Mechanics
- ODEs

Polysymplectic geometry

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Physics motivation

Two types of field theory

Minkowski

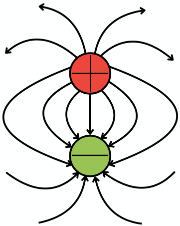
Euclidean

Physics motivation

Two types of field theory

Minkowski

- Electromagnetism
- Wave equation
- Hyperbolic PDEs



Euclidean

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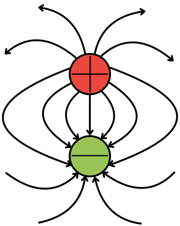
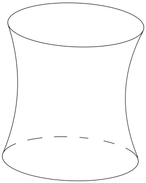


Euclidean

- Electrostatics
- Laplace equation
- Elliptic PDEs



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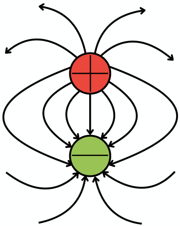

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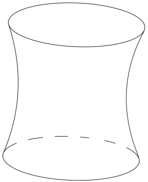

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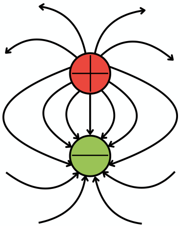

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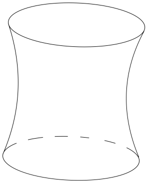

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Given a function $H : M \to \mathbb{R}$ a map $Z : \mathbb{R}^d \to M$ is called a *solution* if

$$\sum_{i=1}^{d} \eta_i(\cdot, \partial_i Z) = dH.$$

Example

Let $d = 2$ and $M = \mathbb{R}^3 \ni (q, p_1, p_2)$ with

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Example *(De Donder-Weyl equations)*

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The flow of these vector fields preserve Ω.

The good and the bad

✓ Geometric framework for PDEs

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- ✓ Hamiltonian and Lagrangian formalism

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Laplace equation

$$-(\partial_1^2 + \partial_2^2)q = \nabla V(q) \qquad q = (q_1, q_2) : \mathbb{R}^2 \to \mathbb{R}^{2n}$$

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$$\not{\partial} Z := \begin{pmatrix} 0 & -2\partial_{\bar{t}} \\ 2\partial_t & 0 \end{pmatrix} Z = \nabla H(Z),$$

where

$$Z = (q, p) : \mathbb{R}^2 \to \mathbb{R}^{4n}$$

$$H(q, p) = \frac{1}{2}|p|^2 + V(q)$$

Polysymplectic formulation

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The Dirac equation is equivalent to

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Note

- Both ω_1 and ω_2 are symplectic forms related by $\omega_2 = -\omega_1(\cdot, I\cdot)$, where

$$I = \begin{pmatrix} i & 0 \\ 0 & i^* \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

on $\mathbb{R}^{4n} = \mathbb{R}^{2n} \times (\mathbb{R}^{2n})^*$.

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  on $\mathbb{R}^{4n} = \mathbb{R}^{2n} \times (\mathbb{R}^{2n})^*$.
- Let $\omega^{\mathbb{C}} = dp \wedge dq = \omega_1 + i\omega_2$, then

$$\Omega = \omega^{\mathbb{C}} \otimes \partial_t + \bar{\omega}^{\mathbb{C}} \otimes \partial_{\bar{t}}.$$

Definition (B.-F. '24)

Let (W, I) a complex manifold. An $\mathbb{R}^2$-valued polysymplectic form $\Omega = \omega_1 \otimes \partial_1 + \omega_2 \otimes \partial_2$ on W is called *complex-regularized* if $\omega_2 = -\omega_1(\cdot, I\cdot)$.

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The triple (W, I, Ω) is called a *complex-regularized polysymplectic (CRPS) manifold*.

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Then $\Omega = d\theta_1 \otimes \partial_1 - d\theta_2 \otimes \partial_2$ is CRPS on (W, I).

First properties

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Let $(W, I, \Omega = \omega_1 \otimes \partial_1 + \omega_2 \otimes \partial_2)$ a CRPS manifold. Then both ω_1 and ω_2 are symplectic forms.

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So Ω being non-degenerate implies $\ker \omega_1^\flat = \ker \omega_2^\flat = 0$. $\square$

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Lemma

Let $\psi : (W, I, \Omega) \to (W', I', \Omega')$ a diffeomorphism between CRPS manifolds, such that

$$\psi^* \Omega' = \Omega.$$

Then ψ is holomorphic with respect to I and I'.

Quick recap

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A holomorphic Hamiltonian system is a tuple $(W, I, \omega^{\mathbb{C}}, F)$, where $F : W \to \mathbb{C}$ is holomorphic. It induces a holomorphic vector field $\mathcal{X}_F$ by

$$\omega^{\mathbb{C}}(\mathcal{X}_F, \cdot) = dF.$$

Relation to CRPS manifolds

Proposition

For a complex manifold (W, I) there is a bijection

$$\{\text{CRPS forms}\} \xleftrightarrow{1-1} \{\text{holomorphic symplectic forms}\}$$

Given by

$$\omega_1 \otimes \partial_1 + \omega_2 \otimes \partial_2 \mapsto \omega_1 + i\omega_2$$
$$\omega^{\mathbb{C}} \otimes \partial_t + \bar{\omega}^{\mathbb{C}} \otimes \partial_{\bar{t}} \leftarrow\!\shortmid \omega^{\mathbb{C}}$$

CRPS geometry

Holomorphic symplectic geometry

CRPS geometry

- Real valued Hamiltonians

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CRPS geometry
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Recall: $F : (W, I, \Omega) \to \mathbb{R}^2$ is a current if there exists X_F such that $\Omega(X_F, \cdot) = dF$.

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Darboux theorem

Corollary

Let (W, I, Ω) a CRPS manifold. Around every point in W there exist coordinates $\{q_1^\alpha, q_2^\alpha, p_1^\alpha, p_2^\alpha\}$ where $\alpha = 1, \dots, n$ such that

$$I\frac{\partial}{\partial q_1^\alpha} = \frac{\partial}{\partial q_2^\alpha} \qquad\qquad I\frac{\partial}{\partial p_1^\alpha} = -\frac{\partial}{\partial p_2^\alpha}$$

and

$$\omega_1 = \sum_\alpha \left(dp_1^\alpha \wedge dq_1^\alpha + dp_2^\alpha \wedge dq_2^\alpha\right)$$
$$\omega_2 = \sum_\alpha \left(dp_1^\alpha \wedge dq_2^\alpha - dp_2^\alpha \wedge dq_1^\alpha\right).$$

Darboux theorem

Corollary

$$I\frac{\partial}{\partial q_1^\alpha} = \frac{\partial}{\partial q_2^\alpha} \qquad I\frac{\partial}{\partial p_1^\alpha} = -\frac{\partial}{\partial p_2^\alpha}$$

and

$$\omega_1 = \sum_\alpha (dp_1^\alpha \wedge dq_1^\alpha + dp_2^\alpha \wedge dq_2^\alpha)$$
$$\omega_2 = \sum_\alpha (dp_1^\alpha \wedge dq_2^\alpha - dp_2^\alpha \wedge dq_1^\alpha)\,.$$

Proof: Follows from the Darboux theorem for holomorphic symplectic manifolds (see thesis of Wagner for proof). □

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For $Z : \mathbb{T}^2 \to W$ define $\tilde{Z} = Z \times \mathrm{id} : \mathbb{T}^2 \to W \times \mathbb{T}^2$ and

$$\begin{aligned} \mathcal{A}(Z) &= \int_{\mathbb{T}^2} \tilde{Z}^* \tilde{\Theta} \\ &= \int_{\mathbb{T}^2} \big(\theta_1(\partial_1 Z) + \theta_2(\partial_2 Z) \big) \, d\mathcal{V} \end{aligned}$$

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Arnold conjecture

Theorem (B.-F. '24)

Let $Q = \mathbb{T}^{2n}$ and $W = T^*Q$ with the standard CRPS form $\omega_1 \otimes \partial_1 + \omega_2 \otimes \partial_2$.

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Let $Q = \mathbb{T}^{2n}$ and $W = T^*Q$ with the standard CRPS form $\omega_1 \otimes \partial_1 + \omega_2 \otimes \partial_2$.
If $H : \mathbb{T}^2 \times W \to \mathbb{R}$ is given by $H(t, q, p) = \frac{1}{2}|p|^2 + h(t, q, p)$ for $h : \mathbb{T}^2 \times W \to \mathbb{R}$ smooth with finite C^2-norm,

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Corollary

When $V : Q \to \mathbb{R}$ has finite C^2-norm then $-\Delta q = \nabla V(q)$ has at least $(2n+1)$ solutions.

Overview of proof of Arnold conjecture

- Follows from studying Floer curves

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- Rest of the proof follows the line of reasoning of the cuplength results from Albers-Hein.

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Let $(W, I, \Omega = \omega_1 \otimes \partial_1 + \omega_2 \otimes \partial_2)$ a CRPS manifold and $\psi : W \to W$ a diffeomorphism.

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Theorem

Let $\psi : \mathbb{R}^{4n} \to \mathbb{R}^{4n}$ a diffeomorphism preserving Ω such that

$$\psi(B_r^{4n}) \subseteq B_R^2 \times \mathbb{R}^{4n-2}$$

where B_R^2 is the R-ball in the (q_i^α, p_j^α)-plane for some $i, j \in \{1, 2\}$ and $\alpha \in \{1, \ldots, n\}$.
Then $r \leq R$.

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Corollary

Let $\psi_\nu : W \to W$ a sequence of diffeomorphisms preserving Ω that converge to a diffeomorphism $\psi : W \to W$ in the C^0-limit. Then $\psi^*\Omega = \Omega$.

Holomorphic Lagrangians $\to$ harmonic maps

Question: Given a holomorphic symplectic manifold $(W^{4n}, \omega^{\mathbb{C}})$ and a closed complex manifold L^{2n}, is there a holomorphic embedding $\iota : L \hookrightarrow W$ such that $\iota^*\omega^{\mathbb{C}} = 0$.

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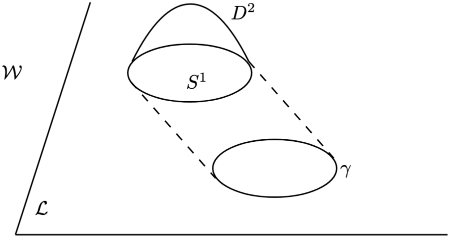
Morse theory of geodesics on L



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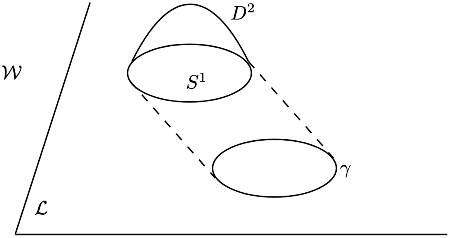

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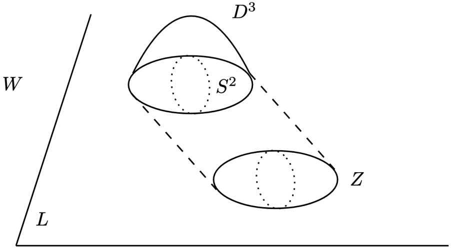
Morse theory of minimal spheres on L



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Boundary fixed to the conormal bundle $N^*\Gamma \subseteq T^*Q \rightarrow$ holomorphic Lagrangian if Γ is a complex submanifold

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Thank you!