

Symplectic : Contact

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Poisson : Jacobi

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Affine : Projective

based on joint work with M.A. Salazar (arXiv:1406.2138)  
& on ongoing joint work with C. Angelo & M.A. Salazar.

# The Main Characters (I)

Symplectic  $(S, \omega)$

$\omega \in \Omega^2(M)$  s.t.  $d\omega = 0$  &

$$\omega^b : TM \xrightarrow{\sim} T^*M$$

Contact  $(C, H)$

$H \subset TC$  codim 1 s.t.

$$[\cdot, \cdot] \Big|_{H \text{ mod } H} \text{ non-deg.}$$

$$L = TC / H \rightarrow C$$

$$H \times H \rightarrow L$$

## The Main Characters (2)

Poisson  $(P, \{ \cdot, \cdot \})$

$\{ \cdot, \cdot \}$  Lie bracket on  $C^\infty(P)$

s.t.

$\forall f \in C^\infty(P) \quad \{f, -\} \in \mathfrak{X}(P)$

$\left\{ \text{g}^*, \{ \cdot, \cdot \}_{\text{Lie}} \right\}$

Jacobi  $(L \rightarrow M, \{ \cdot, \cdot \})$

$L \rightarrow M$  line bundle &

$\{ \cdot, \cdot \}$  local Lie bracket on  $\Gamma(L)$

$\forall u, v \in \Gamma(L)$

$\text{Supp } \{u, v\} \subset \text{Supp } u \cap \text{Supp } v.$

$(O(1) \rightarrow P(\text{g}^*), \{ \cdot, \cdot \})$

## Interlude : $(G, X)$ structures

$G \curvearrowright X$  rigid if  $U \subset X$  open &  $g|_U = \text{id}|_U \Rightarrow g = e$ .

$(G, X)$  structure on  $B$ : atlas  $\mathcal{A} = \{(U_\alpha, \phi_\alpha : U_\alpha \rightarrow \phi_\alpha(U_\alpha) \subset X)\}$  s.t.  
 $\forall \alpha, \beta \text{ with } U_\alpha \cap U_\beta \neq \emptyset \exists g \in G \text{ s.t. } \phi_\beta \circ \phi_\alpha^{-1} = g|_{\phi_\alpha(U_\alpha \cap U_\beta)}$ .

Equivalently :

(i) local diffeo dev:  $\tilde{B} \rightarrow X$ ,

(ii)  $\rho: \pi_1 B \rightarrow G$  s.t. dev  $\pi_1 B$ -equivariant.

# The Main Characters (3)

$\mathbb{Z}$ -Affine ( $B, \mathcal{A}$ )

$\mathcal{A} = (\text{Aff}_{\mathbb{Z}}(\mathbb{R}^n), \mathbb{R}^n)$  str.

$$\text{Aff}(\mathbb{R}^n) = \text{GL}(n; \mathbb{R}^n) \times \mathbb{R}^n$$

$$\text{Aff}_{\mathbb{Z}}(\mathbb{R}^n) = \text{GL}(n; \mathbb{Z}) \times \mathbb{R}^n$$

$\mathbb{Z}$ -Projective ( $B, \mathcal{A}$ )

$\mathcal{A} = (\text{PGL}_{\mathbb{Z}}^{+}(\mathbb{R}^{n+1}), \mathbb{RP}^n)$  str.

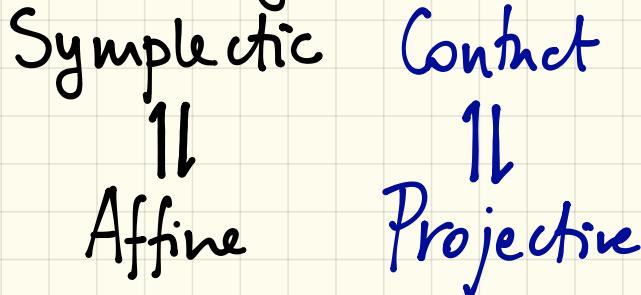
$$\text{PGL}^{+}(\mathbb{R}^{n+1}) = \text{GL}(n+1; \mathbb{R}) / \mathbb{R}_{>0}$$

$$\text{PGL}_{\mathbb{Z}}^{+}(\mathbb{R}^{n+1}) = \text{GL}(n+1; \mathbb{Z})$$

$$\text{dev}^* O(1) \rightarrow \tilde{B}$$

$$\text{dev} : \tilde{B} \xrightarrow{\sim} \mathbb{RP}^n$$

Why?



- Arnol'd (1989):

- Recent interest in Contact/Jacobi

- "Compactness easier in Contact/Jacobi/Projective"

→ Borun, Eliashberg, Murphy

→ Contact : (Milnor)  $\Sigma^2$  closed affine ( $\Rightarrow \chi(\Sigma) = 0$ ).  
(Benzoni)  $\wedge \Sigma^2$  closed  $(\Sigma, A)$  proj.

$T^*C$   
 $\cup$

## Homogenisation

$$(L^* \setminus 0, \omega_{can}|_{L^* \setminus 0}) \xleftarrow{} (C, H) \quad L = TC/H$$

$$(L^* \setminus 0, \{\cdot, \cdot\}) \xleftarrow{} (L \rightarrow M, \{\cdot, \cdot\})$$

$$(L^* \setminus 0, \hat{A}) \xleftarrow{} (B, A)$$

$$p: \pi_1 B \rightarrow PGL^+(\mathbb{R}^{n+1})$$

$$\pi_1(L^* \setminus 0)$$

$$Aff(\mathbb{R}^{n+1})$$

$$\tilde{L} := dw^* O(1)$$

$$\tilde{L}^* \setminus 0 \cong dw^*(O(-1) \setminus 0)$$

$$dw: \tilde{L}^* \setminus 0 \rightarrow O(-1) \setminus 0 \\ \rightarrow \mathbb{R}^{n+1} \setminus 0$$

## Inclusions

$$(P, \{\cdot, \cdot\}) \longrightarrow (P \times \mathbb{R} \rightarrow P, \{\cdot, \cdot\})$$

$$(B, A) \longrightarrow (B, A')$$

$$\text{Aff}(\mathbb{R}^n) \longrightarrow GL(n+1; \mathbb{R})$$

$$(A, \underline{b}) \hookrightarrow \left( \begin{array}{c|c} A & \underline{b} \\ \hline 0 & 1 \end{array} \right)$$

$$\mathbb{R}^n \hookrightarrow \mathbb{R}P^n$$
$$x \mapsto [x : 1].$$

$$\text{Aff}_{\mathbb{Z}}(\mathbb{R}^n) \not\hookrightarrow GL(n+1; \mathbb{Z})$$

## Pre-quantisation (?)

$$(S, \omega), [\omega] \in H^2(S; \mathbb{Z}) \longrightarrow (C, H = \text{Ker } \theta)$$

$S' \rightarrow C \rightarrow S$   $[\omega]$ ,  $\theta \in \Omega^1(C)$  connection 1-form

$$(B, A) \text{ strong } \mathbb{Z}\text{-affine} \longrightarrow (B, A') \mathbb{Z}\text{-projective}$$

$$(\text{Aff}(\mathbb{Z}^n), \mathbb{R}^n)$$

$$\text{Aff}(\mathbb{Z}^n) = GL(n; \mathbb{Z}) \times \mathbb{Z}^n$$

## A simple but nice observation

Theorem (Ovsienko)  $B$  simply connected,  $\dim B = k$

- $K = 2n : B$  affine  $\Leftrightarrow \exists f_1, \dots, f_{2n} \in C^\infty(B)$  s.t.  $\sum_{j=1}^n df_{2j-1} \wedge df_{2j}$  sym.  
dvw:  $B \rightarrow \mathbb{R}^m$
- $K = 2n+1 : B$  proj  $\Leftrightarrow \exists f_1, \dots, f_{2n+2} \in C^\infty(B)$  s.t.  $\text{Ker} \left( \sum_{j=1}^{n+1} f_{2j-1} df_{2j} - f_{2j} df_{2j-1} \right)$  contact.  
dvw:  $B \rightarrow \mathbb{RP}^m$
- $B$  proj  $\Leftrightarrow \exists f_1, \dots, f_{k+1} \in C^\infty(B)$  s.t.  $\sum_{j=1}^{k+1} (-1)^j f_j df_1 \wedge \dots \wedge \hat{df_j} \wedge \dots \wedge df_{k+1}$   
Volume

# Lagrangian foliations

Lagrangian foliation  $\mathcal{F}$  on  $(S, \omega)$  :  $\forall x \in S \quad T_x \mathcal{F}$  Lagrangian.

Theorem (Weinstein)

B leaf of Lagrangian foliation  $\Leftrightarrow$  B affine.

Proof.

$(\Rightarrow)$  : Local model  $T^*V \rightarrow V$   $V$  vector space.  
 $(\Leftarrow)$  : Lag foliat  $\mathcal{F}_0$  on  $T^*\mathbb{R}^n \cong \mathbb{K}^n \times (\mathbb{R}^n)^*$   
 $\rightarrow$  Lg fl  $\tilde{\mathcal{F}}$  on  $T^*\tilde{B} \cong \text{d}v^*T^*\mathbb{R}^n \quad \mathbb{R}^n \times U$ .

## Interlude : Legendrians, jets & projectivisation (I)

$Y \xleftarrow{i} (C, H)$  Legendrian :  $D_i(TY) \subset H$  &  $D_i(TY)$  Lag.

Local model :  $L \rightarrow Y$  line bundle,  $J^1 L \rightarrow Y$  first jet bundle,

$Y \xleftarrow{o} (J^1 L, H_{\text{can}})$  Cartan contact structure

$$J^1 L = \{ j_x^1 u : u \in \pi(L) \}.$$

$$L = Y \times \mathbb{R} \rightarrow Y \quad H_{\text{can}} = \ker(dt - \lambda_{\text{can}})$$

## Interlude : Legendrians, jets & projectivisation (2)

Given  $Y \rightsquigarrow P(T^*Y) \rightarrow Y$  projectivisation of  $T^*Y \rightarrow Y$

$(P(T^*Y), H_{can} = "Ker \lambda_{can}")$  contact &

$\forall x \in Y \quad P(T_x^*Y) \hookrightarrow (P(T^*Y), H_{can})$  Legend.

# Legendrian foliations

Legendrian foliation  $\mathcal{F}$  on  $(C, H)$ :  $\forall x \in C \quad T_x \mathcal{F}$  Legendrian

Theorem (Pang, S.?)

B leaf of Legendrian foliation  $\iff$  B projective  
Proof.

$(\Rightarrow)$  local model is  $P(T^*V) \rightarrow V$

$(\Leftarrow)$  same as before:  $J^1 O(1) \cong RP^n \times \mathbb{R}^{n+1}$   
 $\downarrow_{RP^n} \quad \{RP^n \times dt.\}$

# Isotropic realisations of Poisson manifolds (I)

$(S, \omega)$

$\downarrow \phi$

$(P, \{ \cdot, \cdot \})$

$\dots$

$B = P/F$

isotropic realisation :  $\phi$  Poisson, surj subm & iso fibres

$\Rightarrow (P, \{ \cdot, \cdot \})$  regular

## (S, ω) Isotropic realisations of Poisson manifolds (2)

↓ φ isotropic realisation : φ Poisson, surj subm & iso fibres.

$$(P, \{ \cdot, \cdot \}) \xrightarrow{\quad} (P, \{ \cdot, \cdot \}) \text{ regular}$$

$$B = P/F$$

Theorem (Duistermaat, Dazord & Delzant, ...)

If fibres of φ compact & connected

(a) "B = P/F  $\mathbb{Z}$ -affine"

(b)  $\mathbb{Z}$ -affine structure + coh class  $\rightsquigarrow$  classification

# Isotropic realisations of Jacobi manifolds (I)

$(C, H)$

$\downarrow \phi$  isotropic realisation:  $\phi$  Jacobi, surj w/bm.,  $\forall x \in M$   
 $(L \rightarrow M, \{ \cdot, \cdot \})$   $\forall y \in \phi^{-1}(x) \quad T_y \phi^{-1}(x) + H_y = T_y C \text{ &}$

$B = M/F$

$T_y \phi^{-1}(x) \cap H_y \text{ iso.}$

$G \cap (C, H)$  w/ ft for  $\exists \phi: C \rightarrow P(\mathfrak{g}^*)$  s.t.  
Selctor & s.

## Isotropic realisations of Jacobi manifolds (2)

isotropic realisation :  $\phi$  Jacobi, surj nbm. , ...

Theorem (Salazar & S.)

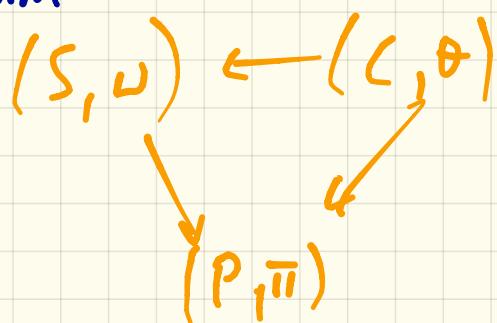
(1)  $(L \rightarrow M, \{ \cdot, \cdot \})$  regular & leaves even  $\dim^L$

If fibres of  $\phi$  compact & connected

(2) "  $B = M/F$   $\mathbb{Z}$ -projective "

(3)  $\mathbb{Z}$ -projective str. + coh class  $\Rightarrow$  classification

(4) Pre-quantisation "commutes" with iso realisations.



# PMCTs (I)

$(S, \omega)$



$(P, \{ \cdot, \cdot \})$



$B = P/F$

PMCT:  $\exists$  "compact" symplectic integration.

Examples: compact symplectic,  $\mathbb{Z}$ -affine,

duals of compact Lie algebras, ...

## PMCTs (2)

PMCT:  $\exists$  "compact" symplectic integration.

Theorem (Crainic, Fernandes, Martínez-Torres)

$(P, \{ \cdot, \cdot \})$  regular PMCT.

(a)  $B = P/F$   $\mathbb{Z}$ -affine orbifold

(b)  $\mathbb{Z}$ -affine  $\Leftarrow$  Poisson structure

(c) Symplectic gerbe  $\dots \rightarrow$  existence iso realisations of  $(P, \{ \cdot, \cdot \})$ .

# Why JMCTs?

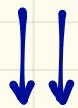
## Theorem (Marwt)

$\mathfrak{g}$  compact, semisimple. Complete local description of

Poisson  $(\mathcal{S}(g^*), \{\cdot, \cdot\}_{\mathcal{S}(g^*)})$ .

- Obs. 1)  $(\mathcal{S}(g^*), \{\cdot, \cdot\}_{\mathcal{S}(g^*)})$  not integrable in general
- 2) But  $(\mathcal{S}(T^*G), \text{tr}_\omega) \rightrightarrows (\mathcal{S}(g^*), \{\cdot, \cdot\}_{\mathcal{S}(g^*)})$   
compact integration.

$(C, H)$



$(L \rightarrow M, \{., .\})$



$B = M/F$

## JMCTs (I)

JMCT :  $\exists$  "compact" contact integration

Examples :  $\mathbb{Z}$  compact symplectic manifolds,

$\mathbb{Z}$ -projective manifolds, projectivisation of  
duals of compact Lie algebras, PMCT with

$\mathbb{Z}$  symplectic form, ...

## JMCTs (2)

JMCT : } "compact" contact integration

Theorem (Angulo & Salazar & S.)

(1)  $(L \rightarrow M, \{ \cdot, \cdot \})$  JMCT  $\Rightarrow (L \rightarrow M, \{ \cdot, \cdot \})^{\text{"}} \cong \text{Poisson}$

(2)  $(L \rightarrow M, \{ \cdot, \cdot \})$  regular JMCT  $\Rightarrow B = M/F$   $\mathbb{Z}$ -projective  
orbifold.

## Next steps & Questions

- Pre-quantisation "commutes" with being of compact type?
- Rigidity results?
- Construction of examples of JMCTs?
- Geometry of Legendrian foliations?