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PMATH 351

REAL ANALYSIS

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TABLE OF CONTENTS

1	Chains and Zorn's Lemma	1
2	Cardinal arithmetic	1
3	2017-09-18	3
3.1	Last class: C.B.S Theorem	3
4	2017-09-22	5
4.1	Metric Spaces	5
5	2017-09-25	6
5.1	The ℓ_p -spaces	8
6	2017-09-29	9
7	2017-10-02	10
7.1	Continuity	11
8	2017-10-06	12
9	2017-10-16	12
9.1	Characterizations of Completeness	12
10	2017-10-18	13
11	2017-10-20	16
11.1	Completeness of Metric Spaces	17
12	2017-10-23	18
12.1	Compactness	19
13	2017-10-25	20
14	2017-10-27	21
15	2017-10-30	22
16	2017-11-01	25
17	2017-11-03	27

18	2017-11-06	29
19	2017-11-08	31
19.1	Baire Category Theorem	31
20	2017-11-10	33
21	2017-11-13	35
21.1	Baire-1 Functions	35
22	2017-11-15	37
22.1	On the Banach spaces $C(X)$, X compact	37
23	2017-11-17	39
24	2017-11-20	41
24.1	Towards Stone-Weierstrauss Theorem	42
25	2017-11-22	43
26	2017-11-24	44
27	2017-11-27	46
28	2017-11-29	47
29	2017-12-01	48

LIST OF THEOREMS

2.1	Theorem (Cantor-Bernstein-Schroder Theorem)	2
3.1	Proposition (surjectivity)	3
3.1	Theorem (Comparison Theorem)	3
3.2	Proposition	4
3.1	Corollary	5
3.2	Theorem (Cantor)	5
4.1	Lemma	5
5.1	Theorem (Minkowski's Inequality)	6
5.1	Corollary	7
5.2	Theorem	8
5.3	Theorem	8
6.1	Proposition	9
6.2	Proposition (characterizations of interior)	10
6.3	Proposition	10
7.1	Theorem (characterization of the closure)	11
7.2	Theorem (characterization of continuity at a point)	12
8.1	Corollary	12
8.1	Theorem	12
9.1	Theorem	12
9.1	Proposition	13
9.2	Theorem (Nested set characterization of completeness)	13
10.1	Theorem	13
10.2	Theorem (abstract M -test)	14
11.1	Corollary	16
11.1	Theorem	17
12.1	Proposition	19
12.2	Proposition	19
12.1	Theorem (continuous image of compact is compact)	19
12.1	Corollary (Extreme Value Theorem)	19
12.2	Theorem (finite intersection property)	20
13.1	Theorem (Characterizations of compact metric spaces)	20
14.1	Corollary	22
15.1	Theorem (sequential characterization of uniform continuity)	23
15.2	Theorem (continuous on compact is uniformly continuous)	23
15.3	Theorem	24
15.1	Corollary	24
16.1	Proposition	25
17.1	Theorem (Banach's Contraction Mapping Theorem)	27
17.2	Theorem (Picard-Lindelof Theorem)	28
18.1	Theorem (Edelstein)	30
19.1	Lemma	32
19.1	Theorem (Baire Category Theorem)	32
20.1	Corollary	33
20.2	Corollary	34
20.1	Theorem (Uniform Boundedness Principle)	34
20.3	Corollary (Banach-Stenhaus Theorem)	34
21.1	Theorem (Baire)	35
21.1	Corollary	37
22.1	Corollary	37
22.1	Lemma	37
22.1	Theorem (Weierstrauss approximation theorem)	38
23.1	Corollary	40

23.2 Corollary	40
23.1 Theorem (nowhere differentiable functions are generic)	40
24.1 Theorem	41
24.1 Proposition	42
24.2 Theorem (Stone)	42
25.1 Corollary	43
25.1 Theorem (Stone-Weierstrauss Theorem)	44
26.1 Corollary (Stone-Weierstrauss without constant functions)	45
26.2 Corollary	45
27.1 Proposition (Properties of relatively compact subsets)	46
27.1 Lemma	47
28.1 Theorem (Arzela-Ascoli Theorem)	47
29.1 Theorem (Peano's Theorem)	48

Abstract

These notes are intended as a resource for myself; past, present, or future students of this course, and anyone interested in the material. The goal is to provide an end-to-end resource that covers all material discussed in the course displayed in an organized manner. If you spot any errors or would like to contribute, please contact me directly.

1 CHAINS AND ZORN'S LEMMA

Let (X, \leq) be a poset. A chain is any subset $C \subseteq X$ such that (C, \leq) is totally ordered.

Office hours:

1. Today 2:30 - 3:20
2. Wednesday next week 2:30 - 4:30

Or, email nspronk@uwaterloo.ca

2 CARDINAL ARITHMETIC

i. \mathbb{R}

ii. $\mathbb{R} \underset{f}{\sim} (-1, 1), f(x) = x/|x| + 1$ (exercise: exhibit f^{-1})

iii. $a < b$ in $\mathbb{R}. (0, 1) \underset{g}{\sim} (a, b), g(x) = a + x(b - a)$

Notation: $\aleph_0 = |\mathbb{N}|$ ("aleph naught"), $c = |\mathbb{R}|$ ("continuous")

Arithmetic: Let A, B be sets.

$$|A| + |B| = |A \sqcup B|$$

$$|A||B| = |A \times B|$$

$$|A|^{|B|} = |A^B| (B \neq \emptyset, A^B = \{f : B \rightarrow A \mid \text{function}\})$$

$A \sqcup A$ is two copies of A , $\sim A \times \{1, 2\}$

Properties

- (commutativity) $|A| + |B| = |B| + |A|$, $|A||B| = |B||A|$
- (distributivity) $|A|(|B| + |C|) = |A||B| + |A||C|$

$$A \times (B \sqcup C) \sim (A \times B) \sqcup (A \times C)$$

- (Exponential laws)

$$|A|^{|B|+|C|} = |A|^{|B|}|A|^{|C|}, |A|^{|B||C|} = (|A|^{|B|})^{|C|}$$

$$(B \neq \emptyset \neq C)$$

$$A^{B \sqcup C} \sim A^B \times A^C \text{ via } \varphi \mapsto (\varphi|_B, \varphi|_C)$$

$$A^{B \times C} \sim (A^B)^C \text{ via } \varphi \mapsto (\varphi(b, \cdot) : C \rightarrow A)$$

Now, for sets A, B , define $A \preceq B$ if there is an injection $f : A \rightarrow B$.

Sometimes write $A \preceq B$. As above:

$$\text{(reflexivity)} \quad A \preceq A$$

$$\text{(transitivity)} \quad A \preceq B, B \preceq C \implies A \preceq C$$

Seems reasonable to write $|A| \leq |B|$, in this case.

Question: Is \leq in cardinal numbers anti-symmetric?

Theorem 2.1 (Cantor-Bernstein-Schroder Theorem). If, for non-empty set A, B we have $A \preceq B, B \preceq A$, then $A \sim B$. I.e. if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Proof. Our assumption is that we have injections $A \xrightarrow{\varphi} B, B \xrightarrow{\psi} A$.

To avoid triviality, let us suppose that neither φ nor ψ is surjective. Thus $\varphi(A) \subsetneq B, \psi(B) \subsetneq A$.

Let $A_0 = A, A_1 = \psi(B), A_2 = \psi \circ \varphi(A)$ and we inductively define $A_{n+2} = g(A_n), g = \psi \circ \varphi$.

Then $A_2 \subsetneq A_1 \subsetneq A_0$, so by applying injection g ,

$$\begin{aligned} A_2 &\subsetneq A_1 \subsetneq A_0 \\ &\vdots \\ A_{n+1} &\subsetneq A_n \subsetneq A_{n-1} \end{aligned}$$

Hence, we may decompose

$$\begin{aligned} A &= A_0 = (A_0 \setminus A_1) \cup A_1 \\ &= (A_0 \setminus A_1) \cup (A_1 \setminus A_2) \cup A_2 \\ &\vdots \\ &= \bigcup_{n=1}^{\infty} (A_{n-1} \setminus A_n) \cup A_{\infty} \end{aligned}$$

where $A_{\infty} = \bigcap_{n=1}^{\infty} A_n = \bigcap_{n=2}^{\infty} A_n$, we likewise observe

$$A_1 = \bigcup_{n=2}^{\infty} (A_{n-1} \setminus A_n) \cup A_{\infty}.$$

Picture:

$$\underbrace{\overbrace{A_0 \setminus A_1} \quad \overbrace{A_1 \setminus A_2} \quad \dots \quad \overbrace{A_{\infty}}} \quad A_0$$

Using definitions of the sets A_n ($n \geq 2$), we have $g(A_{n-1} \setminus A_n) = A_{n+1} \setminus A_{n+2}$. Define

$$h : A_0 \rightarrow A_1, h(x) = \begin{cases} g(x), & \text{if } x \in A_{n-1} \setminus A_n, n \text{ odd} \\ x, & \text{otherwise} \end{cases}$$

Then h is a bijection. Thus

$$A = A_0 \xrightarrow{h} A_1 = \psi(B), B \xrightarrow{\psi} \psi(B)$$

so we conclude that $A \sim B$. □

Examples:

1. Let $a < b$ in \mathbb{R} . Then $[a, b] \preceq \mathbb{R}$ (obvious)
 $\mathbb{R} \sim (-1, 1) \sim (0, 1) \sim (a, b) \preceq [a, b]$
 I.e. $[a, b] \preceq \mathbb{R}$ and $\mathbb{R} \preceq [a, b]$ so $\mathbb{R} \sim [a, b]$

3 2017-09-18

3.1 LAST CLASS: C.B.S THEOREM

If $A \preceq B$ and $B \preceq A$ then $A \sim B$.

Examples:

- (i) $\mathcal{P}(\mathbb{N}) \sim \mathbb{R}$, i.e. $|\mathcal{P}(\mathbb{N})| = c$.

$$\mathcal{P}(\mathbb{N}) \sim \{0, 1\}^{\mathbb{N}}, \text{ via } A \mapsto \chi_A \text{ where } \chi_A(n) \begin{cases} 1 & , n \in A \\ 0 & , n \notin A \end{cases} \text{ ("characteristic indicator")}$$

$$\{0, 1\}^{\mathbb{N}} \sim \mathcal{P}(\mathbb{N}), \text{ via } (x_k)_{k=1}^{\infty} \xrightarrow[\text{injective}]{\quad} \chi_A \text{ where } \sum_{k=1}^{\infty} \frac{x_k}{3^k} = 0.x_1x_2x_3\ldots \text{ (ternary representation)}$$

$$[0, 1] \sim \{0, 1\}^{\mathbb{N}}, 0.x_1x_2x_3\ldots = \sum_{k=1}^{\infty} \frac{x_k}{2^k} \text{ (binary representation) (never allow } 0.111\ldots = 1!) \mapsto (x_k)_{k=1}^{\infty}$$

$$\mathcal{P}(\mathbb{N}) \sim \{0, 1\}^{\mathbb{N}} \preceq [0, 1] \preceq \{0, 1\}^{\mathbb{N}} \sim \mathcal{P}(\mathbb{N})$$

so, by C.B.S. Theorem, we have $|\mathcal{P}(\mathbb{N})| = |[0, 1]| = c = |\mathbb{R}|$.

(ii)

2nd lecture:

- (iii) $\mathbb{N} \sim \mathbb{Q} \sim \mathbb{N}^2$

$$\mathbb{N} \preceq \mathbb{Q}$$

$$\mathbb{Q} \preceq \mathbb{Z} \times \mathbb{N}, \text{ via } \frac{m}{n} \mapsto (m, n) \text{ (gcd}(m, n) = 1)$$

$$\mathbb{Z} \times \mathbb{N} \sim \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}, \text{ as } \mathbb{Z} \sim \mathbb{N}$$

$$\mathbb{N}^2 \preceq \mathbb{N}, \text{ via } (m, n) \mapsto 2^m 3^n$$

Hence $\mathbb{N} \preceq \mathbb{Q} \preceq \mathbb{Z} \times \mathbb{N} \sim \mathbb{N}^2 \preceq \mathbb{N}$ so, by C.B.S. Theorem, $\mathbb{N} \sim \mathbb{Q} \sim \mathbb{N}^2$.

Notation: We say that a set A is

- countable if $A \preceq \mathbb{N}$, i.e. $|A| \leq \aleph_0$
- denumerable if $A \sim \mathbb{N}$, i.e. $|A| = \aleph_0$

Proposition 3.1 (surjectivity). Suppose X and Y are non-empty sets and there is a surjection $g : X \rightarrow Y$. Then $Y \preceq X$.

Proof. Let $f : \mathcal{P}(X) \setminus \{\emptyset\} \rightarrow X$ be a choice function (AC). For each $y \in Y$, we have $g^{-1}(\{y\}) = \{x \in X : g(x) = y\} \neq \emptyset$, as g is surjective. Define $h : Y \rightarrow X$ be given by $h(y) = f(g^{-1}(\{y\}))$ and h is injective, as if $y_1 \neq y_2$, $\{y_1\} \cap \{y_2\} = \emptyset$, so we see that $g^{-1}(\{y_1\}) \cap g^{-1}(\{y_2\}) = \emptyset$ too. \square

Theorem 3.1 (Comparison Theorem). Let X, Y be sets. Then either $X \preceq Y$ or $Y \preceq X$.

Proof. If $X \neq \emptyset$, then $X \preceq Y$; likewise if $Y = \emptyset$. Hence assume $X \neq \emptyset \neq Y$. We let

$$\Delta = \{(A, f) : A \in \mathcal{P}(X) \setminus \{\emptyset\}, f \in Y^A \text{ is an injection mapping from } A \text{ to } Y\}$$

We observe that $\Delta \neq \emptyset$. If $x \in A, y \in Y$, then $(\{x\}, x \mapsto y) \in \Delta$. On Δ let

$$(A, f) \preceq (B, g) \iff A \subseteq B \subseteq X, g|_A = f$$

Notice that \preceq is reflexive, anti-symmetric, and transitive, hence is a partial order on Δ . Let $\Gamma\{(A_i, f_i)\}_{i \in I}$ be a chain in (Δ, \preceq) . We let $A = \bigcup_{i \in I} A_i$ and $f \in Y^A$ be given by $f(x) = f_i(x)$ provided $x \in A_i$.

Notice that f is well-defined. Say $x \in A_i$ and $x \in A_j$, then, since Γ is a chain, $A_i \subseteq A_j$, say, and $f_j|_{A_i} = f_i$.

Furthermore, if $x_1 \neq x_2$ in A , then $x_1 \in A_{i_1}, x_2 \in A_{i_2}$, and we may suppose $A_{i_1} \subseteq A_{i_2}$. Then $f(x_1) = f_{i_1}(x_1) = f_{i_2}(x_1) \neq f_{i_2}(x_2) = f(x_2)$, so f is an injection. Thus $(A, f) \in \Delta$, and is an upper bound of Γ .

Thus, there is a maximal element $(M, g) \in \Delta$, by Zorn's Lemma.

Case #1: $M = X$. Then $X = M \preceq_g Y$.

Case #2: $M \subsetneq X$. We wish to see that g must be surjective. Suppose not, i.e., there is $y_0 \in Y \setminus g(M)$. Since $M \subsetneq X$, there is $x_0 \in X \setminus M$. Define $h : M \cup \{x_0\} \rightarrow Y$ by

$$h(x) = \begin{cases} g(x) & x \in M \\ y_0 & x = x_0 \end{cases} \text{ injective!}$$

Then $(M \cup \{x_0\}, h) \in \Delta$, and $(M, g) \not\preceq (M \cup \{x_0\}, h)$, contradicting maximality of (M, g) . Thus, we have that g is surjective. Thus $Y \underbrace{\preceq}_{g^{-1}} X$.

□

Proposition 3.2. Let A be a set. Then TFAE:

- (i) $n \leq |A|$ for all $n \in \mathbb{N}$
- (ii) $\aleph_0 \leq |A|$ (A is infinite)
- (iii) there is $B \subsetneq A$ s.t. $|B| = |A|$
- (iv) $1 + |A| = |A|$ (Hilbert hotel)
- (v) $\aleph_0 + |A| = |A|$

Proof. (i) \Rightarrow (ii) We have that for each n in \mathbb{N} there is an injection $\varphi_n : \{1, \dots, n\} \rightarrow A$. Inductively, define $f : \mathbb{N} \rightarrow A$ by

$$f(1) = \varphi_1(1)$$

$$f(n+1) = \varphi_{n+1}(k)$$

where $k = \min j \in \{1, \dots, n+1\} : \varphi_{n+1}(j) \notin \{f(1), \dots, f(n)\}$.

Then f is injective by construction.

(ii) \Rightarrow (iii) We have $\mathbb{N} \preceq_f A$. Let $B = A \setminus \{f(1)\}$. Define $g : A \rightarrow B$ by

$$g(x) = \begin{cases} f(n+1) & \text{if } x = f(n), n \in \mathbb{N} \\ x & \text{otherwise} \end{cases}$$

Then $A \sim_g B$, i.e., $|A| = |B|$.

(iii) \Rightarrow (iv) We suppose there is $x_0 \in A \setminus B$ and $B \sim A$. Thus $A \sim B \preceq B \cup \{x_0\} \preceq A$ so by C.B.S. Theorem $A \sim B$ and

$A \sim B \cup \{x_0\} \sim A \sqcup \{1\}$, i.e. $|A| = |A| + 1$.

(iv) \Rightarrow (i) We have $\{1\} \sqcup A \sim_\varphi A$. Then $\varphi(A) \subsetneq A$. Thus $\varphi \circ \varphi(A) \subsetneq \varphi(A) \subsetneq A$, and, by induction,

$$\varphi^{\circ n}(A) \subsetneq \varphi^{\circ n-1}(A) \subsetneq \cdots \subsetneq A$$

$$\underbrace{\varphi \circ \cdots \circ \varphi}_{n \text{ times}}$$

Hence $|A| \geq |A \setminus \varphi^{\circ n}(A)| \geq n$ (at each stage above, we gain at least one point).

(ii) \Rightarrow (v) We have $\mathbb{N} \preceq_f A$. Let $g : \mathbb{N} \sqcup A \rightarrow A$,

$$g(x) = \begin{cases} f(2n) & \text{if } x = n, n \in \mathbb{N} \\ f(2n+1) & \text{if } x = f(n) \in A, n \in \mathbb{N} \\ x & \text{otherwise} \end{cases}$$

(v) \Rightarrow (ii) $\aleph_0 \leq \aleph_0 + |A| = |A|$ by assumption. □

Corollary 3.1. If $A \in \mathcal{P}(\mathbb{N})$, then either A is finite or denumerable.

Proof. Either $n \leq |A|$ for all n , or $|A| < n$ (Comparison lemma). □

Theorem 3.2 (Cantor). For any set X , $|X| < |\mathcal{P}(X)|$.

Proof. : (□

Cantor's paradox: There is no “set” of all sets.

4 2017-09-22

4.1 METRIC SPACES

Example (French railroad / metro metric): Suppose we have a set $X \neq \emptyset$, and a function $f : X \rightarrow [0, \infty)$ which satisfies $f^{-1}(\{0\}) = \{p_0\}$. Notice, then, that $f(x) > 0$ if $x \in X \setminus \{p_0\}$.

$$d_f : X \times X \rightarrow [0, \infty), \quad d_f(x, y) = f(x) + f(y)$$

if $x \neq y$, 0 if $x = y$.

Easy exercise: this is a metric.

(Belongs to family of weighted graph metrics.)

$$\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$$

$$x^p = \begin{cases} e^{p \log x} & x > 0 \\ 0 & x = 0 \end{cases}$$

Lemma 4.1. Let $\alpha, \beta \geq 0$ in \mathbb{R} , $1 < p < \infty$ and q is chosen so that $\frac{1}{p} + \frac{1}{q} = 1$ (ie $q = \frac{p}{p-1}$) then

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$$

with equality when $\alpha^p = \beta^q$.

Proof. Consider the graph of $y = x^{p-1}$ (assume $p \geq 2$).

$$x = y^1 p - 1 = y^q p = y^{q-1}$$

Then

$$\alpha\beta \leq \underbrace{\int_0^\alpha x^{p-1} dx}_{A_1} + \underbrace{\int_0^\beta y^{q-1} dy}_{A_2}$$

(Equality holds only if $\beta = \alpha^{p-1} \Rightarrow \beta^1 q - 1 \Rightarrow \beta^q = \alpha^p$)

$$= \frac{\alpha^p}{p} + \frac{\beta^q}{q}$$

Holder's Inequality

□

5 2017-09-25

Lemma: $\alpha, \beta \geq 0$ in \mathbb{R} , $1 < p < \infty$ with q satisfying $\frac{1}{p} + \frac{1}{q} \Rightarrow \alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$

Holder's Inequality: If $x, y \in \mathbb{R}^n$, $1 < p < \infty$ and q satisfies $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\left| \sum_{j=1}^n x_j y_j \right| \underbrace{\leq}_{1\text{-ineq. of } |\cdot|} \sum_{j=1}^n |x_j| |y_j| \leq \left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^n |y_j|^q \right)^{\frac{1}{q}} := \|x\|_p \|y\|_q$$

Proof. If $\|x\|_p \|y\|_q = 0$, then $x = 0$ or $y = 0$ and the inequality is trivial. Assume $\|x\|_p \|y\|_q \neq 0$. For $j = 1, \dots, n$, let

$$\alpha_j = \frac{|x_j|}{\|x\|_p}, \quad \beta_j = \frac{|y_j|}{\|y\|_q}.$$

Then

$$\begin{aligned} \frac{1}{\|x\|_p \|y\|_q} \sum_{j=1}^n |x_j| |y_j| &= \sum_{j=1}^n \alpha_j \beta_j \\ &\leq \sum_{j=1}^n \left[\frac{\alpha_j^p}{p} + \frac{\beta_j^q}{q} \right] \text{ by lemma} \\ &= \frac{1}{p} \sum_{j=1}^n \alpha_j^p + \frac{1}{q} \sum_{j=1}^n \beta_j^q \\ &= \frac{1}{p \|x\|_p^p} \sum_{j=1}^n |x_j|^p + \frac{1}{q \|y\|_q^q} \sum_{j=1}^n |y_j|^q \\ &= \frac{1}{p} + \frac{1}{q} \\ &= 1 \end{aligned}$$

□

Theorem 5.1 (Minkowski's Inequality). Let $x, y \in \mathbb{R}^n$ and $1 < p < \infty$. Then

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

Proof. If $x + y = 0$ then this is trivial, so suppose $x + y \neq 0$.

$$\begin{aligned}
\|x + y\|_p^p &= \sum_{j=1}^n |x_j + y_j|^p \\
&= \sum_{j=1}^n |x_j + y_j| |x_j + y_j|^{p-1} \\
&\leq \sum_{j=1}^n (|x_j| + |y_j|) |x_j + y_j|^{p-1} \\
&= \sum_{j=1}^n |x_j| |x_j + y_j|^{p-1} + \sum_{j=1}^n |y_j| |x_j + y_j|^{p-1} \\
&\leq \left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^n |x_j + y_j|^{(p-1)q} \right)^{\frac{1}{q}} + \left(\sum_{j=1}^n |y_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^n |x_j + y_j|^{(p-1)q} \right)^{\frac{1}{q}} \\
&= (\|x\|_p + \|y\|_p) \left(\sum_{j=1}^n |x_j + y_j|^{(p-1)q} \right)^{\frac{1}{q}}
\end{aligned}$$

We have

$$\frac{1}{p} + \frac{1}{q} = 1 \implies \frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p} \implies p = q(p-1)$$

and thus

$$\begin{aligned}
\|x + y\|_p^p &\leq (\|x\|_p + \|y\|_p) \left(\sum_{j=1}^n |x_j + y_j|^p \right)^{\frac{1}{q}} \\
&= (\|x\|_p + \|y\|_p) \|x + y\|_p^{\frac{p}{q}}
\end{aligned}$$

Now, divide $\|x + y\|_p^{\frac{p}{q}} \neq 0$ to get

$$\begin{aligned}
\|x + y\|_p &= \|x + y\|_p^{p - \frac{p}{q}} \\
&\leq \|x\|_p + \|y\|_p
\end{aligned}$$

(since $p - \frac{p}{q} = p(1 - \frac{1}{q}) = 1$). □

Corollary 5.1. Given $1 < p < \infty$, $\|\cdot\|_p$ is a norm on \mathbb{R}^n .

Proof. Clearly $\|\cdot\|_p$ is non-negative and non-degenerate. If $\alpha \in \mathbb{R}, x \in \mathbb{R}^n$ then

$$\begin{aligned}
\|\alpha x\|_p &= \left(\sum_{j=1}^n |\alpha x_j|^p \right)^{\frac{1}{p}} \\
&= |\alpha| \left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} \\
&= |\alpha| \|x\|_p
\end{aligned}$$

Finally, subadditivity is provided by Minkowski's inequality. □

$$|x|^p = e^{p \log |x|}$$

5.1 THE ℓ_p -SPACES

Consider $\mathbb{R}^N = \{x = (x_k)_{k=1}^\infty : x_k \in \mathbb{R}\}$ which is a \mathbb{R} -vector space:

$$(x_k)_{k=1}^\infty + (y_k)_{k=1}^\infty = (x_k + y_k)_{k=1}^\infty, \alpha(x_k)_{k=1}^\infty = (\alpha x_k)_{k=1}^\infty.$$

We let for $1 \leq p < \infty$

$$\ell_p = \{x = (x_k)_{k=1}^\infty \in \mathbb{R}^N : \sum_{k=1}^\infty |x_k|^p = \lim_{n \rightarrow \infty} \sum_{k=1}^n |x_k|^p < \infty\}$$

and

$$\ell_\infty = \{x = (x_k)_{k=1}^\infty : \sup_{k \in \mathbb{N}} |x_k| < \infty\}.$$

On ℓ_p we define

$$\|x\|_p = \begin{cases} (\sum_{j=1}^n |x_j|^p)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty \\ \sum_{k \in \mathbb{N}} |x_k|, & \text{if } p = \infty \end{cases}$$

Theorem 5.2. Let $1 \leq p < \infty$. Then ℓ_p is a \mathbb{R} -subspace of \mathbb{R}^N and $\|\cdot\|_p$ is a norm.

Proof. We prove these together. Suppose that $x, y \in \ell_p$. Then

$$\begin{aligned} \|x + y\|_p &= \left(\sum_{k=1}^\infty |x_k + y_k|^p \right)^{\frac{1}{p}} \quad \text{if } \infty, \text{ treat } \infty^{\frac{1}{p}} = \infty \\ &= \left(\lim_{n \rightarrow \infty} \sum_{k=1}^n |x_k + y_k|^p \right)^{\frac{1}{p}} \\ &= \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n |x_k + y_k|^p \right)^{\frac{1}{p}} \quad x \mapsto x^{\frac{1}{p}} \text{ is continuous on } [0, \infty), \text{ if } x \rightarrow \infty, x^{\frac{1}{p}} \rightarrow \infty \\ &\leq \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n |x_k|^p \right)^{\frac{1}{p}} + \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n |y_k|^p \right)^{\frac{1}{p}} \quad \text{Minkowski applied on each } n \\ &= \left(\lim_{n \rightarrow \infty} \sum_{k=1}^n |x_k|^p \right)^{\frac{1}{p}} + \left(\lim_{n \rightarrow \infty} \sum_{k=1}^n |y_k|^p \right)^{\frac{1}{p}} \quad \text{continuity again} \\ &= \left(\sum_{k=1}^\infty |x_k|^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^\infty |y_k|^p \right)^{\frac{1}{p}} \\ &= \|x\|_p + \|y\|_p \\ &< \infty \end{aligned}$$

Thus $x + y \in \ell_p$, and we get subadditivity of $\|\cdot\|_p$.

We note that non-negativity and non-degeneracy of $\|\cdot\|_p$ are obvious. Likewise, the $|\cdot|$ -homogeneity is straightforward. \square

Theorem 5.3. $(\ell_\infty, \|\cdot\|_\infty)$ is a normed vector space.

Proof. If $x, y \in \ell_\infty$ then

$$\begin{aligned}
 \|x + y\|_\infty &= \sup_{k \in \mathbb{N}} |x_k + y_k| \\
 &\leq \sup_{k \in \mathbb{N}} (|x_k| + |y_k|) \\
 &\leq \sup_{j, k \in \mathbb{N}} (|x_j| + |y_k|) \\
 &= \sup_{j \in \mathbb{N}} |x_j| + \sup_{k \in \mathbb{N}} |y_k| \\
 &= \|x\|_\infty + \|y\|_\infty
 \end{aligned}$$

Other properties are very easy. □

6 2017-09-29

i) $X \neq \emptyset$ s.t. $|X| \geq 2$

$$\text{discrete metric } d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

For $x_0 \in X$,

$$\begin{aligned}
 B(x, \varepsilon) &= \begin{cases} \{x_0\} & 0 < \varepsilon \leq 1 \\ x & \varepsilon > 1 \end{cases} \\
 B[x, \varepsilon] &= \begin{cases} \{x_0\} & 0 < \varepsilon < 1 \\ x & \varepsilon \geq 1 \end{cases}
 \end{aligned}$$

ii) (geometry of balls in \mathbb{R}^2)

$$1 \leq p \leq \infty, B_p(0, 1) = \{x \in \mathbb{R}^2 : d_p(0, x) = \|x\|_p < 1\}$$

Proposition 6.1. (X, d) a metric space.

i) X, \emptyset are both open and closed.

ii) If $\{U_i\}_{i \in I}$ is a family of open sets, then $\bigcup_{i \in I} U_i$ is open.

iii) If $\{U_1, \dots, U_n\}$ is a finite family of open sets, then $\bigcap_{i=1}^n U_i$ is open.

iv) If $\{F_i\}_{i \in I}$ is a family of closed sets, then $\bigcap_{i \in I} F_i$ is closed.

v) If $\{U_1, \dots, U_n\}$ is a finite family of closed sets, then $\bigcup_{i=1}^n U_i$ is closed.

Proof. i) Let $x \in X$, then $x \in B(x, 1) \subseteq X$, so X is open. So $\emptyset = X \setminus X$, $X = X \setminus \emptyset$ are closed.

ii) Let $x \in U = \bigcup_{i \in I} U_i$. Then there is some i_0 in I s.t. $x \in U_{i_0}$, which is open, so there is $\varepsilon_x > 0$ s.t. $x \in B(x, \varepsilon_x) \subseteq U_{i_0} \subseteq U$.

iii) Let $x \in V = \bigcap_{i=1}^n U_i$. Then for each $i = 1, \dots, n$, there is $\varepsilon_i > 0$ s.t. $B(x, \varepsilon_i) \subseteq U_i$. Let $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\} \implies B(x, \varepsilon) \subseteq \bigcap_{i=1}^n B(x, \varepsilon_i) \subseteq V$.

iv), v) De Morgan's Laws. □

Given a metric space (X, d) , $A \subseteq X$, we define the boundary of A :

$$\partial A = \{x \in X : \forall \varepsilon > 0, B(x, \varepsilon) \cap A \neq \emptyset, B(x, \varepsilon) \setminus A \neq \emptyset\}.$$

Remark: $\partial A = \partial(X \setminus A)$.

Interior of A :

$$A^\circ = \bigcup \{U \subseteq X : U \subseteq A \text{ and } U \text{ is open}\}.$$

Proposition 6.2 (characterizations of interior). If $(X, d), A$ are as above then

$$\begin{aligned} A^\circ &= \{x \in X : \exists \varepsilon_x > 0 \text{ s.t. } B(x, \varepsilon_x) \subseteq A\} \\ &= A \setminus \partial A. \end{aligned}$$

Proof. Let $x \in A$. Then either:

- for some $\varepsilon_x > 0$, $B(x, \varepsilon_x) \subseteq A \implies x \in A^\circ$, or
- $\forall \varepsilon > 0, B(x, \varepsilon) \setminus A \neq \emptyset \implies$ since $x \in A \cap B(x, \varepsilon)$, $x \in \partial A$.

Since $A^\circ \subseteq A$, the proposition holds. □

Def: (X, d) a metric space, $(x_n)_{n=1}^\infty \subseteq X$ and $x_0 \in X$. Say $(x_n)_{n=1}^\infty$ converges to x_0 , i.e. $\lim_{n \rightarrow \infty} x_n = x_0$ or $x_n \xrightarrow{n \rightarrow \infty} x_0$ if $\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$ s.t. $n \geq n_\varepsilon \implies d(x_0, x_n) < \varepsilon$.

Remark: The limit, if it exists, is unique. Suppose $x_0 = \lim_{n \rightarrow \infty} x_n, y_0 = \lim_{n \rightarrow \infty} x_n$, then given $\varepsilon > 0, \exists n_\varepsilon, n_{\varepsilon'}$ in \mathbb{N} s.t.

$$\begin{aligned} n \geq n_\varepsilon &\implies d(x_0, x_n) < \varepsilon \\ n \geq n_{\varepsilon'} &\implies d(y_0, x_n) < \varepsilon. \end{aligned}$$

Now if $n \geq \max\{n_\varepsilon, n_{\varepsilon'}\}$, then

$$\begin{aligned} d(x_0, y_0) &\leq d(x_0, x_n) + d(x_n, y_0) < \varepsilon \\ &\implies d(x_0, y_0) = 0, \text{ so } x_0 = y_0. \end{aligned}$$

Example: Let $(V, \|\cdot\|)$ be a normed vector space. A subset $\{e_n\}_{n=1}^\infty \subseteq V$ is a Schauder basis if for each $x \in V, \exists$ a unique sequence $\{x_n\}_{n=1}^\infty$ s.t. $x = \lim_{n \rightarrow \infty} \sum_{k=1}^n x_k e_k$ in V .

In $\ell_p, 1 \leq p < \infty$, let $e_n = (0, \dots, 0, \underbrace{1}_{n\text{-th place}}, 0, \dots)$.

Let, for $(X, d), A$ as above, the set of accumulation points (cluster points) be given as

$$A' = \{x \in X : \forall \varepsilon > 0, \underbrace{B(x, \varepsilon) \setminus \{x\}}_{\text{punctured ball}} \cap A \neq \emptyset\}$$

Call elements of $A \setminus A'$ isolated points.

Proposition 6.3. Given $(X, d), A$ as above, we have

$$A' = \{x \in X : x = \lim_{n \rightarrow \infty} x_n, (x_n)_{n=1}^\infty \subseteq A \setminus \{x\}\}$$

Proof. If $x \in A'$, let $x_1 \in (B(x, 1) \setminus \{x\}) \cap A$, and $x_{n+1} \in (B(x, \varepsilon_n) \setminus \{x\}) \cap A$, where $\varepsilon_n = \min\{\frac{1}{n}, d(x, x_n)\}$.

Then $x = \lim_{n \rightarrow \infty} x_n$ while $(x_n)_{n=1}^\infty \subseteq A \setminus \{x\}$. Note x_1, x_2, \dots are distinct.

Converse direction: definition of limits. □

7 2017-10-02

Def: Given a metric space (X, d) and $A \subseteq X$, define the closure of A by

$$\bar{A} = \bigcap \{F \subseteq X : A \subseteq F, F \text{ is closed in } X\}$$

Of course $A^\circ \subseteq A \subseteq \bar{A}$.

Theorem 7.1 (characterization of the closure). Given a metric space (X, d) , $A \subseteq X$, the following sets are the same:

$$\bar{A}, A \cup \partial A, A \cup A'$$

("meet" set) $A_M = \{x \in X : \text{for any } \varepsilon > 0, B(x, \varepsilon) \cap A \neq \emptyset\}$

("limit" set) $A_L = \{x \in X : x = \lim_{n \rightarrow \infty} x_n, \text{ where } (x_n)_{n=1}^\infty \subseteq A\}$

(The notations A_L, A_M will not be used afterwards; we shall use \bar{A} .)

Proof. We have

$$\begin{aligned} \bar{A} &= \cap \{F \subseteq X : A \subseteq F, F \text{ closed}\} \\ &= \cap \{X \subseteq U : U \subseteq X \setminus A, U \text{ open in } X\} \\ &= X \setminus \cup \{U : U \subseteq X \setminus A, U \text{ open in } X\} \\ &= X \setminus [(X \setminus A)^\circ] \text{ complement of interior} \\ &= X \setminus [(X \setminus A) \setminus \partial(X \setminus A)] \text{ characterization of } (X \setminus A)^\circ \\ &= X \setminus [(X \setminus A) \setminus \partial A] \\ &= A \cup \partial A \end{aligned}$$

$$(\cap_{i \in I} (X \setminus U_i) = X \setminus \cup_{i \in I} U_i)$$

We thus have $\bar{A} = A \cup \partial A$.

Now if $x \in A \cup \partial A$, then for each $\varepsilon > 0$, we have that $B(x, \varepsilon) \cap A \neq \emptyset$ [i.e. either $x \in A$ so $x \in A \cap B(x, \varepsilon)$, or $x \in \partial A$, so $B(x, \varepsilon) \cap A \neq \emptyset$]. Thus $A \cup \partial A \subseteq A_M$. Conversely, if $x \in A_M$, then, either

- there is $\varepsilon > 0$ so $B(x, \varepsilon) \subset A \implies x \in A^\circ \subseteq A$, or
- for every $\varepsilon > 0$ we have $B(x, \varepsilon) \setminus A \neq \emptyset$ in which case $x \in \partial A$.

Hence, $x \in A_M \implies x \in A \cup \partial A$ so $A_M \subseteq A \cup \partial A$.

If $x \in A \cup A'$, then for each $\varepsilon > 0$, we have $B(x, \varepsilon) \cap A \neq \emptyset$. Indeed, as above, either $x \in A$, so for any $\varepsilon > 0$, $x \in B(x, \varepsilon) \cap A$, or $x \in A'$, so $B(x, \varepsilon) \cap A \supseteq (B(x, \varepsilon) \setminus \{x\}) \cap A \neq \emptyset$. Hence $A \cup A' \subseteq A_M$.

The definition of the limit of a sequence shows that $A_M = A_L$.

Finally, consider

$$\begin{aligned} X \setminus (A \cup A') &\subseteq \{x \in X : \text{there exists } \varepsilon_x > 0 \text{ s.t. } B(x, \varepsilon_x) \cap A = \emptyset, B(x, \varepsilon_x) \subseteq X \setminus A\} \\ &= (X \setminus A)^\circ \implies X \setminus [(X \setminus A)^\circ] \subseteq X \setminus [X \setminus (A \cup A')] \end{aligned}$$

Hence

$$\begin{aligned} \bar{A} &= X \setminus [(X \setminus A)^\circ] \subseteq X \setminus [X \setminus (A \cup A')] \\ &= A \cup A'. \end{aligned}$$

Hence $\bar{A} \subseteq A \cup A' \subseteq A_M = \bar{A}$, so $\bar{A} = A \cup A'$. □

7.1 CONTINUITY

Def: Let (X, d_X) and (Y, d_Y) be metric spaces $f : X \rightarrow Y$ and $x_0 \in X$. We say that f is continuous at x_0 if given $\varepsilon > 0$, there is $\delta > 0$ s.t. $d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \varepsilon$. (★)

We say that f is continuous on X if it is continuous at each point.

Note:

$$\begin{aligned} (\star) &\iff f(B(x_0, \delta)) \subseteq B(f(x_0), \varepsilon) \\ &\iff B(x, \delta) \subseteq f^{-1}(B(f(x_0), \varepsilon)) \end{aligned}$$

Notation: In a metric space, a set N is a neighbourhood of a point x_0 if $x_0 \in N^\circ$ (interior).

Theorem 7.2 (characterization of continuity at a point). If $(X, d_X), (Y, d_Y), f : X \rightarrow Y, x \in X$ are as above, then TFAE:

- (i) f is continuous at x_0
- (ii) for any neighbourhood N of $f(x_0)$ in (Y, d_Y) , we have $f^{-1}(N)$ is a neighbourhood of x_0 in (X, d_X)
- (iii) if $x_0 = \lim_{n \rightarrow \infty} x_n$ in $(X, d_X) \implies f(x_0) = \lim_{n \rightarrow \infty} f(x_n)$ in (Y, d_Y) .

Proof. (i) \implies (ii) Given a neighbourhood of $f(x_0)$, there exists $\varepsilon > 0$ for which $B(f(x_0), \varepsilon) \subseteq N$. By assumption of continuity, there is $\delta > 0$ s.t.

$$\begin{aligned} B(x_0, \delta) &\subseteq f^{-1}(B(f(x_0), \varepsilon)) \\ &\subseteq f^{-1}(N), \text{ from above.} \end{aligned}$$

Thus $f^{-1}(N)$ is a neighbourhood of x_0 .

(ii) \implies (i) \implies (iii) Given $\varepsilon > 0$, $B(f(x_0), \varepsilon)$ is a neighbourhood of $f(x_0)$, so $f^{-1}(B(f(x_0), \varepsilon))$ is a neighbourhood of x_0 and hence there is $\delta > 0$ s.t. $B(x_0, \delta) \subseteq f^{-1}(B(f(x_0), \varepsilon))$, which gives (i).

Now, if $x_0 = \lim_{n \rightarrow \infty} x_n$ in (X, d_X) then there is n_δ in \mathbb{N} s.t. if $n \leq n_\delta, x_n \in B(x_0, \delta)$. But then for $n \leq n_\delta$, we have

$$f(x_n) \in f(B(x_0, \delta)) \subseteq B(f(x_0), \varepsilon)$$

and hence $f(x_0) = \lim_{n \rightarrow \infty} f(x_n)$.

(iii) \implies (i) (contrapositive) If (i) fails, then there exists $\varepsilon > 0$ s.t. for any $\delta > 0$, $B(x_0, \delta) \not\subseteq f^{-1}(B(f(x_0), \varepsilon))$.

Hence for each $n \in \mathbb{N}$ we may find $x_n \in B(x_0, \frac{1}{n}) \setminus f^{-1}(B(f(x_0), \varepsilon))$. Given $\varepsilon' > 0$, let $n_{\varepsilon'}$ satisfy $n_{\varepsilon'} \leq \frac{1}{\varepsilon'}$, thus $\lim_{n \rightarrow \infty} x_n = x_0$. However, each $f(x_n) \notin B(f(x_0), \varepsilon)$, so $f(x)$ does not go to. \square

8 2017-10-06

Corollary 8.1. A metric space is complete if whenever for any Cauchy sequence, we may find a converging subsequence.

Nested Intervals Theorem, Bolzano-Weierstrauss Theorem

Theorem 8.1. $(\ell_p, \|\cdot\|_p)$ ($1 \leq p < \infty$) is complete as a metric space.

Def: A normed space $(V, \|\cdot\|)$ is called a Banach space provided that V is complete w.r.t. metric $d(x, y) = \|x - y\|$. $(\ell_p, \|\cdot\|_p)$ is a Banach space.

9 2017-10-16

Theorem 9.1. The space of continuous bounded functions under the uniform metric, $(C_b(f), \|\cdot\|_\infty)$, is a Banach space.

Proof. (I) For $x \in X$, $(f_n(x))_{n=1}^\infty$ is Cauchy and admits a limit, so this defines $f : X \rightarrow \mathbb{R}$. The hard part is showing that f is continuous.

Next, show f is bounded, so $f \in C_b(X)$.

(II) $\lim_{n \rightarrow \infty} \|f - f_n\|_\infty = 0$, ie. $\lim_{n \rightarrow \infty} f_n = f$ uniformly in $C_b(X)$. \square

9.1 CHARACTERIZATIONS OF COMPLETENESS

Def: If (X, d) is a metric space, $\emptyset \neq A \subseteq X$, we let the diameter of A be given by

$$\text{diam}(A) = \sum_{x, y \in A} d(x, y) \text{ (may be } \infty)$$

Proposition 9.1. If (X, d) , A are as above then $\text{diam}(\bar{A}) = \text{diam}(A)$.

Proof. If $x, y \in \bar{A}$, $\varepsilon > 0$, then there are $x', y' \in A$ s.t. $d(x, x') < \frac{\varepsilon}{2}$, $d(y, y') < \frac{\varepsilon}{2}$ (using meet set characterization of \bar{A}). Then

$$\begin{aligned} d(x, y) &\leq d(x, x') + d(x', y') + d(y', y) \\ &\leq \frac{\varepsilon}{2} + \text{diam}(A) + \frac{\varepsilon}{2} \\ &= \text{diam}(A) + \varepsilon. \quad (\text{Assume } \text{diam}(A) < \infty). \end{aligned}$$

Thus, since $\varepsilon > 0$ is arbitrary, $d(x, y) \leq \text{diam}(A) \implies \text{diam}(\bar{A}) = \sup_{x, y \in \bar{A}} d(x, y) \leq \text{diam}(A)$. Since $A \subseteq \bar{A}$, $\text{diam}(A) \leq \text{diam}(\bar{A})$. \square

Theorem 9.2 (Nested set characterization of completeness). Let (X, d) be a metric space. Then (X, d) is complete \iff whenever we have closed sets,

- $F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$
- $\text{diam } F_n \xrightarrow{n \rightarrow \infty} 0$

then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

Proof. (\implies) For each n , choose $x_n \in F_n$. Given $\varepsilon > 0$, choose n_ε in \mathbb{N} s.t. $n \geq n_\varepsilon \implies \text{diam}(F_n) < \varepsilon$. Now, if $n, m \geq n_\varepsilon$ we have

$$x_n \in F_n \subseteq F_{n_\varepsilon}, x_m \in F_m \subseteq F_{n_\varepsilon} \implies d(x_n, x_m) \leq \text{diam}(F_{n_\varepsilon}) < \varepsilon$$

so $(x_n)_{n=1}^{\infty}$ is Cauchy, and has limit $x = \lim_{n \rightarrow \infty} x_n$. Since each $F_m = \bar{F}_m$ (closed), and we have for $n \geq m$, $x_n \in F_m$, $x = \lim_{n \rightarrow \infty} x_n \in F_m$ for all m . Hence $x \in \bigcap_{m=1}^{\infty} F_m$ (ie. $\neq \emptyset$).

(\impliedby) Let $(x_n)_{n=1}^{\infty} \subset X$ be Cauchy, let for n in \mathbb{N} , $F_n = \{x_k\}_{k \geq n}$. Then each F_n is closed and $F_n \supseteq F_{n+1}$ for each n . Further, $\text{diam } F_n = \text{diam}\{x_k\}_{k \geq n}$ (last proposition). Given $\varepsilon > 0$, there is n_ε in \mathbb{N} so $n, m \geq n_\varepsilon \implies d(x_n, x_m) < \varepsilon$. So for $n \geq n_\varepsilon$, we have $\text{diam}\{x_k\}_{k \geq n} = \sup_{k, l \geq n} d(x_k, x_l) < \varepsilon$. \square

10 2017-10-18

Continuing the proof that $(C_b(f), \|\cdot\|_\infty)$ is a Banach space from last time:

Theorem 10.1. The space of continuous bounded functions under the uniform metric, $(C_b(f), \|\cdot\|_\infty)$, is a Banach space.

Proof. (I) For $x \in X$, $(f_n(x))_{n=1}^{\infty}$ is Cauchy and admits a limit, so this defines $f : X \rightarrow \mathbb{R}$. f is continuous: let $x \in X$, and let $\varepsilon > 0$. Choose $n_\varepsilon \in \mathbb{N}$ so that

$$n, m \geq n_\varepsilon \implies |f_n(x) - f(x)| < \frac{\varepsilon}{4} \text{ and } \|f_n - f_m\|_\infty < \frac{\varepsilon}{4}.$$

Choose $\delta > 0$ so that for $x, y \in X$,

$$d(x, y) < \delta \implies |f_{n_\varepsilon}(x) - f_{n_\varepsilon}(y)| < \frac{\varepsilon}{4}.$$

Then, given $y \in B(x, \delta)$, let $n_y \in \mathbb{N}$ so that $n_y \geq n_\varepsilon$ and

$$n \geq n_y \implies |f_n(y) - f(y)| < \frac{\varepsilon}{4}.$$

Then for $n \geq n_y \geq n_\varepsilon$ we have

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_{n_\varepsilon}(x)| + |f_{n_\varepsilon}(x) - f_{n_\varepsilon}(y)| + |f_{n_\varepsilon}(y) - f_n(y)| + |f_n(y) - f(y)| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ &= \varepsilon. \end{aligned}$$

Also, f is bounded because

$$\begin{aligned} |f(x)| &\leq |f(x) - f_n(x)| + |f_n(x)| \\ &\leq |f(x) - f_n(x)| + \|f_n\|_\infty \\ &= o(1) + M. \end{aligned}$$

(II) Show that this is actually the limit (i.e. $\lim_{n \rightarrow \infty} \|f - f_n\|_\infty = 0$).

Let $\varepsilon > 0$. Choose $n_\varepsilon \in \mathbb{N}$ so that $m, n \geq n_\varepsilon \implies \|f_m - f_n\|_\infty < \frac{\varepsilon}{2}$. Also, given $x \in X$, choose $n_x \geq n_\varepsilon$ so that $n \geq n_x \implies |f_n(x) - f(x)| < \frac{\varepsilon}{2}$. Then, for $n \geq n_\varepsilon$, find $m \geq n_x \geq n_\varepsilon$ and observe that

$$\begin{aligned} |f(x) - f_n(x)| &\leq |f(x) - f_m(x)| + |f_m(x) - f_n(x)| \\ &< \frac{\varepsilon}{2} + \|f_m - f_n\|_\infty \\ &= \varepsilon. \end{aligned}$$

□

Example: Consider $(\ell_p, \|\cdot\|_p)$, $1 \leq p < \infty$. Let $e_n = (0, \dots, 0, \underbrace{1}_{n\text{-th place}}, 0, \dots)$ and let $F_n = \{e_k\}_{k \geq n} \subseteq \ell_p$.

- Each F_n is closed (easy exercise)
- $F_1 \supseteq F_2 \supseteq \dots$
- $\text{diam } F_n = 2^{\frac{1}{p}}$ (easy computation) (Finite diameter is not sufficient for Nested set characterization)

Notice that $\bigcap_{n=1}^\infty F_n = \emptyset$.

Theorem 10.2 (abstract M -test). Let $(V, \|\cdot\|)$ be a normed vector space. Then $(V, \|\cdot\|)$ is a Banach space \iff for every $(x_k)_{k=1}^\infty \subset V$ with $\sum_{k=1}^\infty \|x_k\| = \lim_{n \rightarrow \infty} \sum_{k=1}^n \|x_k\|$ converging, has that $\sum_{k=1}^\infty x_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n x_k$ converges in $(V, \|\cdot\|)$ [ie. V satisfies that “absolute convergence” \implies convergence.]

Proof. (\implies) Suppose $\sum_{k=1}^\infty \|x_k\|$ converges. Consider $(\sum_{k=1}^n x_k)_{n=1}^\infty \subset V$. We have for $m < n$ that

$$\left\| \sum_{k=1}^n x_k - \sum_{k=1}^m x_k \right\| \leq \underbrace{\sum_{k=m+1}^n \|x_k\|}_{\text{partial tail of converging series in } \mathbb{R}}$$

and hence $(\sum_{k=1}^n x_k)_{n=1}^\infty$ is Cauchy in $(V, \|\cdot\|)$, and thus converges.

(\Leftarrow) Suppose $(x_n)_{n=1}^\infty$ is a Cauchy seq in $(V, \|\cdot\|)$. Let n_1 in \mathbb{N} be so $m, n \geq n_1 \implies \|x_m - x_n\| < 1$, and, inductively, choose n_{k+1} in \mathbb{N} s.t. $n_{k+1} \geq n_k$ and $m, n \geq n_{k+1} \implies \|x_n - x_m\| < \frac{1}{2^k}$.

Let $y_0 = x_{n_1}$, $y_j = x_{n_{j+1}} - x_{n_j}$, $j \in \mathbb{N}$.

Then, each $\|y_j\| = \|x_{n_{j+1}} - x_{n_j}\| < \frac{1}{2^{j-1}}$, as $n_{j+1} > n_j \geq n_1$, so

$$\sum_{i=0}^\infty \|y_j\| = \|y_0\| + \sum_{j=1}^\infty \frac{1}{2^{j-1}},$$

which converges. (\star)

Now

$$\begin{aligned}
 x_{n_k} &= x_{n_1} + \sum_{j=1}^{k-1} (x_{n_{j+1}} - x_{n_j}) \\
 &= y_0 + \sum_{j=1}^{k-1} y_j \\
 &\xrightarrow{k \rightarrow \infty} y_0 + \sum_{j=1}^{\infty} y_j \text{ (by assumption and } (\star))
 \end{aligned}$$

In other words, $(x_{n_k})_{k=1}^{\infty}$ converges, hence $(x_n)_{n=1}^{\infty}$ converges as well. □

Application: a continuous nowhere differentiable function on \mathbb{R} .

Facts: $C_b(\mathbb{R})$ is complete; M -test.

Construction: Let $\varphi : \mathbb{R} \rightarrow [0, 1]$

$$\varphi(t) = \begin{cases} t - 2k & 2k \leq t < 2k + 1 \\ 2k + 2 - t & 2k + 1 \leq t < 2k + 2 \end{cases}$$

Picture: sawtooth function with zeros at $\dots, -4, -2, 0, 2, 4, \dots$

Then

- (i) φ is continuous and bounded
- (ii) φ is 2-periodic, ie. $\varphi(t + 2) = \varphi(t)$ for $t \in \mathbb{R}$
- (iii) $\varphi(2k) = 0, \varphi(2k + 1) = 1$ for $k \in \mathbb{Z}$
- (iv) if $k \leq s, t \leq k + 1$ ($k \in \mathbb{Z}$), then

$$|\varphi(s) - \varphi(t)| = |s - t|$$

Let for $t \in \mathbb{R}$

$$f(t) = \sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^k \underbrace{\varphi(4^k t)}_{\in [0,1]}$$

However, note that each $\varphi(4^k) \in C_b(\mathbb{R})$, $\|\varphi(4^k)\|_{\infty} = 1$, so by the M -test, $f \in C_b(\mathbb{R})$. Fix $t \in \mathbb{R}$. We show that f cannot be differentiable at t . Let $\ell_m = \lfloor 4^m t \rfloor$ ($m \in \mathbb{N}$) so

$$\begin{aligned}
 \ell_m &\leq 4^m t < \ell_m + 1 \\
 \implies p_m = \frac{\ell_m}{4^m} &\leq t < \frac{\ell_m + 1}{4^m} = q_m
 \end{aligned}$$

We compute

$$\begin{aligned}
& |f(p_m) - f(q_m)| \\
&= \left| \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^k [\varphi(4^k p_m) - \varphi(4^k q_m)] \right| \\
&= \left| \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^k [\varphi(4^{k-m} \ell_m) - \varphi(4^{k-m}(\ell_m + 1))] \right| \\
&= \left| \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^k [\varphi(4^{k-m} \ell_m) - \varphi(4^{k-m}(\ell_m + 1))] \right|, \text{ by (ii) (2-periodicity)} \\
&\text{(key step)} \geq \frac{3^m}{4} - \sum_{k=1}^{m-1} \frac{3^k}{4^k} \underbrace{|\varphi(4^{k-m} \ell_m) - \varphi(4^{k-m}(\ell_m + 1))|}_{=4^{k-m}, \text{ by (iv)}} \\
&= \frac{3^k}{4^k} - \frac{1}{4^m} \sum_{k=1}^{m-1} 3^k \\
&= \frac{1}{4^m} [3^m - \sum_{k=1}^{m-1} 3^k] \\
&= \frac{1}{4^m} \left[\frac{2 \cdot 3^m - 3^m + 1}{2} \right] \\
&= \frac{1}{4^m} \left(\frac{3^m + 1}{2} \right)
\end{aligned}$$

Since $|p_m - q_m| = \frac{1}{4^m}$, we have

$$\begin{aligned}
\frac{f(p_m) - f(q_m)}{p_m - q_m} &\geq \frac{3^m + 1}{2} \\
\left(p_m = \frac{\lfloor 4^m t \rfloor}{4^m} \right)
\end{aligned}$$

If $t = \frac{\ell}{4^{m_0}}$ ($\ell \in \mathbb{Z}$), then $t = p_m$ for $m \geq m_0$ and hence for $m \geq m_0$,

$$\left| \frac{f(t) - f(q_m)}{t - q_m} \right| \geq \frac{3^m + 1}{2}$$

while $\lim_{m \rightarrow \infty} q_m = t$, so $f'(t)$ does not exist.

$$\begin{aligned}
\frac{f(p_m) - f(q_m)}{p_m - q_m} &\leq \frac{|f(p_m) - f(t)| + |f(t) - f(q_m)|}{|p_m - q_m|} \\
&\leq \frac{|f(p_m) - f(t)|}{|p_m - t|} + \frac{|f(t) - f(q_m)|}{|t - q_m|}
\end{aligned}$$

Hence, for some $r_m \in \{p_m, q_m\}$, $\frac{|f(t) - f(r_m)|}{|t - r_m|} \geq \frac{3^m + 1}{2 \cdot 2}$.

We have $\left| \frac{f(t) - f(r_m)}{t - r_m} \right| \geq \frac{3^m + 1}{4}$ while $r_m \rightarrow t$.

11 2017-10-20

Corollary 11.1. $(\ell_\infty, \|\cdot\|_\infty)$ is a Banach space.

Proof. $\ell_\infty = C_b(\mathbb{N})$ with usual $|\cdot|$ metric on \mathbb{N} . If $f : \mathbb{N} \rightarrow \mathbb{R}$ is bounded, $U \subseteq \mathbb{R}$ open, then $f^{-1}(U) \in \mathcal{P}(\mathbb{N})$ is open (all subsets of \mathbb{N} are open) $\implies f$ is continuous.

If $(x_n)_{n=1}^\infty \in \ell_\infty$, define $f : \mathbb{N} \rightarrow \mathbb{R}$, $f(n) = x_n$, $f \in C_b(\mathbb{N})$, $\|f\|_\infty = \|(x_n)_{n=1}^\infty\|_\infty$. □

Eg. $(C[0, 2], \|\cdot\|_p)$, $\|f\|_p = (\int_0^2 |f|^p)^{\frac{1}{p}}$, $1 \leq p < \infty$.
 NOT a Banach space!

Let

$$f_n(t) = \begin{cases} 1 & 0 \leq t \leq \frac{1}{2} \\ n(\frac{1}{2} + \frac{1}{n} - t) & \frac{1}{2} < t \leq \frac{1}{2} + \frac{1}{n} \\ 0 & \frac{1}{2} + \frac{1}{n} < t \end{cases}.$$

Then for $m < n \in \mathbb{N}$,

$$\begin{aligned} \|f_n - f_m\|_p &= \left(\int_0^2 |f_n - f_m|^p \right)^{\frac{1}{p}} \\ &= \left(\underbrace{\int_0^{\frac{1}{2}} |f_n - f_m|^p}_0 + \underbrace{\int_{\frac{1}{2}}^{\frac{1}{2} + \frac{1}{m}} |f_n - f_m|^p}_{\leq \frac{1}{m}} + \underbrace{\int_{\frac{1}{2} + \frac{1}{m}}^2 |f_n - f_m|^p}_0 \right)^{\frac{1}{p}} \\ &\leq \frac{1}{m^{\frac{1}{p}}}. \end{aligned}$$

Hence $(f_n)_{n=1}^\infty$ is Cauchy in $(C[0, 2], \|\cdot\|_p)$.

Consider

$$\chi_{[0, \frac{1}{2}]}(t) = \begin{cases} 1 & 0 \leq t \leq \frac{1}{2} \\ 0 & \frac{1}{2} < t \end{cases}.$$

$\chi_{[0, \frac{1}{2}]}$ is bounded, piecewise continuous, so Riemann integrable.

$$\begin{aligned} \|f_n - \chi_{[0, \frac{1}{2}]} \|_p &= \left(\int_0^2 |f_n - \chi_{[0, \frac{1}{2}]}|^p \right)^{\frac{1}{p}} \leq \frac{1}{n^{\frac{1}{p}}} \\ \implies \lim_{n \rightarrow \infty} \|f_n - \chi_{[0, \frac{1}{2}]} \|_p &= 0. \end{aligned}$$

If $g \in C[0, 1]$ s.t. $\lim_{n \rightarrow \infty} \|f_n - g\|_p = 0$, then $\|g - \chi_{[0, \frac{1}{2}]} \|_p = 0$.

Using Riemann integration theory,

$$g(t) = \begin{cases} 1 & 0 \leq t \leq \frac{1}{2} \\ 0 & \frac{1}{2} < t \end{cases}.$$

Then $\lim_{t \rightarrow \frac{1}{2}} g$ does not exist!

11.1 COMPLETENESS OF METRIC SPACES

(X, d) metric space.

Remark: $|d(x, z) - d(y, z)| \leq d(x, y)$.

If $x = \lim_{n \rightarrow \infty} x_n$, $y = \lim_{n \rightarrow \infty} y_n$ in (X, d) , then $\lim_{n \rightarrow \infty} d(x_n, y_n) = d(x, y)$. (See solution to A3Q2).

Def: $(X, d_X), (Y, d_Y)$ metric spaces. $i : X \rightarrow Y$ is an isometry if $d_Y(i(x), i(y)) = d_X(x, y) \forall x, y \in X$.

Notes: An isometry is injective. Consider $i : X \rightarrow i(X) \subseteq Y \implies i^{-1} : i(X) \rightarrow X$ isometry.

Theorem 11.1. (X, d) metric space.

i) Existence of completion: there exists a metric space $(\overline{X}, \overline{d})$ s.t.

- a) $(\overline{X}, \overline{d})$ is complete
- b) \exists isometry $\bar{i} : X \rightarrow \overline{X}$
- c) $\overline{i(X)} = \overline{X}$; i.e. $i(X)$ is dense in \overline{X}

- ii) Uniqueness up to isometry: if (\tilde{X}, \tilde{d}) is a metric space with map $\tilde{i} : X \rightarrow \tilde{X}$ s.t. $(\tilde{X}, \tilde{d}), \tilde{i}$ satisfy (a),(b),(c), then \exists a surjective isometry $\varphi : \tilde{X} \rightarrow \bar{X}$ s.t. $\varphi \circ \tilde{i} = \bar{i}$.

Proof. 1. Fix $x_0 \in X$. For $u \in X$, let $f_u : X \rightarrow \mathbb{R}$, $f_u(x) = d(x, u) - d(x, x_0)$

$\implies f_u$ is continuous and $|f_u(x)| \leq d(u, x_0)$

$\implies \|f_u\|_\infty = \sup_{x \in X} |f_u(x)| \leq d(u, x_0) < \infty \implies f_u$ is bounded

$\implies f_u \in C_b(X)$.

For $u, v \in X, x \in X$,

$$|f_u(x) - f_v(x)| = |d(x, u) - d(x, v)| \leq d(u, v).$$

Thus $\|f_u - f_v\|_\infty \leq d(u, v)$. Finally,

$$\begin{aligned} |f_u(u) - f_v(u)| &= |d(u, u) - d(u, x_0) - d(u, v) + d(u, x_0)| \\ &= d(u, v). \end{aligned}$$

Thus $\|f_u - f_v\|_\infty \geq d(u, v) \implies \|f_u - f_v\|_\infty = d(u, v)$.

Define $\tau : X \rightarrow C_b(X)$, $\tau(u) = f_u$, τ isometry.

Let $\bar{X} = \overline{\tau(X)} = \{\overline{f_u} : u \in X\} \subseteq C_b(X)$.

By A3Q2(a), (\bar{X}, \bar{d}) is complete, where \bar{d} is relativized from the metric on $C_b(X)$.

2. Let $\varphi_0 = \tau \circ \tau^{-1} : \tau(X) \rightarrow \tau(X)$. φ_0 an isometry \implies uniformly continuous. Hence it admits an extension $\varphi = \overline{\varphi_0} : \bar{X} = \overline{\tau(X)} \rightarrow \bar{X} = \overline{\tau(X)}$.

Verify φ is an isometry:

If $\tilde{x}, \tilde{y} \in \bar{X}$, let $\tilde{x} = \lim_{n \rightarrow \infty} \tau(x_n), \tilde{y} = \lim_{n \rightarrow \infty} \tau(y_n), x_n, y_n \in X$. Then

$$\varphi(\tilde{x}) = \lim_{n \rightarrow \infty} \varphi_0(\tau(x_n)) = \lim_{n \rightarrow \infty} \tau(x_n).$$

Hence

$$\begin{aligned} \bar{d}(\varphi(\tilde{x}), \varphi(\tilde{y})) &= \lim_{n \rightarrow \infty} \bar{d}(\tau(x_n), \tau(y_n)) \\ &= \lim_{n \rightarrow \infty} d(x_n, y_n) \\ &= \lim_{n \rightarrow \infty} \tilde{d}(\tau(x_n), \tau(y_n)) = \tilde{d}(\tilde{x}, \tilde{y}). \end{aligned}$$

$\implies \varphi$ is an isometry. $\varphi \circ \tau = \tau$ comes for free.

□

12 2017-10-23

Assignment discussion – the completion vs A4,Q1:

Suppose $(V, \|\cdot\|)$ is a non-complete normed vector space, eg. $(C[0, 2], \|\cdot\|_p)$ ($1 \leq p < \infty$). Consider the map

$$\tau : V \rightarrow C_b(V)$$

$$\tau(v) \in C_b(V), \tau(v)(x) = \|x - v\| - \|x\|$$

We saw that τ is an isometry, hence we let

$$\bar{V} = \overline{\tau(V)}_{\text{complete}} \subseteq C_b(V)$$

Problem: τ is not linear, $\overline{\tau(V)}$ not evidently a subspace of $C_b(V)$.

A4, Q1 shows that an addition and a scalar multiplication may be imposed on $\bar{V} = \overline{\tau(V)}$ which makes it a Banach (complete normed vector) space. These two operations are not the same as addition and scalar multiplication in $C_b(V)$. (The only linear property that τ enjoys seems to be that it takes 0 to 0.)

12.1 COMPACTNESS

Let (X, d) be a metric space, and $K \subseteq X$. We say that K is compact if given a family of open sets $\{U_i\}_{i \in I}$ for which

$$K \subseteq \bigcup_{i \in I} U_i \text{ -- we say } \{U_i\}_{i \in I} \text{ is an "open cover"}$$

there is a finite subfamily $\{U_{i_1}, \dots, U_{i_n}\}$ such that

$$K \subseteq \bigcup_{k=1}^n U_{i_k} \text{ -- we say } \{U_i\}_{i \in I} \text{ admits a "finite subcover" .}$$

If $X = K$ itself is compact, we will call (X, d) a compact metric space.

Remark: If $K \subseteq X$ is compact, the relativized metric space (K, d_K) is a compact metric space.

Proposition 12.1. Let (X, d) be a metric space and $K \subseteq X$. If K is compact, then it must be closed.

Proof. Let us suppose, for sake of contradiction that there is $x \in \overline{K} \setminus K$. Then for n in \mathbb{N} ,

$$B(x, \frac{1}{n}) \cap K \neq \emptyset \implies B[x, \frac{1}{n}] \cap K \neq \emptyset. \quad (\star)$$

Further, $\bigcap_{n=1}^{\infty} B[x, \frac{1}{n}] = \{x\}$. Let $U_n = X \setminus B[x, \frac{1}{n}]$, which is open.

We have that

$$\bigcup_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} (X \setminus B[x, \frac{1}{n}]) = X \setminus \bigcap_{n=1}^{\infty} B[x, \frac{1}{n}] = X \setminus \{x\} \supseteq K.$$

But, for any finite m we have

$$\bigcup_{n=1}^m U_n = X \setminus \bigcap_{n=1}^m B[x, \frac{1}{n}] = X \setminus B[x, \frac{1}{m}] \not\supseteq K$$

by (\star) . Hence if $\overline{K} \setminus K \neq \emptyset$, K cannot be compact. So we are done. \square

Proposition 12.2. Let (X, d) be a compact metric space and $C \subseteq X$ is closed. Then C is compact.

Proof. Suppose $\{U_i\}_{i \in I}$ is an open cover of C . Then $\{U_i\}_{i \in I} \cup \{X \setminus C\}$ is an open cover of X . Hence X admits finite subcover $\{U_{i_1}, \dots, U_{i_n}\} \cup \{X \setminus C\}$, hence, $\{U_{i_1}, \dots, U_{i_n}\}$ is a finite subcover of C . \square

Theorem 12.1 (continuous image of compact is compact). Let (X, d_X) be a compact metric space, (Y, d_Y) be a metric space, and $f : X \rightarrow Y$ be continuous. Then $f(X) = \{f(x) : x \in X\}$ is compact.

Proof. Let $\{V_i\}_{i \in I}$ be an open cover of $f(X)$. Then $U_i = f^{-1}(V_i)$ is open, and $\{U_i\}_{i \in I}$ is an open cover of X . Hence there is a finite subcover, $X \subseteq \bigcup_{k=1}^n U_{i_k}$ so $f(X) \subseteq \bigcup_{k=1}^n f(U_{i_k}) = \bigcup_{k=1}^n V_{i_k}$, so $\{V_{i_1}, \dots, V_{i_n}\}$ is a finite subcover of $f(X)$. \square

Corollary 12.1 (Extreme Value Theorem). If (X, d) is a compact metric space, $f : X \rightarrow \mathbb{R}$ is continuous, then there are $x_{\min}, x_{\max} \in X$ for which

$$f(x_{\min}) \leq f(x) \leq f(x_{\max}) \quad \forall x \in X.$$

Proof. We have $f(X) \subseteq \mathbb{R}$ is compact. Hence $f(X)$ is closed. Also $\{(-n, n)\}_{n=1}^{\infty}$ (open intervals), then $f(X) \subseteq \mathbb{R} = \bigcup_{n=1}^{\infty} (-n, n)$ admits a finite subcover, $\{(-1, 1), \dots, (-n, n)\}$ and hence $f(X) \subseteq (-n, n)$. Thus we have $\inf(f(X)), \sup(f(X))$ exist.

Since $f(X)$ is closed we have

$$\inf(f(X)), \sup(f(X)) \in f(X)$$

(use meet-set of closure). Let x_{\min}, x_{\max} be so $f(x_{\min}) = \inf(f(X)), f(x_{\max}) = \sup(f(X))$. \square

– Assignment line –

Theorem 12.2 (finite intersection property). Let (X, d) be a metric space. Then (X, d) is compact \iff for any family $\{F_i\}_{i \in I}$ of closed subsets of X for which $\bigcap_{k=1}^n F_{i_k} \neq \emptyset$, $\{i_1, \dots, i_n\}$ finite in I , we must have $\bigcap_{i \in I} F_i \neq \emptyset$.

Proof. (\implies) (contrapositive) Let us suppose that $\{F_i\}_{i \in I}$ is a family of closed subsets with $\bigcap_{i \in I} F_i = \emptyset$. Then if $U_i = X \setminus F_i$, we have that $\{U_i\}_{i \in I}$ is an open cover (De Morgan's law) and hence admits finite subcover $\{U_{i_1}, \dots, U_{i_n}\}$. Again, by DeMorgan's law, $\bigcap_{k=1}^n F_{i_k} = \emptyset$. Hence we are done.

(\impliedby) Very similar, interchange roles of U_i s and $F_i = X \setminus U_i$. \square

Example: Let $X = B[0, 1]$ in ℓ_p ($1 \leq p \leq \infty$).

Let $e_n = (0, \dots, 0, \underbrace{1}_{n\text{-th place}}, 0, \dots)$ and let $F_n = \{e_k\}_{k \geq n}$ (seen before on Oct 18).

Each F_n is closed. Also

$$\bigcap_{n=1}^{\infty} F_n = \emptyset$$

$$\bigcap_{n=1}^m F_n = F_m \neq \emptyset$$

Conclusion: $(B[0, 1], d_p)$ ($d_p(x, y) = \|x - y\|_p$) is not compact.

13 2017-10-25

Def: Let (X, d) be a metric space. Then we say it is

- bounded if there are x_0 in X , and $R > 0$ such that $X \subseteq B[x_0, R]$ (of course “=” holds) (equivalently, for any $x \in X$, there is $R_x > 0$ such that $X \subseteq B[x, R_x]$; or, equivalently, $\text{diam}(X) < \infty$)
- totally bounded if, for any $\varepsilon > 0$, there are $x_1, \dots, x_n \in X$ such that $X \subseteq \bigcup_{k=1}^n B[x_k, \varepsilon]$

Totally bounded \implies bounded. [with $\varepsilon > 0$, x_1, \dots, x_n in defn, check that $\bigcup_{k=1}^n B[x_k, \varepsilon] \subseteq B[x_1, \varepsilon + \max_{k=2, \dots, n} d(x_1, x_k)]$]

Example: (bounded $\not\Rightarrow$ totally bounded)

In ℓ_p ($1 \leq p \leq \infty$), $e_n = (0, \dots, 0, \underbrace{1}_{n\text{-th place}}, 0, \dots)$, $F_n = \{e_k\}_{k \geq n} \subseteq \ell_p$,

F_n int, $F_n \subseteq B[0, 1] \subseteq B[e, 2]$ so F_n is bounded. But $n \neq m$, $d(e_n, e_m) = \begin{cases} 2^{\frac{1}{p}} & 1 \leq p < \infty \\ 1 & \text{otherwise} \end{cases} =: R$.

If $0 < \varepsilon < \frac{1}{2}R$, we see that $F_n \not\subseteq \bigcup_{k=1}^n B[e_k, \varepsilon]$ for any n .

Theorem 13.1 (Characterizations of compact metric spaces). Let (X, d) be a metric space. TFAE:

- (X, d) is compact,
- any sequence $(x_n)_{n=1}^{\infty} \subseteq X$ admits a subsequence which converges in X
- (X, d) is complete and totally bounded

Proof. (i) \implies (ii): Let $F_n = \overline{\{x_k\}_{k=n}^{\infty}}$. Then each F_n is closed, and $F_1 \supseteq F_2 \supseteq \dots$, so if $n_1 < n_2 < \dots < n_m$, then $\bigcap_{j=1}^m F_{n_j} = F_{n_m} \neq \emptyset$. Thus, by finite intersection property, we have that $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$. Let $x \in \bigcap_{n=1}^{\infty} F_n$. Now let

$$n_1 = \min\{k : x_k \in B(x, 1)\} \text{ (exists by meet-set closure definition)}$$

and, inductively,

$$n_{m+1} = \min\{k : k > n_m \text{ and } x_k \in B(x, \frac{1}{m+1})\}.$$

Then, as is easy to check, $\lim_{m \rightarrow \infty} x_{n_m} = x$.

(ii) \implies (iii): If $(x_n)_{n=1}^{\infty} \subseteq X$ is Cauchy, it admits a converging subsequence (by assumption), and hence itself converges

(earlier proposition). Thus (X, d) is complete.

Let us suppose that (X, d) is not totally bounded.

Thus, there exists $\varepsilon > 0$ so no finite collection of closed ε -balls covers X . Let

$$x_1 \in X \setminus B[x_1, \varepsilon], \dots, x_{n+1} \in X \setminus \bigcup_{k=1}^n B[x_k, \varepsilon] \text{ (always possible by assumption).}$$

Thus $d(x_n, x_m) > \varepsilon$ for $n \neq m$. Thus, this sequence $(x_n)_{n=1}^\infty$ admits no Cauchy subsequences, hence no subsequences which converge, violating assumption (ii). Thus (ii) $\implies (X, d)$ is totally bounded.

(iii) \implies (ii): We first use total boundedness. Given n in \mathbb{N} , there exist $y_{n1}, \dots, y_{nm_n} \in X$ such that the closed balls

$$B_{n1} = B[y_{n1}, \frac{1}{n}], \dots, B_{nm_n} = B[y_{nm_n}, \frac{1}{n}]$$

satisfy that $X \subseteq \bigcup_{k=1}^{m_n} B_{nk}$. Let

- B_1 be a ball from B_{11}, \dots, B_{1m_1} such that

$$|\{n \in \mathbb{N} : x_n \in B_1\}| = \aleph_0 \text{ (pigeonhole principle)}$$

• \vdots

- B_k be a ball from B_{11}, \dots, B_{1m_1} such that

$$|\{n \in \mathbb{N} : x_n \in \bigcap_{j=1}^k B_j\}| = \aleph_0$$

(we've covered X by 1-balls, B_1 by $\frac{1}{2}$ -balls, then $B_2 \cap B_1$ covered by $\frac{1}{3}$ -balls, ...)

Now we use completeness. Let $F_n = \bigcap_{k=1}^n B_k$ so each F_n is closed.

- $F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$
- $\text{diam}(F_n) \leq \text{diam}(B_n) = \frac{2}{n} \xrightarrow{n \rightarrow \infty} 0$

Thus, by nested sets theorem, $\bigcap_{n=1}^\infty F_n \neq \emptyset$.

Let $n_1 = \min\{k \in \mathbb{N} : x_k \in F_1\}$, inductively, $n_{m+1} = \min\{k \in \mathbb{N} : k > n_m \text{ and } x_k \in F_k\}$.

Then, if $x \in \bigcap_{n=1}^\infty F_n$, $d(x, x_m) \leq \text{diam}(F_m) \leq \text{diam}(B_m) = \frac{2}{m} \xrightarrow{m \rightarrow \infty} 0$ so $x = \lim_{n \rightarrow \infty} x_{n_m}$. □

14 2017-10-27

Office hours:

Mon 2:30 – 4:30

Tue 2 – 3:30

Proof. Continuing theorem from last time:

So far we did (i) $\xRightarrow{\text{F.I.P.}}$ (ii) $\xRightarrow{\text{routine}}$ (iii) $\xRightarrow{\text{harder, nested sets thm}}$ (ii)

(ii) \implies (i): Let $\{U_i\}_{i \in I}$ be an open cover of X .

(LN) There exists $r > 0$ s.t. for any x in X there exists i in I so $B(x, r) \subseteq U_i$.

(This number r is sometimes called the “Lebesgue number” of the covering; its existence is based on (ii).)

Suppose (LN) fails. Then for choice of $r = \frac{1}{n}$, there exists x_n in X s.t. $B(x, \frac{1}{n}) \not\subseteq U_i$ for all i in I .

Our assumption is that $(x_n)_{n=1}^\infty \subseteq X$ admits a subsequence $(x_{n_k})_{k=1}^\infty$ such that $x_0 = \lim_{k \rightarrow \infty} x_{n_k}$ exists.

Then $x_0 \in U_{i_0}$ for some i_0 , so there is $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U_{i_0}$. Now, there is k_ε in \mathbb{N} so $k \geq k_\varepsilon \implies x_{n_k} \in B(x_0, \frac{\varepsilon}{2})$. Hence, let us choose $k \geq k_\varepsilon$ and $\frac{1}{n_k} < \frac{\varepsilon}{2}$. Thus, if $x \in B(x_{n_k}, \frac{1}{n_k})$, we have

$$d(x, x_0) \leq d(x, x_{n_k}) + d(x_{n_k}, x_0) < \frac{1}{n_k} + \frac{\varepsilon}{2} < \varepsilon$$

and hence $B(x_{n_k}, \frac{1}{n_k}) \subseteq B(x_0, \varepsilon) \subseteq U_{i_0}$, contradiction the choice of the elements x_n .

Hence, we must conclude that (LN) holds.

We saw in (ii) \implies (iii) above, that our assumption gives total boundedness of (X, d) . Hence there are y_1, \dots, y_m such that $X \subseteq \bigcup_{j=1}^m B[y_j, \frac{r}{2}] \subseteq \bigcup_{j=1}^m B(y_j, r)$. Now, for each $j = 1, \dots, m$, (LN) tells us that there is $i_j \in I$ so $B(y_j, r) \subseteq U_{i_j}$.

Thus $X \subseteq \bigcup_{i=1}^n B(y_j, r) \subseteq U_{i_j}$, so $\{U_{i_1}, \dots, U_{i_m}\}$ is a finite subcover.

Remark: On \mathbb{R}^n , norms $\|\cdot\|_p$ ($1 \leq p \leq \infty$) are equivalent, and from A2, each gives the same open sets, and hence the same compact sets.

Corollary 14.1.

(i) (Bolzano-Weierstrauss Theorem for \mathbb{R}^n)

If $(x^{(n)})_{n=1}^\infty \subseteq [-R, R]^n = B_\infty[0, R]$, then it admits a converging subsequence.

(ii) (Heine-Borel Theorem)

A subset $K \subseteq \mathbb{R}^n$ is compact $\iff K$ is closed & K is bounded (with respect to any $\|\cdot\|_\infty$).

Proof. (i) We consider, first $(x_1^{(n)})_{n=1}^\infty \subseteq [-R, R] \subseteq \mathbb{R}$. By Bolzano-Weierstrauss for \mathbb{R} , this admits converging subsequence $(x_1^{(n_k)})_{n=1}^\infty$. Then $(x_2^{(n)})_{n=1}^\infty \subseteq [-R, R] \subseteq \mathbb{R}$ admits a converging subsequence $(x_2^{(n_k)})_{n=1}^\infty$. Etc. Hence, after finitely many (n) iterations, we get a subsequence of $(x^{(n)})_{n=1}^\infty$ which converges ($\mathbb{R}^n, \|\cdot\|_\infty$).

(ii) If K is compact, then K is closed by a result at the beginning of the section, and totally bounded by last theorem, hence bounded. Conversely, if K is closed and bounded, $K \subseteq [-R, R]^n$ for some $R > 0$. Let us consider a sequence $(x^{(n)})_{n=1}^\infty \subseteq K$. First, $(x^{(n)})_{n=1}^\infty$ admits a converging subsequence, by (i). Since K is closed, the limit of the subsequence is in K . □

Example: $P = \prod_{k=1}^\infty \{0, \frac{1}{2^k}\} \subseteq \ell_1$ is compact in $(\ell_1, \|\cdot\|_1)$.

First soln: The Cantor set C is closed and bounded in \mathbb{R} , so thus compact. And there is a continuous function $f : C \rightarrow \ell_1$ with $f(C) = P$ (A4, Q3), so P is compact. [In fact f is a bijection from C to P so $f^{-1} : P \rightarrow C$ is also continuous.]

Second soln: P is closed (A3). Hence the relativised metric space (P, d_P) is complete. Let us show total boundedness.

Let $\varepsilon > 0$, and n be so $\frac{1}{2^n} < \varepsilon$. For $(b_1, \dots, b_m) \in \{0, 1\}^n$, let $x_{b_1 \dots b_m} = \sum_{k=1}^\infty \frac{b_k}{2^k} e_k \in P$. If $b = (b_1, b_2, \dots) \in \{0, 1\}^\mathbb{N}$, then $x_b = \sum_{k=1}^\infty \frac{b_k}{2^k} e_k \in P$ (generic element of P).

Then for $b = (b_1, b_2, \dots)$ as above,

$$\|x_b - x_{b_1 \dots b_n}\|_1 = \sum_{k=n+1}^\infty \frac{1}{2^k} b_k \leq \sum_{k=n+1}^\infty \frac{1}{2^k} = \frac{1}{2^n} \leq \varepsilon.$$

Thus, $P \subseteq \bigcup_{(b_1, \dots, b_n) \in \{0, 1\}^n} B[x_{b_1 \dots b_n}, \varepsilon]$. □

– MIDTERM CUTOFF –

15 2017-10-30

Midterm: Wed evening

See info sheet on website

Office hours:

– 2:30 - 4:30

– 1:30 - 3:30

A5 - will be posted Friday

Theorem 15.1 (sequential characterization of uniform continuity). Let (X, d_X) and (Y, d_Y) be metric spaces, $f : X \rightarrow Y$. Then

$$f \text{ is uniformly continuous} \iff \text{whenever } d_X(x_n, y_n) \xrightarrow{n \rightarrow \infty} 0, \ x_n, y_n \in X, \\ \text{we must have } d_Y(f(x_n), f(y_n)) \xrightarrow{n \rightarrow \infty} 0.$$

Proof. (\implies) Given $\varepsilon > 0$, there is $\delta > 0$ such that $d_X(x, y) < \delta \ (x, y \text{ in } X) \implies d_Y(f(x), f(y)) < \varepsilon$. Now suppose $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty \subseteq X$ such that $\lim_{n \rightarrow \infty} d_X(x_n, y_n) = 0$. Then there is n_ε in \mathbb{N} such that

$$n \geq n_\varepsilon \implies d_X(x_n, y_n) < \delta \\ \implies d_Y(f(x_n), f(y_n)) < \varepsilon.$$

I.e. $\lim_{n \rightarrow \infty} d_Y(f(x_n), f(y_n)) = 0$.

(\impliedby) (contrapositive) Suppose f is not uniformly continuous, so there exists $\varepsilon > 0$ such that for all $\delta > 0$ there are x, y in X with $d_X(x, y) < \delta$ but $d_Y(f(x), f(y)) \geq \varepsilon$. For each choice $\delta = \frac{1}{n}$, let x_n, y_n in X so $d_X(x_n, y_n) < \frac{1}{n}$ for which $d_Y(f(x_n), f(y_n)) \geq \varepsilon$.

Plainly, $\lim_{n \rightarrow \infty} d_X(x_n, y_n) = 0$ while $\lim_{n \rightarrow \infty} d_Y(f(x_n), f(y_n)) \neq 0$ (if the limit exists).

Ex: Let $f(x) = x^2$ on \mathbb{R} . Let $x_n = n, y_n = n + \frac{1}{n}$. Then $|x_n - y_n| = \frac{1}{n} \xrightarrow{n \rightarrow \infty} 0$, while $|f(x_n) - f(y_n)| = 2 + \frac{1}{n^2} \not\xrightarrow{n \rightarrow \infty} 0$. Hence f is not uniformly continuous. \square

Theorem 15.2 (continuous on compact is uniformly continuous). Let $(X, d_X), (Y, d_Y)$ be metric spaces, with (X, d_X) compact, and $f : X \rightarrow Y$ continuous. Then f is uniformly continuous.

Proof. Let us suppose not. Then there is $\varepsilon > 0$ and $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty \subseteq X$ such that $d_X(x_n, y_n) \xrightarrow{n \rightarrow \infty} 0$ while $d_Y(f(x_n), f(y_n)) \geq \varepsilon$. Let $(x_{n_k})_{k=1}^\infty$ be a converging subsequence. Then let $(y_{n_k})_{k=1}^\infty$ be a sequence in X , hence admits converging subsequence $(y_{n_{k_\ell}})_{\ell=1}^\infty$. Then if $x = \lim_{k \rightarrow \infty} x_{n_k} = \lim_{\ell \rightarrow \infty} x_{n_{k_\ell}}$ then

$$d_X(x, y_{n_{k_\ell}}) \leq d_X(x, x_{n_{k_\ell}}) + d_X(x_{n_{k_\ell}}, y_{n_{k_\ell}}) \\ \xrightarrow{\ell \rightarrow \infty} 0$$

so $x = \lim_{\ell \rightarrow \infty} y_{n_{k_\ell}}$. Then we have $f(x) = \lim_{\ell \rightarrow \infty} f(y_{n_{k_\ell}})$, by continuity, so

$$0 = d_Y(f(x), f(x)) = \lim_{\ell \rightarrow \infty} d_Y(f(x_{n_{k_\ell}}), f(y_{n_{k_\ell}}))$$

contradicts (\star) . Thus, we conclude that f is uniformly continuous. \square

Definition: A map $f : X \rightarrow Y$ ($(X, d_X), (Y, d_Y)$) is called Lipschitz if there is $L \geq 0$ such that

$$d_Y(f(x), f(y)) \leq L d_X(x, y) \text{ for all } x, y \in X.$$

Notice that

$$\sup_{x, y \in X, x \neq y} \frac{d_Y(f(x), f(y))}{d_X(x, y)} = \inf \{ L \geq 0 : (\text{Lip}) \text{ is satisfied} \}$$

so there exists a minimum L satisfying (Lip). We call this the “Lipschitz constant”.

Remark: Lipschitz $\xrightarrow{\text{exercise}}$ uniform continuity \implies continuity
 Lipschitz $\not\xrightarrow{\text{assignment}}$ uniform continuity $\not\implies$ continuity

Theorem 15.3. Any two norms on \mathbb{R}^n are equivalent, i.e. if $\|\cdot\|, \|\cdot\|$ on \mathbb{R}^n satisfy $\|\cdot\| \approx \|\cdot\|$, i.e., there are $m, M > 0$ for which $m\|x\| \leq \|x\| \leq M\|x\|$.

Proof. Let $\|\cdot\|$ be a norm on \mathbb{R}^n . We will see that $\|\cdot\| \approx \|\cdot\|_1$ ($\|x\|_1 = \sum_{j=1}^n |x_j|$). Since \approx is an equivalence relation, we get $\|\cdot\| \approx \|\cdot\|_1$ so $\|\cdot\| \approx \|\cdot\|$.

Let $\{e_1, \dots, e_n\}$ be the standard basis, so if $x \in \mathbb{R}^n$, $x = \sum_{j=1}^n x_j e_j$. Then

$$\|x\| = \left\| \sum_{j=1}^n x_j e_j \right\| \underbrace{\leq}_{\text{properties of norm}} \sum_{j=1}^n |x_j| \|e_j\| \leq M \|x\|_1 \text{ where } M = \max_{j=1, \dots, n} \|e_j\|.$$

Notice, then, for x, y in \mathbb{R}^n we have

$$\left| \|x\| - \|y\| \right| \underbrace{\leq}_{\text{standard } \leq \text{ (shown before completeness of } C_b(X))} \|x - y\| \leq M \|x - y\|_1$$

so $\|\cdot\| : \mathbb{R}^n \rightarrow \mathbb{R}$ is Lipschitz with respect to $d_1(x, y) = \|x - y\|_1$ and thus continuous.

Let $S_1 = \{x \in \mathbb{R}^n : \|x\|_1 = 1\} = B_1[0, 1] \setminus \underbrace{B_1(0, 1)}_{\subseteq B_1[0, 1]}$ so S_1 is closed in $B_1[0, 1]$. Hence by Heine-Borel Theorem, it is compact.

Hence, by Extreme Value Theorem, there is x_{\min} in S_1 such that

$$\|x_{\min}\| = \inf\{\|x\| : x \in S_1\}.$$

Let $m = \|x_{\min}\| > 0$ (as $x_{\min} \neq 0$, since $\|x_{\min}\|_1 = 1 \neq 0$).

Now, if $x \in \mathbb{R}^n \setminus \{0\}$, then

$$m \leq \left\| \frac{1}{\underbrace{\|x\|_1}_{\in S_1}} x \right\| \implies m \|x\|_1 \leq \|x\| \quad (\ddagger)$$

Then (\dagger) and (\ddagger) show that $\|\cdot\| \approx \|\cdot\|_1$. □

Corollary 15.1. If $\|\cdot\|$ is a norm on \mathbb{R}^n , $\|\cdot\|$ on \mathbb{R}^m and $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear. Then A is Lipschitz from $(\mathbb{R}^n, \|\cdot\|)$ to $(\mathbb{R}^m, \|\cdot\|)$, and hence continuous.

Proof. Let $\{e_1, \dots, e_n\}$ be the standard basis of \mathbb{R}^n , $\{e_1, \dots, e_m\}$ be the standard basis of \mathbb{R}^m . Then there is a matrix $[a_{ij}]$ such that $Ae_j = \sum_{i=1}^m a_{ij} e_i$.

Then for $x = \sum_{j=1}^n x_j e_j$ in \mathbb{R}^n we have

$$\begin{aligned} Ax &= \sum_{j=1}^n x_j Ae_j \\ &= \sum_{j=1}^n x_j \sum_{i=1}^m a_{ij} e_i \\ &= \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} x_j \right) e_i \in \mathbb{R}^m \end{aligned}$$

so

$$\begin{aligned}
\|Ax\| &\leq \sum_{j=1}^n \left| \sum_{i=1}^n a_{ij}x_j \right| \|e_i\|, & M &= \max_{j=1,\dots,n} \|e_j\| \\
&\leq M \sum_{j=1}^n \sum_{i=1}^m |a_{ij}| |x_j|, & \|A\|_\infty &= \max_{i=1,\dots,m, j=1,\dots,n} |a_{ij}| \\
&= M \sum_{i=1}^m \sum_{j=1}^n |a_{ij}| |x_j| \\
&\leq M \sum_{i=1}^m |A|_\infty |x|_1 \\
&= M \|x\|_1 \leq M
\end{aligned}$$

$$\|x\|_1 \leq M \|x\|$$

□

16 2017-11-01

Proposition 16.1. Let $(V, \|\cdot\|_V)$, $(W, \|\cdot\|_W)$ be normed linear spaces, $A : V \rightarrow W$ be linear. Then TFAE:

1. A is continuous
2. $\|A\| := \sup\{\|Ax\|_W : x \in \underbrace{B_V[0,1]}_{\text{closed ball, center 0 in } V}\} < \infty$
3. A is Lipschitz map with Lipschitz constant $\|A\|$

Moreover, in the case of (ii) (hence (iii)), above, $\|Ax\|_W \leq \|A\| \|x\|_V$ for any x in V .

Proof. (i) \implies (ii) A is continuous at 0 in V . Thus, letting $\varepsilon = 1$, there is $\delta > 0$ s.t. $A(B_V(0, \delta)) \subseteq B_W(0, 1)$. Now, if $x \in B_V[0, 1]$, then $\frac{\delta}{2}x \in B_V(0, \delta)$, so

$$\|Ax\|_W = \frac{2}{\delta} \left\| \underbrace{A\left(\frac{\delta}{2}x\right)}_{\in B_W(0,1)} \right\|_W < \frac{2}{\delta} 1 = \frac{2}{\delta} < \infty$$

so $\|A\| = \sup_{x \in B_V[0,1]} \|Ax\|_W \leq \frac{2}{\delta} < \infty$.

(ii) \implies (iii) If $x \in V \setminus \{0\}$, so $\frac{1}{\|x\|_V}x \in B_V[0, 1]$ and

$$(\star) \quad \|Ax\|_W = \|x\|_V \underbrace{\left\| A\left(\frac{1}{\|x\|_V}x\right) \right\|_W}_{\leq \|A\|} \leq \|A\| \|x\|_V.$$

Clearly, (\star) holds for $x = 0$ in V . Hence if $x, y \in V$,

$$\|Ax - Ay\|_W = \|A(x - y)\|_W \leq \|A\| \|x - y\|_V.$$

Thus A is Lipschitz and “Moreover...” holds. Furthermore, by (\star) ,

$$\|A\| = \sup_{x \in V \setminus \{0\}} \frac{\|Ax\|_W}{\|x\|_V} = \sup_{x \neq y \text{ in } V} \frac{\|Ax - Ay\|_W}{\|x - y\|_V}$$

which is the definition of the Lipschitz constant.

(iii) \implies (i) Obvious.

□

Remark: Let $B(V, W) = \{A : V \rightarrow W \mid A \text{ is linear and continuous}\}$. Notice that (ii) above shows that A must be bounded on $B_V[0, 1]$ and we call A a “bounded linear operator”.

$B(V, W)$ is a \mathbb{R} -vector space (pointwise addition and scalar multiplication) and $\|\cdot\|$ is a norm on $B(V, W)$, called “bounded operator norm”. (Exercise.)

Question: Is continuity automatic for linear operators?

Example: Consider the vector space $C[0, 1]$ of continuous \mathbb{R} -valued functions on $[0, 1]$. Let

$$\varphi : C[0, 1] \rightarrow \mathbb{R}, \quad \varphi(f) = f\left(\frac{1}{2}\right) \text{ (evaluation at } \frac{1}{2}\text{)}.$$

Then φ is linear: let $f, g \in C[0, 1]$, $\alpha \in \mathbb{R}$, then

$$\begin{aligned} \varphi(f + \alpha g) &= f\left(\frac{1}{2}\right) + \alpha g\left(\frac{1}{2}\right) \\ &= \varphi(f) + \alpha \varphi(g) \end{aligned}$$

(i) Consider $(C[0, 1], \|\cdot\|_\infty)$. Then

$$|\varphi(f)| = |f\left(\frac{1}{2}\right)| \leq \max_{t \in [0, 1]} |f(t)| = \|f\|_\infty.$$

Thus $\|\varphi\| \leq 1$ (easy to show that $\|\varphi\| = 1$), i.e., $\varphi \in B((C[0, 1], \|\cdot\|_\infty), \mathbb{R})$.

(ii) Now consider $(C[0, 1], \|\cdot\|_p)$ ($1 \leq p < \infty$). Let

$$f_n(t) = \begin{cases} 0 & \text{if } t \leq \frac{1}{2} - \frac{1}{n^{2p}} \\ n^{2p+1}(t - \frac{1}{2} + \frac{1}{n^{2p}}) & \text{if } \frac{1}{2} - \frac{1}{n^{2p}} < t \leq \frac{1}{2} \\ n^{2p+1}(\frac{1}{2} + \frac{1}{n^{2p}} - t) & \text{if } \frac{1}{2} < t \leq \frac{1}{2} + \frac{1}{n^{2p}} \\ 0 & \text{if } t > \frac{1}{2} + \frac{1}{n^{2p}} \end{cases}$$

[triangular spike at $[\frac{1}{2} - \frac{1}{n^{2p}}, \frac{1}{2} + \frac{1}{n^{2p}}]$ with peak at $\frac{1}{2}$ having value n .] Notice

$$\varphi(f_n) = f_n\left(\frac{1}{2}\right) = n$$

while

$$\begin{aligned} \|f_n\|_p &= \left(\int_0^1 f_n^p \right)^{\frac{1}{p}} \\ &= \left(\int_{\frac{1}{2} - \frac{1}{n^{2p}}}^{\frac{1}{2} + \frac{1}{n^{2p}}} \underbrace{f_n^p}_{0 \leq f_n^p \leq n^p} \right)^{\frac{1}{p}} \\ &\leq \left(\int_{\frac{1}{2} - \frac{1}{n^{2p}}}^{\frac{1}{2} + \frac{1}{n^{2p}}} \underbrace{n^p}_{\text{constant}} \right)^{\frac{1}{p}} \\ &= \left(n^p \frac{2}{n^{2p}} \right)^{\frac{1}{p}} = \frac{2^{\frac{1}{p}}}{n}. \end{aligned}$$

Thus

$$\frac{|\varphi(f_n)|}{\|f_n\|_p} = \frac{n}{\frac{2^{\frac{1}{p}}}{n}} = \frac{n^2}{2^{\frac{1}{p}}} \xrightarrow{n \rightarrow \infty} \infty.$$

Hence

$$\varphi \notin B((C[0, 1], \|\cdot\|_p), \mathbb{R}).$$

Example: (Axiom of choice) If $(V, \|\cdot\|)$ is an infinite dimensional normed vector space, then it admits an infinite linearly independent family $\{v_n\}_{n=1}^\infty$. There exists a basis $\{w_i\}_{i \in I}$ s.t. $\{v_n\}_{n=1}^\infty \subseteq \{w_i\}_{i \in I}$.

Define $f : V \rightarrow \mathbb{R}$

$$f(w_i) = \begin{cases} \frac{n}{\|v_n\|} & \text{if } w_i = v_n \\ 0 & \text{otherwise} \end{cases}$$

and extend uniquely to a linear operator on V .

Check that $f \notin B(V, \mathbb{R})$.

Why isn't $B[0, 1]$ in $(C[0, 1], \|\cdot\|_\infty)$ compact?

Reason: existence of subsequence with no converging subsequence [similar holds on $(\ell_p, \|\cdot\|_p)$].

Picture: [triangle spike to height $f_n(t) = 1$ on $[\frac{1}{n+1}, \frac{1}{n}]$, 0 elsewhere.]

Calculate that if $m \neq n$, $\|f_n - f_m\|_\infty = 1$. Conclude that $(f_n)_{n=1}^\infty \subset B[0, 1]$ admits no converging subsequence.

17 2017-11-03

Theorem 17.1 (Banach's Contraction Mapping Theorem). Let (X, d) be a complete metric space and let $\Gamma : X \rightarrow X$ be a strict contraction, i.e., there is $0 < c < 1$ s.t. $d(\Gamma(x), \Gamma(y)) < cd(x, y)$ for x, y in X (Γ is c -Lipschitz). Then

- (i) there is a unique fixed point x_{fix} for Γ , i.e. $\Gamma(x_{\text{fix}}) = x_{\text{fix}}$,
- (ii) given any x_0 in X , if we define a sequence by $x_n = \Gamma(x_{n-1})$, $n \in \mathbb{N}$, then it satisfies

$$d(x_n, x_{\text{fix}}) \leq \frac{c^n}{1-c} d(x_0, \Gamma(x_0))$$

and hence $\lim_{n \rightarrow \infty} x_n = x_{\text{fix}}$.

Proof. Let $x_0 \in X$. We define $(x_n)_{n=1}^\infty \subseteq X$ as in (ii), above. We note that $d(x_1, x_2) = d(\Gamma(x_0), \Gamma(x_1)) \leq cd(x_0, x_1) = cd(x_0, \Gamma(x_0))$.

Now, if

$$(\star) \quad d(x_n, x_{n+1}) \leq c^n d(x_0, \Gamma(x_0)),$$

then

$$d(x_{n+1}, x_{n+2}) = d(\Gamma(x_n), \Gamma(x_{n+1})) \leq cd(x_n, x_{n+1}) \leq c^{n+1} d(x_0, \Gamma(x_0))$$

so (\star) holds generally. Thus, if $m < n$ in \mathbb{N} we have

$$\begin{aligned} d(x_m, x_n) &\leq \sum_{j=m}^{n-1} d(x_j, x_{j+1}) \\ &\leq \sum_{j=m}^{n-1} c^j d(x_0, \Gamma(x_0)), \text{ by } (\star) \\ &\leq \sum_{j=m}^{\infty} c^j d(x_0, \Gamma(x_0)), \text{ by } (\star) = \frac{c^m}{1-c} d(x_0, \Gamma(x_0)). \end{aligned}$$

It follows that $(x_n)_{n=1}^\infty$ is Cauchy, and hence $x_{\text{fix}} = \lim_{n \rightarrow \infty} x_n$ exists. Then

$$x_{\text{fix}} = \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \Gamma(x_n) \underset{\substack{\Gamma \text{ Lipschitz} \implies \text{continuous}}}{=} \Gamma(\lim_{n \rightarrow \infty} x_n) = \Gamma(x_{\text{fix}}).$$

Hence x_{fix} is a fixed point. If y_{fix} is any other fixed point then

$$\begin{aligned} d(x_{\text{fix}}, y_{\text{fix}}) &= d(\Gamma(x_{\text{fix}}), \Gamma(y_{\text{fix}})) \\ &\leq cd(x_{\text{fix}}, y_{\text{fix}}) \\ &< d(x_{\text{fix}}, y_{\text{fix}}), \text{ if } d(x_{\text{fix}}, y_{\text{fix}}) > 0 \end{aligned}$$

so we must have $d(x_{\text{fix}}, y_{\text{fix}}) = 0$, i.e. $x_{\text{fix}} = y_{\text{fix}}$. Thus (i) holds.
Also we have for m, n , as above,

$$d(x_m, x_n) \leq \frac{c^m}{1-c} d(x_0, \Gamma(x_0)) \implies d(x_n, x_{\text{fix}}) = \lim_{n \rightarrow \infty} d(x_m, x_n) \leq \frac{c^m}{1-c} d(x_0, \Gamma(x_0))$$

so (ii) holds. □

Application: Some differentiable equations

Let $F : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous, and $y_0 \in \mathbb{R}$. We consider the following initial value problem:
Want $f \in C[a, b]$, with $\underbrace{f(a) = y_0}_{\text{initial value}}$ and $\underbrace{f'(t) = F(t, f(t))}_{\text{differential equation}}$ (IVP).

We use the Fundamental Theorem of Calculus to convert this to an integral equation:

Want $f \in C[a, b]$, $f(t) = y_0 + \int_a^t F(s, f(s)) ds$ (IE).

Theorem 17.2 (Picard-Lindelof Theorem). Let F, y_0 be as above and suppose that F is Lipschitz in the second variable: for all $t \in [a, b]$, $y, z \in \mathbb{R}$,

$$|F(t, y) - F(t, z)| \leq L|y - z|, \text{ for some } L > 0.$$

Then (IVP) admits a unique solution, f_{sol} in $C[a, b]$.

Proof. (I) Let us assume that $(b-a)L < 1$. Define $\Gamma : C[a, b] \rightarrow C[a, b]$ by, for $t \in [a, b]$,

$$\Gamma(F)(t) = y_0 + \int_a^t F(s, f(s)) ds.$$

Then for $f, g \in C[a, b]$, and $t \in [a, b]$, then

$$\begin{aligned} |\Gamma(f)(t) - \Gamma(g)(t)| &= \left| \int_a^t [F(s, f(s)) - F(s, g(s))] ds \right| \\ &\leq \int_a^t \underbrace{|F(s, f(s)) - F(s, g(s))|}_{\leq L|f(s) - g(s)|} ds \\ &\leq L \int_a^t \underbrace{|f(s) - g(s)|}_{\leq \|f - g\|_\infty} ds \\ &\leq L \|f - g\|_\infty \int_a^t 1 ds \\ &= L \|f - g\|_\infty (t - a) \leq (b - a)L \|f - g\|_\infty. \end{aligned}$$

In summary,

$$\begin{aligned} \|\Gamma(f) - \Gamma(g)\|_\infty &= \sup_{t \in [a, b]} \|\Gamma(f)(t) - \Gamma(g)(t)\| \\ &\leq \underbrace{(b-a)L}_{< 1} \|f - g\|_\infty. \end{aligned}$$

Hence, by the Contraction Mapping Theorem, applied to Γ on $(C[a, b], \|\cdot\|_\infty)$, there is a unique f_{sol} such that $\Gamma(f_{\text{sol}}) = f_{\text{sol}}$.

(II) Let

$$a = a_1 < a_2 < b_1 < b_3 < b_2 < \cdots < a_n < b_{n-1} < b_n = b$$

so that $(b_j - a_j)L < 1$ for $j = 1, \dots, n$.

Notice that $[a_j, b_j] \cap [a_{j+1}, b_{j+1}] = [a_j, b_{j+1}]$ has non-empty interior.

Let $f_1 \in C[a_1, b_1]$ be the unique solution to (IVP) with $f_1(a) = y_0$, by (I).

Then, let f_2 in $C[a_2, b_2]$ satisfy (IVP) with $f_2(a_2) = f_1(a_2)$. Then, let f_3 in $C[a_3, b_3]$ satisfy (IVP) with $f_3(a_3) = f_2(a_3)$. Etc. Let $f : [a, b] \rightarrow \mathbb{R}$ be given by

$$f(t) = f_j(t) \text{ for } t \in [a_j, b_j], j = 1, \dots, n.$$

Check that this is well-defined. Its value is uniquely determined on each $[a_{j+1}, b_j]$, thanks to uniqueness in (I). \square

18 2017-11-06

Example: (IVP) Want $f \in C[0, 1]$ s.t.

$$f(0) = 1, \quad f'(t) = tf(t).$$

We convert to

$$(IE) \quad f(t) = 1 + \int_0^t sf(s)ds.$$

This fits into Picard-Lindelof Theorem. Let $F(t, y) = ty$, so $f(t) = 1 + \int_0^t F(s, f(s))ds$ with $|F(t, y) - F(t, z)| = \underbrace{|t|}_{\leq 1} |y - z| \leq$

$|y - z|$. (Case (II) of Picard-Lindelof.)

However, let $\Gamma : C[0, 1] \rightarrow C[0, 1]$ by, for $t \in [0, 1]$,

$$\Gamma(f)(t) = 1 + \int_0^t sf(s)ds.$$

Let us see that Γ , itself, is a strict contraction. Let $f, g \in C[0, 1], t \in [0, 1]$,

$$\begin{aligned} |\Gamma(f)(t) - \Gamma(g)(t)| &\leq \int_0^t s \underbrace{|f(s) - g(s)|}_{\leq \|f - g\|_\infty} ds \\ &\leq \int_0^t s ds \|f - g\|_\infty \\ &= \underbrace{\frac{t^2}{2}}_{\leq \frac{1}{2}} \|f - g\|_\infty \\ &\leq \frac{1}{2} \|f - g\|_\infty. \end{aligned}$$

$$(\|\Gamma(f) - \Gamma(g)\|_\infty \leq \frac{1}{2} \|f - g\|_\infty)$$

Hence, contraction mapping theorem tells us that Γ has a unique fixed point, ie (IE) and (IVP) have a unique solution, f_{sol} . Furthermore, if we choose $f_0 \in C[0, 1]$ and let $f_n = \Gamma(f_{n-1})$ ($n \in \mathbb{N}$) then

$$\|f_{\text{sol}} - f_n\|_\infty \leq \underbrace{\frac{(\frac{1}{2})^n}{1 - \frac{1}{2}}}_{=\frac{1}{2^{n-1}}} \|f_0 - \Gamma(f_0)\|_\infty.$$

We can compute f_{sol} .

Let $f_0(t) = 0$ (constant zero).

$$\begin{aligned} f_1(t) &= \Gamma(f_0)(t) = 1 + \int_0^t s \cdot 0 ds = 1 \\ f_2(t) &= \Gamma(f_1)(t) = 1 + \int_0^t s \cdot 1 ds = 1 + \frac{t^2}{2} \\ f_3(t) &= \Gamma(f_2)(t) = 1 + \int_0^t s \left(1 + \frac{t^2}{2}\right) ds = 1 + \frac{t^2}{2} + \frac{t^4}{4 \cdot 2} \end{aligned}$$

(Use induction to check)

$$f_n(t) = 1 + \frac{t^2}{2} + \frac{t^4}{4 \cdot 2} + \cdots + \frac{t^{2(n-1)}}{[2(n-1)][2(n-2)] \cdots 2} = \sum_{k=1}^n \frac{t^{2(k-1)}}{2^{k-1}(k-1)!}.$$

Thus, at any t in $[0, 1]$,

$$f_{\text{sol}} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{t^{2(k-1)}}{2^{k-1}(k-1)!} = \sum_{k=1}^{\infty} \frac{t^{2(k-1)}}{2^{k-1}(k-1)!}.$$

Furthermore, for each n ,

$$\begin{aligned} \|f_{\text{sol}} - f_n\|_{\infty} &= \max_{t \in [0,1]} |f_{\text{sol}}(t) - f_n(t)| \\ &\leq \frac{1}{2^{n-1}} \left\| 0 - \underbrace{\Gamma(0)}_{=1} \right\|_{\infty} = \frac{1}{2^{n-1}}. \end{aligned}$$

Question: Suppose we only knew that

$$d(\Gamma(x), \Gamma(y)) < d(x, y) \text{ for } x \neq y \text{ in } X.$$

(“proper contraction” instead of “strict contraction”)

Does Γ necessarily admit a fixed point?

Answer #1: No.

Example: On $X = [1, \infty) \subset \mathbb{R}$, let $\Gamma(x) = x + \frac{1}{x}$. If $x < y$, we have there is $x < c_{x,y} < y$ such that

$$|\Gamma(x) - \Gamma(y)| = |\Gamma'(c_{x,y})| |x - y| = \left| 1 - \frac{1}{c_{x,y}^2} \right| |x - y| < |x - y|.$$

Notice: if $\Gamma(x) = x$ we'd have $x = x + \frac{1}{x} \implies 0 = \frac{1}{x}$. Hence Γ admits no fixed point in $[1, \infty)$.

Answer #2: Yes, provided we limit (X, d) .

Theorem 18.1 (Edelstein). Let (X, d) be compact, and $\Gamma : X \rightarrow X$ satisfy $d(\Gamma(x), \Gamma(y)) < d(x, y)$ for $x \neq y$ in X . Then

- (i) Γ admits a unique fixed point x_{fix} , and
- (ii) if $x_0 \in X$, and $x_n = \Gamma(x_{n-1})$ ($n \in \mathbb{N}$), then $x_{\text{fix}} = \lim_{n \rightarrow \infty} x_n$.

Proof. (i) Let $f : X \rightarrow \mathbb{R}, f(x) = d(x, \Gamma(x))$. Since Γ is continuous, f is continuous. [Check that f is 2-Lipschitz.] Hence, by EVT, there is x_{min} in X so $f(x_{\text{min}}) = \min f(X)$. Suppose $x_{\text{min}} \neq \Gamma(x_{\text{min}})$, then

$$\begin{aligned} f(\Gamma(x_{\text{min}})) &= d(\Gamma(x_{\text{min}}), \Gamma \circ \Gamma(x_{\text{min}})) \\ &< d(x_{\text{min}}, \Gamma(x_{\text{min}})) = f(x_{\text{min}}) \end{aligned}$$

violating choice of x_{\min} . Hence $x_{\min} = \Gamma(x_{\min})$, so write $x_{\min} = x_{\text{fix}}$.

If, also, $y = \Gamma(y)$ in X , with $y \neq x_{\text{fix}}$, then

$$d(y, x_{\text{fix}}) = d(\Gamma(y), \Gamma(x_{\text{fix}})) < d(y, x_{\text{fix}})$$

which is absurd.

(ii) Let $x_0 \in X, (x_n)_{n=1}^\infty$ be as above. Notice that

$$0 \leq d(x_{\text{fix}}, x_{n+1}) = d(\Gamma(x_{\text{fix}}), \Gamma(x_0)) < d(x_{\text{fix}}, x_0)$$

so $L = \lim_{n \rightarrow \infty} d(x_{\text{fix}}, x_n)$ exists (decreasing, bounded sequence in \mathbb{R}).

Consider any converging subsequence $(x_{n_k})_{k=1}^\infty$ of $(x_n)_{n=1}^\infty$, with $x = \lim_{k \rightarrow \infty} x_{n_k}$. Then $d(x_{\text{fix}}, x) = \lim_{k \rightarrow \infty} d(x_{\text{fix}}, x_{n_k}) = L$.

If $x \neq x_{\text{fix}}$, then

$$\begin{aligned} L &= \lim_{k \rightarrow \infty} d(x_{\text{fix}}, x_{n_k} + 1) = \lim_{k \rightarrow \infty} d(x_{\text{fix}}, \Gamma(x_{n_k})) \\ &= d(x_{\text{fix}}, \Gamma(x)) < d(x_{\text{fix}}, x) = L \end{aligned}$$

which is absurd. Hence the sequence $(x_n)_{n=1}^\infty$ has that x_{fix} is the only possible limit of a subsequence. Thus $\lim_{n \rightarrow \infty} x_n = x_{\text{fix}}$ (check!). \square

19 2017-11-08

Office hours:

Today	2:30-3:30
Tomorrow	2:30-4
Friday	2:30-3:30

19.1 BAIRE CATEGORY THEOREM

Definition: Let (X, d) be a metric space.

- (i) A subset $N \subset X$ is called nowhere dense if $(\overline{N})^\circ = \emptyset$ (ie. the interior of the closure of N is the empty set). [Equivalently, for any $x \in N, \varepsilon > 0, B(x, \varepsilon) \setminus \overline{N} \neq \emptyset$].
- (ii) A set $S \subseteq X$ will be called meager (or is 1st category) if S is a countable union of nowhere dense sets: i.e.

$$S = \bigcup_{n=1}^{\infty} N_n, \text{ each } (\overline{N}_n)^\circ = \emptyset.$$

(ii') $S \subseteq X$ is non-meager (or is 2nd category) provided that it is not meager.

- (iii) A set $R \subseteq X$ is residual if $X \setminus R$ is meager.

Remarks:

nowhere dense \implies meager

residual \implies non-meager (provided (X, d) is complete;

consequence of B.C.T, Baire Category Theorem)

If (X, d) is complete, we think of meager = “small”, non-meager = “not small” \iff residual.

Examples:

- (i) If $x_0 \in X, \{x_0\}$ is nowhere dense $\iff x_0$ is an accumulation point.

(ii) In $(\mathbb{R}^2, \|\cdot\|_2)$, $\mathbb{R} \times \{0\}$ is meager (exercise).

(iii) In $(\mathbb{R}, |\cdot|)$, the Cantor set C is nowhere dense.

Indeed, C is closed. If $t = 0.t_1t_2 \dots \in C$ (ternary representation), then given $\varepsilon > 0$, find k so $\frac{1}{3^k} < \varepsilon$ and then

$$t' = 0.t_1t_2 \dots t_{k-1}1t_{k+1} \dots \in B(t, \varepsilon) \setminus C.$$

(iv) $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$ is meager in $(\mathbb{R}, |\cdot|)$ (using (i)).

(v) $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$ is meager in $(\mathbb{Q}, |\cdot|)$ (using (i)).

Note: if (X, d) is not complete, it may be meager in itself. [meager sets are interesting in complete settings.]

Remark: If (X, d) is a metric space, $U \subseteq X$ is open and $x_0 \in U$, then there is $\varepsilon > 0$, s.t. $B[x, \varepsilon] \subseteq U$ (Indeed, let $\varepsilon' > 0$ be so $B(x, \varepsilon') \subseteq U$, and $\varepsilon \in (0, \varepsilon')$).

Lemma 19.1. Let (X, d) be a metric space, $N \subset X$. Then N is nowhere dense $\iff \overline{X \setminus N} = X$.

Proof.

$$\begin{aligned} N \text{ is nowhere dense} &\iff \text{for any } x \in \overline{N}, \varepsilon > 0, B(x, \varepsilon) \setminus \overline{N} \neq \emptyset \\ &\iff x \in \overline{X \setminus N} \text{ for any } x \in \overline{N} \cup (X \setminus \overline{N}). \end{aligned}$$

□

Theorem 19.1 (Baire Category Theorem). Let (X, d) be a complete metric space.

(i) Suppose $\{U_n\}_{n=1}^\infty$ is a sequence of open sets, each dense in X . Then $\bigcap_{n=1}^\infty U_n$ is dense in X .

(ii) If $M \subset X$ is meager, then $M^\circ = \emptyset$.

Proof. (i) Let $x_0 \in X$ and $\varepsilon_0 > 0$. We wish to show that $B(x_0, \varepsilon_0) \cap \bigcap_{n=1}^\infty U_n \neq \emptyset$.

Since $\overline{U_1} = X$, there is $x_1 \in B(x_0, \varepsilon_0) \cap U_1$ (using meet set characterization of closure). Let $\varepsilon_1 > 0$ be chosen so $B[x_1, \varepsilon_1] \subseteq B(x_0, \varepsilon_0) \cap U_1$.

Since $\overline{U_2} = X$, there is $x_2 \in B(x_1, \varepsilon_1) \cap U_2$.

Let $\varepsilon_2 \in (0, \frac{\varepsilon_1}{2}]$ be so $B[x_2, \varepsilon_2] \subseteq B(x_1, \varepsilon_1) \cap U_2$.

Inductively, having chosen x_n, ε_n , we appeal to the fact that $\overline{U_{n+1}} = X$ to find $x_{n+1} \in B(x_n, \varepsilon_n) \cap U_{n+1}$, then choose $\varepsilon_{n+1} \in (0, \frac{\varepsilon_n}{2}]$ and $B[x_{n+1}, \varepsilon_{n+1}] \subseteq B(x_n, \varepsilon_n) \cap U_{n+1}$.

Thus, we have $(x_n)_{n=1}^\infty \subseteq X$, $(\varepsilon_n)_{n=1}^\infty \subset (0, \infty)$ s.t.

$$(a) \ B[x_{n+1}, \varepsilon_{n+1}] \subseteq B(x_n, \varepsilon_n) \subseteq B[x_n, \varepsilon_n]$$

$$(b) \ \text{diam } B[x_n, \varepsilon_n] = 2\varepsilon_n \leq \varepsilon_{n-1} \leq \frac{\varepsilon_{n-2}}{2} \leq \dots \leq \frac{\varepsilon_1}{2^{n-1}}.$$

$$(c) \ B[x_n, \varepsilon_n] \subseteq U_n \cap B(x_0, \varepsilon_0).$$

Then (a) & (b), with the Nested Sets Theorem, show that $\bigcap_{n=1}^\infty B[x_n, \varepsilon_n] \neq \emptyset$.

Further, (c) shows that $\emptyset \neq \bigcap_{n=1}^\infty B[x_n, \varepsilon_n] \subseteq \bigcap_{n=1}^\infty U_n \cap B(x_0, \varepsilon_0)$.

Hence, for any $x_0 \in X$, $\varepsilon_0 > 0$, $B(x_0, \varepsilon_0) \cap \bigcap_{n=1}^\infty U_n \neq \emptyset$, so $\bigcap_{n=1}^\infty \overline{U_n} = X$.

(ii) Write $M = \bigcup_{n=1}^\infty N_n$, each $(\overline{N_n})^\circ = \emptyset$. Then $U_n = X \setminus \overline{N_n}$ is open, and dense in X , by Lemma.

We have

$$\begin{aligned} X \setminus M &= X \setminus \bigcup_{n=1}^\infty N_n \supseteq X \setminus \bigcup_{n=1}^\infty \overline{N_n} \text{ (as each } N_n \subseteq \overline{N_n}) \\ &= \bigcap_{n=1}^\infty (X \setminus \overline{N_n}) = \bigcap_{n=1}^\infty U_n \end{aligned}$$

so $\overline{X \setminus M} = X$. Thus if $x \in M, \varepsilon > 0$, we have $B(x, \varepsilon) \setminus M = B(x, \varepsilon) \cap (X \setminus M) \neq \emptyset$. Thus $x \notin M^\circ$, i.e. $M^\circ = \emptyset$. \square

Question: Let $\{q_k\}_{k=1}^\infty = \mathbb{Q}$. Let for n in \mathbb{N}

$$U_n = \underbrace{\bigcup_{k=1}^\infty \underbrace{\left(q_k - \frac{1}{2^{kn+1}}, q_k + \frac{1}{2^{kn+1}}\right)}_{\text{length is } \frac{1}{2^{nk}}}}_{U_n \text{ is a union of intervals, sum of lengths is } \sum_{k=1}^\infty \frac{1}{(2^n)^k} = \frac{1}{1 - \frac{1}{2^n}}}$$

Is $\mathbb{Q} = \bigcap_{n=1}^\infty U_n$?

20 2017-11-10

Remark: In particular, a nonempty open subset in a complete metric space is nonmeager. The whole of X is a nonempty open set.

Corollary 20.1. A residual set in a complete metric space is nonmeager.

Proof. Let $R \subset X$ be residual, so $M = X \setminus R$ is meager, so $X \setminus R = \bigcup_{n=1}^\infty N_n$, each $(\overline{N_n})^\circ = \emptyset$. If we had that R was meager, i.e. $R = \bigcup_{n=1}^\infty N'_n$, $(\overline{N'_n})^\circ = \emptyset$, then

$$X = R \cup (X \setminus R) = \underbrace{\bigcup_{n=1}^\infty N'_n \cup \bigcup_{n=1}^\infty N_n}_{\text{countable union of nowhere dense sets}}.$$

But $X^\circ = X$, so this contradicts B.C.T. \square

meager = “small”, residual = “bigness”, “typical elements”

Definition: Let (X, d) be a metric space.

1. $G \subseteq X$ is a G_δ -set if $G = \bigcap_{n=1}^\infty U_n$, each U_n open
2. $F \subseteq X$ is an F_σ -set if $F = \bigcup_{n=1}^\infty F_n$, each F_n closed

Examples:

1. In A4, Q2, we saw that any closed set is G_δ
(i') Any open set $U \subseteq X$ is F_σ (De Morgan's law)

2. $\mathbb{R} \setminus \mathbb{Q}$ is not F_σ .

First, $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$ is F_σ . Second, if $F \subset \mathbb{R} \setminus \mathbb{Q}$ is closed, then F is nowhere dense (this just follows density of \mathbb{Q}). Thus if we had an F_σ realization $\mathbb{R} \setminus \mathbb{Q} = \bigcup_{n=1}^\infty F_n$, $F_n \subset \mathbb{R} \setminus \mathbb{Q}$ closed, then $\mathbb{R} \setminus \mathbb{Q}$ is meager. Thus,

$$\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q}) = \bigcup_{q \in \mathbb{Q}} \{q\} \cup \bigcup_{n=1}^\infty F_n$$

would be meager which violates B.C.T. (Corollary just stated).

(ii') \mathbb{Q} is not G_δ (De Morgan, from (ii)).

In particular

$$\mathbb{Q} \not\subseteq \bigcap_{n=1}^\infty \bigcup_{k=1}^\infty \underbrace{\left(q_k - \frac{1}{2^{kn+1}}, q_k + \frac{1}{2^{kn+1}}\right)}_{U_n}.$$

$$\{q_k\}_{k=1}^\infty = \mathbb{Q}.$$

Corollary 20.2. In a complete metric space, a dense G_δ -subset is residual.

Proof. In complete (X, d) , if $G = \bigcap_{n=1}^\infty U_n$, each U_n open, and $\overline{G} = X$, then each $\overline{U_n} = X$. Thus, by lemma before B.C.T., each $X \setminus U_n$ is nowhere dense hence $X \setminus G = X \setminus \bigcap_{n=1}^\infty U_n = \bigcup_{n=1}^\infty (X \setminus U_n)$ is meager. \square

Theorem 20.1 (Uniform Boundedness Principle). Let (X, d) be a complete metric space and $\{f_i\}_{i \in I} \subset C(X)$ (continuous \mathbb{R} -valued functions) which satisfies for each x

$$\sup_{i \in I} |f_i(x)| < \infty \text{ (pointwise boundedness).}$$

Then there exists an open $\emptyset \neq U \subseteq X$ s.t.

$$\sup_{i \in I} \sup_{x \in U} |f_i(x)| < \infty \text{ (uniform boundedness on } U\text{).}$$

Proof. For n in \mathbb{N} , let

$$F_n = \{x \in X : |f_i(x)| \leq n \text{ for all } i \in I\}.$$

By our pointwise boundedness assumption,

$$X = \bigcup_{n=1}^\infty F_n \quad (\star).$$

Each F_n is closed:

$$F_n = \bigcap_{i \in I} |f_i|^{-1}((-\infty, n]) = \bigcap_{i \in I} \underbrace{(X \setminus \underbrace{|f_i|^{-1}(n, \infty)}_{\substack{\text{open, as } |f_i(\cdot)| \text{ is continuous} \\ \text{closed}}})}_{\text{closed}}$$

But B.C.T. tells us that our complete X is non-meager, so for some n_0 , $F_{n_0}^\circ \neq \emptyset$. Let $U = F_{n_0}^\circ$, and for all $x \in U \subseteq F_n$

$$\begin{aligned} &|f_i(x)| \leq n_0 \text{ for all } i \in I \\ \implies &\sup_{x \in U} |f_i(x)| \leq n_0 \text{ for all } i \in I \\ \implies &\sup_{i \in I} \sup_{x \in U} |f_i(x)| \leq n_0 < \infty. \end{aligned}$$

\square

Corollary 20.3 (Banach-Stenhaus Theorem). Let $(V, \|\cdot\|_V)$ be a Banach space, $(W, \|\cdot\|_W)$ a normed vector space, and $\{T_i\}_{i \in I} \subset B(V, W)$ satisfies

$$\sup_{i \in I} \|T_i x\|_W < \infty \text{ for each } x \in V.$$

Then

$$\sup_{i \in I} \|T_i\| < \infty. \text{ [Recall } \|T_i\| = \sup_{x \in B_V[0,1]} \|T_i x\|_W.]$$

Proof. Let $f_i(x) = \|T_i x\|_W$, for $i \in I, x \in V$, so $\{f_i\}_{i \in I} \subset C(V)$. Our assumption on $\{T_i\}_{i \in I}$, gives pointwise boundedness of $\{f_i\}_{i \in I}$, so U.B.P provides $\emptyset \neq U \subset V$ for which

$$M = \sup_{i \in I} \sup_{x \in U} \|T_i x\| < \infty.$$

As U is open, if $x_0 \in U$, there is $\varepsilon > 0$, $B[x_0, \varepsilon] \subset U$.

Now if $z \in B_V[0, 1]$, then we may write

$$z = \frac{1}{2\varepsilon}(-x_0 + \varepsilon z) + \frac{1}{2\varepsilon}(x_0 + \varepsilon z)$$

and, for i in I , we have

$$\begin{aligned}\|T_i z\|_W &\leq \frac{1}{2\varepsilon} \left\| T_i \left(\underbrace{x_0 - \varepsilon z}_{\in B[x, \varepsilon] \subset U} \right) \right\|_W + \frac{1}{2\varepsilon} \left\| T_i \left(\underbrace{x_0 + \varepsilon z}_{\in B[x, \varepsilon] \subset U} \right) \right\|_W \\ &\leq \frac{1}{2\varepsilon} M + \frac{1}{2\varepsilon} M = \frac{M}{\varepsilon}. \\ \Rightarrow \|T_i\| &= \sup_{z \in B_V[0,1]} \|T_i z\|_W \leq \frac{M}{\varepsilon} < \infty.\end{aligned}$$

□

21 2017-11-13

21.1 BAIRE-1 FUNCTIONS

Def: Let $\emptyset \neq X \subseteq \mathbb{R}$, so (X, d) is a metric space with relativized metric from \mathbb{R} .

A function $f : X \rightarrow \mathbb{R}$ is called Baire-1 if there is a sequence $(f_n)_{n=1}^\infty \subset C(X)$ such that for $t \in X$,

$$f(t) = \lim_{n \rightarrow \infty} f_n(t) \text{ (pointwise limit).}$$

Remark: Unlike uniform limits, pointwise limits of continuous functions need not be continuous.

Example: Let $X = [0, 1]$, $f_n(t) = t^n$. Then

$$\lim_{n \rightarrow \infty} f_n(t) = \begin{cases} 0 & t \in [0, 1) \\ 1 & t = 1. \end{cases}$$

Question: Let for t in $[0, 1]$,

$$f_n(t) = \cos(n! \pi t)^{n!}.$$

If $t = \frac{k}{\ell} \in \mathbb{Q}$, $\ell \in \mathbb{N}$, then $f_n(t) = 1$, if $t \geq \ell + 1$.

Does $\lim_{n \rightarrow \infty} f_n(t) = \chi_{\mathbb{Q} \cap [0, 1]}(t)$ for t in $[0, 1]$?

Answer: No. (Probably the limit does not exist.)

The answer will follow from (corollary to) the next theorem and B.C.T.

Theorem 21.1 (Baire). Let $a < b$, and $f : (a, b) \rightarrow \mathbb{R}$ be a Baire-1 function, then there is t_0 in (a, b) such that f is continuous at t_0 .

$$\chi_{\mathbb{Q}}(t) = \lim_{n \rightarrow \infty} \underbrace{\lim_{m \rightarrow \infty} |\cos(n! \pi t)^m|}_{\chi_{\{\frac{k}{n!}, k \in \mathbb{Z}\}}(t)}$$

Baire-2 = pointwise limit of Baire-1 functions.

At no t_0 is $\chi_{\mathbb{Q}}$ continuous, thus not Baire-1.

Proof. Let $f(t) = \lim_{n \rightarrow \infty} f_n(t)$, $t \in (a, b)$, $(f_n)_{n=1}^\infty \subset C(a, b)$.

(I) Given $\varepsilon > 0$, we will show that there are $\alpha < \beta$ in (a, b) , and N_ε in \mathbb{N} such that for all $n, m \geq N_\varepsilon$,

$$|f_n(t) - f_m(t)| \leq \varepsilon \text{ for } t \in [\alpha, \beta].$$

Let us proceed by contradiction. Hence, there exists t_1 in (a, b) , and $n_1, m_1 \in \mathbb{N}$ such that

$$|f_{n_1}(t_1) - f_{m_1}(t_1)| > \varepsilon.$$

Since each f_{n_1}, f_{m_1} is continuous, there is an open interval $I_1 \subset \overline{I_1} \subset (a, b)$ such that

$$|f_{n_1}(t) - f_{m_1}(t)| > \varepsilon \text{ for } t \in I_1.$$

$[t \mapsto |f_{n_1}(t) - f_{m_1}(t)| \text{ is continuous.}]$

Next, by assumption, there is $t_2 \in I_1$ such that there exist $n_2, m_2 > \max\{n_1, m_1\}$ such that

$$|f_{n_2}(t_2) - f_{m_2}(t_2)| > \varepsilon.$$

Again, as f_{n_2}, f_{m_2} are continuous, there is an open interval $I_2 \subset \overline{I_2} \subset I_1$ such that

$$|f_{n_2}(t) - f_{m_2}(t)| > \varepsilon \text{ for } t \in I_2.$$

Inductively, we obtain

- a sequence of intervals

$$\overline{I_1} \supset I_1 \supset \overline{I_2} \supset I_2 \supset \cdots \supset \overline{I_n} \supset I_n \supset \cdots, \text{ and}$$

- two increasing sequences $(n_k)_{k=1}^\infty, (m_k)_{k=1}^\infty \subseteq \mathbb{N}$ such that

$$|f_{n_k}(t) - f_{m_k}(t)| > \varepsilon \text{ for } t \in I_k.$$

Thus, by Nested Intervals Theorem, there exists

$$t_0 \in \bigcap_{k=1}^\infty \overline{I_k} = \bigcap_{k=2}^\infty \overline{I_k} \subseteq \bigcap_{k=1}^\infty I_k$$

so $t_0 \in I_k$ for each k , so

$$|f_{n_k}(t) - f_{m_k}(t)| > \varepsilon. \quad (\dagger)$$

But, by pointwise convergence, $f(t_0) = \lim_{k \rightarrow \infty} f_k(t_0)$ so $(f_n(t_0))_{n=1}^\infty \subset \mathbb{R}$ is Cauchy. This violates (\dagger) . Hence (I) holds.

(II) We use (I), with $\varepsilon = 1$, to find $\alpha_1 < \beta_1$ in (a, b) and N_1 in \mathbb{N} so

$$|f_n(t) - f_m(t)| \leq 1 \text{ for } t \in [\alpha_1, \beta_1], \text{ if } n, m \geq N_1.$$

We again use (I), with $\varepsilon = \frac{1}{2}$, to find $\alpha_2 < \beta_2$ in (a, b) and N_2 in \mathbb{N} so

$$|f_n(t) - f_m(t)| \leq \frac{1}{2} \text{ for } t \in [\alpha_2, \beta_2], \text{ if } n, m \geq N_2.$$

Inductively, we obtain

- intervals

$$(a, b) \supset [\alpha_1, \beta_1] \supset (\alpha_1, \beta_1) \supset [\alpha_2, \beta_2] \supset (\alpha_2, \beta_2) \supset \cdots \supset [\alpha_n, \beta_n] \supset (\alpha_n, \beta_n) \supset \cdots, \text{ and}$$

- an increasing sequence $(N_k)_{k=1}^\infty \subset \mathbb{N}$ such that

$$|f_n(t) - f_m(t)| \leq \frac{1}{k} \text{ for } t \in [\alpha_k, \beta_k], \text{ if } n, m \geq N_k. \quad (\ddagger)$$

By N.I.T. (Nested Intervals Theorem), there exists

$$t_0 \in \bigcap_{k=1}^\infty [\alpha_k, \beta_k] \subseteq \bigcap_{k=1}^\infty (\alpha_k, \beta_k).$$

Now, given $\varepsilon > 0$, let k in \mathbb{N} so $\frac{1}{k} < \varepsilon$, and then let $\delta = \min\{t_0 - \alpha_k, \beta_k - t_0\} > 0$ so $(t_0 - \delta, t_0 + \delta) \subset (\alpha_k, \beta_k) \subset [\alpha_k, \beta_k]$. Hence by (\ddagger) , we have that

$$|f_n(t) - f_m(t)| \leq \frac{1}{k} < \varepsilon \text{ whenever } t \in (t_0 - \delta, t_0 + \delta), n, m \geq N_k.$$

Hence $(f_n)_{n=1}^\infty$ converges “uniformly at t_0 ” (see Assignment 6), so f is continuous at t_0 (Assignment 6). \square

Corollary 21.1. Let $a < b$ in \mathbb{R} , $f : (a, b) \rightarrow \mathbb{R}$ be a Baire-1 function. The set $G = \{t \in (a, b) : f \text{ is continuous at } t\}$ is a dense G_δ -subset of (a, b) . [By B.C.T., $G \subset [a, b]$ is residual.]

Proof. If $t_0 \in (a, b)$ and $\varepsilon > 0$, then there exists $t \in (t_0 - \varepsilon, t_0 + \varepsilon) \cap (a, b) \cap G$. I.e. $G \cap (t_0 - \varepsilon, t_0 + \varepsilon) \neq \emptyset$, so $\overline{G} = (a, b)$ (relativized topology). Furthermore, the set G is always G_δ (Assignment 6). \square

Example: $\underbrace{\chi_{\mathbb{Q}}}_{\text{nowhere continuous}}$ is not Baire-1 on any interval.

22 2017-11-15

Corollary 22.1. Let $f \in C(a, b)$ ($a < b$ in \mathbb{R}) be right differentiable on (a, b) . Then f'_+ (right derivative) is continuous on a dense G_δ -subset of (a, b) . [In particular, if f is differentiable, f' is continuous on a dense G_δ -subset.]

Proof. Let $h_n(t) = \min\{b - t, \frac{1}{n}\}$ for n in \mathbb{N} , t in (a, b) . Then

$$f_n(t) = \frac{f(t + h_n(t)) - f(t)}{h_n(t)} \quad \left(= \frac{f(t + \frac{1}{n}) - f(t)}{\frac{1}{n}}, n \text{ large} \right)$$

satisfies that each $f_n \in C(a, b)$ and

$$f'_+(t) = \lim_{n \rightarrow \infty} f_n(t) \text{ for each } t \in (a, b).$$

\square

22.1 ON THE BANACH SPACES $C(X)$, X COMPACT

First case $X = [a, b]$, compact interval in \mathbb{R} .

Lemma 22.1. For n in \mathbb{N} let $q_n(t) = c_n(1 - t^2)^n$ where c_n satisfies

$$1 = c_n \int_{-1}^1 (1 - t^2)^n dt.$$

Then

(q1) $q_n(t) \geq 0$ for $t \in [-1, 1]$, n in \mathbb{N} (non-negative)

(q2) $\int_{-1}^1 q_n(t) dt = 1$, n in \mathbb{N} (total mass 1)

(q3) if $\delta \in (0, 1)$, then $\left(\int_{-1}^{-\delta} + \int_{\delta}^1 \right) q_n(t) dt \xrightarrow{n \rightarrow \infty} 0$ (concentration of mass near 0)

Proof. (q1) and (q2) are obvious. Now for $t \in [0, 1]$,

$$\begin{aligned} t^2 \leq t &\implies 1 - t \leq 1 - t^2 \\ &\implies (1 - t)^n \leq (1 - t^2)^n \end{aligned}$$

and hence

$$\begin{aligned} \frac{1}{c_n} &= \int_{-1}^1 (1 - t^2)^n dt = 2 \int_0^1 (1 - t^2)^n dt \\ &\leq 2 \int_0^1 (1 - t)^n dt = \frac{-2}{n+1} (1 - t)^{n+1} \Big|_0^1 = \frac{2}{n+1} \end{aligned}$$

so $c_n \leq \frac{n+1}{2}$. Hence, for $|t| \in (\delta, 1)$, we have

$$\begin{aligned} q_n(t) &= c_n(1-t^2)^n \leq c_n(1-t^2)^n \\ &\leq \frac{n+1}{2} \underbrace{(1-t^2)^n}_{<1} \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Thus

$$\begin{aligned} \left(\int_{-1}^{-\delta} + \int_{\delta}^1 \right) q_n(t) dt &\leq \left(\int_{-1}^{-\delta} + \int_{\delta}^1 \right) \frac{n+1}{2} (1-t^2)^n dt \\ &= (1-\delta)(n+1)(1-\delta^2)^n \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

□

Theorem 22.1 (Weierstrauss approximation theorem). Given $a < b$ in \mathbb{R} , $f \in C[a, b]$, there exists a sequence $(p_n)_{n=1}^{\infty}$ of polynomial functions such that

$$(WA) \|p_n - f\|_{\infty} = \max_{t \in [a, b]} |p_n(t) - f(t)| \xrightarrow{n \rightarrow \infty} 0.$$

Proof. (I) We condition f . Let $\tilde{f} \in C[0, 1]$ be given by

$$\tilde{f}(t) = f(a + t(b-a)) - [f(b) - f(a)]t - f(a).$$

So

- $\tilde{f}(0) = f(b) - f(a) = 0$
- $\tilde{f}(1) = f(b) - [f(b) - f(a)]1 - f(a) = 0.$

If we can find a sequence $(\tilde{p}_n)_{n=1}^{\infty}$ of polynomials,

$$\|\tilde{p}_n - \tilde{f}\|_{\infty} = \sup_{t \in [0, 1]} |\tilde{p}_n(t) - \tilde{f}(t)| \xrightarrow{n \rightarrow \infty} 0$$

we are done. Indeed, if $s \in [a, b]$, then define each $p_n(s) = \tilde{p}_n(\frac{1}{b-a}(s-a)) + \frac{f(b)-f(a)}{b-a}(s-a) + f(a)$; may be easily shown to satisfy (WA).

(II) Let us assume that

$$f \in C[0, 1], f(0) = 0 = f(1).$$

We can extend f to \mathbb{R} by letting $f(t) = 0$ for $t \in (-\infty, 0) \cup (1, \infty)$, so $f \in C_b(\mathbb{R})$, but $f(t) \neq 0$ only possibly for $t \in [0, 1]$, and f is uniformly continuous [any function in $C[0, 1]$ is uniformly continuous].

Let $(q_n)_{n=1}^{\infty}$ be as in the last lemma, and let for each n in \mathbb{N} and each t in $[0, 1]$,

$$p_n(t) = \int_0^1 q_n(s-t)f(s)ds.$$

Let us compute, for each n, t ,

$$\begin{aligned} \frac{d^{2n+1}}{dt^{2n+1}} p_n(t) &= \int_0^1 \frac{\partial^{2n+1}}{\partial t^{2n+1}} \underbrace{q_n(s-t)}_{\text{function is } 2n+2\text{-times continuously differentiable}} f(s) ds \\ &= 0, \text{ since } \deg q_n(t) = \deg(1-t^2)^n = 2n. \end{aligned}$$

$\implies p_n$ is a polynomial, $\deg p_n(t) \leq 2n$.

By change of variable $u = s - t$,

$$\begin{aligned} p_n(t) &= \int_0^1 q_n(s-t)f(s)ds \\ &= \int_{-t}^{1-t} q_n(u)f(u+t)du \\ &= \int_{-1}^1 q_n(u)f(u+t)du, \text{ since } f(u+t) \geq 0 \text{ possibly only on } [-t, 1-t]. \end{aligned}$$

Hence for t in $[0, 1]$,

$$\begin{aligned} |p_n(t) - f(t)| &= \left| \int_{-1}^1 q_n(u)f(u+t)du - \underbrace{\int_{-1}^1 q_n(u)f(t)du}_{\text{property (q2)}} \right| \\ &\leq \int_{-1}^1 q_n(u)|f(u+t) - f(t)|du. \end{aligned}$$

Given $\varepsilon > 0$, let $\delta > 0$ be so $|x - y| < \delta(x, y \in \mathbb{R}) \implies |f(x) - f(y)| < \frac{\varepsilon}{2}$ and then

$$\begin{aligned} |p_n(t) - f(t)| &\leq \int_{-\delta}^{\delta} q_n(u) \underbrace{|f(u+t) - f(t)|}_{< \frac{\varepsilon}{2}, \text{ by choice of } \delta} du + \left(\int_{-1}^{-\delta} + \int_{\delta}^1 \right) q_n(u) \underbrace{|f(u+t) - f(t)|}_{\leq 2\|f\|_{\infty}} du \\ &\leq \frac{\varepsilon}{2} \int_{-1}^1 q_n(u)du + \left(\int_{-1}^{-\delta} + \int_{\delta}^1 \right) q_n(u)2\|f\|_{\infty} du \text{ by (q1)} \xrightarrow{n \rightarrow \infty} \frac{\varepsilon}{2} + 0. \end{aligned}$$

(Continued next lecture.)

23 2017-11-17

We saw p_n is polynomial, i.e. $d^{2n+1}/dt^{2n+1}p_n(t) = 0$. Need approx.

Using (q2) we saw for $t \in [0, 1]$

$$|p_n(t) - f(t)| \leq \int_{-1}^1 \underbrace{q_n(u)}_{(q1)} |f(u+t) - f(t)| du$$

Given $\varepsilon > 0$, use uniform continuity of f to find $\delta > 0$ s.t. $|x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{2}$.

$$\begin{aligned}
|p_n(t) - f(t)| &\leq \int_{-1}^1 q_n(u) |f(u+t) - f(t)| du \\
&= \int_{-\delta}^{\delta} q_n(u) |f(u+t) - f(t)| du + \left(\int_{-1}^{-\delta} + \int_{\delta}^1 \right) q_n(u) \underbrace{|f(u+t) - f(t)|}_{\leq 2\|f\|_{\infty}} du \\
&\leq \int_{-\delta}^{\delta} q_n(u) \frac{\varepsilon}{2} du + \left(\int_{-1}^{-\delta} + \int_{\delta}^1 \right) q_n(u) 2\|f\|_{\infty} du \\
&\leq \underbrace{\frac{\varepsilon}{2} \int_{-\delta}^{\delta} q_n(u) du}_{=1(q2)} + 2\|f\|_{\infty} \left(\int_{-1}^{-\delta} + \int_{\delta}^1 \right) q_n(u) du.
\end{aligned}$$

Hence, if n_{ε} is so $n \geq n_{\varepsilon} \implies \left(\int_{-1}^{-\delta} + \int_{\delta}^1 \right) q_n(u) du \leq \frac{\varepsilon}{2(2\|f\|_{\infty}+1)}$
we have for $n \geq n_{\varepsilon}$,

$$|p_n(t) - f(t)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and we thus have

$$\|p_n - f\|_{\infty} = \max_{t \in [0,1]} |p_n(t) - f(t)| < \varepsilon$$

and we thus see that $\lim_{n \rightarrow \infty} p_n = f$ in $(C[0,1], \|\cdot\|_{\infty})$. □

Corollary 23.1. If $f \in C^1[a,b]$ (differentiable on $[a,b]$, with continuous derivative). Then, given $\varepsilon > 0$, there is a polynomial p s.t.

$$\begin{aligned}
\|p' - f\|_{\infty} &< \varepsilon \\
\|p - f\|_{\infty} &< (b-a)\varepsilon.
\end{aligned}$$

Proof. By Weierstrauss approximation theorem, find a polynomial q s.t. $\|f' - q\|_{\infty} < \varepsilon$. Let $p(t) = f(a) + \int_a^t q(s) ds$. Check that this works. (Remember Fundamental Theorem of Calculus.) □

Corollary 23.2. $(C[a,b], \|\cdot\|_{\infty})$ is separable.

Proof. Let $f \in C[a,b], \varepsilon > 0$.

By Weierstrauss approximation theorem, find polynomial p s.t.

$$\|f - p\|_{\infty} < \frac{\varepsilon}{2}.$$

Write $p(t) = a_0 + a_1 t + \dots + a_n t^n$. For $j = 1, \dots, n$ let $q_j \in \mathbb{Q}$ be such that

$$|a_j - q_j| < \frac{\varepsilon}{2(n+1) \max\{|a|^j, |b|^j\}}$$

then let $r(t) = q_0 + q_1 t + \dots + q_n t^n$.

Check that for each t in $[a,b]$,

$$|p(t) - r(t)| < \frac{\varepsilon}{2}$$

so $\|p - r\|_{\infty} = \max_{t \in [a,b]} |p(t) - r(t)| < \frac{\varepsilon}{2}$,
and thus

$$\|f - r\|_{\infty} \leq \|f - p\|_{\infty} + \|p - r\|_{\infty} < \varepsilon.$$

□

Theorem 23.1 (nowhere differentiable functions are generic). Let $ND[0,1]$ denote the set of f in $C[0,1]$ which are nowhere differentiable. Then $ND[0,1]$ is residual in $C[a,b]$.

Proof. Recall for $M, \delta > 0$,

$$F_{M,\delta} = \{f \in C[0,1] : \text{there is } x \text{ in } [0,1] \text{ so } \frac{|f(x) - f(t)|}{|x - t|} \leq M \\ \text{for all } t \in [0,1] \cap [(x - \delta, x) \cup (x, x + \delta)] \}$$

(A5,Q1).

(I) Let us see that each $F_{M,\delta}$ is nowhere dense in $(C[0,1], \|\cdot\|_\infty)$.

To this end, let $f \in F_{M,\delta}, \varepsilon > 0$.

First, use Weierstrauss approximation to get a polynomial p so $\|f - p\|_\infty < \frac{\varepsilon}{2}$. In particular, p' exists everywhere, let $M' = \sup_{t \in [0,1]} \|p'(t)\|$.

Let

$$\varphi : [0, \infty) \rightarrow [0, 1], \varphi(t) = \begin{cases} t - n & t \in [n, n+1], n \in \{0\} \cup \mathbb{N} \text{ is even} \\ n+1 - t & t \in [n, n+1], n \in \mathbb{N} \text{ is odd} \end{cases}$$

For each k in \mathbb{N} let $\varphi_k(t) = \frac{1}{k} \varphi(k^2 t)$.

For $s, t \in [\frac{n-1}{k^2}, \frac{n}{k^2}], n \in \mathbb{N}$,

$$\frac{|\varphi_k(s) - \varphi_k(t)|}{|s - t|} = k \quad (\dagger).$$

Now let k be so $\frac{1}{k} < \frac{\varepsilon}{2}$ and $k - M' > M, \frac{1}{k^2} < \delta$.

Let $\psi_k = p + \varphi_k$ and we have for s, t satisfying (\dagger) ,

$$\begin{aligned} \frac{|\psi_k(s) - \psi_k(t)|}{|s - t|} &= \left| \frac{p(s) - p(t)}{s - t} - \frac{\varphi_k(s) - \varphi_k(t)}{s - t} \right| \\ &\geq \left| \underbrace{\frac{|\psi_k(s) - \psi_k(t)|}{|s - t|}}_k - \underbrace{\frac{|p(s) - p(t)|}{|s - t|}}_{\leq M', \text{ by Mean Value Theorem}} \right| \\ &\geq |k - M'| = k - M' > M. \end{aligned}$$

Hence $\psi_k \notin F_{M,\delta}$. And $\|f - \psi_k\|_\infty \leq \|f - p\|_\infty + \left\| \underbrace{p - \psi_k}_{-\varphi_k} \right\|_\infty < \frac{\varepsilon}{2} + \frac{1}{k} < \varepsilon$. □

24 2017-11-20

Theorem 24.1. $ND[0,1] = \{f \in C[0,1] : f \text{ is nowhere differentiable}\}$ is a residual set in $(C[0,1], \|\cdot\|_\infty)$.

Proof. We saw:

Each

$$F_{M,\delta} = \{f \in C[0,1] : \exists x \text{ in } [0,1], \frac{|f(x) - f(t)|}{|x - t|} \leq M \text{ for } t \in [0,1] \cap [(x - \delta, x) \cup (x, x + \delta)]\}$$

is closed (A5), nowhere dense (I).

(II) Let $SD[0,1] = C[0,1] \setminus ND[0,1]$ ("somewhere differentiable"). If $f \in SD[0,1]$, in A5, it was shown that $f \in F_{M,\delta}$ for some $M > 0, \delta > 0$. If $n \in \mathbb{N}$, with $n > \max\{M, \frac{1}{\delta}\}$, then $F_{M,\delta} \subseteq F_{n, \frac{1}{n}}$. Then

$$SD[0,1] = \bigcup_{n=1}^{\infty} F_{n, \frac{1}{n}}, \text{ each } F_{n, \frac{1}{n}} \text{ closed, } F_{n, \frac{1}{n}}^\circ = \emptyset.$$

Thus $SD[0,1]$ is meager, so $ND[0,1] = C[0,1] \setminus SD[0,1]$ is residual. □

Remark: Baire Category Theorem tells us that in the complete metric space $(C[0,1], \|\cdot\|_\infty)$.
residual = "large" = "generic"

24.1 TOWARDS STONE-WEIERSTRAUSS THEOREM

Notation: (lattice structure)

Let X be non-empty, $f, g : X \rightarrow \mathbb{R}$. Define

$$\begin{aligned} \text{("join")} \quad & f \vee g : X \rightarrow \mathbb{R}, f \vee g(x) = \max\{f(x), g(x)\} \\ \text{("meet", min)} \quad & f \wedge g : X \rightarrow \mathbb{R}, f \wedge g(x) = \min\{f(x), g(x)\}. \end{aligned}$$

Proposition 24.1. Let (X, d) be a (compact) metric space, $f, g \in C(X)$. Then $f \vee g, f \wedge g \in C(X)$.

Proof. If $a, b \in \mathbb{R}$, then $\max\{a, b\} = \frac{1}{2}(a + b) + \frac{1}{2}|a - b|$.

Hence

$$f \vee g = \frac{1}{2}(f + g) + \frac{1}{2} \underbrace{|f - g|}_{f-g \text{ compact with } |\cdot|} \in C(X).$$

Also $\min\{a, b\} = -\max\{-a, -b\}$, so

$$f \wedge g = -(-f) \vee (-g) \in C(X).$$

□

Notation: A family $\mathcal{L} \subseteq C(X)$ is called a lattice if for each $f, g \in \mathcal{L}$, $f \vee g, f \wedge g \in \mathcal{L}$. Notice if $f_1, \dots, f_n \in \mathcal{L}$,

$$\begin{aligned} f_1 \vee f_2 &\in \mathcal{L} \\ \implies f_1 \vee f_2 \vee f_3 &\in \mathcal{L} \\ &\vdots \text{ (obvious induction)} \\ \implies f_1 \vee \dots \vee f_n &\in \mathcal{L}. \end{aligned}$$

Likewise $f_1 \wedge \dots \wedge f_n \in \mathcal{L}$.

Theorem 24.2 (Stone). Let (X, d) be a compact metric space and let the lattice $\mathcal{L} \subseteq C(X)$ satisfy

- \mathcal{L} is a \mathbb{R} -space
- $1 \in \mathcal{L}$ (contains constant function)
- \mathcal{L} separates points: if $x \neq y$ in X , there exists $\varphi \in \mathcal{L}$, so $\varphi(x) \neq \varphi(y)$.

Then $\overline{\mathcal{L}} = C(X)$ (\mathcal{L} is uniformly dense in $C(X)$).

Proof. Suppose $x \neq y$ in X and $\alpha, \beta \in \mathbb{R}$. Since \mathcal{L} separates points, there is $\varphi \in \mathcal{L}$ with $\varphi(x) \neq \varphi(y)$. Then

$$g = \alpha 1 + \frac{\beta - \alpha}{\varphi(y) - \varphi(x)} [\varphi - \varphi(x)1] \in \mathcal{L} \text{ as } 1 \in \mathcal{L}, \mathcal{L} \text{ is a } \mathbb{R}\text{-subspace}$$

with $g(x) = \alpha, g(y) = \beta$.

Fix $f \in C(X), \varepsilon > 0$.

(I) Fix x in X . For each y in X , letting $\alpha = f(x), \beta = f(y)$, if $y \neq x$, we have that there is

$$g_{x,y} \in \mathcal{L} \text{ s.t. } g_{x,y}(x) = f(x), g_{x,y}(y) = f(y).$$

Since each $f, g_{x,y}$ are continuous (near y), there are $\delta_y > 0$ so that

$$d(z, y) < \delta_y \implies g_{x,y}(z) < f(z) + \varepsilon \text{ i.e. } g_{x,y} < f + \varepsilon \text{ on } B(x, \delta_y)$$

$$\text{(i.e. } g_{x,y} - f \text{ is 0 at } y \text{ so } < \varepsilon \text{ in a neighbourhood of } y)$$

Since $X = \bigcup_{y \in X} B(y, \delta_y)$, by compactness, there are y_1, \dots, y_m s.t. $X = \bigcup_{j=1}^m B(y_j, \delta_{y_j})$. Let

$$g_x = g_{x, y_1} \wedge \dots \wedge g_{x, y_m} \in \mathcal{L}$$

and we have $g_x \leq g_{x, y} < f + \varepsilon 1$.

Notice that $g_x(x) = \min\{f_{x, y_1}(x), \dots, f_{x, y_m}(x)\} = f(x)$. □

25 2017-11-22

Small goof up:

Then we let $g_x = g_{x, y_1} \wedge \dots \wedge g_{x, y_m} \in \mathcal{L}$.

Now, if $z \in X$, then $z \in B(y_j, \delta_{y_j})$ for some $j = 1, \dots, m$ and then

$$g_x(z) = g_{x, y_1} \wedge \dots \wedge g_{x, y_m} \leq g_{x, y_j}(z) < f(z) + \varepsilon, \text{ property of } \delta_{y_j} \text{ w.r.t. } y_j$$

so we have

$$g_x < f + \varepsilon 1, \text{ and } g_x(x) = f(x).$$

(II) For each x in X , we found $g_x \in \mathcal{L}$ s.t. $g_x < f + \varepsilon 1, g_x(x) = f(x)$.

Hence $g_x(x) = f(x) < f(x) + \varepsilon$ at each x , so there is $\delta_x > 0$, s.t.

$$g_x(z) > f(z) - \varepsilon \text{ for } z \in B(x, \delta_x).$$

We have $X = \bigcup_{x \in X} B(x, \delta_x)$ so there are $x_1, \dots, x_n \in X$ so $X = \bigcup_{j=1}^n B(x_j, \delta_{x_j})$. We then let

$$g = g_{x_1} \vee \dots \vee g_{x_n} \in \mathcal{L}.$$

For $z \in X, z \in B(x_j, \delta_{x_j})$ for some $j = 1, \dots, n$, so

$$g(z) \geq g_{x_j}(z) > \dots > f(z) - \varepsilon$$

and thus

$$g > f - \varepsilon 1.$$

Furthermore, each $g_{x_j} < f + \varepsilon 1$, so if $z \in X$, then $g(z) = g_{x_j}(z)$ for some j , so

$$g(z) = g_{x_j}(z) < f(z) + \varepsilon \implies g < f + \varepsilon 1$$

i.e. $f - \varepsilon 1 < g < f + \varepsilon 1$, so $g \in B(f, \varepsilon)$ in $(C(X), \|\cdot\|_\infty)$.

In summary, given $f \in C(X), \varepsilon > 0, B(f, \varepsilon) \cap \mathcal{L} \neq \emptyset$. Hence, $\overline{\mathcal{L}} = C(X)$. □

Corollary 25.1. (i) Let $\mathcal{L} = \{f \in C[a, b] : f \text{ is piecewise affine (A5)}\}$. Then $\overline{\mathcal{L}} = C[a, b]$.

(ii) Let C be the Cantor set and $\mathcal{L} = \{f \in C(C) : |f(C)| < \aleph_0\}$. Then $\overline{\mathcal{L}} = C(C)$.

Definition: Let (X, d) be a (compact) metric space. A subset $A \subseteq C(X)$ is called an algebra if for $f, g \in A, \alpha \in \mathbb{R}$, we have

$$\begin{aligned} f + \alpha g &\in A && (A \text{ is a } \mathbb{R}\text{-subspace}) \\ fg &\in A && (A \text{ is closed under pointwise multiplication}) \end{aligned}$$

(If $f, g \in C(X)$, then $fg \in C(X)$, too.) If $f_1, \dots, f_n \in A, f_1 \cdots f_n \in A$ too.

If $1 \in A$, and $p(t) = \sum_{i=1}^n a_i t^i$, then for $f \in A$,

$$p \circ f = a_0 1 + a_1 f + a_2 f^2 + \dots + a_n f^n \in A.$$

$(f^k(x) = f(x)^k \text{ for } x \in X.)$

Theorem 25.1 (Stone-Weierstrauss Theorem). If (X, d) is a compact metric space, $A \subseteq C(X)$ satisfies

- A is an algebra
- $1 \in A$
- A separates points: for $x \neq y$ in X , there is $g \in A$ so $g(x) \neq g(y)$

Then $\overline{A} = C(X)$ (uniform closure).

Proof. (I) If $f \in A$, then $|f| \in \overline{A}$. First, since (X, d) is compact, f continuous, $f(X) \subset \mathbb{R}$ is compact, hence bounded, so there is $a > 0$ s.t. $f(X) \subseteq [-a, a]$. Now, the Weierstrauss approximation theorem provides $(p_n)_{n=1}^\infty$ of polynomials s.t. $\|p_n - |\cdot|\|_\infty = \max_{t \in [-a, a]} |p_n(t) - |t|| \rightarrow 0$. Hence $\|p_n \circ f - |f|\|_\infty = \max_{x \in X} |p_n(f(x)) - |f(x)|| \rightarrow 0$. Each $p_n \circ f \in A$.

(II) Since A is a \mathbb{R} -subspace, so is \overline{A} (A4 Q1). If $f, g \in \overline{A}$, let $f = \lim_{n \rightarrow \infty} f_n, g = \lim_{n \rightarrow \infty} g_n$ under uniform limits, each $f_n, g_n \in A$. Then

$$\begin{aligned} f \vee g &= \frac{1}{2}(f + g) + \frac{1}{2}|f - g| \\ &= \lim_{n \rightarrow \infty} \underbrace{\frac{1}{2}(f_n + g_n)}_{\in A \subseteq \overline{A}} + \underbrace{\frac{1}{2}|f_n - g_n|}_{\in A \text{ by (I)}} \in \overline{A} \end{aligned}$$

since \overline{A} is closed.

Also, $f \wedge g = -(-f) \vee (-g) \in \overline{A}$ as well.

$\implies \overline{A}$ is a \mathbb{R} -subspace and a lattice. Also, $1 \in A \subseteq \overline{A}$, and A separates points, hence \overline{A} separates points.

Thus \overline{A} is dense in $C(X)$, but is closed, so $\overline{A} = C(X)$. □

26 2017-11-24

Example: Let $I = [a_1, b_1] \times \cdots \times [a_n, b_n]$ be a non-empty compact interval in \mathbb{R}^n . A polynomial on I is any function

$$p(t_1, \dots, t_n) = \sum_{j_1, \dots, j_n=1}^N a_{j_1, \dots, j_n} t_1^{j_1} \cdots t_n^{j_n}$$

where each $a_{j_1, \dots, j_n} \in \mathbb{R}, N \in \mathbb{N}$. By Stone-Weierstrauss Theorem, the family $P(I)$ of polynomial functions is dense in $C(I)$.

Example: Let $(X, d_X), (Y, d_Y)$ be compact metric spaces. Let $\|\cdot\|$ be a norm on \mathbb{R}^2 . Define

$$\rho(X \times Y) \times (X \times Y) \rightarrow [0, \infty) \text{ by}$$

$$\rho((x_1, y_1), (x_2, y_2)) = \|(d_X(x_1, x_2), d_Y(y_1, y_2))\|.$$

It is “obvious” that ρ is a metric on $X \times Y$.

(Usually, $\|\cdot\| = \|\cdot\|_\infty, \|\cdot\|_1, \|\cdot\|_2$ on \mathbb{R}^2 .)

Furthermore, $(X \times Y, \rho)$ is compact. Indeed, let $((x_n, y_n))_{n=1}^\infty \subseteq X \times Y$ be a sequence. Then $(x_n)_{n=1}^\infty \subseteq X$ admits a converging subsequence: let $x = \lim_{k \rightarrow \infty} x_{n_k} \in X$. Then $(y_{n_k})_{k=1}^\infty \subseteq Y$ admits a converging subsequence: let $y = \lim_{\ell \rightarrow \infty} y_{n_{k_\ell}} \in Y$.

Notice that

$$\begin{aligned} &\rho((x, y), (x_{n_{k_\ell}}, y_{n_{k_\ell}})) \\ &= \left\| (d_X(x, x_{n_{k_\ell}}), d_Y(y, y_{n_{k_\ell}})) \right\| \\ &\leq d_X(x, x_{n_{k_\ell}}) \|(1, 0)\| + d_Y(y, y_{n_{k_\ell}}) \|(0, 1)\| \\ &\xrightarrow{\ell \rightarrow \infty} 0. \end{aligned}$$

Hence $((x_{n_{k_\ell}}, y_{n_{k_\ell}}))_{\ell=1}^\infty$ is a converging subsequence of $((x_n, y_n))_{n=1}^\infty$. Suppose that each $A_X \subseteq C(X)$ and $A_Y \subseteq C(Y)$, each satisfy assumptions of Stone-Weierstrauss Theorem. If $f \in A_X, g \in A_Y$,

$$f \otimes g : X \times Y \rightarrow \mathbb{R}, f \otimes g(x, y) = f(x)g(y).$$

Let $A_X \otimes A_Y = \text{span}_{\mathbb{R}}\{f \otimes g : f \in A_X, g \in A_Y\}$. Convince yourself that $A_X \otimes A_Y \subseteq C(X \times Y)$ and satisfies assumptions of Stone-Weierstrauss Theorem.

Hence $\overline{A_X \otimes A_Y} = C(X \times Y)$ (uniform closure).

Corollary 26.1 (Stone-Weierstrauss without constant functions). Let (X, d) be a compact metric space, and $A \subseteq C(X)$ satisfy

- A is an algebra
- A separates points
- there is x_0 in X s.t. $f(x_0) = 0$ for f in A .

Then $\overline{A} = C_{x_0}(X) := \{f \in C(X) : f(x_0) = 0\}$

Proof. First, $C_{x_0}(X)$ is closed in $C(X)$. (Let $\varphi : C(X) \rightarrow \mathbb{R}, \varphi(f) = f(x_0)$, which is linear and continuous: $\|\varphi\| \leq 1$ (seen before). Then $C_{x_0}(X) = \varphi^{-1}(\{0\}) = C(X) \setminus \underbrace{\varphi^{-1}(\mathbb{R} \setminus \{0\})}_{\substack{\text{open} \\ \text{open}}} \underbrace{\phantom{C(X) \setminus \varphi^{-1}(\mathbb{R} \setminus \{0\})}}_{\text{closed}}$. Since $A \subseteq C_{x_0}(X) \implies \overline{A} \subseteq C_{x_0}(X)$.)

Second, note that $\mathbb{R}1 + A = \{\alpha 1 + f : \alpha \in \mathbb{R}, f \in A\}$ satisfies $\overline{\mathbb{R}1 + A} = C(X)$. If $g \in \mathbb{R}1 + A$, write $g = \alpha 1 + h, \alpha \in \mathbb{R}, h \in A$, and $g(x_0) = \alpha + h(x_0) = \alpha$ so $g = g(x_0)1 + h$.

Now, if $f \in C_{x_0}(X)$, there exists $(g_n)_{n=1}^\infty \subseteq \mathbb{R}1 + A$ s.t. $\|f - g\|_\infty \xrightarrow{n \rightarrow \infty} 0$ (Stone-Weierstrauss Theorem). Write each $g_n = g_n(x_0)1 + h_n$ where $h_n \in A$. Notice that $0 = f(x_0) = \lim_{n \rightarrow \infty} g_n(x_0)$. Hence

$$\begin{aligned} \|f - h_n\|_\infty &\leq \|f - (g_n(x_0)1 + h_n)\|_\infty + \|g_n(x_0)1\|_\infty \\ &= \|f - g_n\|_\infty + |g_n(x_0)| \quad (\|1\|_\infty = 1) \\ &\xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Thus $C_{x_0}(X) \subseteq \overline{A}$. □

Def: Let $C_\infty(\mathbb{R}) = \{\bar{f} \in C(\mathbb{R}) : \lim_{|t| \rightarrow \infty} f(t) = 0\}$. Then $C_\infty(\mathbb{R}) \subseteq C_b(\mathbb{R})$ and is a closed subspace. ($L_\pm : C_b(\mathbb{R}) \rightarrow \mathbb{R}, L_\pm(f) = \lim_{t \rightarrow \pm\infty} f(t)$, then L_+, L_- are linear and with $\|L_\pm\| \leq 1$. Then $C_\infty(\mathbb{R}) = L_+^{-1}(\{0\}) \cap L_-^{-1}(\{0\})$ is closed.)

Corollary 26.2. Let $A \subseteq C_\infty(\mathbb{R})$ satisfy that

- A is an algebra
- A separates points
- for each t of \mathbb{R} , there is $f \in A$ s.t. $f(t) \neq 0$.

Then $\overline{A} = C_\infty(\mathbb{R})$ (uniform closure).

Proof. (Sketch of proof) $\psi : \mathbb{R} \rightarrow (-1, 1), \psi(t) = \frac{t}{|t|+1}$, then ψ is continuous and bijective with $\psi^{-1}(-1, 1) \rightarrow \mathbb{R}$ continuous. Let $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

$$\begin{aligned} \varphi(-1, 1) &\rightarrow S \setminus \{(-1, 0)\} \\ \varphi(s) &= (\cos(\pi s), \sin(\pi s)) \end{aligned}$$

so φ is a continuous bijection with continuous inverse. Hence, $\varphi \circ \psi : \mathbb{R} \rightarrow S \setminus \{(-1, 0)\}$ is a homeomorphism, i.e. continuous bijection with continuous inverse.

Define

$$\begin{aligned}\Psi : C_\infty(\mathbb{R}) &\rightarrow C_{(-1,0)}(S) \\ \Psi(f)(x, y) &= f(\psi^{-1} \circ \varphi^{-1}(x, y)).\end{aligned}$$

Check that Ψ is a surjective isometry, between $(C_\infty(\mathbb{R}), \|\cdot\|_\infty)$ and $(C_{(-1,0)}(S), \|\cdot\|_\infty)$, and hence has isometric inverse.

We have $\Psi(A) \subseteq C_{(-1,0)}(S)$ satisfies assumptions of last corollary, so $\overline{\Psi(A)} = C_{(-1,0)}(S)$ but it follows that $\overline{A} = \Psi^{-1}(\overline{\Psi(A)}) = C_\infty(\mathbb{R})$. \square

27 2017-11-27

Today's subject: towards Arzela-Ascoli Theorem (by guest lecturer)

Def: Let (X, d) be a complete metric space. Let $F \subseteq X$ be a subset. We say F is relatively compact if \overline{F} is compact. (Here \overline{F} means the closure of F .)

Proposition 27.1 (Properties of relatively compact subsets). Let (X, d) be a metric space, $F \subseteq X$. TFAE:

1. F is relatively compact
2. Every sequence (x_n) admits a Cauchy subsequence (x_{n_k})
3. F is totally bounded

Proof. (i) \implies (ii) Let (x_n) be a sequence in F . Since (x_n) is in \overline{F} and \overline{F} is compact, (x_n) has a Cauchy subsequence (x_{n_k}) (that may converge to a point in $\overline{F} \setminus F$).

(ii) \implies (i) Let (x_n) be a sequence in \overline{F} . We want to show there is a subsequence (x_{n_k}) converging to a point in \overline{F} (note this is nonempty by characterization of the closure).

Now, by (ii), there is a Cauchy subsequence (y_{n_k}) .

Claim: (x_{n_k}) is Cauchy.

For $k, \ell \geq 1$,

$$\begin{aligned}d(x_{n_k}, x_{n_\ell}) &\leq d(x_{n_k}, y_{n_k}) + d(x_{n_k}, y_{n_\ell}) + d(x_{n_\ell}, y_{n_\ell}) \\ &\leq \frac{1}{n_k} + d(y_{n_k}, y_{n_\ell}) + \frac{1}{n_\ell} \xrightarrow{k, \ell \rightarrow \infty} 0.\end{aligned}$$

(i) \implies (iii) \overline{F} is totally bounded since it is compact. So for $\frac{\varepsilon}{2} > 0$, there are $x_1, \dots, x_n \in \overline{F}$ s.t. $B(x_i, \frac{\varepsilon}{2})$ covers \overline{F} (i.e. $\bigcup_{i=1}^n B(x_i, \frac{\varepsilon}{2}) \supseteq \overline{F}$).

For each i , choose $y_i \in B(x_i, \frac{\varepsilon}{2}) \cap F$. Then $B(y_i, \varepsilon) \supseteq B(x_i, \frac{\varepsilon}{2})$ so y_1, \dots, y_n is an ε -net for F .

(iii) \implies (i) Since F is totally bounded, there is an ε -net $y_1, \dots, y_n \in F$. So

$$\begin{aligned}F &\subseteq \bigcup_{i=1}^n B(y_i, \varepsilon) \\ \implies \overline{F} &\subseteq \bigcup_{i=1}^n \overline{B(y_i, \varepsilon)} \\ \implies \overline{F} &\subseteq \bigcup_{i=1}^n B(y_i, 2\varepsilon).\end{aligned}$$

So \overline{F} is totally bounded. \square

Def: [Equicontinuity] Let (X, d) be a (compact) metric space. A subset $F \subseteq C(X)$ is equicontinuous if for $\varepsilon > 0$ and $x \in X$ there is $\delta > 0$ s.t. if $d(x, y) < \delta$ then $|f(y) - f(x)| < \varepsilon \forall f \in F$ (holds for all f simultaneously).

Lemma 27.1. If (X, d) is compact and $F \subseteq C(X)$ then F is equicontinuous $\iff F$ is uniformly equicontinuous meaning for $\varepsilon > 0$ there is $\delta > 0$ s.t. if $x, y \in X$ and $d(x, y) < \delta$ then $|f(x) - f(y)| < \varepsilon \forall f \in F$.

Proof. If F is uniformly equicontinuous it is clearly equicontinuous.

For the other direction, fix $\varepsilon > 0$. For each x there is δ_x s.t. if $d(x, y) < \delta_x$ then $|f(y) - f(x)| < \varepsilon/2 \forall f \in F$. Then $(B(x, \delta_x))_{x \in X}$ is an open cover. Let $\delta > 0$ be the corresponding Lebesgue covering number. So for any $y \in X$, $B(y, \delta) \subseteq B(x, \delta_x)$ for some $x \in X$. So if $y, z \in X$ with $d(y, z) < \delta$, choose $x \in X$ s.t. $B(y, \delta) \subseteq B(x, \delta_x)$, then

$$\begin{aligned} |f(y) - f(z)| &\leq |f(y) - f(x)| + |f(x) - f(z)| \quad (z \in B(x, \delta_x)) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

□

Ex: Let F be a set of differentiable functions from $[0, 1]$ to \mathbb{R} s.t. $|f'(x)| \leq M \forall f \in F, x \in [0, 1]$ for some M . By the MVT, for $x, y \in [0, 1]$ there is $z \in [0, 1]$ s.t. $M \geq |f'(z)| = \frac{|f(y) - f(x)|}{|y - x|}$.

$$|f(y) - f(x)| \leq M|y - x| \forall y, x \in [0, 1], \forall f \in F.$$

Now take $\delta = \frac{\varepsilon}{M}$. Then if $|x - y| < \delta$ then

$$\begin{aligned} |f(x) - f(y)| &\leq M|x - y| \\ &< M \frac{\delta}{M} = \varepsilon. \end{aligned}$$

28 2017-11-29

Office Hours:

Today: 2:30-4:30

Tomorrow: 2-4 pm

Last time:

In complete (X, d) , TFAE:

- (i) relative compactness
- (ii) every sequence admits a Cauchy subsequence
- (iii) total boundedness

Discussed for $F \subset C(X)$:

- equicontinuity \implies uniform equicontinuity if (X, d) compact
- pointwise boundedness

Theorem 28.1 (Arzela-Ascoli Theorem). Let (X, d) be a compact metric space, $F \subset C(X)$. Then

F is relatively compact in $(C(X), \|\cdot\|_\infty) \iff F$ is both equicontinuous and pointwise bounded.

Proof. (\implies) F is totally bounded. In particular, F is bounded: $\sup_{f \in F} \|f\|_\infty < \infty$ (totally bounded \implies bounded). Hence for x in X , $\sup_{f \in F} |f(x)| < \sup_{f \in F} \sup_{x \in X} |f(x)| = \sup_{f \in F} \|f\|_\infty < \infty$.

Given $\varepsilon > 0$, let $f_1, \dots, f_n \in F$ s.t. $F \subseteq \bigcup_{j=1}^n B[f_j, \frac{\varepsilon}{3}]$. Let for $j = 1, \dots, n$, $\delta_j > 0$ be so for x, y in X , $d(x, y) < \delta_j \implies |f_j(x) - f_j(y)| < \frac{\varepsilon}{3}$ (uniform continuity of f_j). Then let $\delta = \min\{\delta_1, \dots, \delta_n\}$ and then for x, y in X , $d(x, y) < \delta$, we have for f in F , then $f \in B[f_j, \frac{\varepsilon}{3}]$ for some j . Then

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f(y)| \\ &< \|f - f_j\|_\infty + \frac{\varepsilon}{3} + \|f - f_j\|_\infty \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Hence, F is (uniformly) equicontinuous, thus equicontinuous.

(\Leftarrow) Let $(x_n)_{n=1}^\infty \subset X$ satisfy that there are $n_1 < n_2 < n_3 < \dots$ for which

$$X = \bigcup_{k=1}^\infty \bigcup_{j=1}^{n_k} B[x_j, \frac{1}{k}] \quad (\dagger)$$

(assignment 5, (X, d) compact $\implies (X, d)$ separable).

Now, let $(f_n)_{n=1}^\infty \subseteq F$. We wish to extract a uniformly Cauchy subsequence, hence showing F is relatively compact.

(I) Let us extract a candidate Cauchy subsequence. This technique is a variant of “Cantor’s diagonalization argument”. First, $(f_n(x_1))_{n=1}^\infty \subset \mathbb{R}$ is bounded (pointwise bounded assumption) so by Bolzano-Weierstrauss admits a Cauchy subsequence $(f_{n_k}(x_1))_{k=1}^\infty \subset \mathbb{R}$. Let $f_{1,k} = f_{n_k}$ for each k . Second, $(f_{1,n}(x_2))_{n=1}^\infty \subset \mathbb{R}$ is bounded, and again admits a Cauchy subsequence $(f_{1,n_k}(x_2))_{k=1}^\infty \subset \mathbb{R}$. Let $f_{2,k} = f_{1,n_k}$.

Inductively, we continue. We build sequences $(f_{1,k})_{k=1}^\infty, (f_{2,k})_{k=1}^\infty, \dots, (f_{n,k})_{k=1}^\infty, \dots \subseteq F$ which satisfy

- $m < n$, $(f_{n,k})_{k=1}^\infty$ is a subsequence of $(f_{m,k})_{k=1}^\infty$
- $(f_{n,k}(x_n))_{k=1}^\infty \subset \mathbb{R}$ is Cauchy.

We now let

$$g_n = f_{n,n}.$$

Then $(g_n)_{n=m}^\infty$ is a subsequence of $(f_{m,n})_{n=1}^\infty$ so $(g_n(x_m))_{n=1}^\infty$ is Cauchy in \mathbb{R} , (being a subsequence of $(f_{m,n}(x_m))_{n=1}^\infty$). Thus $(g_n(x_m))_{m=1}^\infty$ is Cauchy for each m in \mathbb{N} , and $(g_k)_{k=1}^\infty$ is a subsequence of $(f_n)_{n=1}^\infty$.

(II) Let us show that $(g_n)_{n=1}^\infty$ is Cauchy in $(C(X), \|\cdot\|_\infty)$, i.e., Cauchy in $\|\cdot\|_\infty$.

Given $\varepsilon > 0$, our set F , being equicontinuous on compact (X, d) , is uniformly equicontinuous (lemma Monday), so there is $\delta > 0$ s.t. $|f(x) - f(y)| < \frac{\varepsilon}{3}$ whenever $x, y \in X$, $d(x, y) < \delta$ and $f \in F$.

Now, let k in \mathbb{N} satisfy $\frac{1}{k} < \delta$, and we have from (\dagger) that $X = \bigcup_{j=1}^{n_k} B[x_j, \frac{\varepsilon}{3}]$.

Now, for $j = 1, \dots, n_k$, let N_j in \mathbb{N} be s.t. $m, n \geq N_j \implies |g_m(x_j) - g_n(x_j)| < \frac{\varepsilon}{3}$ (i.e. $(g_n(x_j))_{n=1}^\infty$ is Cauchy). Let $N = \max\{N_1, \dots, N_{n_k}\}$. If $x \in X$, so $x \in B[x_j, \frac{\varepsilon}{3}]$ for some $j = 1, \dots, n_k$, and we have for $m, n \geq N$ that

$$\begin{aligned} |g_m(x) - g_n(x)| &\leq |g_m(x) - g_m(x_j)| + |g_m(x_j) - g_n(x_j)| + |g_n(x_j) - g_n(x)| \\ &< \underbrace{\frac{\varepsilon}{3}}_{\text{thanks to uniform equicontinuity of } F; g_n \in F} + \underbrace{\frac{\varepsilon}{3}}_{n, m \geq N \geq N_j \text{ Cauchy at } x_j} \\ &\quad + \underbrace{\frac{\varepsilon}{3}}_{\text{thanks to uniform equicontinuity of } F; g_n \in F} = \varepsilon. \end{aligned}$$

Hence $\|g_m - g_n\|_\infty = \max_{x \in X} |g_m(x) - g_n(x)| < \varepsilon$.

– END OF FINAL LINE (except Assignment 7) –

□

29 2017-12-01

Theorem 29.1 (Peano’s Theorem). Let $D \subset \mathbb{R}^2$ be open and $F : D \rightarrow \mathbb{R}$ be continuous, and $(t_0, y_0) \in D$. Then there are $a < b$ in \mathbb{R} so $t_0 \in (a, b)$ for which

$$(IVP) \quad f'(t) = F(t, f(t)), f(t_0) = y_0, t \in (a, b)$$

admits a solution.

(This is stronger than Picard-Lindelof, which required a Lipschitz condition on the second variable of a two variable function.) The solution here may not be unique.

Proof. (Most of proof):

(I) (Get $a < b$.) Let $R = [a_1, b_1] \times [a_2, b_2] \subset D$ (compact interval) so $(t_0, y_0) \in R^\circ$ (interior), and let $M = \max_{(t,y) \in R} |F(t, y)|$.

We let

$$W = \{(t, y) \in D : |y - y_0| \leq M|t - t_0|\}$$

and $a < b$ in \mathbb{R} so

$$([a, b] \times \mathbb{R}) \cap W \subset R.$$

(II) (Work on $[t_0, b]$, find a particular family of piecewise affine functions.) Given $\varepsilon > 0$, the uniform continuity of F on R provides $\delta > 0$ such that

$$\begin{aligned} (s, x), (t, y) \in R \text{ with } \max\{|s - t|, |x - y|\} = \|(s, x) - (t, y)\|_\infty < \delta \\ \implies |F(s, x) - F(t, y)| < \varepsilon. \end{aligned}$$

We partition $[t_0, b]$, $t_0 < t_1 < \dots < t_n = b$, so $\max_{j=1, \dots, n} (t_j - t_{j-1}) < \frac{\delta}{M+1}$ (let $M = 0$).

We define $f_\varepsilon : [t_0, b] \rightarrow \mathbb{R}$ inductively by

$$f_\varepsilon(t) = \begin{cases} y_0 + F(t_0, y_0)(t - t_0) & t \in [t_0, t_1] \\ f_\varepsilon(t_1) + F(t_1, f_\varepsilon(t_1))(t - t_1) & t \in (t_1, t_2] \\ \vdots & \\ f_\varepsilon(t_{n-1}) + F(t_{n-1}, f_\varepsilon(t_{n-1}))(t - t_{n-1}) & t \in (t_{n-1}, t_n] \end{cases}.$$

Two nice properties (exercise):

- graph of f_ε on $[t_0, b]$ is in R , so $\max_{t \in [t_0, b]} |f_\varepsilon(t)| \leq \max\{|a_2|, |b_2|\}$
- if $s < t$ in $[t_0, b]$, then $|f_\varepsilon(t) - f_\varepsilon(s)| \leq M|t - s|$ (\dagger).

These estimates are independent of ε . I.e. if we form $K = \{f_\varepsilon\}_{\varepsilon \in (0, \infty)}$ it is

- pointwise bounded & equi-Lipschitz \implies (uniformly) equicontinuous.

Hence K is relatively compact.

(III) (Relate $K = \{f_\varepsilon\}_{\varepsilon \in (0, \infty)}$ to the (IVP).) Fix $f_\varepsilon, \varepsilon$ and δ as in $(\varepsilon - \delta)$ above. If $t \in (t_j, t_{j+1})$, $j = 0, \dots, n-1$ then

$$f'_\varepsilon(t) = F(t_j, f_\varepsilon(t_j)). \quad (\star)$$

Also, for such t as above, then $|t - t_j| < \frac{\delta}{M+1}$ so by (\dagger)

$$|f_\varepsilon(t) - f_\varepsilon(t_j)| \leq M|t - t_j| \leq \delta \frac{M}{M+1} < \delta$$

so, by choice of δ ,

$$\begin{aligned} |F(t, f_\varepsilon(t)) - F(t_j, f_\varepsilon(t_j))| &< \varepsilon \\ (\text{using } (\star)) \implies |F(t, f_\varepsilon(t)) - f'_\varepsilon(t)| &< \varepsilon \quad (\star\star). \end{aligned}$$

Thus for $t \in [t_0, b]$ we have

$$\begin{aligned} f_\varepsilon(t) &= y_0 + \int_{t_0}^t f'_\varepsilon(s) ds \text{ (piecing together F.T. of C., as } f'_\varepsilon(t) \text{ exists except at } t_1, \dots, t_{n-1}) \\ &= y_0 + \int_{t_0}^t F(s, f_\varepsilon(s)) ds + \int_{t_0}^t [f'_\varepsilon(s) - F(s, f_\varepsilon(s))] ds \end{aligned}$$

Let $\tilde{f}_\varepsilon(t) = y_0 + \int_{t_0}^t F(s, f_\varepsilon(s))ds$, and we have for $t \in [t_0, b]$

$$|f_\varepsilon(t) - \tilde{f}_\varepsilon(t)| \leq \int_{t_0}^t \underbrace{|f'_\varepsilon(s) - F(s, f_\varepsilon(s))|}_{< \varepsilon} ds$$

$$(\star \star \star) \quad \leq (t - t_0)\varepsilon \leq (b - t_0)\varepsilon.$$

We now consider a sequence $(f_{\frac{1}{n}})_{n=1}^\infty \subseteq K$. By relative compactness, we get a uniformly Cauchy, hence uniformly converging subsequence $(f_{\frac{1}{n_k}})_{k=1}^\infty, f = \lim_{k \rightarrow \infty} f_{\frac{1}{n_k}}$ (uniform limit). Let $\tilde{f}(t) = y_0 + \int_{t_0}^t F(s, f(s))ds$.

We have

$$\|f - \tilde{f}\|_\infty \leq \|f - f_{\frac{1}{n_k}}\|_\infty + \|f_{\frac{1}{n_k}} - \tilde{f}_{\frac{1}{n_k}}\|_\infty + \|\tilde{f}_{\frac{1}{n_k}} - \tilde{f}\|_\infty$$

We have $\lim_{k \rightarrow \infty} f_{\frac{1}{n_k}}(s) = f(s)$ uniformly for $s \in [t_0, b]$, so, by uniform continuity $\lim_{k \rightarrow \infty} |F(s, f_{\frac{1}{n_k}}(s)) - F(s, f(s))| = 0$ uniformly for s in $[t_0, b]$, and thus $(\ddagger) \xrightarrow{k \rightarrow \infty} 0$. In conclusion

$$\|f - \tilde{f}\|_\infty \leq \|\tilde{f}_{\frac{1}{n_k}}\|_\infty + (b - t_0)\frac{1}{n_k} + (\ddagger)$$

$\implies f(t) = \tilde{f}(t) = y_0 + \int_{t_0}^t F(s, f(s))ds$, i.e. f satisfies (IE) \implies (IVP).

□