https://github.com/friedeggs

PMATH 351
Real Analysis

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Table of Contents

List of Theorems

Abstract

These notes are intended as a resource for myself; past, present, or future students of this course, and anyone interested in the material. The goal is to provide an end-to-end resource that covers all material discussed in the course displayed in an organized manner. If you spot any errors or would like to contribute, please contact me directly.

1 Chains and Zorn's Lemma

Let $(X, \leq)$ be a poset. A chain is any subset $C \subseteq X$ such that $(C, \leq)$ is totally ordered.

Office hours:

1. Today 2:30 - 3:20
2. Wednesday next week 2:30 - 4:30

Or, email `nspronk@uwaterloo.ca`

2 Cardinal arithmetic

i. :(

ii. $\mathbb{R} \underbrace{\sim}_{f} (-1, 1), f(x) = x/|x| + 1$ (exercise: exhibit f^{-1})

iii. $a < b$ in $\mathbb{R}.(0, 1) \underbrace{\sim}_{g} (a, b), g(x) = a + x(b - a)$

Notation: $\mathcal{N}_0 = |\mathbb{N}|$ ("aleph naught"), $c = |\mathbb{R}|$ ("continuous")
Arithmetic: Let A, B be sets.

$$\begin{aligned} |A| + |B| &= |A \sqcup B| \\ |A||B| &= |A \times B| \\ |A|^{|B|} &= |A^B| (B \neq \varnothing, A^B = \{f : B \to A \mid \text{ function } \}) \end{aligned}$$

$A \sqcup A$ is two copies of A, $\sim A \times \{1, 2\}$

Properties

- (commutativity) $|A| + |B| = |B| + |A|,\ |A||B| = |B||A|$
- (distributivity) $|A|(|B| + |C|) = |A||B| + |A||C|$

$$A \times (B \sqcup C) \sim (A \times B) \sqcup (A \times C)$$

- (Exponential laws)

$$|A|^{|B|+|C|} = |A|^{|B|}|A|^{|C|},\ |A|^{|B||C|} = (|A|^{|B|})^{|C|}$$

$(B \neq \varnothing \neq C)$

$$\begin{aligned} A^{B \sqcup C} &\sim A^B \times A^C \text{ via } \varphi \longmapsto (\varphi|_B, \varphi|_C) \\ A^{B \times C} &\sim (A^B)^C \text{ via } \varphi \longmapsto (\varphi(b, \cdot) : C \to A) \end{aligned}$$

Now, for sets A, B, define $A \preceq B$ if there is an injection $f : A \to B$.
Sometimes write $A \underbrace{\preceq}_{f} B$. As above:

(reflexivity) $A \underbrace{\preceq}_{\text{id}} A$

(transitivity) $A \preceq B, B \preceq C \Longrightarrow A \preceq C$

Seems reasonable to write $|A| \leq |B|$, in this case.

Question: Is $\leq$ in cardinal numbers anti-symmetric?

Theorem 2.1 (Cantor-Bernstein-Schroder Theorem)**.** If, for non-empty set A, B we have $A \preceq B, B \preceq A$, then $A \sim B$. Ie. if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Proof. Our assumption is that we have injections $A \underbrace{\preceq}_{\varphi} B$, $B \underbrace{\preceq}_{\psi} A$.

To avoid triviality, let us suppose that neither φ nor ψ is surjective. Thus $\varphi(A) \subsetneq B, \qquad \psi \circ \varphi(A) \subsetneq \psi(B) \subsetneq A$.
Let $A_0 = A, A_1 = \psi(B), A_2 = \psi \circ \varphi(A)$ and we inductively define $A_{n+2} = g(A_n), g = \psi \circ \varphi$.
Then $A_2 \subsetneq A_1 \subsetneq A_0$, so by applying injection g,

$$\begin{aligned} &A_2 \subsetneq A_1 \subsetneq A_0 \\ &\vdots \\ &A_{n+1} \subsetneq A_n \subsetneq A_{n-1} \end{aligned}$$

Hence, we may decompose

$$\begin{aligned} A = A_0 &= (A_0 \setminus A_1) \cup A_1 \\ &= (A_0 \setminus A_1) \cup (A_1 \setminus A_2) \cup A_2 \\ &\vdots \\ &= \bigcup_{n=1}^{\infty} (A_{n-1} \setminus A_n) \cup A_\infty \end{aligned}$$

where $A_\infty = \bigcap_{n=1}^{\infty} A_n = \bigcap_{n=2}^{\infty} A_n$, we likewise observe
$A_1 = \bigcup_{n=2}^{\infty} (A_{n-1} \setminus A_n) \cup A_\infty$.
Picture:

$$\underbrace{\underbrace{A_0 \setminus A_1}\ \underbrace{\underbrace{A_1 \setminus A_2}\ \underbrace{\ldots}\ \underbrace{A_\infty}}_{A_1}}_{A_0}$$

Using definitions of the sets A_n $(n \geq 2)$, we have $g(A_{n-1} \setminus A_n) = A_{n+1} \setminus A_{n+2}$. Define

$$h : A_0 \to A_1, h(x) = \begin{cases} g(x), & \text{if } x \in A_{n-1} \setminus A_n, n \text{ odd} \\ x, & \text{otherwise} \end{cases}$$

Then h is a bijection. Thus

$$A = A_0 \underbrace{\sim}_{h} A_1 = \psi(B), B \underbrace{\sim}_{\psi} \psi(B)$$

so we conclude that $A \sim B$. □

Examples:

1. Let $a < b$ in $\mathbb{R}$. Then $[a,b) \preceq \mathbb{R}$ (obvious)
   $\mathbb{R} \sim (-1,1) \sim (0,1) \sim (a,b) \preceq [a,b)$
   Ie. $[a,b) \preceq \mathbb{R}$ and $\mathbb{R} \preceq [a,b)$ so $\mathbb{R} \sim [a,b)$

3 2017-09-18

3.1 Last class: C.B.S Theorem

If $A \preceq B$ and $B \preceq A$ then $A \sim B$.
Examples:

(i) $\mathcal{P}(\mathbb{N}) \sim \mathbb{R}$, i.e. $|\mathcal{P}(\mathbb{N})| = c$.

$$\mathcal{P}(\mathbb{N}) \sim \{0,1\}^{\mathbb{N}},\ \text{via } A \longmapsto \chi_A \text{ where } \chi_A(n) \begin{cases} 1 & , n \in A \\ 0 & , n \notin A \end{cases} \quad (\text{"characteristic indicator"})$$

$$\{0,1\}^{\mathbb{N}} \sim \mathcal{P}(\mathbb{N}),\ \text{via } (x_k)_{k=1}^{\infty} \underbrace{\longmapsto}_{\text{injective}} \chi_A \text{ where } \sum_{k=1}^{\infty} \frac{x_k}{3^k} = 0.x_1x_2x_3\ldots \ (\text{ternary representation})$$

$$[0,1) \sim \{0,1\}^{\mathbb{N}},\ 0.x_1x_2x_3\cdots = \sum_{k=1}^{\infty} \frac{x_k}{2^k} \ (\text{binary representation}) \ (\text{never allow } 0.111\cdots = 1!) \longmapsto (x_k)_{k=1}^{\infty}$$

$$\mathcal{P}(\mathbb{N}) \sim \{0,1\}^{\mathbb{N}} \preceq [0,1) \preceq \{0,1\}^{\mathbb{N}} \sim \mathcal{P}(\mathbb{N})$$

so, by C.B.S. Theorem, we have $|\mathcal{P}(\mathbb{N})| = |[0,1)| = c = |\mathbb{R}|$.

(ii)

2nd lecture:

(iii) $\mathbb{N} \sim \mathbb{Q} \sim \mathbb{N}^2$

$$\mathbb{N} \preceq \mathbb{Q}$$
$$\mathbb{Q} \preceq \mathbb{Z} \times \mathbb{N},\ \text{via } \frac{m}{n} \longmapsto (m,n) \ (\gcd(m,n) = 1)$$
$$\mathbb{Z} \times \mathbb{N} \sim \mathbb{N}^2 = \mathbb{N} \times \mathbb{N},\ \text{as } \mathbb{Z} \sim \mathbb{N}$$
$$\mathbb{N}^2 \preceq \mathbb{N},\ \text{via } (m,n) \longmapsto 2^m 3^n$$

Hence $\mathbb{N} \preceq \mathbb{Q} \preceq \mathbb{Z} \times \mathbb{N} \sim \mathbb{N}^2 \preceq \mathbb{N}$ so, by C.B.S. Theorem, $\mathbb{N} \sim \mathbb{Q} \sim \mathbb{N}^2$.

Notation: We say that a set A is

- countable if $A \preceq \mathbb{N}$, i.e. $|A| \leq \aleph_0$
- denumerable if $A \sim \mathbb{N}$, i.e. $|A| = \aleph_0$

Proposition 3.1 (surjectivity)**.** Suppose X and Y are non-empty sets and there is a surjection $g : X \to Y$. Then $Y \preceq X$.

Proof. Let $f : \mathcal{P}(X) \setminus \{\varnothing\} \to X$ be a choice function (AC). For each $y \in Y$, we have $g^{-1}(\{y\}) = \{x \in X : g(x) = y\} \neq \varnothing$, as g is surjective. Define $h : Y \to X$ be given by $h(y) = f(g^{-1}(\{y\}))$ and h is injective, as if $y_1 \neq y_2, \{y_1\} \cap \{y_2\} = \varnothing$, so we see that $g^{-1}(\{y_1\}) \cap g^{-1}(\{y_2\}) = \varnothing$ too. □

Theorem 3.1 (Comparison Theorem)**.** Let X, Y be sets. Then either $X \preceq Y$ or $Y \preceq X$.

Proof. If $X \neq \varnothing$, then $X \preceq Y$; likewise if $Y = \varnothing$. Hence assume $X \neq \varnothing \neq Y$. We let

$$\Delta = \{(A, f) : A \in \mathcal{P}(X) \setminus \{\varnothing\}, f \in Y^A \text{ is an injection mapping from } A \text{ to } Y\}$$

We observe that $\Delta \neq \varnothing$. If $x \in A, y \in Y$, then $(\{x\}, x \longmapsto y) \in \Delta$. On Δ let

$$(A, f) \preceq (B, g) \iff A \subseteq B \subseteq X,\ g\mid_A = f$$

Notice that $\preceq$ is reflexive, anti-symmetric, and transitive, hence is a partial order on Δ. Let $\Gamma\{(A_i, f_i)\}_{i \in I}$ be a chain in $(\Delta, \preceq)$. We let $A = \bigcup_{i \in I} A_i$ and $f \in Y^A$ be given by $f(x) = f_i(x)$ provided $x \in A_i$.
Notice that f is well-defined. Say $x \in A_i$ and $x \in A_j$, then, since Γ is a chain, $A_i \subseteq A_j$, say, and $f_j\mid_{A_i} = f_i$.
Furthermore, if $x_1 \neq x_2$ in A, then $x_1 \in A_{i_1}, x_2 \in A_{i_2}$, and we may suppose $A_{i_1} \subseteq A_{i_2}$. Then $f(x_1) = f_{i_1}(x_1) = f_{i_2}(x_1) \neq f_{i_2}(x_2) = f(x_2)$, so f is an injection. Thus $(A, f) \in \Delta$, and is an upper bound of Γ.
Thus, there is a maximal element $(M, g) \in \Delta$, by Zorn's Lemma.

Case #1: $M = X$. Then $X = M \underset{g}{\preceq} Y$.

Case #2: $M \subsetneq X$. We wish to see that g must be surjective. Suppose not, i.e., there is $y_0 \in Y \setminus g(M)$. Since $M \subsetneq X$, there is $x_0 \in X \setminus M$. Define $h : M \cup \{x_0\} \to Y$ by

$$h(x) = \begin{cases} g(x) & x \in M \\ y_0 & x = x_0 \end{cases} \quad \text{injective!}$$

Then $(M \cup \{x_0\}, h) \in \Delta$, and $(M, g) \npreceq (M \cup \{x_0\}, h)$, contradicting maximality of (M, g). Thus, we have that that g is surjective. Thus $Y \underbrace{\preceq}_{g^{-1}} X$.

□

Proposition 3.2. Let A be a set. Then TFAE:

(i) $n \leq |A|$ for all $n \in \mathbb{N}$

(ii) $\aleph_0 \leq |A|$ (A is infinite)

(iii) there is $B \subsetneq A$ s.t. $|B| = |A|$

(iv) $1 + |A| = |A|$ (Hilbert hotel)

(v) $\aleph_0 + |A| = |A|$

Proof. (i) $\Rightarrow$ (ii) We have that for each n in $\mathbb{N}$ there is an injection $\varphi_N : \{1, \ldots, n\} \to A$. Inductively, define $f : \mathbb{N} \to A$ by

$$f(1) = \varphi_1(1)$$

$$f(n+1) = \varphi_{n+1}(k)$$

where $k = \min j \in \{1, \ldots, n+1\} : \varphi_{n+1}(j) \notin \{f(1), \ldots, f(n)\}$.
Then f is injective by construction.
(ii) $\Rightarrow$ (iii) We have $\mathbb{N} \preceq_f A$. Let $B = A \setminus \{f(1)\}$. Define $g : A \to B$ by

$$g(x) = \begin{cases} f(n+1) & \text{if } x = f(n), n \in \mathbb{N} \\ x & \text{otherwise} \end{cases}$$

Then $A \sim_g B$, i.e., $|A| = |B|$.
(iii) $\Rightarrow$ (iv) We suppose there is $x_0 \in A \setminus B$ and $B \sim A$. Thus $A \sim B \preceq B \cup \{x_0\} \preceq A$ so by C.B.S. Theorem $A \sim B$ and

$A \sim B \cup \{x_0\} \sim A \sqcup \{1\}$, i.e. $|A| = |A| + 1$.
(iv) $\Rightarrow$ (i) We have $\{1\} \sqcup A \sim_\varphi A$. Then $\varphi(A) \subsetneq A$. Thus $\varphi \circ \varphi(A) \subsetneq \varphi(A) \subsetneq A$, and, by induction,

$$\varphi^{\circ n}(A) \subsetneq \varphi^{\circ n-1}(A) \subsetneq \cdots \subsetneq A$$

$$\underbrace{\varphi \circ \cdots \circ \varphi}_{n \text{ times}}$$

Hence $|A| \geq |A \setminus \varphi^{\circ n}(A)| \geq n$ (at each stage above, we gain at least one point).
(ii) $\Rightarrow$ (v) We have $\mathbb{N} \preceq_f A$. Let $g : \mathbb{N} \sqcup A \to A$,

$$g(x) = \begin{cases} f(2n) & \text{if } x = n, n \in \mathbb{N} \\ f(2n+1) & \text{if } x = f(n) \in A, n \in \mathbb{N} \\ x & \text{otherwise} \end{cases}$$

(v) $\Rightarrow$ (ii) $\aleph_0 \leq \aleph_0 + |A| = |A|$ by assumption. □

Corollary 3.1. If $A \in \mathcal{P}(\mathbb{N})$, then either A is finite or denumerable.

Proof. Either $n \leq |A|$ for all n, or $|A| < n$ (Comparison lemma). □

Theorem 3.2 (Cantor)**.** For any set X, $|X| < |\mathcal{P}(X)|$.

Proof. :(□

Cantor's paradox: There is no "set" of all sets.

4 2017-09-22

4.1 Metric Spaces

Example (French railroad / metro metric): Suppose we have a set $X \neq \varnothing$, and a function $f : X \to [0, \infty)$ which satisfies $f^{-1}(\{0\}) = \{p_0\}$. Notice, then, that $f(x) > 0$ if $x \in X \setminus \{p_0\}$.

$$d_f : X \times X \to [0, \infty),\ d_f(x, y) = f(x) + f(y)$$

if $x \neq y$, 0 if $x = y$.
Easy exercise: this is a metric.
(Belongs to family of weighted graph metrics.)

$$||x||_p = (|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$$

$$x^p = \begin{cases} e^{p \log x} & \text{x} > 0 \\ 0 & \text{x} = 0 \end{cases}$$

Lemma 4.1. Let $\alpha, \beta \geq 0$ in $\mathbb{R}$, $1 < p < \infty$ and q is chosen so that $\frac{1}{p} + \frac{1}{q} = 1$ (ie $q = \frac{p}{p-1}$) then

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$$

with equality when $\alpha^p = \beta^q$.

Proof. Consider the graph of $y = x^{p-1}$ (assume $p \geq 2$).

$$x = y^1 p - 1 = y^q p = y^{q-1}$$

Then

$$\alpha\beta \leq \underbrace{\int_0^\alpha x^{p-1}dx}_{A_1} + \underbrace{\int_0^\beta y^{q-1}dy}_{A_2}$$

(Equality holds only if $\beta = \alpha^{p-1} \Rightarrow \beta^1 q - 1 \Rightarrow \beta^q = \alpha^p$)

$$= \frac{\alpha^p}{p} + \frac{\beta^q}{q}$$

Holder's Inequality

□

5 2017-09-25

Lemma: $\alpha, \beta \geq 0$ in $\mathbb{R}, 1 < p < \infty$ with q satisfying $\frac{1}{p} + \frac{1}{q} \Longrightarrow \alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$

Holder's Inequality: If $x, y \in \mathbb{R}^n, 1 < p < \infty$ and q satisfies $\frac{1}{p} + \frac{1}{q} = 1$, then

$$|\sum_{j=1}^n x_j y_j| \underbrace{\leq}_{\text{1-ineq. of } |\cdot|} \sum_{j=1}^n |x_j||y_j| \leq \left(\sum_{j=1}^n |x_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^n |y_j|^p\right)^{\frac{1}{p}} := ||x||_p ||y||_q$$

Proof. If $||x||_p||y||_q = 0$, then $x = 0$ or $y = 0$ and the inequality is trivial. Assume $||x||_p||y||_q \neq 0$. For $j = 1, \ldots, n$, let

$$\alpha_j = \frac{|x_j|}{||x||_p}, \quad \beta_j = \frac{|y_j|}{||y||_q}.$$

Then

$$\begin{aligned}
\frac{1}{||x||_p||y||_q} \sum_{j=1}^n |x_j||y_j| &= \sum_{j=1}^n \alpha_j \beta_j \\
&\leq \sum_{j=1}^n \left[\frac{\alpha_j^p}{p} + \frac{\beta_j^q}{q}\right] \text{ by lemma} \\
&= \frac{1}{p}\sum_{j=1}^n \alpha_j^p + \frac{1}{q}\sum_{j=1}^n \beta_j^q \\
&= \frac{1}{p||x||_p^p}\sum_{j=1}^n |x_j|^p + \frac{1}{q||x||_q^q}\sum_{j=1}^n |y_j|^q \\
&= \frac{1}{p} + \frac{1}{q} \\
&= 1
\end{aligned}$$

□

Theorem 5.1 (Minkowski's Inequality)**.** Let $x, y \in \mathbb{R}^n$ and $1 < p < \infty$. Then

$$||x + y||_p \leq ||x||_p + ||y||_p.$$

Proof. If $x+y=0$ then this is trivial, so suppose $x+y\neq 0$.

$$\begin{aligned}
||x+y||_p^p &= \sum_{j=1}^n |x_j+y_j|^p \\
&= \sum_{j=1}^n |x_j+y_j||x_j+y_j|^{p-1} \\
&\leq \sum_{j=1}^n (|x_j|+|y_j|)(|x_j+y_j|^{p-1}) \\
&= \sum_{j=1}^n |x_j||x_j+y_j|^{p-1} + \sum_{j=1}^n |y_j||x_j+y_j|^{p-1} \\
&\leq \left(\sum_{j=1}^n |x_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^n |x_j+y_j|^{(p-1)q}\right)^{\frac{1}{q}} + \left(\sum_{j=1}^n |y_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^n |x_j+y_j|^{(p-1)q}\right)^{\frac{1}{q}} \\
&= (||x||_p + ||y||_p) \left(\sum_{j=1}^n |x_j+y_j|^{(p-1)q}\right)^{\frac{1}{q}}
\end{aligned}$$

We have

$$\frac{1}{p}+\frac{1}{q}=1 \Longrightarrow \frac{1}{q} = 1-\frac{1}{p} = \frac{p-1}{p} \Longrightarrow p = q(p-1)$$

and thus

$$\begin{aligned}
||x+y||_p^p &\leq (||x||_p+||y||_p)\left(\sum_{j=1}^n |x_j+y_j|^p\right)^{\frac{1}{q}} \\
&= (||x||_p+||y||_p)||x+y||_p^{\frac{p}{q}}
\end{aligned}$$

Now, divide $||x+y||_p^{\frac{p}{q}} \neq 0$ to get

$$\begin{aligned}
||x+y||_p &= ||x+y||_p^{p-\frac{p}{q}} \\
&\leq ||x||_p + ||y||_p
\end{aligned}$$

(since $p-\frac{p}{q} = p(1-\frac{1}{q}) = 1$). □

Corollary 5.1. Given $1<p<\infty, ||\cdot||_p$ is a norm on $\mathbb{R}^n$.

Proof. Clearly $||\cdot||_p$ is non-negative and non-degenerate. If $\alpha\in\mathbb{R}, x\in\mathbb{R}^n$ then

$$\begin{aligned}
||\alpha x||_p &= (\sum_{j=1}^n |\alpha x_j|^p)^{\frac{1}{p}} \\
&= |\alpha|(\sum_{j=1}^n |x_j|^p)^{\frac{1}{p}} \\
&= |\alpha|||x||_p
\end{aligned}$$

Finally, subadditivity is provided by Minkowski's inequality. □

$|x|^p = e^{p\log|x|}$

5.1 The ℓ_p-spaces

Consider $\mathbb{R}^N = \{x = (x_k)_{k=1}^\infty : x_k \in \mathbb{R}\}$ which is a $\mathbb{R}$-vector space:

$$(x_k)_{k=1}^\infty + (y_k)_{k=1}^\infty = (x_k + y_k)_{k=1}^\infty, \alpha(x_k)_{k=1}^\infty = (\alpha x_k)_{k=1}^\infty.$$

We let for $1 \leq p < \infty$

$$\ell_p = \{x = (x_k)_{k=1}^\infty \in \mathbb{R}^N : \sum_{k=1}^\infty |x_k|^p = \lim_{n\to\infty} \sum_{k=1}^n |x_k|^p < \infty\}$$

and

$$\ell_\infty = \{x = (x_k)_{k=1}^\infty \sup_{k\in\mathbb{N}} |x_k| < \infty\}.$$

On ℓ_p we define

$$||x||_p = \begin{cases} (\sum_{j=1}^n |x_j|^p)^{\frac{1}{p}} & \text{, if } 1 \leq p < \infty \\ \sum_{k\in\mathbb{N}} |x_k| & \text{, if } p = \infty \end{cases}$$

Theorem 5.2. Let $1 \leq p < \infty$. Then ℓ_p is a $\mathbb{R}$-subspace of $\mathbb{R}^\mathbb{N}$ and $||\cdot||_p$ is a norm.

Proof. We prove these together. Suppose that $x, y \in \ell_p$. Then

$$\begin{aligned}
||x+y||_p &= \left(\sum_{k=1}^\infty |x_k + y_k|^p\right)^{\frac{1}{p}} \text{ if } \infty\text{, treat } \infty^{\frac{1}{p}} = \infty \\
&= \left(\lim_{n\to\infty} \sum_{k=1}^n |x_k + y_k|^p\right)^{\frac{1}{p}} \\
&= \lim_{n\to\infty} \left(\sum_{k=1}^n |x_k + y_k|^p\right)^{\frac{1}{p}} \qquad x \longmapsto x^{\frac{1}{p}} \text{ is continuous on } [0,\infty),\ \text{if } x\to\infty, x^{\frac{1}{p}} \to \infty \\
&\leq \lim_{n\to\infty} \left(\sum_{k=1}^n |x_k|^p\right)^{\frac{1}{p}} + \lim_{n\to\infty} \left(\sum_{k=1}^n |y_k|^p\right)^{\frac{1}{p}} \text{ Minkowski applied on each } n \\
&= \left(\lim_{n\to\infty} \sum_{k=1}^n |x_k|^p\right)^{\frac{1}{p}} + \left(\lim_{n\to\infty} \sum_{k=1}^n |y_k|^p\right)^{\frac{1}{p}} \text{ continuity again} \\
&= \left(\sum_{k=1}^\infty |x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^\infty |y_k|^p\right)^{\frac{1}{p}} \\
&= ||x||_p + ||y||_p \\
&< \infty
\end{aligned}$$

Thus $x + y \in \ell_p$, and we get subadditivity of $||\cdot||_p$.
We note that non-negativity and non-degeneracy of $||\cdot||_p$ are obvious. Likewise, the $|\cdot|$-homogeneity is straightforward. □

Theorem 5.3. $(\ell_\infty, ||\cdot||_\infty)$ is a normed vector space.

Proof. If $x, y \in \ell_\infty$ then

$$\begin{aligned} ||x+y||_\infty &= \sup_{k\in\mathbb{N}} |x_k + y_k| \\ &\le \sup_{k\in\mathbb{N}} (|x_k| + |y_k|) \\ &\le \sup_{j,k\in\mathbb{N}} (|x_j| + |y_k|) \\ &= \sup_{j\in\mathbb{N}} |x_j| + \sup_{k\in\mathbb{N}} |y_k| \\ &= ||x||_\infty + ||y||_\infty \end{aligned}$$

Other properties are very easy. □

6 2017-09-29

i) $X \neq \varnothing$ s.t. $|X| \geq 2$
discrete metric $d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$
For $x_0 \in X$,

$$B(x, \varepsilon) = \begin{cases} \{x_0\} & 0 < \varepsilon \le 1 \\ x & \varepsilon > 1 \end{cases}$$
$$B[x, \varepsilon] = \begin{cases} \{x_0\} & 0 < \varepsilon < 1 \\ x & \varepsilon \ge 1 \end{cases}$$

ii) (geometry of balls in $\mathbb{R}^2$)
$1 \le p \le \infty, B_p(0,1) = \{x \in \mathbb{R}^2 : d_p(0,x) = \|x\|_p < 1\}$

Proposition 6.1. (X, d) a metric space.

i) $X, \varnothing$ are both open and closed.

ii) If $\{U_i\}_{i\in I}$ is a family of open sets, then $\bigcup_{i\in I} U_i$ is open.

iii) If $\{U_1, \dots, U_n\}$ is a finite family of open sets, then $\bigcap_{i=1}^n U_i$ is open.

iv) If $\{F_i\}_{i\in I}$ is a family of closed sets, then $\bigcap_{i\in I} U_i$ is closed.

v) If $\{U_1, \dots, U_n\}$ is a finite family of closed sets, then $\bigcup_{i=1}^n U_i$ is closed.

Proof. i) Let $x \in X$, then $x \in B(x, 1) \subseteq X$, so X is open. So $\varnothing = X \setminus X$, $X = X \setminus \varnothing$ are closed.

ii) Let $x \in U = \bigcup_{i\in I} U_i$. Then there is some i_0 in I s.t. $x \in U_{i_0}$, which is open, so there is $\varepsilon_x > 0$ s.t. $x \in B(x, \varepsilon_x) \subseteq U_{i_0} \subseteq U$.

iii) Let $x \in V = \bigcap_{i=1}^n U_i$. Then for each $i = 1, \dots, n$, there is $\varepsilon_i > 0$ s.t. $B(x, \varepsilon_i) \subseteq U_i$. Let $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\} \implies B(x, \varepsilon) \subseteq \bigcap_{i=1}^n B(x, \varepsilon_i) \subseteq V$.

iv), v) De Morgan's Laws.

□

Given a metric space (X, d), $A \subseteq X$, we define the boundary of A:

$$\partial A = \{x \in X : \forall \varepsilon > 0, B(x, \varepsilon) \cap A \neq \varnothing, B(x, \varepsilon) \setminus A \neq \varnothing\}.$$

Remark: $\partial A = \partial(X \setminus A)$.
Interior of A :

$$A^\circ = \bigcup \{U \subseteq X : U \subseteq A \text{ and } U \text{ is open}\}.$$

Proposition 6.2 (characterizations of interior)**.** If $(X, d), A$ are as above then

$$\begin{aligned} A^\circ &= \{x \in X : \exists \varepsilon_x > 0 \text{ s.t. } B(x, \varepsilon_x) \subseteq A\} \\ &= A \setminus \partial A. \end{aligned}$$

Proof. Let $x \in A$. Then either:

- for some $\varepsilon_x > 0$, $B(x, \varepsilon_x) \subseteq A \implies x \in A^\circ$, or
- $\forall \varepsilon > 0, B(x, \varepsilon) \setminus A \neq \varnothing \implies$ since $x \in A \cap B(x, \varepsilon)$, $x \in \partial A$.

Since $A^\circ \subseteq A$, the proposition holds. □

Def: (X, d) a metric space, $(x_n)_{n=1}^\infty \subseteq X$ and $x_0 \in X$. Say $(x_n)_{n=1}^\infty$ converges to x_0, i.e. $\lim_{n\to\infty} x_n = x_0$ or $x_n \xrightarrow{n\to\infty} x_0$ if $\forall \varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$ s.t. $n \geq n_\varepsilon \implies d(x_0, x_n) < \varepsilon$.
Remark: The limit, if it exists, is unique. Suppose $x_0 = \lim_{n\to\infty} x_n, y_0 = \lim_{n\to\infty} x_n$, then given $\varepsilon > 0$, $\exists n_\varepsilon, n_{\varepsilon'}$ in $\mathbb{N}$ s.t.

$$\begin{aligned} n \geq n_\varepsilon &\implies d(x_0, x_n) < \varepsilon \\ n \geq n_{\varepsilon'} &\implies d(y_0, x_n) < \varepsilon. \end{aligned}$$

Now if $n \geq \max\{n_\varepsilon, n_{\varepsilon'}\}$, then

$$\begin{aligned} &d(x_0, y_0) \leq d(x_0, x_n) + d(x_n, y_0) < \varepsilon \\ &\implies d(x_0, y_0) = 0, \text{ so } x_0 = y_0. \end{aligned}$$

Example: Let $(V, \|\cdot\|)$ be a normed vector space. A subset $\{e_n\}_{n=1}^\infty \subseteq V$ is a Schauder basis if for each $x \in V$, $\exists$ a unique sequence $\{x_n\}_{n=1}^\infty$ s.t. $x = \lim_{n\to\infty} \sum_{k=1}^n x_k e_k$ in V.
In $\ell_p, 1 \leq p < \infty$, let $e_n = (0, \dots, 0, \underbrace{1}_{n\text{-th place}}, 0, \dots)$.

Let, for $(X, d), A$ as above, the set of accumulation points (cluster points) be given as

$$A' = \{x \in X : \forall \varepsilon > 0, \underbrace{B(x, \varepsilon) \setminus \{x\}}_{\text{punctured ball}} \cap A \neq \varnothing.\}$$

Call elements of $A \setminus A'$ isolated points.

Proposition 6.3. Given $(X, d), A$ as above, we have

$$A' = \{x \in X : x = \lim_{n\to\infty} x_n, \ (x_n)_{n=1}^\infty \subseteq A \setminus \{x\}.\}$$

Proof. If $x \in A'$, let $x_1 \in (B(x, 1) \setminus \{x\}) \setminus A$, and $x_{n+1} \in (B(x, \varepsilon_n) \setminus \{x\}) \setminus A$, where $\varepsilon_n = \min\{\frac{1}{n}, d(x, x_n)\}$.
Then $x = \lim_{n\to\infty} x_n$ while $(x_n)_{n=1}^\infty \subseteq A \setminus \{x\}$. Note $x_1, x_2, \dots$ are distinct.
Converse direction: definition of limits. □

7 2017-10-02

Def: Given a metric space (X, d) and $A \subseteq X$, define the closure of A by

$$\bar{A} = \cap \{F \subseteq X : A \subseteq F, F \text{ is closed in } X.\}$$

Of course $A^\circ \subseteq A \subseteq \bar{A}$.

Theorem 7.1 (characterization of the closure)**.** Given a metric space $(X, d), A \subseteq X$, the following sets are the same:

$$\bar{A}, A \cup \partial A, A \cup A'$$

("meet" set) $A_M = \{x \in X : \text{ for any } \varepsilon > 0, B(x, \varepsilon) \cap A \neq \varnothing\}$
("limit" set) $A_L = \{x \in X : x = \lim_{n\to\infty} x_n, \text{ where } (x_n)_{n=1}^\infty \subseteq A\}$
(The notations A_L, A_M will not be used afterwards; we shall use $\bar{A}$.)

Proof. We have

$$\begin{aligned}\bar{A} &= \cap\{F \subseteq X : A \subseteq F, F \text{ closed }\}\\ &= \cap\{X \subseteq U : U \subseteq X \setminus A, U \text{ open in } X\}\\ &= X \setminus U\{U : U \subseteq X \setminus A, U \text{ open in } X\}\\ &= X \setminus [(X \setminus A)^o] \text{ complement of interior}\\ &= X \setminus [(X \setminus A) \setminus \partial(X \setminus A)] \text{ characterization of } (X \setminus A)^o\\ &= X \setminus [(X \setminus A) \setminus \partial A]\\ &= A \cup \partial A\end{aligned}$$

$(\cap_{i\in I}(X \setminus U_i) = X \setminus \cup_{i\in I} U_i)$
We thus have $\bar{A} = A \cup \partial A$.
Now if $x \in A \cup \partial A$, then for each $\varepsilon > 0$, we have that $B(x, \varepsilon) \cap A \neq \varnothing$ [i.e. either $x \in A$ so $x \in A \cap B(x, \varepsilon)$, or $x \in \partial A$, so $B(x, \varepsilon) \cap A \neq \varnothing$]. Thus $A \cup \partial A \subseteq A_M$. Conversely, if $x \in A_M$, then, either

- there is $\varepsilon > 0$ so $B(x, \varepsilon) \subset A \Longrightarrow x \in A^o \subseteq A$, or
- for every $\varepsilon > 0$ we have $B(x, \varepsilon) \setminus A \neq \varnothing$ in which case $x \in \partial A$.

Hence, $x \in A_M \Longrightarrow x \in A \cup \partial A$ so $A_M \subseteq A \cup \partial A$.
If $x \in A \cup A'$, then for each $\varepsilon > 0$, we have $B(x, \varepsilon) \cap A \neq \varnothing$. Indeed, as above, either $x \in A$, so for any $\varepsilon > 0, x \in B(x, \varepsilon) \cap A$, or $x \in A'$, so $B(x, \varepsilon) \cap A \supseteq (B(x, \varepsilon) \setminus \{x\}) \cap A \neq \varnothing$. Hence $A \cup A' \subseteq A_M$.
The definition of the limit of a sequence shows that $A_M = A_L$.
Finally, consider

$$\begin{aligned}X \setminus (A \cup A') &\subseteq \{x \in X : \text{ there exists } \varepsilon_x > 0 \text{ s.t. } B(x, \varepsilon_x) \cap A = \varnothing, B(x, \varepsilon_x) \subseteq X \setminus A\}\\ &= (X \setminus A)^o \Longrightarrow X \setminus [(X \setminus A)^o] \subseteq X \setminus [X \setminus (A \cup A')].\end{aligned}$$

Hence

$$\begin{aligned}\bar{A} = X \setminus [(X \setminus A)^o] &\subseteq X \setminus [X \setminus (A \cup A')]\\ &= A \cup A'.\end{aligned}$$

Hence $\bar{A} \subseteq A \cup A' \subseteq A_M = \bar{A}$, so $\bar{A} = A \cup A'$. □

7.1 Continuity

Def: Let (X, d_X) and (Y, d_Y) be metric spaces $f : X \to Y$ and $x_0 \in X$. We say that f is continuous at x_0 if given $\varepsilon > 0$, there is $\delta > 0$ s.t. $d_X(x, x_0) < \delta \Longrightarrow d_Y(f(x), f(x_0)) < \varepsilon$. $(\star)$
We say that f is continuous on X if it is continuous at each point.
Note:

$$\begin{aligned}(\star) &\iff f(B(x_0, \delta)) \subseteq B(f(x_0), \varepsilon)\\ &\iff B(x, \delta) \subseteq f^{-1}(B(f(x_0), \varepsilon))\end{aligned}$$

Notation: In a metric space, a set N is a neighbourhood of a point x_0 if $x_0 \in N^o$ (interior).

Theorem 7.2 (characterization of continuity at a point)**.** If $(X, d_X), (Y, d_Y), f : X \to Y, x \in X$ are as above, then TFAE:

(i) f is continuous at x_0

(ii) for any neighbourhood N of $f(x_0)$ in (Y, d_Y), we have $f^{-1}(N)$ is a neighbourhood of x_0 in (X, d_X)

(iii) if $x_0 = \lim_{n\to\infty} x_n$ in $(X, d_X) \implies f(x_0) = \lim_{n\to\infty} f(x_n)$ in (Y, d_Y).

Proof. (i) $\implies$ (ii) Given a neighbourhood of $f(x_0)$, there exists $\varepsilon > 0$ for which $B(f(x_0), \varepsilon) \subseteq N$. By assumption of continuity, there is $\delta > 0$ s.t.

$$\begin{aligned} B(x_0, \delta) &\subseteq f^{-1}(B(f(x_0), \varepsilon)) \\ &\subseteq f^{-1}(N), \text{ from above.} \end{aligned}$$

Thus $f^{-1}(N)$ is a neighbourhood of x_0.
(ii) $\implies$ (i) $\implies$ (iii) Given $\varepsilon > 0, B(f(x_0), \varepsilon)$ is a neighbourhood of $f(x_0)$, so $f^{-1}(B(f(x_0), \varepsilon))$ is a neighbourhood of x_0 and hence there is $\delta > 0$ s.t. $B(x_0, \delta) \subseteq f^{-1}(B(f(x_0), \varepsilon))$, which gives (i).
Now, if $x_0 = \lim_{n\to\infty} x_n$ in (X, d_X) then there is n_δ in $\mathbb{N}$ s.t. if $n \leq n_\delta, x_n \in B(x_0, \delta)$. But then for $n \leq n_\delta$, we have

$$f(x_n) \in f(B(x, \delta)) \subseteq B(f(x_0), \varepsilon)$$

and hence $f(x_0) = \lim_{n\to\infty} f(x_n)$.
(iii) $\implies$ (i) (contrapositive) If (i) fails, then there exists $\varepsilon > 0$ s.t. for any $\delta > 0$, $B(x_0, \delta) \not\subseteq f^{-1}(B(f(x_0), \varepsilon))$.
Hence for each $n \in \mathbb{N}$ we may find $x_n \in B(x_0, \frac{1}{n}) \backslash f^{-1}(B(f(x_0), \varepsilon))$. Given $\varepsilon' > 0$, let $n_{\varepsilon'}$ satisfy $n_{\varepsilon'} \leq \frac{1}{\varepsilon}$, thus $\lim_{n\to\infty} x_n = x_0$. However, each $f(x_n) \notin B(f(x_0), \varepsilon)$, so $f(x)$ does not go to. □

8 2017-10-06

Corollary 8.1. A metric space is complete if whenever for any Cauchy sequence, we may find a converging subsequence.

Nested Intervals Theorem, Bolzano-Weierstrauss Theorem

Theorem 8.1. $(\ell_p, \|\cdot\|_p)$ $(1 \leq p < \infty)$ is complete as a metric space.

Def: A normed space $(V, \|\cdot\|)$ is called a Banach space provided that V is complete w.r.t. metric $d(x, y) = \|x - y\|$.
$(\ell_p, \|\cdot\|_p)$ is a Banach space.

9 2017-10-16

Theorem 9.1. The space of continuous bounded functions under the uniform metric, $(C_b(f), \|\cdot\|_\infty)$, is a Banach space.

Proof. (I) For $x \in X$, $(f_n(x))_{n=1}^\infty$ is Cauchy and admits a limit, so this defines $f : X \to \mathbb{R}$. The hard part is showing that f is continuous.
Next, show f is bounded, so $f \in C_b(X)$.
(II) $\lim_{n\to\infty} \|f - f_n\|_\infty = 0$, ie. $\lim_{n\to\infty} f_n = f$ uniformly in $C_b(X)$. □

9.1 Characterizations of Completeness

Def: If (X, d) is a metric space, $\varnothing \neq A \subseteq X$, we let the diameter of A be given by

$$\operatorname{diam}(A) = \sum_{x,y \in A} d(x, y) \text{ (may be } \infty)$$

Proposition 9.1. If $(X,d), A$ are as above then $\operatorname{diam}(\bar{A}) = \operatorname{diam}(A)$.

Proof. If $x, y \in \bar{A}, \varepsilon > 0$, then there are x', y' in A s.t. $d(x,x') < \frac{\varepsilon}{2}, d(y,y') < \frac{\varepsilon}{2}$ (using meet set characterization of $\bar{A}$). Then

$$\begin{aligned} d(x,y) &\leq d(x,x') + d(x',y') + d(y',y) \\ &\leq \frac{\varepsilon}{2} + \operatorname{diam}(A) + \frac{\varepsilon}{2} \\ &= \operatorname{diam}(A) + \varepsilon. \text{ (Assume } \operatorname{diam}(A) < \infty). \end{aligned}$$

Thus, since $\varepsilon > 0$ is arbitrary, $d(x,y) \leq \operatorname{diam}(A) \implies \operatorname{diam}(\bar{A}) = \sup_{x,y\in A} d(x,y) \leq \operatorname{diam}(A)$. Since $A \subseteq \bar{A}, \operatorname{diam}(A) \leq \operatorname{diam}(\bar{A})$. □

Theorem 9.2 (Nested set characterization of completeness)**.** Let (X,d) be a metric space. Then (X,d) is complete $\iff$ whenever we have closed sets,

- $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$
- $\operatorname{diam} F_n \xrightarrow{n\to\infty} 0$

then $\bigcap_{n=1}^{\infty} F_n \neq \varnothing$.

Proof. $(\implies)$ For each n, choose $x_n \in F_n$. Given $\varepsilon > 0$, choose n_ε in $\mathbb{N}$ s.t. $n \geq n_\varepsilon \implies \operatorname{diam}(F_n) < \varepsilon$. Now, if $n, m \geq n_\varepsilon$ we have

$$x_n \in F_n \subseteq F_{n_\varepsilon}, x_m \in F_m \subseteq F_{n_\varepsilon} \implies d(x_n, x_m) \leq \operatorname{diam}(F_{n_\varepsilon}) < \varepsilon$$

so $(x_n)_{n=1}^{\infty}$ is Cauchy, and has limit $x = \lim_{n\to\infty} x_n$. Since each $F_m = \bar{F}_m$ (closed), and we have for $n \geq m, x_n \in F_m, x = \lim_{n\to\infty} x_m \in F_m$ for all m. Hence $x \in \bigcap_{m=1}^{\infty} F_m$ (ie. $\neq \varnothing$).
$(\impliedby)$ Let $(x_n)_{n=1}^{\infty} \subset X$ be Cauchy, let for n in $\mathbb{N}$, $F_n = \overline{\{x_k\}_{k\geq n}}$. Then each F_n is closed and $F_n \supseteq F_{n+1}$ for each n. Further, $\operatorname{diam} F_n = \operatorname{diam}\{x_k\}_{k\geq n}$ (last proposition). Given $\varepsilon > 0$, there is n_ε in $\mathbb{N}$ so $n, m \geq n_\varepsilon \implies d(x_n, x_m) < \varepsilon$. So for $n \geq n_\varepsilon$, we have $\operatorname{diam}\{x_k\}_{k\geq n} = \sup_{k,l\geq n} d(x_k, x_l) < \varepsilon$. □

10 2017-10-18

Continuing the proof that $(C_b(f), \|\cdot\|_\infty)$ is a Banach space from last time:

Theorem 10.1. The space of continuous bounded functions under the uniform metric, $(C_b(f), \|\cdot\|_\infty)$, is a Banach space.

Proof. (I) For $x \in X$, $(f_n(x))_{n=1}^{\infty}$ is Cauchy and admits a limit, so this defines $f : X \to \mathbb{R}$.
f is continuous: let $x \in X$, and let $\varepsilon > 0$. Choose $n_\varepsilon \in N$ so that

$$n, m \geq n_\varepsilon \implies |f_n(x) - f(x)| < \frac{\varepsilon}{4} \text{ and } \|f_n - f_m\|_\infty < \frac{\varepsilon}{4}.$$

Choose $\delta > 0$ so that for $x, y \in X$,

$$d(x,y) < \delta \implies |f_{n_\varepsilon}(x) - f_{n_\varepsilon}(y)| < \frac{\varepsilon}{4}.$$

Then, given $y \in B(x,\delta)$, let $n_y \in \mathbb{N}$ so that $n_y \geq n_\varepsilon$ and

$$n \geq n_y \implies |f_n(y) - f(y)| < \frac{\varepsilon}{4}.$$

Then for $n \geq n_y \geq n_\varepsilon$ we have

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_{n_\varepsilon}(x)| + |f_{n_\varepsilon}(x) - f_{n_\varepsilon}(y)| + |f_{n_\varepsilon}(y) - f_n(y)| + |f_n(y) - f(y)| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ &= \varepsilon. \end{aligned}$$

Also, f is bounded because

$$\begin{aligned}|f(x)| &\leq |f(x)-f_n(x)|+|f_n(x)| \\ &\leq |f(x)-f_n(x)|+\|f_n\|_\infty \\ &= o(1)+M.\end{aligned}$$

(II) Show that this is actually the limit (i.e. $\lim_{n\to\infty}\|f-f_n\|_\infty = 0$).
Let $\varepsilon > 0$. Choose $n_\varepsilon \in \mathbb{N}$ so that $m,n \geq n_\varepsilon \implies \|f_m - f_n\|_\infty < \frac{\varepsilon}{2}$. Also, given $x \in X$, choose $n_x \geq n_\varepsilon$ so that $n \geq n_x \implies |f_n(x)-f(x)| < \frac{\varepsilon}{2}$. Then, for $n \geq n_\varepsilon$, find $m \geq n_x \geq n_\varepsilon$ and observe that

$$\begin{aligned}|f(x)-f_n(x)| &\leq |f(x)-f_m(x)|+|f_m(x)-f_n(x)| \\ &< \frac{\varepsilon}{2}+\|f_m-f_n\|_\infty \\ &= \varepsilon.\end{aligned}$$

□

Example: Consider $(\ell_p, \|\cdot\|_p)$, $1 \leq p < \infty$. Let $e_n = (0,\dots,0,\underbrace{1}_{n\text{-th place}},0,\dots)$ and let $F_n = \{e_k\}_{k\geq n} \subseteq \ell_p$.

- Each F_n is closed (easy exercise)
- $F_1 \supseteq F_2 \supseteq \cdots$
- $\operatorname{diam} F_n = 2^{\frac{1}{p}}$ (easy computation) (Finite diameter is not sufficient for Nested set characterization)

Notice that $\bigcap_{n=1}^\infty F_n = \varnothing$.

Theorem 10.2 (abstract M-test)**.** Let $(V,\|\cdot\|)$ be a normed vector space. Then $(V,\|\cdot\|)$ is a Banach space $\iff$ for every $(x_k)_{k=1}^\infty \subset V$ with $\sum_{k=1}^\infty\|x_k\| = \lim_{n\to\infty}\sum_{k=1}^n\|x_k\|$ converging, has that $\sum_{k=1}^\infty x_n = \lim_{n\to\infty}\sum_{k=1}^n x_k$ converges in $(V,\|\cdot\|)$ [ie. V satisfies that "absolute convergence" $\implies$ convergence.]

Proof. ($\implies$) Suppose $\sum_{k=1}^\infty\|x_k\|$ converges. Consider $(\sum_{k=1}^n x_k)_{n=1}^\infty \subset V$. We have for $m < n$ that

$$\left\|\sum_{k=1}^n x_k - \sum_{k=1}^m x_k\right\| \leq \underbrace{\sum_{k=m+1}^n \|x_k\|}_{\text{partial tail of converging series in } \mathbb{R}}$$

and hence $(\sum_{k=1}^n x_k)_{n=1}^\infty$ is Cauchy in $(V,\|\cdot\|)$, and thus converges.
($\impliedby$) Suppose $(x_n)_{n=1}^\infty$ is a Cauchy seq in $(V,\|\cdot\|)$. Let n_1 in $\mathbb{N}$ be so $m,n \geq n_1 \implies \|x_m - x_n\| < 1$, and, inductively, choose n_{k+1} in $\mathbb{N}$ s.t. $n_{k+1} \geq n_k$ and $m,n \geq n_{k+1} \implies \|x_n - x_m\| < \frac{1}{2^k}$.
Let $y_0 = x_{n_1},\ y_j = x_{n_{j+1}} - x_{n_j},\ j \in \mathbb{N}$.
Then, each $\|y_j\| = \left\|x_{n_{j+1}} - x_{n_j}\right\| < \frac{1}{2^{j-1}}$, as $n_{j+1} > n_j \geq n$, so

$$\sum_{i=0}^\infty \|y_j\| = \|y_0\| + \sum_{j=1}^\infty \frac{1}{2^{j-1}},$$

which converges. ($\star$)

Now

$$\begin{aligned} x_{n_k} &= x_{n_1} + \sum_{j=1}^{k-1}(x_{n_{j+1}} - x_{n_j}) \\ &= y_0 + \sum_{j=1}^{k-1} y_j \\ &\xrightarrow{k\to\infty} y_0 + \sum_{j=1}^{\infty} y_j \text{ (by assumption and } (\star)) \end{aligned}$$

In other words, $(x_{n_k})_{k=1}^{\infty}$ converges, hence $(x_n)_{n=1}^{\infty}$ converges as well. $\square$

Application: a continuous nowhere differentiable function on $\mathbb{R}$.

Facts: $C_b(\mathbb{R})$ is complete; M-test.
Construction: Let $\varphi : \mathbb{R} \to [0, 1]$

$$\varphi(t) = \begin{cases} t - 2k & 2k \le t < 2k+1 \\ 2k+2-t & 2k+1 \le t < 2k+2 \end{cases}$$

Picture: sawtooth function with zeros at $\dots, -4, -2, 0, 2, 4, \dots$.
Then

(i) φ is continuous and bounded

(ii) φ is 2-periodic, ie. $\varphi(t+2) = \varphi(t)$ for $t \in \mathbb{R}$

(iii) $\varphi(2k) = 0, \varphi(2k+1) = 1$ for $k \in \mathbb{Z}$

(iv) if $k \le s, t \le k+1$ ($k \in \mathbb{Z}$), then

$$|\varphi(s) - \varphi(t)| - |s-t|$$

Let for $t \in \mathbb{R}$

$$f(t) = \sum_{k=1}^{\infty} (\tfrac{3}{4})^k \underbrace{\varphi(4^k t)}_{\in[0,1]}$$

However, note that each $\varphi(4^k) \in C_b(\mathbb{R})$, $\left\|\varphi(4^k)\right\|_\infty = 1$, so by the M-test, $f \in C_b(\mathbb{R})$. Fix $t \in \mathbb{R}$. We show that f cannot be differentiable at t. Let $\ell_m = \lfloor 4^m t \rfloor$ ($m \in \mathbb{N}$) so

$$\begin{aligned} &\ell_m \le 4^m t < \ell_m + 1 \\ &\implies p_m = \tfrac{\ell_m}{4^m} \le t < \tfrac{\ell_m+1}{4^m} = q_m \end{aligned}$$

We compute

$$\begin{aligned}
& |f(p_m) - f(q_m)| \\
&= |\lim_{n\to\infty} \sum_{k=1}^{\infty} (\tfrac{3}{4})^k [\varphi(4^k p_m) - \varphi(4^k q_m)] \\
&= |\lim_{n\to\infty} \sum_{k=1}^{\infty} (\tfrac{3}{4})^k [\varphi(4^{k-m} \ell_m) - \varphi(4^{k-m}(\ell_m + 1))] \\
&= |\lim_{n\to\infty} \sum_{k=1}^{\infty} (\tfrac{3}{4})^k [\varphi(4^{k-m} \ell_m) - \varphi(4^{k-m}(\ell_m + 1))], \text{ by (ii) (2-periodicity)} \\
\text{(key step)} \quad &\geq \tfrac{3}{4}^m 1 - \sum_{k=1}^{m-1} \tfrac{3^k}{4^k} |\underbrace{\varphi(4^{k-m}\ell_m) - \varphi(4^{k-m}(\ell_m+1))}_{=4^{k-m}\text{, by (iv)}}| \\
&= \tfrac{3^k}{4^k} - \tfrac{1}{4^m} \sum_{k=1}^{m-1} 3^k \\
&= \tfrac{1}{4^m} [3^m - \sum_{k=1}^{m-1} 3^k] \\
&= \tfrac{1}{4^m} [\tfrac{2\cdot 3^m - 3^m + 1}{2}] \\
&= \tfrac{1}{4^m} (\tfrac{3^m + 1}{2})
\end{aligned}$$

Since $|p_m - q_m| = \frac{1}{4^m}$, we have

$$\frac{f(p_m) - f(q_m)}{p_m - q_m} \geq \frac{3^m + 1}{2}.$$

$$\left(p_m = \frac{\lfloor 4^m t \rfloor}{4^m} \right)$$

If $t = \frac{\ell}{4^{m_0}}$ $(\ell \in \mathbb{Z})$, then $t = p_m$ for $m \geq m_0$ and hence for $m \geq m_0$,

$$\left| \frac{f(t) - f(q_m)}{t - q_m} \right| \geq \frac{3^m + 1}{2}$$

while $\lim_{m\to\infty} q_m = t$, so $f'(t)$ does not exist.

$$\begin{aligned}
\frac{f(p_m) - f(q_m)}{p_m - q_m} &\leq \frac{|f(p_m) - f(t)| + |f(t) - f(q_m)|}{|p_m - q_m|} \\
&\leq \frac{|f(p_m) - f(t)|}{|p_m - t|} + \frac{|f(t) - f(q_m)|}{|t - q_m|}
\end{aligned}$$

Hence, for some $r_m \in \{p_m, q_m\}$, $\frac{|f(t) - f(r_m)|}{|t - r_m|} \geq \frac{3^m+1}{2\cdot 2}$.
We have $|\frac{f(t)-f(r_m)}{t - r_m}| \geq \frac{3^m+1}{4}$ while $r_m \to t$.

11 2017-10-20

Corollary 11.1. $(\ell_\infty, \|\cdot\|_\infty)$ is a Banach space.

Proof. $\ell_\infty = C_b(\mathbb{N})$ with usual $|\cdot|$ metric on $\mathbb{N}$. If $f : \mathbb{N} \to \mathbb{R}$ is bounded, $U \subseteq \mathbb{R}$ open, then $f^{-1}(U) \in \mathcal{P}(\mathbb{N})$ is open (all subsets of $\mathbb{N}$ are open) $\implies f$ is continuous.
If $(x_n)_{n=1}^\infty \in \ell_\infty$, define $f : \mathbb{N} \to \mathbb{R}$, $f(n) = x_n$, $f \in C_b(\mathbb{N})$, $\|f\|_\infty = \|(x_n)_{n=1}^\infty\|_\infty$. $\square$

Eg. $(C[0,2], \|\cdot\|_p)$, $\|f\|_p = (\int_0^2 |f|^p)^{\frac{1}{p}}$, $1 \le p < \infty$.
NOT a Banach space!
Let

$$f_n(t) = \begin{cases} 1 & 0 \le t \le \frac{1}{2} \\ n(\frac{1}{2} + \frac{1}{n} - t) & \frac{1}{2} < t \le \frac{1}{2} + \frac{1}{n} \\ 0 & \frac{1}{2} + \frac{1}{n} < t \end{cases}.$$

Then for $m < n \in \mathbb{N}$,

$$\begin{aligned} \|f_n - f_m\|_p &= (\int_0^2 |f_n - f_m|^p)^{\frac{1}{p}} \\ &= \Bigg(\underbrace{\int_0^{\frac{1}{2}} |f_n - f_m|^p}_{0} + \underbrace{\int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{m}} \overbrace{|f_n - f_m|}^{\le 1}}_{\le \frac{1}{m}} + \underbrace{\int_{\frac{1}{2}+\frac{1}{m}}^{2} |f_n - f_m|^p}_{0} \Bigg)^{\frac{1}{p}} \\ &\le \frac{1}{m^{\frac{1}{p}}}. \end{aligned}$$

Hence $(f_n)_{n=1}^\infty$ is Cauchy in $(C[0,2], \|\cdot\|_p)$.
Consider

$$\chi_{[0,\frac{1}{2}]}(t) = \begin{cases} 1 & 0 \le t \le \frac{1}{2} \\ 0 & \frac{1}{2} < t \end{cases}.$$

$\chi_{[0,\frac{1}{2}]}$ is bounded, piecewise continuous, so Riemann integrable.

$$\begin{aligned} &\left\| f_n - \chi_{[0,\frac{1}{2}]} \right\|_p = (\int_0^2 |f_n - \chi_{[0,\frac{1}{2}]}|^p)^{\frac{1}{p}} \le \frac{1}{n^{\frac{1}{p}}} \\ &\implies \lim_{n\to\infty} \left\| f_n - \chi_{[0,\frac{1}{2}]} \right\|_p = 0. \end{aligned}$$

If $g \in C[0,1]$ s.t. $\lim_{n\to\infty} \|f_n - g\|_p$, then $\left\| g - \chi_{[0,\frac{1}{2}]} \right\|_p = 0$.
Using Riemann integration theory,

$$g(t) = \begin{cases} 1 & 0 \le t \le \frac{1}{2} \\ 0 & \frac{1}{2} < t \end{cases}.$$

Then $\lim_{t\to\frac{1}{2}} g$ does not exist!

11.1 Completeness of Metric Spaces

(X, d) metric space.
Remark: $|d(x,z) - d(y,z)| \le d(x,y)$.
If $x = \lim_{n\to\infty} x_n$, $y = \lim_{n\to\infty} y_n$ in (X,d), then $\lim_{n\to\infty} d(x_n, y_n) = d(x,y)$. (See solution to A3Q2).
Def: $(X, d_X), (Y, d_Y)$ metric spaces. $i : X \to Y$ is an isometry if $d_Y(i(x), i(y)) = d_X(x,y) \forall x, y \in X$.
Notes: An isometry is injective. Consider $i : X \to i(X) \subseteq Y \implies i^{-1} : i(X) \to X$ isometry.

Theorem 11.1. (X, d) metric space.

i) Existence of completion: there exists a metric space $(\overline{X}, \overline{d})$ s.t.
   a) $(\overline{X}, \overline{d})$ is complete
   b) $\exists$ isometry $\overline{i} : X \to \overline{X}$
   c) $\overline{i(X)} = \overline{X}$; i.e. $i(X)$ is dense in $\overline{X}$

ii) Uniqueness up to isometry: if $(\widetilde{X},\widetilde{d})$ is a metric space with map $\widetilde{i}:X\to\widetilde{X}$ s.t. $(\widetilde{X},\widetilde{d}),\widetilde{i}$ satisfy (a),(b),(c), then $\exists$ a surjective isometry $\varphi:\widetilde{X}\to\overline{X}$ s.t. $\varphi\circ\widetilde{i}=\overline{i}$.

Proof. 1. Fix $x_0\in X$. For $u\in X$, let $f_u:X\to\mathbb{R}$, $f_u(x)=d(x,u)-d(x,x_0)$
$\implies f_u$ is continuous and $|f_u(x)|\le d(u,x_0)$
$\implies \|f_u\|_\infty=\sup_{x\in X}|f_n(x)|\le d(u,x_0)<\infty \implies f_u$ is bounded
$\implies f_u\in C_b(X)$.
For $u,v\in X, x\in X$,

$$|f_u(x)-f_v(x)|=|d(x,u)-d(x,v)|\le d(u,v).$$

Thus $\|f_u-f_v\|_\infty\le d(u,v)$. Finally,

$$\begin{aligned}|f_u(u)-f_v(u)|&=|d(u,u)-d(u,x_0)-d(u,v)+d(u,x_0)|\\&=d(u,v).\end{aligned}$$

Thus $\|f_u-f_v\|_\infty\ge d(u,v)\implies\|f_u-f_v\|_\infty=d(u,v)$.
Define $\tau:X\to C_b(X),\tau(u)=f_u,\tau$ isometry.
Let $\overline{X}=\overline{\tau(X)}=\overline{\{f_u:u\in X\}}\subseteq C_b(X)$.
By A3Q2(a), $(\overline{X},\overline{d})$ is complete, where $\overline{d}$ is relativized from the metric on $C_b(X)$.

2. Let $\varphi_0=\tau\circ\tau^{-1}:\tau(X)\to\tau(X)$. φ_0 an isometry $\implies$ uniformly continuous. Hence it admits an extension $\varphi=\overline{\varphi_0}:\widetilde{X}=\overline{i(X)}\to\overline{X}=\overline{\tau(X)}$.
Verify φ is an isometry:
If $\widetilde{x},\widetilde{y}\in\widetilde{X}$, let $\widetilde{x}=\lim_{n\to\infty}\tau(x_n),\widetilde{y}=\lim_{n\to\infty}\tau(y_n)$, $x_n,y_n\in X$. Then

$$\varphi(\widetilde{x})=\lim_{n\to\infty}\varphi_0(\tau(x_n))=\lim_{n\to\infty}\tau(x_n).$$

Hence

$$\begin{aligned}\overline{d}(\varphi(\widetilde{x}),\varphi(\widetilde{y}))&=\lim_{n\to\infty}\overline{d}(\tau(x_n),\tau(y_n))\\&=\lim_{n\to\infty}d(x_n,y_n)\\&=\lim_{n\to\infty}\widetilde{d}(\tau(x_n),\tau(y_n))=\widetilde{d}(\widetilde{x},\widetilde{y}).\end{aligned}$$

$\implies\varphi$ is an isometry. $\varphi\circ\tau=\tau$ comes for free.

□

12 2017-10-23

Assignment discussion – the completion vs A4,Q1:

Suppose $(V,\|\cdot\|)$ is a non-complete normed vector space, eg. $(C[0,2],\|\cdot\|_p)$ $(1\le p<\infty)$. Consider the map

$$\tau:V\to C_b(V)$$

$$\tau(v)\in C_b(V),\ \tau(v)(x)=\|x-y\|-\|x\|$$

We saw that τ is an isometry, hence we let

$$\overline{V}=\overline{\overline{\tau(V)}}_{\text{complete}}\subseteq C_b(V)$$

Problem: τ is not linear, $\overline{\tau(V)}$ not evidently a subspace of $C_b(V)$.
A4, Q1 shows that an addition and a scalar multiplication may be imposed on $\overline{V}=\overline{\tau(V)}$ which makes it a Banach (complete normed vector) space. These two operations are not the same as addition and scalar multiplication in $C_b(V)$. (The only linear property that τ enjoys seems to be that it takes 0 to 0.)

12.1 Compactness

Let (X,d) be a metric space, and $K \subseteq X$. We say that K is compact if given a family of open sets $\{U_i\}_{i\in I}$ for which

$$K \subseteq \bigcup_{i\in I} U_i \text{ – we say } \{U_i\}_{i\in I} \text{ is an “open cover”}$$

there is a finite subfamily $\{U_{i_1},\dots,U_{i_n}\}$ such that

$$K \subseteq \bigcup_{k=1}^{n} U_{i_k} \text{ – we say } \{U_i\}_{i\in I} \text{ admits a “finite subcover” .}$$

If $X = K$ itself is compact, we will call (X,d) a compact metric space.

Remark: If $K \subseteq X$ is compact, the relativized metric space (K, d_K) is a compact metric space.

Proposition 12.1. Let (X,d) be a metric space and $K \subseteq X$. If K is compact, then it must be closed.

Proof. Let us suppose, for sake of contradiction that there is $x \in \overline{K}\setminus K$. Then for n in $\mathbb{N}$,

$$B(x,\tfrac{1}{n})\cap K \neq \varnothing \Longrightarrow B[x,\tfrac{1}{n}]\cap K \neq \varnothing. \qquad (\star)$$

Further, $\cap_{n=1}^{\infty} B[x,\frac{1}{n}] = \{x\}$. Let $U_n = X \setminus B[x,\frac{1}{n}]$, which is open.
We have that

$$\bigcup_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty}(X\setminus B[x,\tfrac{1}{n}]) = X \setminus \bigcap_{n=1}^{\infty} B[x,\tfrac{1}{n}] = X\setminus\{x\} \supseteq K.$$

But, for any finite m we have

$$\bigcup_{n=1}^{m} U_n = X \setminus \bigcap_{n=1}^{m} B[x,\tfrac{1}{n}] = X\setminus B[x,\tfrac{1}{m}] \not\supseteq K$$

by $(\star)$. Hence if $\overline{K}\setminus K \neq \varnothing$, K cannot be compact. So we are done. □

Proposition 12.2. Let (X,d) be a compact metric space and $C \subseteq X$ is closed. Then C is compact.

Proof. Suppose $\{U_i\}_{i\in I}$ is an open cover of C. Then $\{U_i\}_{i\in I}\cup\{X\setminus C\}$ is an open cover of X. Hence X admits finite subcover $\{U_{i_1},\dots,U_{i_n}\}\cup\{X\setminus C\}$, hence, $\{U_{i_1},\dots,U_{i_n}\}$ is a finite subcover of C. □

Theorem 12.1 (continuous image of compact is compact)**.** Let (X,d_X) be a compact metric space, (Y,d_Y) be a metric space, and $f : X \to Y$ be continuous. Then $f(X) = \{f(x) : x \in X\}$ is compact.

Proof. Let $\{V_i\}_{i\in I}$ be an open cover of $f(X)$. Then $U_i = f^{-1}(V_i)$ is open, and $\{U_i\}_{i\in I}$ is an open cover of X. Hence there is a finite subcover, $X \underbrace{\subseteq}_{“=”} \bigcup_{k=1}^{n} U_{i_k}$ so $f(X) \subseteq \bigcup_{k=1}^{n} f(U_{i_k}) = \bigcup_{k=1}^{n} V_{i_k}$, so $\{V_{i_1},\dots,V_{i_n}\}$ is a finite subcover of $f(X)$. □

Corollary 12.1 (Extreme Value Theorem)**.** If (X,d) is a compact metric space, $f : X \to \mathbb{R}$ is continuous, then there are $x_{\min}, x_{\max} \in X$ for which

$$f(x_{\min}) \le f(x) \le f(x_{\max}) \ \forall x \in X.$$

Proof. We have $f(X) \subseteq \mathbb{R}$ is compact. Hence $f(X)$ is closed. Also $\{(-n,n)\}_{n=1}^{\infty}$ (open intervals), then $f(X) \subseteq \mathbb{R} = \bigcup_{n=1}^{\infty}(-n,n)$ admits a finite subcover, $\{(-1,1),\dots,(-n,n)\}$ and hence $f(X)\subseteq(-n,n)$. Thus we have $\inf(f(X)), \sup(f(X))$ exist.
Since $f(X)$ is closed we have

$$\inf(f(X)), \sup(f(X)) \in f(X)$$

(use meet-set of closure). Let $x_{\min}, x_{\max}$ be so $f(x_{\min}) = \inf(f(X)), f(x_{\max}) = \sup(f(X))$. □

– Assignment line –

Theorem 12.2 (finite intersection property)**.** Let (X,d) be a metric space. Then (X,d) is compact $\iff$ for any family $\{F_i\}_{i\in I}$ of closed subsets of X for which $\bigcap_{k=1}^n F_{i_k} \neq \varnothing$, $\{i_1,\dots,i_n\}$ finite in I, we must have $\bigcap_{i\in I} F_i \neq \varnothing$.

Proof. ($\Longrightarrow$) (contrapositive) Let us suppose that $\{F_i\}_{i\in I}$ is a family of closed subsets with $\bigcap_{i\in I} F_i = \varnothing$. Then if $U_i = X\setminus F_i$, we have that $\{U_i\}_{i\in I}$ is an open cover (De Morgan's law) and hence admits finite subcover $\{U_{i_1},\dots,U_{i_n}\}$. Again, by DeMorgan's law, $\bigcap_{k=1}^n F_{i_k} = \varnothing$. Hence we are done.
($\Longleftarrow$) Very similar, interchange roles of U_is and $F_i = X\setminus U_i$. □

<u>Example:</u> Let $X = B[0,1]$ in ℓ_p $(1\le p\le\infty)$.
Let $e_n = (0,\dots,0,\underbrace{1}_{n\text{-th place}},0,\dots)$ and let $F_n = \{e_k\}_{k\ge n}$ (seen before on Oct 18).
Each F_n is closed. Also

$$\bigcap_{n=1}^{\infty} F_n = \varnothing$$
$$\bigcap_{n=1}^{m} F_n = F_m \neq \varnothing$$

Conclusion: $(B[0,1],d_p)$ $(d_p(x,y)=\|x-y\|_p)$ is <u>not</u> compact.

13 2017-10-25

<u>Def:</u> Let (X,d) be a metric space. Then we say it is

- <u>bounded</u> if there are x_0 in X, and $R>0$ such that $X\subseteq B[x_0,R]$ (of course "=" holds) (equivalently, for any $x\in X$, there is $R_x>0$ such that $X\subseteq B[x,R_x]$; or, equivalently, $\operatorname{diam}(X)<\infty$)
- <u>totally bounded</u> if, for any $\varepsilon>0$, there are $x_1,\dots,x_n\in X$ such that $X\subseteq\bigcup_{k=1}^n B[x_k,\varepsilon]$

Totally bounded $\Longrightarrow$ bounded. [with $\varepsilon>0,\ x_1,\dots,x_n$ in defn, check that $\bigcup_{k=1}^n B[x_k,\varepsilon]\subseteq B[x_1,\varepsilon+\max_{k=2,\dots,n} d(x_1,x_k)]$]

<u>Example:</u> (bounded $\not\Longrightarrow$ totally bounded)
In ℓ_p $(1\le p\le\infty)$, $e_n = (0,\dots,0,\underbrace{1}_{n\text{-th place}},0,\dots)$, $F_n=\{e_k\}_{k\ge n}\subseteq\ell_p$,

F_n int, $F_n\subseteq B[0,1]\subseteq B[e,2]$ so F_n is bounded. But $n\neq m$, $d(e_n,e_m)=\begin{cases}2^{\frac{1}{p}} & 1\le p<\infty\\ 1 & \text{otherwise}\end{cases} =: R.$
If $0<\varepsilon<\frac{1}{2}R$, we see that $F_n\not\subseteq\bigcup_{k=1}^n B[e_k,\varepsilon]$ for any n.

Theorem 13.1 (Characterizations of compact metric spaces)**.** Let (X,d) be a metric space. TFAE:

(i) (X,d) is compact,

(ii) any sequence $(x_n)_{n=1}^\infty\subseteq X$ admits a subsequence which converges in X

(iii) (X,d) is complete and totally bounded

Proof. (i) $\Longrightarrow$ (ii): Let $F_n=\overline{\{x_k\}_{k=n}^\infty}$. Then each F_n is closed, and $F_1\supseteq F_2\supseteq\cdots$, so if $n_1<n_2<\cdots n_m$, then $\bigcap_{j=1}^m F_n = F_{n_m}\neq\varnothing$. Thus, by finite intersection property, we have that $\bigcap_{n=1}^\infty F_n\neq\varnothing$. Let $x\in\bigcap_{n=1}^\infty F_n$.
Now let
$$n_1=\min\{k: x_k\in B(x,1)\}\text{ (exists by meet-set closure definition)}$$
and, inductively,
$$n_{m+1}=\min\{k: k>n_m \text{ and } x_k\in B(x,\tfrac{1}{m+1})\}.$$
Then, as is easy to check, $\lim_{m\to\infty} x_{n_m}=x$.
(ii) $\Longrightarrow$ (iii): If $(x_n)_{n=1}^\infty\subseteq X$ is Cauchy, it admits a converging subsequence (by assumption), and hence itself converges

(earlier proposition). Thus (X, d) is complete.
Let us suppose that (X, d) is <u>not</u> totally bounded.
Thus, there exists $\varepsilon > 0$ so no finite collection of closed ε-balls covers X. Let

$$x_1 \in X \setminus B[x_1, \varepsilon], \ldots, x_{n+1} \in X \setminus \bigcup_{k=1}^{n} B[x_k, \varepsilon] \text{ (always possible by assumption).}$$

Thus $d(x_n, x_m) > \varepsilon$ for $n \neq m$. Thus, this sequence $(x_n)_{n=1}^{\infty}$ admits no Cauchy subsequences, hence no subsequences which converge, violating assumption (ii). Thus (ii) $\implies (X, d)$ is totally bounded.
(iii) $\implies$ (ii): We first use total boundedness. Given n in $\mathbb{N}$, there exist $y_{n1}, \ldots, y_{nm_n} \in X$ such that the closed balls

$$B_{n1} = B[y_{n1}, \tfrac{1}{n}], \ldots, B_{nm_n} = B[y_{nm_n}, \tfrac{1}{n}]$$

satisfy that $X \subseteq \bigcup_{k=1}^{m_n} B_{nk}$. Let

- B_1 be a ball from $B_{11}, \ldots, B_{1m_1}$ such that

$$|\{n \in \mathbb{N} : x_n \in B_1\}| = \aleph_0 \text{ (pigeonhole principle)}$$

- $\vdots$

- B_k be a ball from $B_{11}, \ldots, B_{1m_1}$ such that

$$|\{n \in \mathbb{N} : x_n \in \bigcap_{j=1}^{k} B_j\}| = \aleph_0$$

$$\text{(we've covered } X \text{ by 1-balls, } B_1 \text{ by } \frac{1}{2}\text{-balls, then } B_2 \cap B_1 \text{ covered by } \frac{1}{3}\text{-balls, } \ldots)$$

Now we use completeness. Let $F_n = \bigcap_{k=1}^{n} B_k$ so each F_n is closed.

- $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$
- $\operatorname{diam}(F_n) \leq \operatorname{diam}(B_n) = \frac{2}{n} \xrightarrow{n \to \infty} 0$

Thus, by nested sets theorem, $\bigcap_{n=1}^{\infty} F_n \neq \varnothing$.
Let $n_1 = \min\{k \in \mathbb{N} : x_k \in F_1\}$, inductively, $n_{m+1} = \min\{k \in \mathbb{N} : k > n_m \text{ and } x_k \in F_k\}$.
Then, if $x \in \bigcap_{n=1}^{\infty} F_n$, $d(x, x_m) \leq \operatorname{diam}(F_m) \leq \operatorname{diam}(B_m) = \frac{2}{m} \xrightarrow{n \to \infty} 0$ so $x = \lim_{n \to \infty} x_{n_m}$. □

14 2017-10-27

<u>Office hours:</u>
Mon 2:30 – 4:30
Tue 2 – 3:30

Proof. Continuing theorem from last time:
So far we did (i) $\underbrace{\implies}_{\text{F.I.P}}$ (ii) $\underbrace{\implies}_{\text{routine}}$ (iii) $\underbrace{\implies}_{\text{harder, nested sets thm}}$ (ii)
(ii) $\implies$ (i): Let $\{U_i\}_{i \in I}$ be an open cover of X.
(LN) There exists $r > 0$ s.t. for any x in X there exists i in I so $B(x, r) \subseteq U_i$.
(This number r is sometimes called the "Lebesgue number" of the covering; its existence is based on (ii).)

Suppose (LN) fails. Then for choice of $r = \frac{1}{n}$, there exists x_n in X s.t. $B(x, \frac{1}{n}) \not\subseteq U_i$ for all i in I.
Our assumption is that $(x_n)_{n=1}^{\infty} \subseteq X$ admits a subsequence $(x_{n_k})_{k=1}^{\infty}$ such that $x_0 = \lim_{k \to \infty} x_{n_k}$ exists.

Then $x_0 \in U_{i_0}$ for some i_0, so there is $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U_{i_0}$. Now, there is k_ε in $\mathbb{N}$ so $k \geq k_\varepsilon \implies x_{n_k} \in B(x_0, \frac{\varepsilon}{2})$. Hence, let us choose $k \geq k_\varepsilon$ and $\frac{1}{n_k} < \frac{\varepsilon}{2}$. Thus, if $x \in B(x_{n_k}, \frac{1}{n_k})$, we have

$$d(x, x_0) \leq d(x, x_{n_k}) + d(x_{n_k}, x_0) < \frac{1}{n_k} + \frac{\varepsilon}{2} < \varepsilon$$

and hence $B(x_{n_k}, \frac{1}{n_k}) \subseteq B(x_0, \varepsilon) \subseteq U_{i_0}$, contradiction the choice of the elements x_n.
Hence, we must conclude that (LN) holds.
We saw in (ii) $\implies$ (iii) above, that our assumption gives total boundedness of (X, d). Hence there are $y_1, \ldots, y_m$ such that $X \subseteq \bigcup_{j=1}^m B[y_j, \frac{r}{2}] \subseteq \bigcup_{j=1}^m B(y_j, r)$. Now, for each $j = 1, \ldots, m$, (LN) tells us that there is $i_j \in I$ so $B(y_j, r) \subseteq U_{i_j}$.
Thus $X \subseteq \bigcup_{i=1}^n B(y_j, r) \subseteq U_{i_j}$, so $\{U_{i_1}, \ldots, U_{i_m}\}$ is a finite subcover.

Remark: On $\mathbb{R}^n$, norms $\|\cdot\|_p$ $(1 \leq p \leq \infty)$ are equivalent, and from A2, each gives the same open sets, and hence the same compact sets.

Corollary 14.1.

(i) (Bolzano-Weierstrauss Theorem for $\mathbb{R}^n$)
If $(x^{(n)})_{n=1}^\infty \subseteq [-R, R]^n = B_\infty[0, R]$, then it admits a converging subsequence.

(ii) (Heine-Borel Theorem)
A subset $K \subseteq \mathbb{R}^n$ is compact $\iff$ K is closed & K is bounded (with respect to any $\|\cdot\|_\infty$).

Proof. (i) We consider, first $(x_1^{(n)})_{n=1}^\infty \subseteq [-R, R] \subseteq \mathbb{R}$. By Bolzano-Weierstrauss for $\mathbb{R}$, this admits converging subsequence $(x_1^{(n_k)})_{n=1}^\infty$. Then $(x_2^{(n)})_{n=1}^\infty \subseteq [-R, R] \subseteq \mathbb{R}$ admits a converging subsequence $(x_2^{(n_k)})_{n=1}^\infty$. Etc. Hence, after finitely many (n) iterations, we get a subsequence of $(x^{(n)})_{n=1}^\infty$ which converges $(\mathbb{R}^n, \|\cdot\|_\infty)$.

(ii) If K is compact, then K is closed by a result at the beginning of the section, and totally bounded by last theorem, hence bounded. Conversely, if K is closed and bounded, $K \subseteq [-R, R]^n$ for some $R > 0$. Let us consider a sequence $(x^{(n)})_{n=1}^\infty \subseteq K$. First, $(x^{(n)})_{n=1}^\infty$ admits a converging subsequence, by (i). Since K is closed, the limit of the subsequence is in K.

□

Example: $P = \prod_{k=1}^\infty \{0, \frac{1}{2^k}\} \subseteq \ell_1$ is compact in $(\ell_1, \|\cdot\|_1)$.
First soln: The Cantor set C is closed and bounded in $\mathbb{R}$, so thus compact. And there is a continuous function $f : C \to \ell_1$ with $f(C) = P$ (A4,Q3), so P is compact. [In fact f is a bijection from C to P so $f^{-1} : P \to C$ is also continuous.]
Second soln: P is closed (A3). Hence the relativised metric space (P, d_P) is complete. Let us show total boundedness.
Let $\varepsilon > 0$, and n be so $\frac{1}{2^n} < \varepsilon$. For $(b_1, \ldots, b_m) \in \{0, 1\}^n$, let $x_{b_1 \ldots b_m} = \sum_{k=1}^\infty \frac{b_k}{2^k} e_k \in P$. If $b = (b_1, b_2, \ldots) \in \{0, 1\}^{\mathbb{N}}$, then $x_b = \sum_{k=1}^\infty \frac{b_k}{2^k} e_k \in P$ (generic element of P).
Then for $b = (b_1, b_2, \ldots)$ as above,

$$\|x_b - x_{b_1 \ldots b_n}\|_1 = \sum_{k=n+1}^\infty \frac{1}{2^k} b_k \leq \sum_{k=n+1}^\infty \frac{1}{2^k} = \frac{1}{2^n} \leq \varepsilon.$$

Thus, $P \subseteq \bigcup_{(b_1, \ldots, b_n) \in \{0,1\}^n} B[x_{b_1 \ldots b_n}, \varepsilon]$. □

– MIDTERM CUTOFF –

15 2017-10-30

Midterm: Wed evening
See info sheet on website

Office hours:
– 2:30 - 4:30
– 1:30 - 3:30

A5 - will be posted Friday

Theorem 15.1 (sequential characterization of uniform continuity)**.** Let (X, d_X) and (Y, d_Y) be metric spaces, $f : X \to Y$. Then

$$f \text{ is uniformly continuous} \iff \text{whenever } d_X(x_n, y_n) \xrightarrow{n\to\infty} 0,\ x_n, y_n \in X,$$
$$\text{we must have } d_Y(f(x_n), f(y_n)) \xrightarrow{n\to\infty} 0.$$

Proof. ($\Longrightarrow$) Given $\varepsilon > 0$, there is $\delta > 0$ such that $d_X(x,y) < \delta$ (x, y in X) $\Longrightarrow d_Y(f(x), f(y)) < \varepsilon$. Now suppose $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty \subseteq X$ such that $\lim_{n\to\infty} d_X(x_n, y_n) = 0$. Then there is n_ε in $\mathbb{N}$ such that

$$\begin{aligned} n \geq n_\varepsilon &\Longrightarrow d_X(x_n, y_n) < \delta \\ &\Longrightarrow d_Y(f(x_n), f(y_n)) < \varepsilon. \end{aligned}$$

I.e. $\lim_{n\to\infty} d_Y(f(x_n), f(y_n)) = 0$.
($\Longleftarrow$) (contrapositive) Suppose f is not uniformly continuous, so there exists $\varepsilon > 0$ such that for all $\delta > 0$ there are x, y in X with $d_X(x, y) < \delta$ but $d_Y(f(x), f(y)) \geq \varepsilon$. For each choice $\delta = \frac{1}{n}$, let x_n, y_n in X so $d_X(x_n, y_n) < \frac{1}{n}$ for which $d_Y(f(x_n), f(y_n)) \geq \varepsilon$.
Plainly, $\lim_{n\to\infty} d_X(x_n, y_n) = 0$ while $\lim_{n\to\infty} d_Y(f(x_n), f(y_n)) \neq 0$ (if the limit exists).

Ex: Let $f(x) = x^2$ on $\mathbb{R}$. Let $x_n = n,\ y_n = n + \frac{1}{n}$. Then $|x_n - y_n| = \frac{1}{n} \xrightarrow{n\to\infty} 0$, while $|f(x_n) - f(y_n)| = 2 + \frac{1}{n^2} \not\xrightarrow{n\to\infty} 0$.
Hence f is not uniformly continuous. □

Theorem 15.2 (continuous on compact is uniformly continuous)**.** Let (X, d_X), (Y, d_Y) be metric spaces, with (X, d_X) compact, and $f : X \to Y$ continuous. Then f is uniformly continuous.

Proof. Let us suppose not. Then there is $\varepsilon > 0$ and $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty \subseteq X$ such that $d_X(x_n, y_n) \xrightarrow{n\to\infty} 0$ while $d_Y(f(x_n), f(y_n)) \geq \varepsilon$. Let $(x_{n_k})_{k=1}^\infty$ be a converging subsequence. Then let $(y_{n_k})_{k=1}^\infty$ be a sequence in X, hence admits converging subsequence $(y_{n_{k_\ell}})_{\ell=1}^\infty$. Then if $x = \lim_{k\to\infty} x_{n_k} = \lim_{\ell\to\infty} x_{n_{k_\ell}}$ then

$$\begin{aligned} d_X(x, y_{n_{k_\ell}}) &\leq d_X(x, x_{n_{k_\ell}}) + d_X(x_{n_{k_\ell}}, y_{n_{k_\ell}}) \\ &\xrightarrow{\ell\to\infty} 0 \end{aligned}$$

so $x = \lim_{\ell\to\infty} y_{n_{k_\ell}}$. Then we have $f(x) = \lim_{\ell\to\infty} f(y_{n_{k_\ell}})$, by continuity, so

$$0 = d_Y(f(x), f(x)) = \lim_{\ell\to\infty} d_Y(f(x_{n_{k_\ell}}), f(y_{n_{k_\ell}}))$$

contradicts ($\star$). Thus, we conclude that f is uniformly continuous. □

Definition: A map $f : X \to Y$ $((X, d_X), (Y, d_Y))$ is called Lipschitz if there is $L \geq 0$ such that

$$d_Y(f(x), f(y)) \leq L d_X(x, y) \text{ for all } x, y \in X.$$

Notice that

$$\sup_{x,y\in X,\ x\neq y} \frac{d_Y(f(x), f(y))}{d_X(x, y)} = \inf\{L \geq 0 : \text{ (Lip) is satisfied }\}$$

so there exists a minimum L satisfying (Lip). We call this the "Lipschitz constant".

Remark: Lipschitz $\xRightarrow{\text{exercise}}$ uniform continuity $\Longrightarrow$ continuity
Lipschitz $\overset{\text{assignment}}{\not\Longleftarrow}$ uniform continuity $\not\Longleftarrow$ continuity

Theorem 15.3. Any two norms on $\mathbb{R}^n$ are equivalent, i.e. if $\|\cdot\|, ⦀\cdot⦀$ on $\mathbb{R}^n$ satisfy $\|\cdot\| \approx ⦀\cdot⦀$, i.e., there are $m, M > 0$ for which $m\|x\| \leq ⦀x⦀ \leq M\|x\|$.

Proof. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. We will see that $\|\cdot\| \approx \|\cdot\|_1$ ($\|x\|_1 = \sum_{j=1}^n |x_j|$). Since $\approx$ is an equivalence relation, we get $\|\cdot\| \approx \|\cdot\|_1$ so $\|\cdot\| \approx \|\cdot\|$.
Let $\{e_1, \dots, e_n\}$ be the standard basis, so if $x \in \mathbb{R}^n$, $x = \sum_{j=1}^n x_j e_j$. Then

$$\|x\| = \left\| \sum_{j=1}^n x_j e_j \right\| \underbrace{\leq}_{\text{properties of norm}} \sum_{j=1}^n |x_j| \|e_j\| \leq M\|x\|_1 \text{ where } M = \max_{j=1,\dots,n} \|e_j\|.$$

Notice, then, for x, y in $\mathbb{R}^n$ we have

$$|\|x\| - \|y\|| \underbrace{\leq}_{\text{standard} \leq \text{ (shown before completeness of } C_b(X))} \|x - y\| \leq M\|x - y\|_1$$

so $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz with respect to $d_1(x, y) = \|x - y\|_1$ and thus continuous.
Let $S_1 = \{x \in \mathbb{R}^n : \|x\|_1 = 1\} = B_1[0, 1] \setminus \underbrace{B_1(0, 1)}_{\subseteq B_1[0,1]}$ so S_1 is closed in $B_1[0, 1]$. Hence by Heine-Borel Theorem, it is compact.
Hence, by Extreme Value Theorem, there is $x_{\min}$ in S_1 such that

$$\|x_{\min}\| = \inf\{\|x\| : x \in S_1\}.$$

Let $m = \|x_{\min}\| > 0$ (as $x_{\min} \neq 0$, since $\|x_{\min}\|_1 = 1 \neq 0$).
Now, if $x \in \mathbb{R}^n \setminus \{0\}$, then

$$m \leq \left\| \underbrace{\frac{1}{\|x\|_1} x}_{\in S_1} \right\| \Longrightarrow m\|x\|_1 \leq \|x\| \qquad (\ddagger)$$

Then (†) and (‡) show that $\|\cdot\| \approx \|\cdot\|_1$. ☐

Corollary 15.1. If $\|\cdot\|$ is a norm on $\mathbb{R}^n$, $⦀\cdot⦀$ on $\mathbb{R}^m$ and $A : \mathbb{R}^n \to \mathbb{R}^m$ is linear. Then A is Lipschitz from $(\mathbb{R}^n, \|\cdot\|)$ to $(\mathbb{R}^m, ⦀\cdot⦀)$, and hence continuous.

Proof. Let $\{e_1, \dots, e_n\}$ be the standard basis of $\mathbb{R}^n$, $\{e_1, \dots, e_m\}$ be the standard basis of $\mathbb{R}^m$. Then there is a matrix $[a_{ij}]$ such that $Ae_j = \sum_{i=1}^n a_{ij} e_i$.
Then for $x = \sum_{j=1}^n x_j e_j$ in $\mathbb{R}^m$ we have

$$\begin{aligned} Ax &= \sum_{j=1}^n x_j A e_j \\ &= \sum_{j=1}^n x_j \sum_{i=1}^n a_{ij} e_j \\ &= \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} x_i \right) e_i \in \mathbb{R}^m \end{aligned}$$

so

$$\begin{aligned}
\|Ax\| &\leq \sum_{j=1}^{n} |\sum_{j=1}^{n} a_{ij}x_j| \|e_i\|, \qquad M = \max_{j=1,\ldots,n} \|e_i\| \\
&\leq M \sum_{j=1}^{n}\sum_{i=1}^{m} |a_{ij}||x_j|, \qquad \|A\|_\infty = \max_{i=1,\ldots,m,\ j=1,\ldots,n} |a_{ij}| \\
&= M \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}||x_j| \\
&\leq M \sum_{i=1}^{m} |A|_\infty |x|_1 \\
&= M\|x\|_1 \leq M
\end{aligned}$$

$\|x\|_1 \leq M\|x\|$ □

16 2017-11-01

Proposition 16.1. Let $(V, \|\cdot\|_V)$, $(W, \|\cdot\|_W)$ be normed linear spaces, $A : V \to W$ be linear. Then TFAE:

1. A is continuous
2. $\|A\| := \sup\{\|Ax\|_W : x \in \underbrace{B_V[0,1]}_{\text{closed ball, center 0 in } V}\} < \infty$
3. A is Lipschitz map with Lipschitz constant $\|A\|$

Moreover, in the case of (ii) (hence (iii)), above, $\|Ax\|_W \leq \|A\|\|x\|_V$ for any x in V.

Proof. (i) $\implies$ (ii) A is continuous at 0 in V. Thus, letting $\varepsilon = 1$, there is $\delta > 0$ s.t. $A(B_V(0,\delta)) \subseteq B_W(0,1)$. Now, if $x \in B_V[0,1]$, then $\frac{\delta}{2}x \in B_V(0,\delta)$, so

$$\|Ax\|_W = \frac{2}{\delta}\left\|\underbrace{A(\tfrac{\delta}{2}x)}_{\in B(0,1)}\right\|_W < \frac{2}{\delta}1 = \frac{2}{\delta} < \infty$$

so $\|A\| = \sup_{x\in B_V[0,1]}\|Ax\|_W \leq \frac{2}{\delta} < \infty$.
(ii) $\implies$ (iii) If $x \in V \setminus \{0\}$, so $\frac{1}{\|x\|_V}x \in B_V[0,1]$ and

$$(\star) \qquad \|Ax\|_W = \|x\|_V \underbrace{\left\|A\left(\frac{1}{\|x\|_V}x\right)\right\|_W}_{\leq \|A\|} \leq \|A\|\|x\|_V.$$

Clearly, $(\star)$ holds for $x = 0$ in V. Hence if $x, y \in V$,

$$\|Ax - Ay\|_W = \|A(x-y)\|_W \leq \|A\|\|x-y\|_V.$$

Thus A is Lipschitz and "Moreover..." holds. Furthermore, by $(\star)$,

$$\|A\| = \sup_{x\in V\setminus\{0\}} \frac{\|Ax\|_W}{\|x\|_V} = \sup_{x\neq y \text{ in } V} \frac{\|Ax-Ay\|_W}{\|x-y\|_V}$$

which is the definition of the Lipschitz constant.
(iii) $\implies$ (i) Obvious. □

Remark: Let $B(V,W) = \{A : V \to W \mid A \text{ is linear and continuous}\}$. Notice that (ii) above shows that A must be bounded on $B_V[0,1]$ and we call A a "bounded linear operator".
$B(V,W)$ is a $\mathbb{R}$-vector space (pointwise addition and scalar multiplication) and $\|\cdot\|$ is a norm on $B(V,W)$, called "bounded operator norm". (Exercise.)

Question: Is continuity automatic for linear operators?
Example: Consider the vector space $C[0,1]$ of continuous $\mathbb{R}$-valued functions on $[0,1]$. Let

$$\varphi : C[0,1] \to \mathbb{R},\ \varphi(f) = f(\tfrac{1}{2}) \text{ (evaluation at } \tfrac{1}{2}).$$

Then φ is linear: let $f, g \in C[0,1],\ \alpha \in \mathbb{R}$, then

$$\begin{aligned}\varphi(f+\alpha g) &= f(\tfrac{1}{2}) + \alpha g(\tfrac{1}{2}) \\ &= \varphi(f) + \alpha\varphi(g)\end{aligned}$$

(i) Consider $(C[0,1], \|\cdot\|_\infty)$. Then
$$|\varphi(f)| = |f(\tfrac{1}{2})| \leq \max_{t\in[0,1]} |f(t)| = \|f\|_\infty.$$
Thus $\|\varphi\| \leq 1$ (easy to show that $\|\varphi\| = 1$), i.e., $\varphi \in B((C[0,1], \|\cdot\|_\infty), \mathbb{R})$.

(ii) Now consider $(C[0,1], \|\cdot\|_p)$ $(1 \leq p < \infty)$. Let
$$f_n(t) = \begin{cases} 0 & \text{if } t \leq \frac{1}{2} - \frac{1}{n^{2p}} \\ n^{2p+1}(t - \frac{1}{2} + \frac{1}{n^{2p}}) & \text{if } \frac{1}{2} - \frac{1}{n^{2p}} < t \leq \frac{1}{2} \\ n^{2p+1}(\frac{1}{2} + \frac{1}{n^{2p}} - t) & \text{if } \frac{1}{2} < t \leq \frac{1}{2} + \frac{1}{n^{2p}} \\ 0 & t > \frac{1}{2} + \frac{1}{n^{2p}} \end{cases}$$
[triangular spike at $[\frac{1}{2} - \frac{1}{n^{2p}}, \frac{1}{2} + \frac{1}{n^{2p}}$ with peak at $\frac{1}{2}$ having value n.] Notice
$$\varphi(f_n) = f_n(\tfrac{1}{2}) = n$$
while
$$\begin{aligned}\|f_n\|_p &= \left(\int_0^1 f_n^p\right)^{\frac{1}{p}} \\ &= \left(\int_{\frac{1}{2}-\frac{1}{n^{2p}}}^{\frac{1}{2}+\frac{1}{n^{2p}}} \underbrace{f_n^p}_{0\leq f_n^p \leq n^p}\right)^{\frac{1}{p}} \\ &\leq \left(\int_{\frac{1}{2}-\frac{1}{n^{2p}}}^{\frac{1}{2}+\frac{1}{n^{2p}}} \underbrace{n^p}_{\text{constant}}\right)^{\frac{1}{p}} \\ &= \left(n^p \frac{2}{n^{2p}}\right)^{\frac{1}{p}} = \frac{2^{\frac{1}{p}}}{n}.\end{aligned}$$
Thus
$$\frac{|\varphi(f_n)|}{\|f_n\|_p} = \frac{n}{\frac{2^{\frac{1}{p}}}{n}} = \frac{n^2}{2^{\frac{1}{p}}} \xrightarrow{n\to\infty} \infty.$$
Hence
$$\varphi \notin B((C[0,1], \|\cdot\|_p), R).$$

Example: (Axiom of choice) If $(V, \|\cdot\|)$ is an infinite dimensional normed vector space, then it admits an infinite linearly independent family $\{v_n\}_{n=1}^\infty$. There exists a basis $\{w_i\}_{i\in I}$ s.t. $\{v_n\}_{n=1}^\infty \subseteq \{w_i\}_{i\in I}$.

Define $f : V \to \mathbb{R}$

$$f(w_i) = \begin{cases} \frac{n}{\|v_n\|} & \text{if } w_i = v_n \\ 0 & \text{otherwise} \end{cases}$$

and extend uniquely to a linear operator on V.
Check that $f \notin B(V, \mathbb{R})$.
Why isn't $B[0,1]$ in $(C[0,1], \|\cdot\|_\infty)$ compact?
Reason: existence of subsequence with no converging subsequence [similar holds on $(\ell_p, \|\cdot\|_p)$].

Picture: [triangle spike to height $f_n(t) = 1$ on $[\frac{1}{n+1}, \frac{1}{n}]$, 0 elsewhere.]
Calculate that if $m \neq n$, $\|f_n - f_m\|_\infty = 1$. Conclude that $(f_n)_{n=1}^\infty \subset B[0,1]$ admits no converging subsequence.

17 2017-11-03

Theorem 17.1 (Banach's Contraction Mapping Theorem)**.** Let (X, d) be a complete metric space and let $\Gamma : X \to X$ be a strict contraction, i.e., there is $0 < c < 1$ s.t. $d(\Gamma(x), \Gamma(y)) < cd(x, y)$ for x, y in X (Γ is c-Lipschitz). Then

(i) there is a unique fixed point x_{fix} for Γ, i.e. $\Gamma(x_{\text{fix}}) = x_{\text{fix}}$,

(ii) given any x_0 in X, if we define a sequence by $x_n = \Gamma(x_{n-1}), n \in \mathbb{N}$, then it satisfies

$$d(x_n, x_{\text{fix}}) \leq \frac{c^n}{1-c} d(x_0, \Gamma(x_0))$$

and hence $\lim_{n\to\infty} x_n = x_{\text{fix}}$.

Proof. Let $x_0 \in X$. We define $(x_n)_{n=1}^\infty \subseteq X$ as in (ii), above. We note that $d(x_1, x_2) = d(\Gamma(x_0), \Gamma(x_1)) \leq cd(x_0, x_1) = cd(x_0, \Gamma(x_0))$.
Now, if

$$(\star) \qquad d(x_n, x_{n+1}) \leq c^n d(x_0, \Gamma(x_0)),$$

then

$$d(x_{n+1}, x_{n+2}) = d(\Gamma(x_n), \Gamma(x_{n+1})) \leq cd(x_n, x_{n+1}) \leq c^{n+1} d(x_0, \Gamma(x_0))$$

so $(\star)$ holds generally. Thus, if $m < n$ in $\mathbb{N}$ we have

$$\begin{aligned} d(x_m, x_n) &\leq \sum_{j=m}^{n-1} d(x_j, x_{j+1}) \\ &\leq \sum_{j=m}^{n-1} c^j d(x_0, \Gamma(x_0)), \text{ by } (\star) \\ &\leq \sum_{j=m}^{\infty} c^j d(x_0, \Gamma(x_0)), \text{ by } (\star) = \frac{c^m}{1-c} d(x_0, \Gamma(x_0)). \end{aligned}$$

It follows that $(x_n)_{n=1}^\infty$ is Cauchy, and hence $x_{\text{fix}} = \lim_{n\to\infty} x_n$ exists. Then

$$x_{\text{fix}} = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \Gamma(x_n) \underbrace{=}_{\Gamma \text{ Lipschitz} \implies \text{continuous}} \Gamma(\lim_{n\to\infty} x_n) = \Gamma(x_{\text{fix}}).$$

Hence x_{fix} is a fixed point. If y_{fix} is any other fixed point then

$$\begin{aligned} d(x_{\text{fix}}, y_{\text{fix}}) &= d(\Gamma(x_{\text{fix}}), \Gamma(y_{\text{fix}})) \\ &\leq cd(x_{\text{fix}}, y_{\text{fix}}) \\ &< d(x_{\text{fix}}, y_{\text{fix}}), \text{ if } d(x_{\text{fix}}, y_{\text{fix}}) > 0 \end{aligned}$$

so we must have $d(x_{\text{fix}}, y_{\text{fix}}) = 0$, i.e. $x_{\text{fix}} = y_{\text{fix}}$. Thus (i) holds.
Also we have for m, n, as above,

$$d(x_m, x_n) \leq \frac{c^m}{1-c} d(x_0, \Gamma(x_0)) \Longrightarrow d(x_n, x_{\text{fix}}) = \lim_{n\to\infty} d(x_m, x_n) \leq \frac{c^m}{1-c} d(x_0, \Gamma(x_0))$$

so (ii) holds. □

Application: Some differentiable equations

Let $F : [a, b] \times \mathbb{R} \to \mathbb{R}$ be continuous, and $y_0 \in \mathbb{R}$. We consider the following initial value problem:
Want $f \in C[a, b]$, with $\underbrace{f(a) = y_0}_{\text{initial value}}$ and $\underbrace{f'(t) = F(t, f(t))}_{\text{differential equation}}$ (IVP).
We use the Fundamental Theorem of Calculus to convert this to an integral equation:
Want $f \in C[a, b], f(t) = y_0 + \int_a^t F(s, (f(s)))ds$ (IE).

Theorem 17.2 (Picard-Lindelof Theorem)**.** Let F, y_0 be as above and suppose that F is Lipschitz in the second variable: for all $t \in [a, b], y, z \in \mathbb{R}$,

$$|F(t, y) - F(t, z)| \leq L|y - z|, \text{ for some } L > 0.$$

Then (IVP) admits a unique solution, f_{sol} in $C[a, b]$.

Proof. (I) Let us assume that $(b - a)L < 1$. Define $\Gamma : C[a, b] \to C[a, b]$ by, for $t \in [a, b]$,

$$\Gamma(F)(t) = y_0 + \int_a^t F(s, f(s))ds.$$

Then for $f, g \in C[a, b]$, and $t \in [a, b]$, then

$$\begin{aligned}
|\Gamma(f)(t) - \Gamma(g)(t)| &= |\int_a^t [F(s, f(s)) - F(s, g(s))]ds| \\
&\leq \int_a^t \underbrace{|F(s, f(s)) - F(s, g(s))|}_{\leq L|f(s)-g(s)|} ds \\
&\leq L \int_a^t \underbrace{|f(s) - g(s)|}_{\leq \|f-g\|_\infty} ds \\
&\leq L\|f - g\|_\infty \int_a^t 1ds \\
&= L\|f - g\|_\infty (t - a) \leq (b - a)L\|f - g\|_\infty.
\end{aligned}$$

In summary,

$$\begin{aligned}
\|\Gamma(f) - \Gamma(g)\|_\infty &= \sup_{t\in[a,b]} \|\Gamma(f)(t) - \Gamma(g)(t)\| \\
&\leq \underbrace{(b - a)L}_{<1} \|f - g\|_\infty.
\end{aligned}$$

Hence, by the Contraction Mapping Theorem, applied to Γ on $(C[a, b], \|\cdot\|_\infty)$, there is a unique f_{sol} such that $\Gamma(f_{\text{sol}}) = f_{\text{sol}}$.
(II) Let

$$a = a_1 < a_2 < b_1 < b_3 < b_2 < \cdots < a_n < b_{n-1} < b_n = b$$

so that $(b_j - a_j)L < 1$ for $j = 1, \ldots, n$.
Notice that $[a_j, b_j] \cap [a_{j+1}, b_{j+1}] = [a_j, b_{j+1}]$ has non-empty interior.
Let $f_1 \in C[a_1, b_1]$ be the unique solution to (IVP) with $f_1(a) = y_0$, by (I).

Then, let f_2 in $C[a_2, b_2]$ satisfy (IVP) with $f_2(a_2) = f_1(a_2)$. Then, let f_3 in $C[a_3, b_3]$ satisfy (IVP) with $f_3(a_3) = f_2(a_3)$. Etc. Let $f : [a, b] \to \mathbb{R}$ be given by

$$f(t) = f_j(t) \text{ for } t \in [a_j, b_j], j = 1, \ldots, n.$$

Check that this is well-defined. Its value is uniquely determined on each $[a_{j+1}, b_j]$, thanks to uniqueness in (I). $\square$

18 2017-11-06

Example: (IVP) Want $f \in C[0, 1]$ s.t.

$$f(0) = 1, \qquad f'(t) = tf(t).$$

We convert to

$$\text{(IE) } f(t) = 1 + \int_0^t sf(s)ds.$$

This fits into Picard-Lindelof Theorem. Let $F(t, y) = ty$, so $f(t) = 1 + \int_0^t F(s, f(s))ds$ with $|F(t, y) - F(t, z)| = \underbrace{|t|}_{\le 1} |y - z| \le |y - z|$. (Case (II) of Picard-Lindelof.)
However, let $\Gamma : C[0, 1] \to C[0, 1]$ by, for $t \in [0, 1]$,

$$\Gamma(f)(t) = 1 + \int_0^t sf(s)ds.$$

Let us see that Γ, itself, is a strict contraction. Let $f, g \in C[0, 1], t \in [0, 1]$,

$$\begin{aligned}
|\Gamma(f)(t) - \Gamma(g)(t)| &\le \int_0^t s \underbrace{|f(s) - g(s)|}_{\le \|f-g\|_\infty} ds \\
&\le \int_0^t sds\|f - g\|_\infty \\
&= \underbrace{\frac{t^2}{2}}_{\le \frac{1}{2}} \|f - g\|_\infty \\
&\le \frac{1}{2}\|f - g\|_\infty.
\end{aligned}$$

$$(\|\Gamma(f) - \Gamma(g)\|_\infty \le \frac{1}{2}\|f - g\|_\infty)$$

Hence, contraction mapping theorem tells us that Γ has a unique fixed point, ie (IE) and (IVP) have a unique solution, f_{sol}. Furthermore, if we choose $f_0 \in C[0, 1]$ and let $f_n = \Gamma(f_{n-1})$ $(n \in \mathbb{N})$ then

$$\|f_{\text{sol}} - f_n\|_\infty \le \underbrace{\frac{(\frac{1}{2})^n}{1 - \frac{1}{2}}}_{= \frac{1}{2^{n-1}}} \|f_0 - \Gamma(f_0)\|_\infty.$$

We can compute f_{sol}.

Let $f_0(t) = 0$ (constant zero).

$$
\begin{aligned}
f_1(t) &= \Gamma(f_0)(t) = 1 + \int_0^t s0ds = 1 \\
f_2(t) &= \Gamma(f_1)(t) = 1 + \int_0^t s1ds = 1 + \frac{t^2}{2} \\
f_3(t) &= \Gamma(f_2)(t) = 1 + \int_0^t s(1 + \frac{t^2}{2})ds = 1 + \frac{t^2}{2} + \frac{t^4}{4 \cdot 2}
\end{aligned}
$$

(Use induction to check)

$$
f_n(t) = 1 + \frac{t^2}{2} + \frac{t^4}{4 \cdot 2} + \cdots + \frac{t^{2(n-1)}}{[2(n-1)][2(n-2)]\cdots 2} = \sum_{k=1}^{n} \frac{t^{2(k-1)}}{2^{k-1}(k-1)!}.
$$

Thus, at any t in $[0, 1]$,

$$
f_{\mathrm{sol}} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{t^{2(k-1)}}{2^{k-1}(k-1)!} = \sum_{k=1}^{\infty} \frac{t^{2(k-1)}}{2^{k-1}(k-1)!}.
$$

Furthermore, for each n,

$$
\begin{aligned}
\|f_{\mathrm{sol}} - f_n\|_\infty &= \max_{t\in[0,1)} |f_{\mathrm{sol}}(t) - f_n(t)| \\
&\le \frac{1}{2^{n-1}} \left\| 0 - \underbrace{\Gamma(0)}_{=1} \right\|_\infty = \frac{1}{2^{n-1}}.
\end{aligned}
$$

Question: Suppose we only knew that

$$
d(\Gamma(x), \Gamma(y)) < d(x, y) \text{ for } x \ne y \text{ in } X.
$$

("proper contraction" instead of "strict contraction")
Does Γ necessarily admit a fixed point?

Answer #1: No.
Example: On $X = [1, \infty) \subset R$, let $\Gamma(x) = x + \frac{1}{x}$. If $x < y$, we have there is $x < c_{x,y} < y$ such that

$$
|\Gamma(x) - \Gamma(y)| = |\Gamma'(c_{x,y})||x - y| = |1 - \frac{1}{c_{x,y}^2}||x - y| < |x - y|.
$$

Notice: if $\Gamma(x) = x$ we'd have $x = x + \frac{1}{x} \implies 0 = \frac{1}{x}$. Hence Γ admits no fixed point in $[1, \infty)$.

Answer #2: Yes, provided we limit (X, d).

Theorem 18.1 (Edelstein)**.** Let (X, d) be compact, and $\Gamma : X \to X$ satisfy $d(\Gamma(x), \Gamma(y)) < d(x, y)$ for $x \ne y$ in X. Then

(i) Γ admits a unique fixed point x_{fix}, and

(ii) if $x_0 \in X$, and $x_n = \Gamma(x_{n-1})$ $(n \in \mathbb{N})$, then $x_{\mathrm{fix}} = \lim_{n\to\infty} x_n$.

Proof. (i) Let $f : X \to \mathbb{R}, f(x) = d(x, \Gamma(x))$. Since Γ is continuous, f is continuous. [Check that f is 2-Lipschitz.] Hence, by EVT, there is $x_{\min}$ in X so $f(x_{\min}) = \min f(X)$. Suppose $x_{\min} \ne \Gamma(x_{\min})$, then

$$
\begin{aligned}
f(\Gamma(x_{\min})) &= d(\Gamma(x_{\min}), \Gamma \circ \Gamma(x_{\min})) \\
&< d(x_{\min}, \Gamma(x_{\min})) = f(x_{\min})
\end{aligned}
$$

violating choice of $x_{\min}$. Hence $x_{\min} = \Gamma(x_{\min})$, so write $x_{\min} = x_{\text{fix}}$.
If, also, $y = \Gamma(y)$ in X, with $y \neq x_{\text{fix}}$, then

$$d(y, x_{\text{fix}}) = d(\Gamma(y), \Gamma(x_{\text{fix}})) < d(y, x_{\text{fix}})$$

which is absurd.

(ii) Let $x_0 \in X, (x_n)_{n=1}^{\infty}$ be as above. Notice that

$$0 \le d(x_{\text{fix}}, x_{n+1}) = d(\Gamma(x_{\text{fix}}), \Gamma(x_0)) < d(x_{\text{fix}}, x_0)$$

so $L = \lim_{n\to\infty} d(x_{\text{fix}}, x_n)$ exists (decreasing, bounded sequence in $\mathbb{R}$).
Consider any converging subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$, with $x = \lim_{k\to\infty} x_{n_k}$. Then $d(x_{\text{fix}}, x) = \lim_{k\to\infty} d(x_{\text{fix}}, x_{n_k}) = L$.
If $x \neq x_{\text{fix}}$, then

$$\begin{aligned} L &= \lim_{k\to\infty} d(x_{\text{fix}}, x_{n_k+1}) = \lim_{k\to\infty} d(x_{\text{fix}}, \Gamma(x_{n_k})) \\ &= d(x_{\text{fix}}, \Gamma(x)) < d(x_{\text{fix}}, x) = L \end{aligned}$$

which is absurd. Hence the sequence $(x_n)_{n=1}^{\infty}$ has that x_{fix} is the only possible limit of a subsequence. Thus $\lim_{n\to\infty} x_n = x_{\text{fix}}$ (check!). $\square$

19 2017-11-08

Office hours:

Today	2:30-3:30
Tomorrow	2:30-4
Friday	2:30-3:30

19.1 Baire Category Theorem

Definition: Let (X, d) be a metric space.

(i) A subset $N \subset X$ is called nowhere dense if $(\overline{N})^\circ = \varnothing$ (ie. the interior of the closure of N is the empty set). [Equivalently, for any $x \in N, \varepsilon > 0, B(x,\varepsilon) \setminus \overline{N} \neq \varnothing$].

(ii) A set $S \subseteq X$ will be called meager (or is 1st category) if S is a countable union of nowhere dense sets: i.e.

$$S = \bigcup_{n=1}^{\infty} N_n, \text{ each } (\overline{N}_n)^\circ = \varnothing.$$

(ii') $S \subseteq X$ is non-meager (or is 2nd category) provided that it is not meager.

(iii) A set $R \subseteq X$ is residual if $X \setminus R$ is meager.
Remarks:

$$\begin{aligned} \text{nowhere dense} &\implies \text{meager} \\ \text{residual} &\implies \text{non-meager (provided } (X,d) \text{ is complete;} \\ &\qquad\qquad \text{consequence of B.C.T, Baire Category Theorem)} \end{aligned}$$

If (X, d) is complete, we think of meager = "small", non-meager = "not small" $\Longleftarrow$ residual.

Examples:

(i) If $x_0 \in X$, $\{x_0\}$ is nowhere dense $\iff$ x_0 is an accumulation point.

(ii) In $(\mathbb{R}^2, \|\cdot\|_2)$, $\mathbb{R} \times \{0\}$ is meager (exercise).

(iii) In $(\mathbb{R}, |\cdot|)$, the Cantor set C is nowhere dense.
Indeed, C is closed. If $t = 0.t_1 t_2 \cdots \in C$ (ternary representation), then given $\varepsilon > 0$, find k so $\frac{1}{3^k} < \varepsilon$ and then

$$t' = 0.t_1 t_2 \dots t_{k-1} 1 t_{k+1} \cdots \in B(t, \varepsilon) \setminus C.$$

(iv) $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$ is meager in $(\mathbb{R}, |\cdot|)$ (using (i)).

(v) $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$ is meager in $(\mathbb{Q}, |\cdot|)$ (using (i)).

Note: if (X, d) is not complete, it may be meager in itself. [meager sets are interesting in complete settings.]

Remark: If (X, d) is a metric space, $U \subseteq X$ is open and $x_0 \in U$, then there is $\varepsilon > 0$, s.t. $B[x, \varepsilon] \subseteq U$ (Indeed, let $\varepsilon' > 0$ be so $B(x, \varepsilon') \subseteq U$, and $\varepsilon \in (0, \varepsilon')$).

Lemma 19.1. Let (X, d) be a metric space, $N \subset X$. Then N is nowhere dense $\iff$ $\overline{X \setminus \overline{N}} = X$.

Proof.

$$\begin{aligned} N \text{ is nowhere dense} &\iff \text{for any } x \in \overline{N}, \varepsilon > 0, B(x, \varepsilon) \setminus \overline{N} \neq \varnothing \\ &\iff x \in \overline{X \setminus \overline{N}} \text{ for any } x \in \overline{N} \cup (X \setminus \overline{N}). \end{aligned}$$

□

Theorem 19.1 (Baire Category Theorem)**.** Let (X, d) be a complete metric space.

(i) Suppose $\{U\}_{n=1}^\infty$ is a sequence of open sets, each dense in X. Then $\bigcap_{n=1}^\infty U_n$ is dense in X.

(ii) If $M \subset X$ is meager, then $M^\circ = \varnothing$.

Proof. (i) Let $x_0 \in X$ and $\varepsilon_0 > 0$. We wish to show that $B(x_0, \varepsilon_0) \cap \bigcap_{n=1}^\infty U_n \neq \varnothing$.
Since $\overline{U_1} = X$, there is $x_1 \in B(x_0, \varepsilon_0) \cap U_1$ (using meet set characterization of closure). Let $\varepsilon_1 > 0$ be chosen so $B[x_1, \varepsilon_1] \subseteq B(x_0, \varepsilon_0) \cap U_1$.
Since $\overline{U_2} = X$, there is $x_2 \in B(x_1, \varepsilon_1) \cap U_2$.
Let $\varepsilon_2 \in (0, \frac{\varepsilon_1}{2}]$ be so $B[x_2, \varepsilon_2] \subseteq B(x_1, \varepsilon_1) \cap U_2$.
Inductively, having chosen x_n, ε_n, we appeal to the fact that $\overline{U_{n+1}} = X$ to find $x_{n+1} \in B(x_n, \varepsilon_n) \cap U_{n+1}$, then choose $\varepsilon_{n+1} \in (0, \frac{\varepsilon_n}{2}]$ and $B[x_{n+1}, \varepsilon_{n+1}] \subseteq B(x_n, \varepsilon_n) \cap U_{n+1}$.
Thus, we have $(x_n)_{n=1}^\infty \subseteq X, (\varepsilon_n)_{n=1}^\infty \subset (0, \infty)$ s.t.

(a) $B[x_{n+1}, \varepsilon_{n+1}] \subseteq B(x_n, \varepsilon_n) \subseteq B[x_n, \varepsilon_n]$

(b) $\operatorname{diam} B[x_n, \varepsilon_n] = 2\varepsilon_n \leq \varepsilon_{n-1} \leq \frac{\varepsilon_{n-2}}{2} \leq \cdots \leq \frac{\varepsilon_1}{2^{n-1}}$.

(c) $B[x_n, \varepsilon_n] \subseteq U_n \cap B(x_0, \varepsilon_0)$.

Then (a) & (b), with the Nested Sets Theorem, show that $\bigcap_{n=1}^\infty B[x_n, \varepsilon_n] \neq \varnothing$.
Further, (c) shows that $\varnothing \neq \bigcap_{n=1}^\infty B[x_n, \varepsilon_n] \subseteq \bigcap_{n=1}^\infty U_n \cap B(x_0, \varepsilon_0)$.
Hence, for any $x_0 \in X, \varepsilon_0 > 0, B(x_0, \varepsilon_0) \cap \bigcap_{n=1}^\infty U_n \neq \varnothing$, so $\overline{\bigcap_{n=1}^\infty U_n} = X$.

(ii) Write $M = \bigcup_{n=1}^\infty N_n$, each $(\overline{N_n})^\circ = \varnothing$. Then $U_n = X \setminus \overline{N_n}$ is open, and dense in X, by Lemma.
We have

$$\begin{aligned} X \setminus M = X \setminus \bigcup_{n=1}^\infty N_n &\supseteq X \setminus \bigcup_{n=1}^\infty \overline{N_n} (\text{as each } N_n \subseteq \overline{N_n}) \\ &= \bigcap_{n=1}^\infty (X \setminus \overline{N_n}) = \bigcap_{n=1}^\infty U_n \end{aligned}$$

so $\overline{X \setminus M} = X$. Thus if $x \in M, \varepsilon > 0$, we have $B(x,\varepsilon) \setminus M = B(x,\varepsilon) \cap (X \setminus M) \neq \varnothing$. Thus $x \notin M^\circ$, i.e. $M^\circ = \varnothing$. □

Question: Let $\{q_k\}_{k=1}^\infty = \mathbb{Q}$. Let for n in $\mathbb{N}$

$$U_n = \underbrace{\bigcup_{k=1}^\infty \underbrace{(q_k - \frac{1}{2^{kn+1}}, q_k + \frac{1}{2^{kn+1}})}_{\text{length is } \frac{1}{2^{nk}}}}_{U_n \text{ is a union of intervals, sum of lengths is } \sum_{k=1}^\infty \frac{1}{(2^n)^k} = \frac{\frac{1}{2^n}}{1-\frac{1}{2^n}}}$$

Is $\mathbb{Q} = \bigcap_{n=1}^\infty U_n$?

20 2017-11-10

Remark: In particular, a nonempty open subset in a complete metric space is nonmeager. The whole of X is a nonempty open set.

Corollary 20.1. A residual set in a complete metric space is nonmeager.

Proof. Let $R \subset X$ be residual, so $M = X \setminus R$ is meager, so $X \setminus R = \bigcup_{n=1}^\infty N_n$, each $(\overline{N_n})^\circ = \varnothing$. If we had that R was meager, i.e. $R = \bigcup_{n=1}^\infty N'_n$, $(\overline{N'_n}^\circ) = \varnothing$, then

$$X = R \cup (X \setminus R) = \underbrace{\bigcup_{n=1}^\infty N'_n \cup \bigcup_{n=1}^\infty N_n}_{\text{countable union of nowhere dense sets}} .$$

But $X^\circ = X$, so this contradicts B.C.T. □

meager = "small", residual = "bigness", "typical elements"

Definition: Let (X,d) be a metric space.

1. $G \subseteq X$ is a G_δ-set if $G = \bigcap_{n=1}^\infty U_n$, each U_n open
2. $F \subseteq X$ is an F_σ-set if $F = \bigcup_{n=1}^\infty F_n$, each F_n closed

Examples:

1. In A4,Q2, we saw that any closed set is G_δ
   (i') Any open set $U \subseteq X$ is F_σ (De Morgan's law)
2. $\mathbb{R} \setminus \mathbb{Q}$ is not F_σ.
   First, $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$ is F_σ. Second, if $F \subset \mathbb{R} \setminus \mathbb{Q}$ is closed, then F is nowhere dense (this just follows density of $\mathbb{Q}$). Thus if we had an F_σ realization $\mathbb{R} \setminus \mathbb{Q} = \bigcup_{n=1}^\infty F_n, F_n \subset \mathbb{R} \setminus \mathbb{Q}$ closed, then $\mathbb{R} \setminus \mathbb{Q}$ is meager. Thus,

$$\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q}) = \bigcup_{q \in \mathbb{Q}} \{q\} \cup \bigcup_{n=1}^\infty F_n$$

   would be meager which violates B.C.T. (Corollary just stated).
   (ii') $\mathbb{Q}$ is not G_δ (De Morgan, from (ii)).
   In particular

$$\mathbb{Q} \not\subseteq \bigcap_{n=1}^\infty \bigcup_{k=1}^\infty \underbrace{(q_k - \frac{1}{2^{kn+1}}, q_k + \frac{1}{2^{kn+1}})}_{U_n}.$$

$\{q_k\}_{n=1}^{\infty} = \mathbb{Q}$.

Corollary 20.2. In a complete metric space, a dense G_δ-subset is residual.

Proof. In complete (X,d), if $G = \bigcap_{n=1}^{\infty} U_n$, each U_n open, and $\overline{G} = X$, then each $\overline{U_n} = X$. Thus, by lemma before B.C.T., each $X \setminus U_n$ is nowhere dense hence $X \setminus G = X \setminus \bigcap_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty}(X \setminus U_n)$ is meager. □

Theorem 20.1 (Uniform Boundedness Principle)**.** Let (X,d) be a complete metric space and $\{f_i\}_{i\in I} \subset C(X)$ (continuous $\mathbb{R}$-valued functions) which satisfies for each x

$$\sup_{i\in I} |f_i(x)| < \infty \text{ (pointwise boundedness).}$$

Then there exists an open $\varnothing \neq U \subseteq X$ s.t.

$$\sup_{i\in I}\sup_{x\in U} |f_i(x)| < \infty \text{ (uniform boundedness on } U).$$

Proof. For n in $\mathbb{N}$, let

$$F_n = \{x \in X : |f_i(x)| \leq n \text{ for all } i \in I\}.$$

By our pointwise boundedness assumption,

$$X = \bigcup_{n=1}^{\infty} F_n \qquad (\star).$$

Each F_n is closed:

$$F_n = \bigcap_{i\in I} |f_i|^{-1}((-\infty, n]) = \bigcap_{i\in I} \underbrace{(X \setminus \underbrace{|f_i|^{-1}(n,\infty)}_{\text{open, as } |f_i(\cdot)| \text{ is continuous}})}_{\text{closed}}$$

But B.C.T. tells us that our complete X is non-meager, so for some n_0, $F_{n_0}^\circ \neq \varnothing$. Let $U = F_{n_0}^\circ$, and for all $x \in U \subseteq F_n$

$$\begin{aligned} &|f_i(x)| \leq n_0 \text{ for all } i \in I \\ \Longrightarrow &\sup_{x\in U} |f_i(x)| \leq n_0 \text{ for all } i \in I \\ \Longrightarrow &\sup_{i\in I}\sup_{x\in U} |f_i(x)| \leq n_0 < \infty. \end{aligned}$$

□

Corollary 20.3 (Banach-Stenhaus Theorem)**.** Let $(V, \|\cdot\|_V)$ be a Banach space, $(W, \|\cdot\|_W)$ a normed vector space, and $\{T_i\}_{i\in I} \subset B(V,W)$ satisfies

$$\sup_{i\in I} \|T_i x\|_W < \infty \text{ for each } x \in V.$$

Then

$$\sup_{i\in I} \|\!|T_i|\!\| < \infty. \text{ [Recall } \|\!|T_i|\!\| = \sup_{x\in B_V[0,1]} \|T_i x\|_W.]$$

Proof. Let $f_i(x) = \|T_i x\|_W$, for $i \in I, x \in V$, so $\{f_i\}_{i\in I} \subset C(V)$. Our assumption on $\{T_i\}_{i\in I}$, gives pointwise boundedness of $\{f_i\}_{i\in I}$, so U.B.P provides $\varnothing \neq U \subset V$ for which

$$M = \sup_{i\in I}\sup_{x\in U} \|T_i x\| < \infty.$$

As U is open, if $x_0 \in U$, there is $\varepsilon > 0, B[x_0, \varepsilon] \subset U$.
Now if $z \in B_V[0,1]$, then we may write

$$z = \frac{1}{2\varepsilon}(-x_0 + \varepsilon z) + \frac{1}{2\varepsilon}(x_0 + \varepsilon z)$$

and, for i in I, we have

$$\begin{aligned}
\|T_i z\|_W &\le \frac{1}{2\varepsilon}\Bigg\|T_i(\underbrace{x_0-\varepsilon z}_{\in B[x,\varepsilon]\subset U})\Bigg\|_W + \frac{1}{2\varepsilon}\Bigg\|T_i(\underbrace{x_0+\varepsilon z}_{\in B[x,\varepsilon]\subset U})\Bigg\|_W \\
&\le \frac{1}{2\varepsilon}M + \frac{1}{2\varepsilon}M = \frac{M}{\varepsilon}. \\
\implies \|T_i\| &= \sup_{z\in B_V[0,1]} \|T_i z\|_W \le \frac{M}{\varepsilon} < \infty.
\end{aligned}$$

□

21 2017-11-13

21.1 Baire-1 Functions

Def: Let $\varnothing \neq X \subseteq \mathbb{R}$, so (X,d) is a metric space with relativized metric from $\mathbb{R}$.
A function $f : X \to \mathbb{R}$ is called Baire-1 if there is a sequence $(f_n)_{n=1}^{\infty} \subset C(X)$ such that for $t \in X$,

$$f(t) = \lim_{n\to\infty} f_n(t) \text{ (pointwise limit).}$$

Remark: Unlike uniform limits, pointwise limits of continuous functions need not be continuous.

Example: Let $X = [0,1]$, $f_n(t) = t^n$. Then

$$\lim_{n\to\infty} f_n(t) = \begin{cases} 0 & t \in [0,1) \\ 1 & t = 1. \end{cases}$$

Question: Let for t in $[0,1]$,

$$f_n(t) = \cos(n!\pi t)^{n!^{n!}}.$$

If $t = \frac{k}{\ell} \in \mathbb{Q}, \ell \in \mathbb{N}$, then $f_n(t) = 1$, if $t \ge \ell + 1$.
Does $\lim_{n\to\infty} f_n(t) = \chi_{\mathbb{Q}\cap[0,1]}(t)$ for t in $[0,1]$?
Answer: No. (Probably the limit does not exist.)
The answer will follow from (corollary to) the next theorem and B.C.T.

Theorem 21.1 (Baire)**.** Let $a < b$, and $f : (a,b) \to \mathbb{R}$ be a Baire-1 function, then there is t_0 in (a,b) such that f is continuous at t_0.

$$\chi_{\mathbb{Q}}(t) = \lim_{n\to\infty} \underbrace{\lim_{m\to\infty} |\cos(n!\pi t)^m|}_{\chi_{\{\frac{k}{n!},k\in\mathbb{Z}\}}(t)}$$

Baire-2 = pointwise limit of Baire-1 functions.
At no t_0 is χ_Q continuous, thus <u>not</u> Baire-1.

Proof. Let $f(t) = \lim_{n\to\infty} f_n(t), t \in (a,b), (f_n)_{n=1}^{\infty} \subset C(a,b)$.
(I) Given $\varepsilon > 0$, we will show that there are $\alpha < \beta$ in (a,b), and N_ε in $\mathbb{N}$ such that for all $n, m \ge N_\varepsilon$,

$$|f_n(t) - f_m(t)| \le \varepsilon \text{ for } t \in [\alpha,\beta].$$

Let us proceed by contradiction. Hence, there exists t_1 in (a,b), and $n_1, m_1 \in \mathbb{N}$ such that

$$|f_{n_1}(t_1) - f_{m_1}(t_1)| > \varepsilon.$$

Since each f_{n_1}, f_{m_1} is continuous, there is an open interval $I_1 \subset \overline{I_1} \subset (a,b)$ such that

$$|f_{n_1}(t) - f_{m_1}(t)| > \varepsilon \text{ for } t \in I_1.$$

[$t \longmapsto |f_{n_1}(t) - f_{m_1}(t)|$ is continuous.]
Next, by assumption, there is $t_2 \in I_1$ such that there exist $n_2, m_2 > \max\{n_1, m_1\}$ such that

$$|f_{n_2}(t_2) - f_{m_2}(t_2)| > \varepsilon.$$

Again, as f_{n_2}, f_{m_2} are continuous, there is an open interval $I_2 \subset \overline{I_2} \subset I_1$ such that

$$|f_{n_2}(t) - f_{m_2}(t)| > \varepsilon \text{ for } t \in I_2.$$

Inductively, we obtain

- a sequence of intervals
$$\overline{I_1} \supset I_1 \supset \overline{I_2} \supset I_2 \supset \cdots \supset \overline{I_n} \supset I_n \supset \cdots, \text{ and}$$
- two increasing sequences $(n_k)_{k=1}^\infty, (m_k)_{k=1}^\infty \subseteq \mathbb{N}$ such that
$$|f_{n_k}(t) - f_{m_k}(t)| > \varepsilon \text{ for } t \in I_k.$$

Thus, by Nested Intervals Theorem, there exists

$$t_0 \in \bigcap_{k=1}^\infty \overline{I_k} = \bigcap_{k=2}^\infty \overline{I_k} \subseteq \bigcap_{k=1}^\infty I_k$$

so $t_0 \in I_k$ for each k, so

$$|f_{n_k}(t) - f_{m_k}(t)| > \varepsilon. \qquad (\dagger)$$

But, by pointwise convergence, $f(t_0) = \lim_{k\to\infty} f_k(t_0)$ so $(f_n(t_0))_{n=1}^\infty \subset \mathbb{R}$ is Cauchy. This violates ($\dagger$). Hence (I) holds.
(II) We use (I), with $\varepsilon = 1$, to find $\alpha_1 < \beta_1$ in (a,b) and N_1 in $\mathbb{N}$ so

$$|f_n(t) - f_m(t)| \le 1 \text{ for } t \in [\alpha_1, \beta_1], \text{ if } n, m \ge N_1.$$

We again use (I), with $\varepsilon = \frac{1}{2}$, to find $\alpha_2 < \beta_2$ in (a,b) and N_2 in $\mathbb{N}$ so

$$|f_n(t) - f_m(t)| \le \tfrac{1}{2} \text{ for } t \in [\alpha_2, \beta_2], \text{ if } n, m \ge N_2.$$

Inductively, we obtain

- intervals
$$(a,b) \supset [\alpha_1,\beta_1] \supset (\alpha_1,\beta_1) \supset [\alpha_2,\beta_2] \supset (\alpha_2,\beta_2) \supset \cdots \supset [\alpha_n,\beta_n] \supset (\alpha_n,\beta_n) \supset \cdots, \text{ and}$$
- an increasing sequence $(N_k)_{k=1}^\infty \subset \mathbb{N}$ such that
$$|f_n(t) - f_m(t)| \le \tfrac{1}{k} \text{ for } t \in [\alpha_k, \beta_k], \text{ if } n, m \ge N_k. \qquad (\ddagger)$$

By N.I.T. (Nested Intervals Theorem), there exists

$$t_0 \in \bigcap_{k=1}^\infty [\alpha_k, \beta_k] \subseteq \bigcap_{k=1}^\infty (\alpha_k, \beta_k).$$

Now, given $\varepsilon > 0$, let k in $\mathbb{N}$ so $\frac{1}{k} < \varepsilon$, and then let $\delta = \min\{t_0 - \alpha_k, \beta_k - t_0\} > 0$ so $(t_0 - \delta, t_0 + \delta) \subset (\alpha_k, \beta_k) \subset [\alpha_k, \beta_k]$.
Hence by ($\ddagger$), we have that

$$|f_n(t) - f_m(t)| \le \tfrac{1}{k} < \varepsilon \text{ whenever } t \in (t_0 - \delta, t_0 + \delta), n, m \ge N_k.$$

Hence $(f_n)_{n=1}^{\infty}$ converges "uniformly at t_0" (see Assignment 6), so f is continuous at t_0 (Assignment 6). $\square$

Corollary 21.1. Let $a < b$ in $\mathbb{R}$, $f : (a, b) \to \mathbb{R}$ be a Baire-1 function. The set $G = \{t \in (a, b) : f \text{ is continuous at } t\}$ is a dense G_δ-subset of (a, b). [By B.C.T., $G \subset [a, b]$ is residual.]

Proof. If $t_0 \in (a, b)$ and $\varepsilon > 0$, then there exists $t \in (t_0 - \varepsilon, t_0 + \varepsilon) \cap (a, b) \cap G$. I.e. $G \cap (t_0 - \varepsilon, t_0 + \varepsilon) \neq \varnothing$, so $\overline{G} = (a, b)$ (relativized topology). Furthermore, the set G is always G_δ (Assignment 6). $\square$

Example: $\underbrace{\chi_{\mathbb{Q}}}_{\text{nowhere continuous}}$ is not Baire-1 on any interval.

22 2017-11-15

Corollary 22.1. Let $f \in C(a, b)$ ($a < b$ in $\mathbb{R}$) be right differentiable on (a, b). Then f'_+ (right derivative) is continuous on a dense G_δ-subset of (a, b). [In particular, if f is differentiable, f' is continuous on a dense G_δ-subset.]

Proof. Let $h_n(t) = \min\{b - t, \frac{1}{n}\}$ for n in $\mathbb{N}, t$ in (a, b). Then

$$f_n(t) = \frac{f(t + h_n(t)) - f(t)}{h_n(t)} \qquad \left(= \frac{f(t + \frac{1}{n}) - f(t)}{\frac{1}{n}}, n \text{ large} \right)$$

satisfies that each $f_n \in C(a, b)$ and

$$f'_+(t) = \lim_{n\to\infty} f_n(t) \text{ for each } t \in (a, b).$$

$\square$

22.1 On the Banach spaces $C(X)$, X compact

First case $X = [a, b]$, compact interval in $\mathbb{R}$.

Lemma 22.1. For n in $\mathbb{N}$ let $q_n(t) = c_n(1 - t^2)^n$ where c_n satisfies

$$1 = c_n \int_{-1}^{1} (1 - t^2)^n dt.$$

Then

($q1$) $q_n(t) \geq 0$ for $t \in [-1, 1], n$ in $\mathbb{N}$ (non-negative)

($q2$) $\int_{-1}^{1} q_n(t)dt = 1, n$ in $\mathbb{N}$ (total mass 1)

($q3$) if $\delta \in (0, 1)$, then $\left(\int_{-1}^{-\delta} + \int_{\delta}^{1}\right) q_n(t)dt \xrightarrow{n\to\infty} 0$ (concentration of mass near 0)

Proof. ($q1$) and ($q2$) are obvious. Now for $t \in [0, 1]$,

$$\begin{aligned} t^2 \leq t &\implies 1 - t \leq 1 - t^2 \\ &\implies (1 - t)^n \leq (1 - t^2)^n \end{aligned}$$

and hence

$$\begin{aligned} \frac{1}{c_n} &= \int_{-1}^{1} (1 - t^2)^n dt = 2 \int_{0}^{1} (1 - t^2)^n dt \\ &\leq 2 \int_{0}^{1} (1 - t)^n dt = \frac{-2}{n + 1} (1 - t)^{n+1} \Big|_{0}^{1} = \frac{2}{n + 1} \end{aligned}$$

so $c_n \leq \frac{n+1}{2}$. Hence, for $|t| \in (\delta, 1)$, we have

$$\begin{aligned} q_n(t) = c_n(1-t^2)^n &\leq c_n(1-t^2)^n \\ &\leq \frac{n+1}{2} \underbrace{(1-t^2)^n}_{<1} \xrightarrow{n\to\infty} 0. \end{aligned}$$

Thus

$$\begin{aligned} \left(\int_{-1}^{-\delta} + \int_{\delta}^{1}\right) q_n(t)dt &\leq \left(\int_{-1}^{-\delta} + \int_{\delta}^{1}\right) \frac{n+1}{2}(1-t^2)^n dt \\ &= (1-\delta)(n+1)(1-\delta^2)^n \xrightarrow{n\to\infty} 0. \end{aligned}$$

□

Theorem 22.1 (Weierstrauss approximation theorem)**.** Given $a < b$ in $\mathbb{R}$, $f \in C[a,b]$, there exists a sequence $(p_n)_{n=1}^{\infty}$ of polynomial functions such that

$$\text{(WA) } \|p_n - f\|_\infty = \max_{t\in[a,b]} |p_n(t) - f(t)| \xrightarrow{n\to\infty} 0.$$

Proof. (I) We condition f. Let $\widetilde{f} \in C[0,1]$ be given by

$$\widetilde{f}(t) = f(a + t(b-a)) - [f(b) - f(a)]t - f(a).$$

So

- $\widetilde{f}(0) = f(b) - f(a) = 0$
- $\widetilde{f}(1) = f(b) - [f(b) - f(a)]1 - f(a) = 0.$

If we can find a sequence $(\widetilde{p_n})_{n=1}^{\infty}$ of polynomials,

$$\left\|\widetilde{p_n} - \widetilde{f}\right\|_\infty = \sup_{t\in[0,1]} |\widetilde{p_n}(t) - \widetilde{f}(t)| \xrightarrow{n\to\infty} 0$$

we are done. Indeed, if $s \in [a,b]$, then define each $p_n(s) = \widetilde{p_n}(\frac{1}{b-a}(s-a)) + \frac{f(b)-f(a)}{b-a}(s-a) + f(a)$; may be easily shown to satisfy (WA).
(II) Let us assume that

$$f \in C[0,1], f(0) = 0 = f(1).$$

We can extend f to $\mathbb{R}$ by letting $f(t) = 0$ for $t \in (-\infty, 0) \cup (1, \infty)$, so $f \in C_b(\mathbb{R})$, but $f(t) \neq 0$ only possibly for $t \in [0,1]$, and f is uniformly continuous [any function in $C[0,1]$ is uniformly continuous].
Let $(q_n)_{n=1}^{\infty}$ be as in the last lemma, and let for each n in $\mathbb{N}$ and each t in $[0,1]$,

$$p_n(t) = \int_0^1 q_n(s-t)f(s)ds.$$

Let us compute, for each n, t,

$$\begin{aligned} \frac{d^{2n+1}}{dt^{2n+1}} p_n(t) &= \int_0^1 \frac{\partial^{2n+1}}{\partial t^{2n+1}} \underbrace{q_n(s-t)}_{\text{function is } 2n+2\text{-times continuously differentiable}} f(s)ds \\ &= 0, \text{ since } \deg q_n(t) = \deg(1-t^2)^n = 2n. \end{aligned}$$

$\implies p_n$ is a polynomial, $\deg p_n(t) \leq 2n$.

By change of variable $u = s - t$,

$$\begin{aligned}
p_n(t) &= \int_0^1 q_n(s-t)f(s)ds \\
&= \int_{-t}^{1-t} q_n(u)f(u+t)du \\
&= \int_{-1}^{1} q_n(u)f(u+t)du, \text{ since } f(u+t) \geq 0 \text{ possibly only on } [-t, 1-t].
\end{aligned}$$

Hence for t in $[0,1]$,

$$\begin{aligned}
|p_n(t) - f(t)| &= \left| \int_{-1}^{1} q_n(u)f(u+t)du - \underbrace{\int_{-1}^{1} q_n(u)f(t)du}_{\text{property } (q2)} \right| \\
&\leq \int_{-1}^{1} q_n(u)|f(u+t) - f(t)|du.
\end{aligned}$$

Given $\varepsilon > 0$, let $\delta > 0$ be so $|x-y| < \delta (x, y \in \mathbb{R}) \implies |f(x) - f(y)| < \frac{\varepsilon}{2}$ and then

$$\begin{aligned}
|p_n(t) - f(t)| &\leq \int_{-\delta}^{\delta} q_n(u) \underbrace{|f(u+t) - f(t)|}_{< \frac{\varepsilon}{2}, \text{ by choice of } \delta} du + \left(\int_{-1}^{-\delta} + \int_{\delta}^{1} \right) q_n(u) \underbrace{|f(u+t) - f(t)|}_{\leq 2\|f\|_\infty} du \\
&\leq \frac{\varepsilon}{2} \int_{-1}^{1} q_n(u)du + \left(\int_{-1}^{-\delta} + \int_{\delta}^{1} \right) q_n(u) 2\|f\|_\infty du \text{ by } (q1) \xrightarrow{n\to\infty} \frac{\varepsilon}{2} + 0.
\end{aligned}$$

(Continued next lecture.)

23 2017-11-17

We saw p_n is polynomial, i.e. $d^{2n+1}/dt^{2n+1} p_n(t) = 0$. Need approx.
Using (q2) we saw for $t \in [0,1]$

$$|p_n(t) - f(t)| \leq \int_{-1}^{1} \underbrace{q_n(u)}_{(q1)} |f(u+t) - f(t)|du$$

Given $\varepsilon > 0$, use uniform continuity of f to find $\delta > 0$ s.t. $|x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{2}$.

$$
\begin{aligned}
|p_n(t)-f(t)| &\le \int_{-1}^{1} q_n(u)|f(u+t)-f(t)|du \\
&= \int_{-\delta}^{\delta} q_n(u)|f(u+t)-f(t)|du + \left(\int_{-1}^{-\delta} + \int_{\delta}^{1}\right) q_n(u)\underbrace{|f(u+t)-f(t)|}_{\le 2\|f\|_\infty} du \\
&\le \int_{-\delta}^{\delta} q_n(u)\frac{\varepsilon}{2}du + \left(\int_{-1}^{-\delta} + \int_{\delta}^{1}\right) q_n(u)2\|f\|_\infty du \\
&\le \frac{\varepsilon}{2}\underbrace{\int_{-\delta}^{\delta} q_n(u)du}_{=1(q2)} + 2\|f\|_\infty \left(\int_{-1}^{-\delta} + \int_{\delta}^{1}\right) q_n(u)du.
\end{aligned}
$$

Hence, if n_ε is so $n \ge n_\varepsilon \implies \left(\int_{-1}^{-\delta} + \int_{\delta}^{1}\right) q_n(u)du \le \frac{\varepsilon}{2(2\|f\|_\infty+1)}$
we have for $n \ge n_\varepsilon$,

$$|p_n(t)-f(t)| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and we thus have

$$\|p_n - f\|_\infty = \max_{t\in[0,1]} |p_n(t)-f(t)| < \varepsilon$$

and we thus see that $\lim_{n\to\infty} p_n = f$ in $(C[0,1], \|\cdot\|_\infty)$. □

Corollary 23.1. If $f \in C^1[a,b]$ (differentiable on $[a,b]$, with continuous derivative). Then, given $\varepsilon > 0$, there is a polynomial p s.t.

$$
\begin{aligned}
\|p' - f\|_\infty &< \varepsilon \\
\|p - f\|_\infty &< (b-a)\varepsilon.
\end{aligned}
$$

Proof. By Weierstrauss approximation theorem, find a polynomial q s.t. $\|f' - q\|_\infty < \varepsilon$. Let $p(t) = f(a) + \int_a^t q(s)ds$. Check that this works. (Remember Fundamental Theorem of Calculus.) □

Corollary 23.2. $(C[a,b], \|\cdot\|_\infty)$ is separable.

Proof. Let $f \in C[a,b], \varepsilon > 0$.
By Weierstrauss approximation theorem, find polynomial p s.t.

$$\|f - p\|_\infty < \frac{\varepsilon}{2}.$$

Write $p(t) = a_0 + a_1 t + \cdots + a_n t^n$. For $j = 1, \ldots, n$ let $q_j \in \mathbb{Q}$ be such that

$$|a_j - q_j| < \frac{\varepsilon}{2(n+1)\max\{|a|^j, |b|^j\}}$$

then let $r(t) = q_0 + q_1 t + \cdots + q_n t^n$.
Check that for each t in $[a,b]$,

$$|p(t) - r(t)| < \frac{\varepsilon}{2}$$

so $\|p - r\|_\infty = \max_{t\in[a,b]} |p(t)-r(t)| < \frac{\varepsilon}{2}$,
and thus

$$\|f - r\|_\infty \le \|f - p\|_\infty + \|p - r\|_\infty < \varepsilon.$$

□

Theorem 23.1 (nowhere differentiable functions are generic)**.** Let $ND[0,1]$ denote the set of f in $C[0,1]$ which are nowhere differentiable. Then $ND[0,1]$ is residual in $C[a,b]$.

Proof. Recall for $M, \delta > 0$,

$$F_{M,\delta} = \{f \in C[0,1] : \text{there is } x \text{ in } [0,1] \text{ so } \frac{|f(x)-f(t)|}{|x-t|} \le M \\ \text{for all } t \in [0,1] \cap [(x-\delta,x)\cup(x,x+\delta)]\ \}$$

(A5,Q1).
(I) Let us see that each $F_{M,\delta}$ is nowhere dense in $(C[0,1], \|\cdot\|_\infty)$.
To this end, let $f \in F_{M,\delta}, \varepsilon > 0$.
First, use Weierstrauss approximation to get a polynomial p so $\|f-p\|_\infty < \frac{\varepsilon}{2}$. In particular, p' exists everywhere, let $M' = \sup_{t\in[0,1]} \|p'(t)\|$.
Let

$$\varphi : [0,\infty) \to [0,1], \varphi(t) = \begin{cases} t-n & t \in [n,n+1], n \in \{0\}\cup\mathbb{N} \text{ is even} \\ n+1-t & t \in [n,n+1], n \in \mathbb{N} \text{ is odd} . \end{cases}$$

For each k in $\mathbb{N}$ let $\varphi_k(t) = \frac{1}{k}\varphi(k^2 t)$.
For $s,t \in [\frac{n-1}{k^2}, \frac{n}{k^2}], n \in \mathbb{N}$,

$$\frac{|\varphi_k(s)-\varphi_k(t)|}{|s-t|} = k \qquad (\dagger).$$

Now let k be so $\frac{1}{k} < \frac{\varepsilon}{2}$ and $k - M' > M, \frac{1}{k^2} < \delta$.
Let $\psi_k = p + \varphi_k$ and we have for s,t satisfying $(\dagger)$,

$$\begin{aligned} \frac{|\psi_k(s)-\psi_k(t)|}{|s-t|} &= \left| \frac{p(s)-p(t)}{s-t} - \frac{\varphi_k(s)-\varphi_k(t)}{s-t} \right| \\ &\ge \left| \underbrace{\frac{|\psi_k(s)-\psi_k(t)|}{|s-t|}}_{k} - \underbrace{\frac{|p(s)-p(t)|}{|s-t|}}_{\le M',\text{ by Mean Value Theorem}} \right| \\ &\ge |k-M'| = k - M' > M. \end{aligned}$$

Hence $\psi_k \notin F_{M,\delta}$. And $\|f-\psi_k\|_\infty \le \|f-p\|_\infty + \Bigg\| \underbrace{p-\psi_k}_{-\varphi_k} \Bigg\|_\infty < \frac{\varepsilon}{2} + \frac{1}{k} < \varepsilon$. □

24 2017-11-20

Theorem 24.1. $ND[0,1] = \{f \in C[0,1] : f \text{ is nowhere differentiable}\}$ is a residual set in $(C[0,1], \|\cdot\|_\infty)$.

Proof. We saw:
Each

$$F_{M,\delta} = \{f \in C[0,1] : \exists x \text{ in } [0,1], \frac{|f(x)-f(t)|}{|x-t|} \le M \text{ for } t \in [0,1]\cap[(x-\delta,x)\cup(x,x+\delta)]\}$$

is closed (A5), nowhere dense (I).
(II) Let $SD[0,1] = C[0,1] \setminus ND[0,1]$ ("somewhere differentiable"). If $f \in SD[0,1]$, in A5, it was shown that $f \in F_{M,\delta}$ for some $M > 0, \delta > 0$. If $n \in \mathbb{N}$, with $n > \max\{M, \frac{1}{\delta}\}$, then $F_{M,\delta} \subseteq F_{n,\frac{1}{n}}$. Then

$$SD[0,1] = \bigcup_{n=1}^{\infty} F_{n,\frac{1}{n}}, \text{ each } F_{n,\frac{1}{n}} \text{ closed, } F^\circ_{n,\frac{1}{n}} = \varnothing.$$

Thus $SD[0,1]$ is meager, so $ND[0,1] = C[0,1] \setminus SD[0,1]$ is residual. □

Remark: Baire Category Theorem tells us that in the complete metric space $(C[0,1], \|\cdot\|_\infty)$.
residual = "large" = "generic"

24.1 Towards Stone-Weierstrauss Theorem

Notation: (lattice structure)
Let X be non-empty, $f, g : X \to \mathbb{R}$. Define

$$\begin{aligned} &(\text{“join”}) && f \vee g : X \to \mathbb{R}, f \vee g(x) = \max\{f(x), g(x)\} \\ &(\text{“meet”, min}) && f \wedge g : X \to \mathbb{R}, f \wedge g(x) = \min\{f(x), g(x)\}. \end{aligned}$$

Proposition 24.1. Let (X, d) be a (compact) metric space, $f, g \in C(X)$. Then $f \vee g, f \wedge g \in C(X)$.

Proof. If $a, b \in \mathbb{R}$, then $\max\{a, b\} = \frac{1}{2}(a+b) + \frac{1}{2}|a-b|$.
Hence

$$f \vee g = \frac{1}{2}(f+g) + \frac{1}{2} \underbrace{|f-g|}_{f-g \text{ compact with } |\cdot|} \in C(x).$$

Also $\min\{a, b\} = -\max\{-a, -b\}$, so

$$f \wedge g = -(-f) \vee (-g) \in C(X).$$

□

Notation: A family $\mathcal{L} \subseteq C(X)$ is called a lattice if for each $f, g \in \mathcal{L}, f \vee g, f \wedge g \in \mathcal{L}$. Notice if $f_1, \ldots, f_n \in \mathcal{L}$,

$$\begin{aligned} & f_1 \vee f_2 \in \mathcal{L} \\ & \implies f_1 \vee f_2 \vee f_3 \in \mathcal{L} \\ & \vdots \text{ (obvious induction)} \\ & \implies f_1 \vee \cdots \vee f_n \in \mathcal{L}. \end{aligned}$$

Likewise $f_1 \wedge \cdots \wedge f_n \in \mathcal{L}$.

Theorem 24.2 (Stone)**.** Let (X, d) be a compact metric space and let the lattice $\mathcal{L} \subseteq C(X)$ satisfy

- $\mathcal{L}$ is a $\mathbb{R}$-space
- $1 \in \mathcal{L}$ (contains constant function)
- $\mathcal{L}$ separates points: if $x \neq y$ in X, there exists $\varphi \in \mathcal{L}$, so $\varphi(x) \neq \varphi(y)$.

Then $\overline{\mathcal{L}} = C(X)$ ($\mathcal{L}$ is uniformly dense in $C(X)$).

Proof. Suppose $x \neq y$ in X and $\alpha, \beta \in \mathbb{R}$. Since $\mathcal{L}$ separates points, there is $\varphi \in \mathcal{L}$ with $\varphi(x) \neq \varphi(y)$. Then

$$g = \alpha 1 + \frac{\beta - \alpha}{\varphi(y) - \varphi(x)}[\varphi - \varphi(x)1] \in \mathcal{L} \text{ as } 1 \in \mathcal{L}, \mathcal{L} \text{ is a } \mathbb{R}\text{-subspace}$$

with $g(x) = \alpha, g(x) = \beta$.
Fix $f \in C(X), \varepsilon > 0$.
(I) Fix x in X. For each y in X, letting $\alpha = f(x), \beta = f(y)$, if $y \neq x$, we have that there is

$$g_{x,y} \in \mathcal{L} \text{ s.t. } g_{x,y}(x) = f(x), g_{x,y}(y) = f(y).$$

Since each $f, g_{x,y}$ are continuous (near y), there are $\delta_y > 0$ so that

$$d(z, y) < \delta_y \implies g_{x,y}(z) < f(z) + \varepsilon \text{ i.e. } g_{x,y} < f + \varepsilon \text{ on } B(y, \delta_y)$$

$$(\text{i.e. } g_{x,y} - f \text{ is 0 at } y \text{ so } < \varepsilon \text{ in a neighbourhood of } y)$$

Since $X = \bigcup_{y\in X} B(y,\delta_y)$, by compactness, there are $y_1,\dots,y_m$ s.t. $X = \bigcup_{j=1}^m B(y_j,\delta_{y_j})$. Let

$$g_x = g_{x,y_1} \wedge \cdots \wedge g_{x,y_m} \in \mathcal{L}$$

and we have $g_x \le g_{x,y} < f + \varepsilon 1$.
Notice that $g_x(x) = \min\{f_{x,y_1}(x),\dots,f_{x,y_m}(x)\} = f(x)$. □

25 2017-11-22

Small goof up:
Then we let $g_x = g_{x,y_1} \wedge \cdots \wedge g_{x,y_m} \in \mathcal{L}$.
Now, if $z \in X$, then $z \in B(y_j,\delta_{y_j})$ for some $j = 1,\dots,m$ and then

$$g_x(z) = g_{x,y_1} \wedge \cdots \wedge g_{x,y_n} \le g_{x,y_j}(z) < f(z) + \varepsilon, \text{ property of } \delta_{y_j} \text{ w.r.t. } y_j$$

so we have

$$g_x < f + \varepsilon 1, \text{ and } g_x(x) = f(x).$$

(II) For each x in X, we found $g_x \in \mathcal{L}$ s.t. $g_x < f + \varepsilon 1, g_x(x) = f(x)$.
Hence $g_x(x) = f(x) < f(x) + \varepsilon$ at each x, so there is $\delta_x > 0$, s.t.

$$g_x(z) > f(z) - \varepsilon \text{ for } z \in B(x,\delta_x).$$

We have $X = \bigcup_{x\in X} B(x,\delta_x)$ so there are $x_1,\dots,x_n \in X$ so $X = \bigcup_{j=1}^n B(x_j,\delta_{x_j})$. We then let

$$g = g_{x_1} \vee \cdots \vee g_{x_n} \in \mathcal{L}.$$

For $z \in X, z \in B(x_j,\delta_{x_j})$ for some $j = 1,\dots,n$, so

$$g(z) \ge g_{x_j}(z) > \cdots > f(z) - \varepsilon$$

and thus

$$g > f - \varepsilon 1.$$

Furthermore, each $g_{x_j} < f + \varepsilon 1$, so if $z \in X$, then $g(z) = g_{x_j}(z)$ for some j, so

$$g(z) = g_{x_j}(z) < f(z) + \varepsilon \Longrightarrow g < f + \varepsilon 1$$

i.e. $f - \varepsilon 1 < g < f + \varepsilon 1$, so $g \in B(f,\varepsilon)$ in $(C(X), \|\cdot\|_\infty)$.
In summary, given $f \in C(X), \varepsilon > 0$, $B(f,\varepsilon) \cap \mathcal{L} \ne \varnothing$. Hence, $\overline{\mathcal{L}} = C(X)$. □

Corollary 25.1. (i) Let $\mathcal{L} = \{f \in C[a,b] : f \text{ is piecewise affine (A5)}\}$. Then $\overline{\mathcal{L}} = C[a,b]$.

(ii) Let C be the Cantor set and $\mathcal{L} = \{f \in C(C) : |f(C)| < \aleph_0\}$. Then $\overline{\mathcal{L}} = C(C)$.

Definition: Let (X,d) be a (compact) metric space. A subset $A \subseteq C(X)$ is called an algebra if for $f,g \in A, \alpha \in \mathbb{R}$, we have

$$\begin{aligned} f + \alpha g \in A \qquad & (A \text{ is a } \mathbb{R}\text{-subspace}) \\ fg \in A \qquad & (A \text{ is closed under pointwise multplication}) \end{aligned}$$

(If $f,g \in C(X)$, then $fg \in C(X)$, too.) If $f_1,\dots,f_n \in A$, $f_1\cdots f_n \in A$ too.
If $1 \in A$, and $p(t) = \sum_{i=1}^n a_i t^i$, then for $f \in A$,

$$p \circ f = a_0 1 + a_1 f + a_2 f^2 + \cdots + a_n f^n \in A.$$

$(f^k(x) = f(x)^k$ for $x \in X.)$

Theorem 25.1 (Stone-Weierstrauss Theorem)**.** If (X, d) is a compact metric space, $A \subseteq C(X)$ satisfies

- A is an algebra
- $1 \in A$
- A separates points: for $x \neq y$ in X, there is $g \in A$ so $g(x) \neq g(y)$

Then $\overline{A} = C(X)$ (uniform closure).

Proof. (I) If $f \in A$, then $|f| \in \overline{A}$. First, since (X, d) is compact, f continuous, $f(X) \subset \mathbb{R}$ is compact, hence bounded, so there is $a > 0$ s.t. $f(X) \subseteq [-a, a]$. Now, the Weierstrauss approximation theorem provides $(p_n)_{n=1}^{\infty}$ of polynomials s.t. $\|p_n - |\cdot|\|_\infty = \max_{t\in[-a,a]} |p_n(t) - |t|| \to 0$. Hence $\|p_n \circ f - |f|\|_\infty = \max_{x\in X} |p_n(f(x)) - |f(x)|| \to 0$
Each $p_n \circ f \in A$.
(II) Since A is a $\mathbb{R}$-subspace, so is $\overline{A}$ (A4 Q1). If $f, g \in \overline{A}$, let $f = \lim_{n\to\infty} f_n, g = \lim_{n\to\infty} g_n$ under uniform limits, each $f_n, g_n \in A$. Then

$$\begin{aligned} f \vee g &= \tfrac{1}{2}(f+g) + \tfrac{1}{2}|f-g| \\ &= \lim_{n\to\infty} \underbrace{\tfrac{1}{2}(f_n+g_n)}_{\in A \subseteq \overline{A}} + \underbrace{\tfrac{1}{2}|f_n - g_n|}_{\in A \text{ by (I)}} \in \overline{A} \end{aligned}$$

since $\overline{A}$ is closed.
Also, $f \wedge g = -(-f) \vee (-g) \in \overline{A}$ as well.
$\implies \overline{A}$ is a $\mathbb{R}$-subspace and a lattice. Also, $1 \in A \subseteq \overline{A}$, and A separates points, hence $\overline{A}$ separates points.
Thus $\overline{A}$ is dense in $C(X)$, but is closed, so $\overline{A} = C(X)$.

□

26 2017-11-24

<u>Example:</u> Let $I = [a_1, b_1] \times \cdots \times [a_n, b_n]$ be a non-empty compact interval in $\mathbb{R}^n$. A <u>polynomial</u> on I is any function

$$p(t_1, \ldots, t_n) = \sum_{j_1,\ldots,j_n=1}^{N} a_{j_1,\ldots,j_n} t_1^{j_1} \cdots t_n^{j_n}$$

where each $a_{j_1,\ldots,j_n} \in \mathbb{R}, N \in \mathbb{N}$. By Stone-Weierstrauss Theorem, the family $P(I)$ of polynomial functions is dense in $C(I)$.
<u>Example:</u> Let $(X, d_X), (Y, d_Y)$ be compact metric spaces. Let $\|\cdot\|$ be a norm on $\mathbb{R}^2$. Define

$$\rho(X \times Y) \times (X \times Y) \to [0, \infty) \text{ by}$$

$$\rho((x_1, y_1), (x_2, y_2)) = \|(d_X(x_1, x_2), d_Y(y_1, y_2))\|.$$

It is "obvious" that ρ is a metric on $X \times Y$.
(Usually, $\|\cdot\| = \|\cdot\|_\infty, \|\cdot\|_1, \|\cdot\|_2$ on $\mathbb{R}^2$.)
Furthermore, $(X \times Y, \rho)$ is compact. Indeed, let $((x_n, y_n))_{n=1}^{\infty} \subseteq X \times Y$ be a sequence. Then $(x_n)_{n=1}^{\infty} \subseteq X$ admits a converging subsequence: let $x = \lim_{k\to\infty} x_{n_k} \in X$. Then $(y_{n_k})_{k=1}^{\infty} \subseteq Y$ admits a converging subsequence: let $y = \lim_{\ell\to\infty} y_{n_{k_\ell}} \in Y$.
Notice that

$$\begin{aligned} &\rho((x, y), (x_{n_{k_\ell}}, y_{n_{k_\ell}})) \\ &= \left\|(d_X(x, x_{n_{k_\ell}}), d_Y(y, y_{n_{k_\ell}}))\right\| \\ &\le d_X(x, x_{n_{k_\ell}})\|(1,0)\| + d_Y(y, y_{n_{k_\ell}})\|(0,1)\| \\ &\xrightarrow{\ell\to\infty} 0. \end{aligned}$$

Hence $((x_{n_{k_\ell}}, y_{n_{k_\ell}}))_{\ell=1}^{\infty}$ is a converging subsequence of $((x_n, y_n))_{n=1}^{\infty}$. Suppose that each $A_X \subseteq C(X)$ and $A_Y \subseteq C(Y)$, each satisfy assumptions of Stone-Weierstrauss Theorem. If $f \in A_X, g \in A_Y$,

$$f \otimes g : X \times Y \to \mathbb{R}, f \otimes g(x, y) = f(x)g(y).$$

Let $A_X \otimes A_Y = \operatorname{span}_{\mathbb{R}}\{f \otimes g : f \in A_X, g \in A_Y\}$. Convince yourself that $A_X \otimes A_Y \subseteq C(X \times Y)$ and satisfies assumptions of Stone-Weierstrauss Theorem.
Hence $\overline{A_X \otimes A_Y} = C(X \times Y)$ (uniform closure).

Corollary 26.1 (Stone-Weierstrauss without constant functions)**.** Let (X, d) be a compact metric space, and $A \subseteq C(X)$ satisfy

- A is an algebra
- A separates points
- there is x_0 in X s.t. $f(x_0) = 0$ for f in A.

Then $\overline{A} = C_{x_0}(X) := \{f \in C(X) : f(x_0) = 0\}$.

Proof. First, $C_{x_0}(X)$ is closed in $C(X)$. (Let $\varphi : C(X) \to \mathbb{R}, \varphi(f) = f(x_0)$, which is linear and continuous: $\|\varphi\| \le 1$ (seen before). Then $C_{x_0}(X) = \varphi^{-1}(\{0\}) = \underbrace{C(X) \setminus \underbrace{\varphi^{-1}(\underbrace{\mathbb{R} \setminus \{0\}}_{\text{open}})}_{\text{open}}}_{\text{closed}}$. Since $A \subseteq C_{x_0}(X) \implies \overline{A} \subseteq C_{x_0}(X)$.)

Second, note that $\mathbb{R}1 + A = \{\alpha 1 + f : \alpha \in \mathbb{R}, f \in A\}$ satisfies $\overline{\mathbb{R}1 + A} = C(X)$. If $g \in \mathbb{R}1 + A$, write $g = \alpha 1 + h, \alpha \in \mathbb{R}, h \in A$, and $g(x_0) = \alpha + h(x_0) = \alpha$ so $g = g(x_0)1 + h$.
Now, if $f \in C_{x_0}(X)$, there exists $(g_n)_{n=1}^{\infty} \subseteq \mathbb{R}1 + A$ s.t. $\|f - g\|_\infty \xrightarrow{n\to\infty} 0$ (Stone-Weierstrauss Theorem). Write each $g_n = g_n(x_0)1 + h_n$ where $h_n \in A$. Notice that $0 = f(x_0) = \lim_{n\to\infty} g_n(x_0)$. Hence

$$\begin{aligned}
\|f - h_n\|_\infty &\le \|f - (g_n(x_0)1 + h_0)\|_\infty + \|g_n(x_0)\|_\infty \\
&= \|f - g_n\|_\infty + |g_n(x_0)| \qquad (\|1\|_\infty = 1) \\
&\xrightarrow{n\to\infty} 0.
\end{aligned}$$

Thus $C_{x_0}(X) \subseteq \overline{A}$. $\square$

<u>Def:</u> Let $C_\infty(\mathbb{R}) = \{\overline{f} \in C(\mathbb{R}) : \lim_{|t|\to\infty} f(t) = 0\}$. Then $C_\infty(\mathbb{R}) \underbrace{\subseteq}_{\text{exercise}} C_b(\mathbb{R})$ and is a closed subspace. ($L_\pm : C_b(\mathbb{R}) \to \mathbb{R}, L_\pm(f) = \lim_{t\to\pm\infty} f(t)$, then L_+, L_- are linear and with $\|L_\pm\| \le 1$. Then $C_\infty(\mathbb{R}) = L_+^{-1}(\{0\}) \cap L_-^{-1}(\{0\})$ is closed.)

Corollary 26.2. Let $A \subseteq C_\infty(\mathbb{R})$ satisfy that

- A is an algebra
- A separates points
- for each t of $\mathbb{R}$, there is $f \in A$ s.t. $f(t) \neq 0$.

Then $\overline{A} = C_\infty(\mathbb{R})$ (uniform closure).

Proof. (Sketch of proof) $\psi : \mathbb{R} \to (-1, 1), \psi(t) = \frac{t}{|t|+1}$, then ψ is continuous and bijective with $\psi^{-1}(-1, 1) \to \mathbb{R}$ continuous. Let $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

$$\begin{aligned}
&\varphi(-1, 1) \to S \setminus \{(-1, 0)\} \\
&\quad \varphi(s) = (\cos(\pi s), \sin(\pi s))
\end{aligned}$$

so φ is a continuous bijection with continuous inverse. Hence, $\varphi \circ \psi : \mathbb{R} \to S \setminus \{(-1,0)\}$ is a homeomorphism, i.e. continuous bijection with continuous inverse.
Define

$$\begin{aligned}\Psi &: C_\infty(\mathbb{R}) \to C_{(-1,0)}(S)\\ \Psi(f)(x,y) &= f(\psi^{-1} \circ \varphi^{-1}(x,y)).\end{aligned}$$

Check that Ψ is a surjective isometry, between $(C_\infty(\mathbb{R}), \|\cdot\|_\infty)$ and $(C_{(-1,0)}(S), \|\cdot\|_\infty)$, and hence has isometric inverse.
We have $\Psi(A) \subseteq C_{(-1,0)}(S)$ satisfies assumptions of last corollary, so $\overline{\Psi(A)} = C_{(-1,0)}(S)$ but it follows that $\overline{A} = \Psi^{-1}(\overline{\Psi(A)}) = C_\infty(\mathbb{R})$. □

27 2017-11-27

Today's subject: towards Arzela-Ascoli Theorem (by guest lecturer)

Def: Let (X,d) be a complete metric space. Let $F \subseteq X$ be a subset. We say F is relatively compact if $\overline{F}$ is compact. (Here $\overline{F}$ means the closure of F.)

Proposition 27.1 (Properties of relatively compact subsets)**.** Let (X,d) be a metric space, $F \subseteq X$. TFAE:

1. F is relatively compact
2. Every sequence (x_n) admits a Cauchy subsequence (x_{n_k})
3. F is totally bounded

Proof. (i) $\implies$ (ii) Let (x_n) be a sequence in F. Since (x_n) is in $\overline{F}$ and $\overline{F}$ is compact, (x_n) has a Cauchy subsequence (x_{n_k}) (that may converge to a point in $\overline{F} \setminus F$).
(ii) $\implies$ (i) Let (x_n) be a sequence in $\overline{F}$. We want to show there is a subsequence (x_{n_k}) converging to a point in $\overline{F}$ (note this is nonempty by characterization of the closure).
Now, by (ii), there is a Cauchy subsequence (y_{n_k}).
Claim: (x_{n_k}) is Cauchy.
For $k, \ell \geq 1$,

$$\begin{aligned}d(x_{n_k}, x_{n_\ell}) &\leq d(x_{n_k}, y_{n_k}) + d(x_{n_k}, y_{n_\ell}) + d(x_{n_\ell}, y_{n_\ell})\\ &\leq \frac{1}{n_k} + d(y_{n_k}, y_{n_\ell}) + \frac{1}{n_\ell} \xrightarrow{k,\ell\to\infty} 0.\end{aligned}$$

(i) $\implies$ (iii) $\overline{F}$ is totally bounded since it is compact. So for $\frac{\varepsilon}{2} > 0$, there are $x_1, \ldots, x_n \in \overline{F}$ s.t. $B(x, \frac{\varepsilon}{2})$ covers $\overline{F}$ (i.e. $\bigcup_{i=1}^n B(x_i, \frac{\varepsilon}{2}) \supseteq \overline{F}$.)
For each i, choose $y_i \in B(x_i, \frac{\varepsilon}{2}) \cap F$. Then $B(y_1, \varepsilon) \supseteq B(x_i, \frac{\varepsilon}{2})$ so $y_1, \ldots, y_n$ is an ε-net for F.
(iii) $\implies$ (i) Since F is totally bounded, there is an ε-net $y_1, \ldots, y_n \in F$. So

$$\begin{aligned}F &\subseteq \bigcup_{i=1}^n B(y_i, \varepsilon)\\ \implies \overline{F} &\subseteq \bigcup_{i=1}^n \overline{B(y_i, \varepsilon)}\\ &\implies \bigcup_{i=1}^n B(y_i, 2\varepsilon).\end{aligned}$$

So $\overline{F}$ is totally bounded. □

Def: [Equicontinuity] Let (X,d) be a (compact) metric space. A subset $F \subseteq C(X)$ is equicontinuous if for $\varepsilon > 0$ and $x \in X$ there is $\delta > 0$ s.t. if $d(x,y) < \delta$ then $|f(y) - f(x)| < \varepsilon \forall f \in F$ (holds for all f simultaneously).

Lemma 27.1. If (X, d) is compact and $F \subseteq C(X)$ then F is equicontinuous $\iff$ F is uniformly equicontinuous meaning for $\varepsilon > 0$ there is $\delta > 0$ s.t. if $x, y \in X$ and $d(x, y) < \delta$ then $|f(x) - f(y)| < \varepsilon \forall f \in F$.

Proof. If F is uniformly equicontinuous it is clearly equicontinuous.
For the other direction, fix $\varepsilon > 0$. For each x there is δ_x s.t. if $d(x, y) < \delta_x$ then $|f(y) - f(x)| < \varepsilon/2 \forall f \in F$. Then $(B(x, \delta_x))_{x \in X}$ is an open cover. Let $\delta > 0$ be the corresponding Lebesgue covering number. So for any $y \in X$, $B(y, \delta) \subseteq B(x, \delta_x)$ for some $x \in X$. So if $y, z \in X$ with $d(y, z) < \delta$, choose $x \in X$ s.t. $B(y, \delta) \subseteq B(x, \delta_x)$, then

$$\begin{aligned} |f(y) - f(z)| &\leq |f(y) - f(x)| + |f(x) - f(z)| \qquad (z \in B(x, \delta_x)) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

□

Ex: Let F be a set of differentiable functions from $[0, 1]$ to $\mathbb{R}$ s.t. $|f'(x)| \leq M \forall f \in F, x \in [0, 1]$ for some M. By the MVT, for $x, y \in [0, 1]$ there is $z \in [0, 1]$ s.t. $M \geq |f'(z)| = \frac{|f(y) - f(x)|}{|y - x|}$.

$$|f(y) - f(x)| \leq M|y - x| \forall y, x \in [0, 1], \forall f \in F.$$

Now take $\delta = \frac{\varepsilon}{M}$. Then if $|x - y| < \delta$ then

$$\begin{aligned} |f(x) - f(y)| &\leq M|x - y| \\ &< M\frac{\delta}{M} = \delta. \end{aligned}$$

28 2017-11-29

Office Hours:
Today: 2:30-4:30
Tomorrow: 2-4 pm

Last time:
In complete (X, d), TFAE:

(i) relative compactness

(ii) every sequence admits a Cauchy subsequence

(iii) total boundedness

Discussed for $F \subset C(X)$:

- equicontinuity $\implies$ uniform equicontinuity if (X, d) compact
- pointwise boundedness

Theorem 28.1 (Arzela-Ascoli Theorem)**.** Let (X, d) be a compact metric space, $F \subset C(X)$. Then

$$F \text{ is relatively compact in } (C(X), \|\cdot\|_\infty) \iff F \text{ is both equicontinuous and pointwise bounded.}$$

Proof. ($\implies$) F is totally bounded. In particular, F is bounded: $\sup_{f \in F} \|f\|_\infty < \infty$ (totally bounded $\implies$ bounded). Hence for x in X, $\sup_{f \in F} |f(x)| < \sup_{f \in F} \sup_{x \in X} |f(x)| = \sup_{f \in F} \|f\|_\infty < \infty$.
Given $\varepsilon > 0$, let $f_1, \ldots, f_n \in F$ s.t. $F \subseteq \bigcup_{j=1}^n B[f_j, \frac{\varepsilon}{3}]$. Let for $j = 1, \ldots, n$, $\delta_j > 0$ be so for x, y in X, $d(x, y) < \delta_j \implies |f_j(x) - f_j(y)| < \frac{\varepsilon}{3}$ (uniform continuity of f_j). Then let $\delta = \min\{\delta_1, \ldots, \delta_n\}$ and then for x, y in X, $d(x, y) < \delta$, we have for f in F, then $f \in B[f_j, \frac{\varepsilon}{3}]$ for some j. Then

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f(y)| \\ &< \|f - f_j\|_\infty + \tfrac{\varepsilon}{3} + \|f - f_j\|_\infty \\ &< \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Hence, F is (uniformly) equicontinuous, thus equicontinuous.
($\Longleftarrow$) Let $(x_n)_{n=1}^\infty \subset X$ satisfy that there are $n_1 < n_2 < n_3 < \cdots$ for which

$$X = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{n_k} B[x_j, \tfrac{1}{k}] \qquad (\dagger)$$

(assignment 5, (X,d) compact $\Longrightarrow$ (X,d) separable).
Now, let $(f_n)_{n=1}^\infty \subseteq F$. We wish to extract a uniformly Cauchy subsequence, hence showing F is relatively compact.
(I) Let us extract a candidate Cauchy subsequence. This technique is a variant of "Cantor's diagonalization argument". First, $(f_n(x_1))_{n=1}^\infty \subset \mathbb{R}$ is bounded (pointwise bounded assumption) so by Bolzano-Weierstrauss admits a Cauchy subsequence $(f_{n_k}(x_1))_{k=1}^\infty \subset \mathbb{R}$. Let $f_{1,k} = f_{n_k}$ for each k. Second, $(f_{1,n}(x_2))_{n=1}^\infty \subset \mathbb{R}$ is bounded, and again admits a Cauchy subsequence $(f_{1,n_k}(x_2))_{k=1}^\infty \subset \mathbb{R}$. Let $f_{2,k} = f_{1,n_k}$.
Inductively, we continue. We build sequences $(f_{1,k})_{k=1}^\infty, (f_{2,k})_{k=1}^\infty, \dots, (f_{n,k})_{k=1}^\infty, \dots \subseteq F$ which satisfy

- $m < n$, $(f_{n,k})_{k=1}^\infty$ is a subsequence of $(f_{m,k})_{k=1}^\infty$
- $(f_{n,k}(x_n))_{k=1}^\infty \subset \mathbb{R}$ is Cauchy.

We now let

$$g_n = f_{n,n}.$$

Then $(g_n)_{n=m}^\infty$ is a subsequence of $(f_{m,n})_{n=1}^\infty$ so $(g_n(x_m))_{n=1}^\infty$ is Cauchy in $\mathbb{R}$, (being a subsequence of $(f_{m,n}(x_m))_{n=1}^\infty$). Thus $(g_n(x_m))_{m=1}^\infty$ is Cauchy for each m in $\mathbb{N}$, and $(g_k)_{k=1}^\infty$ is a subsequence of $(f_n)_{n=1}^\infty$.
(II) Let us show that $(g_n)_{n=1}^\infty$ is Cauchy in $(C(X), \|\cdot\|_\infty)$, i.e., Cauchy in $\|\cdot\|_\infty$.
Given $\varepsilon > 0$, our set F, being equicontinuous on compact (X,d), is uniformly equicontinuous (lemma Monday), so there is $\delta > 0$ s.t. $|f(x) - f(y)| < \frac{\varepsilon}{3}$ whenever $x, y \in X$, $d(x,y) < \delta$ and $f \in F$.
Now, let k in $\mathbb{N}$ satisfy $\frac{1}{k} < \delta$, and we have from ($\dagger$) that $X = \bigcup_{j=1}^{n_k} B[x_j, \frac{\varepsilon}{3}]$.
Now, for $j = 1, \dots, n_k$, let N_j in $\mathbb{N}$ be s.t. $m, n \geq N_j \Longrightarrow |g_m(x_j) - g_n(x_j)| < \frac{\varepsilon}{3}$ (i.e. $(g_n(x_j))_{n=1}^\infty$ is Cauchy). Let $N = \max\{N_1, \dots, N_{n_k}\}$. If $x \in X$, so $x \in B[x_j, \frac{\varepsilon}{3}]$ for some $j = 1, \dots, n_k$, and we have for $m, n \geq N$ that

$$\begin{aligned} |g_m(x) - g_n(x)| &\leq |g_m(x) - g_m(x_j)| + |g_m(x_j) - g_n(x_j)| + |g_n(x_j) - g_n(x)| \\ &< \underbrace{\tfrac{\varepsilon}{3}}_{\text{thanks to uniform equicontinuity of } F;\, g_n \in F} + \underbrace{\tfrac{\varepsilon}{3}}_{n,m \geq N \geq N_j \text{ Cauchy at } x_j} \\ &\quad + \underbrace{\tfrac{\varepsilon}{3}}_{\text{thanks to uniform equicontinuity of } F;\, g_n \in F} = \varepsilon. \end{aligned}$$

Hence $\|g_m - g_n\|_\infty = \max_{x \in X} |g_m(x) - g_n(x)| < \varepsilon$.

– END OF FINAL LINE (except Assignment 7) – □

29 2017-12-01

Theorem 29.1 (Peano's Theorem)**.** Let $D \subset \mathbb{R}^2$ be open and $F : D \to \mathbb{R}$ be continuous, and $(t_0, y_0) \in D$. Then there are $a < b$ in $\mathbb{R}$ so $t_0 \in (a,b)$ for which

$$\text{(IVP)} \qquad f'(t) = F(t, f(t)), f(t_0) = y_0, t \in (a,b)$$

admits a solution.

(This is stronger than Picard-Lindelof, which required a Lipschitz condition on the second variable of a two variable function.) The solution here may not be unique.

Proof. (Most of proof):
(I) (Get $a < b$.) Let $R = [a_1, b_1] \times [a_2, b_2] \subset D$ (compact interval) so $(t_0, y_0) \in R^\circ$ (interior), and let $M = \max_{(t,y) \in R} |F(t,y)|$.

We let

$$W = \{(t,y) \in D : |y - y_0| \leq M|t - t_0|\}$$

and $a < b$ in $\mathbb{R}$ so

$$([a,b] \times \mathbb{R}) \cap W \subset R.$$

(II) (Work on $[t_0, b]$, find a particular family of piecewise affine functions.) Given $\varepsilon > 0$, the uniform continuity of F on R provides $\delta > 0$ such that

$$\begin{aligned}
&(s,x),(t,y) \in R \text{ with } \max\{|s-t|,|x-y|\} = \|(s,x)-(t,y)\|_\infty < \delta \\
&\Longrightarrow |F(s,x) - F(t,y)| < \varepsilon.
\end{aligned}$$

We partition $[t_0, b], t_0 < t_1 < \cdots < t_n = b$, so $\max_{j=1,\ldots,n}(t_j - t_{j-1}) < \frac{\delta}{M+1}$ (let $M = 0$).
We define $f_\varepsilon : [t_0, b] \to \mathbb{R}$ inductively by

$$f_\varepsilon(t) = \begin{cases} y_0 + F(t_0, y_0)(t - t_0) & t \in [t_0, t_1] \\ f_\varepsilon(t_1) + F(t_1, f_\varepsilon(t_1))(t - t_1) & t \in (t_1, t_2] \\ \quad\vdots & \\ f_\varepsilon(t_{n-1}) + F(t_{n-1}, f_\varepsilon(t_{n-1}))(t - t_{n-1}) & t \in (t_{n-1}, t_n] \end{cases}.$$

Two nice properties (exercise):

- graph of f_ε on $[t_0, b]$ is in R, so $\max_{t \in [t_0,b]} |f_\varepsilon(t)| \leq \max\{|a_2|, |b_2|\}$
- if $s < t$ in $[t_0, b]$, then $|f_\varepsilon(t) - f_\varepsilon(s)| \leq M|t - s| \qquad (\dagger)$.

These estimates are independent of ε. I.e. if we form $K = \{f_\varepsilon\}_{\varepsilon \in (0,\infty)}$ it is

- pointwise bounded & equi-Lipschitz $\Longrightarrow$ (uniformly) equicontinuous.

Hence K is relatively compact.
(III) (Relate $K = \{f_\varepsilon\}_{\varepsilon \in (0,\infty)}$ to the (IVP).) Fix $f_\varepsilon, \varepsilon$ and δ as in $(\varepsilon - \delta)$ above. If $t \in (t_j, t_{j+1}), j = 0, \ldots, n-1$ then

$$f'_\varepsilon(t) = F(t_j, f_\varepsilon(t_j)). \qquad (\star)$$

Also, for such t as above, then $|t - t_j| < \frac{\delta}{M+1}$ so by $(\dagger)$

$$|f_\varepsilon(t) - f_\varepsilon(t_j)| \leq M|t - t_j| \leq \delta \frac{M}{M+1} < \delta$$

so, by choice of δ,

$$\begin{aligned}
&|F(t, f_\varepsilon(t)) - F(t_j, f_\varepsilon(t_j))| < \varepsilon \\
\text{(using } (\star)) \ \Longrightarrow &|F(t, f_\varepsilon(t)) - f'_\varepsilon(t)| < \varepsilon \qquad (\star\star).
\end{aligned}$$

Thus for $t \in [t_0, b]$ we have

$$\begin{aligned}
f_\varepsilon(t) &= y_0 + \int_{t_0}^t f'_\varepsilon(s)ds \text{ (piecing together F.T. of C., as } f'_\varepsilon(t) \text{ exists except at } t_1, \ldots, t_{n-1}) \\
&= y_0 + \int_{t_0}^t F(s, f_\varepsilon(s))ds + \int_{t_0}^t [f'_\varepsilon(s) - F(s, f_\varepsilon(s))]ds
\end{aligned}$$

Let $\widetilde{f}_\varepsilon(t) = y_0 + \int_{t_0}^t F(s, f_\varepsilon(s))ds$, and we have for $t \in [t_0, b]$

$$\begin{aligned} |f_\varepsilon(t) - \widetilde{f}_\varepsilon(t)| &\leq \int_{t_0}^t |\underbrace{f'_\varepsilon(s) - F(s, f_\varepsilon(s))}_{<\varepsilon}|ds \\ (\star\star\star) \qquad &\leq (t - t_0)\varepsilon \leq (b - t_0)\varepsilon. \end{aligned}$$

We now consider a sequence $(f_{\frac{1}{n}})_{n=1}^\infty \subseteq K$. By relative compactness, we get a uniformly Cauchy, hence uniformly converging subsequence $(f_{\frac{1}{n_k}})_{k=1}^\infty, f = \lim_{k\to\infty} f_{\frac{1}{n_k}}$ (uniform limit). Let $\widetilde{f}(t) = y_0 + \int_{t_0}^t F(s, f(s))ds$.
We have

$$\left\|f - \widetilde{f}\right\|_\infty \leq \left\|f - f_{\frac{1}{n_k}}\right\|_\infty + \left\|f_{\frac{1}{n_k}} - \widetilde{f}_{\frac{1}{n_k}}\right\|_\infty + \left\|\widetilde{f}_{\frac{1}{n_k}} - \widetilde{f}\right\|_\infty$$

We have $\lim_{k\to\infty} f_{\frac{1}{n_k}}(s) = f(s)$ uniformly for $s \in [t_0, b]$, so, by uniform continuity $\lim_{k\to\infty} |F(s, f_{\frac{1}{n_k}}(s)) - F(s, f(s))| = 0$ uniformly for s in $[t_0, b]$, and thus $(\ddagger) \xrightarrow{k\to\infty} 0$. In conclusion

$$\left\|f - \widetilde{f}\right\|_\infty \leq \left\|\widetilde{f}_{\frac{1}{n_k}}\right\| + (b - t_0)\frac{1}{n_k} + (\ddagger)$$

$\implies f(t) = \widetilde{f}(t) = y_0 + \int_{t_0}^t F(s, f(s))ds$, i.e. f satisfies (IE) $\implies$ (IVP).

$\square$