

<https://github.com/friedeggs>

PMATH 351

REAL ANALYSIS

PROF: NICO SPRONK • FALL 2017 • UNIVERSITY OF WATERLOO

Last Revision: December 7, 2017

TABLE OF CONTENTS

| | | |
|-----------|---|-----------|
| 1 | Chains and Zorn's Lemma | 1 |
| 2 | Cardinal arithmetic | 1 |
| 3 | 2017-09-18 | 3 |
| 3.1 | Last class: C.B.S Theorem | 3 |
| 4 | 2017-09-22 | 5 |
| 4.1 | Metric Spaces | 5 |
| 5 | 2017-09-25 | 6 |
| 5.1 | The ℓ_p -spaces | 8 |
| 6 | 2017-09-29 | 9 |
| 7 | 2017-10-02 | 10 |
| 7.1 | Continuity | 11 |
| 8 | 2017-10-06 | 12 |
| 9 | 2017-10-16 | 12 |
| 9.1 | Characterizations of Completeness | 12 |
| 10 | 2017-10-18 | 13 |
| 11 | 2017-10-20 | 16 |
| 11.1 | Completeness of Metric Spaces | 17 |
| 12 | 2017-10-23 | 18 |
| 12.1 | Compactness | 19 |
| 13 | 2017-10-25 | 20 |
| 14 | 2017-10-27 | 21 |
| 15 | 2017-10-30 | 22 |
| 16 | 2017-11-01 | 25 |
| 17 | 2017-11-03 | 27 |

| | | |
|-----------|---|-----------|
| 18 | 2017-11-06 | 29 |
| 19 | 2017-11-08 | 31 |
| 19.1 | Baire Category Theorem | 31 |
| 20 | 2017-11-10 | 33 |
| 21 | 2017-11-13 | 35 |
| 21.1 | Baire-1 Functions | 35 |
| 22 | 2017-11-15 | 37 |
| 22.1 | On the Banach spaces $C(X)$, X compact | 37 |
| 23 | 2017-11-17 | 39 |
| 24 | 2017-11-20 | 41 |
| 24.1 | Towards Stone-Weierstrauss Theorem | 42 |
| 25 | 2017-11-22 | 43 |
| 26 | 2017-11-24 | 44 |
| 27 | 2017-11-27 | 46 |
| 28 | 2017-11-29 | 47 |
| 29 | 2017-12-01 | 48 |

LIST OF THEOREMS

| | | |
|------|---|----|
| 2.1 | Theorem (Cantor-Bernstein-Schroder Theorem) | 2 |
| 3.1 | Proposition (surjectivity) | 3 |
| 3.1 | Theorem (Comparison Theorem) | 3 |
| 3.2 | Proposition | 4 |
| 3.1 | Corollary | 5 |
| 3.2 | Theorem (Cantor) | 5 |
| 4.1 | Lemma | 5 |
| 5.1 | Theorem (Minkowski's Inequality) | 6 |
| 5.1 | Corollary | 7 |
| 5.2 | Theorem | 8 |
| 5.3 | Theorem | 8 |
| 6.1 | Proposition | 9 |
| 6.2 | Proposition (characterizations of interior) | 10 |
| 6.3 | Proposition | 10 |
| 7.1 | Theorem (characterization of the closure) | 11 |
| 7.2 | Theorem (characterization of continuity at a point) | 12 |
| 8.1 | Corollary | 12 |
| 8.1 | Theorem | 12 |
| 9.1 | Theorem | 12 |
| 9.1 | Proposition | 13 |
| 9.2 | Theorem (Nested set characterization of completeness) | 13 |
| 10.1 | Theorem | 13 |
| 10.2 | Theorem (abstract M -test) | 14 |
| 11.1 | Corollary | 16 |
| 11.1 | Theorem | 17 |
| 12.1 | Proposition | 19 |
| 12.2 | Proposition | 19 |
| 12.1 | Theorem (continuous image of compact is compact) | 19 |
| 12.1 | Corollary (Extreme Value Theorem) | 19 |
| 12.2 | Theorem (finite intersection property) | 20 |
| 13.1 | Theorem (Characterizations of compact metric spaces) | 20 |
| 14.1 | Corollary | 22 |
| 15.1 | Theorem (sequential characterization of uniform continuity) | 23 |
| 15.2 | Theorem (continuous on compact is uniformly continuous) | 23 |
| 15.3 | Theorem | 24 |
| 15.1 | Corollary | 24 |
| 16.1 | Proposition | 25 |
| 17.1 | Theorem (Banach's Contraction Mapping Theorem) | 27 |
| 17.2 | Theorem (Picard-Lindelof Theorem) | 28 |
| 18.1 | Theorem (Edelstein) | 30 |
| 19.1 | Lemma | 32 |
| 19.1 | Theorem (Baire Category Theorem) | 32 |
| 20.1 | Corollary | 33 |
| 20.2 | Corollary | 34 |
| 20.1 | Theorem (Uniform Boundedness Principle) | 34 |
| 20.3 | Corollary (Banach-Stenhaus Theorem) | 34 |
| 21.1 | Theorem (Baire) | 35 |
| 21.1 | Corollary | 37 |
| 22.1 | Corollary | 37 |
| 22.1 | Lemma | 37 |
| 22.1 | Theorem (Weierstrauss approximation theorem) | 38 |
| 23.1 | Corollary | 40 |

| | |
|--|----|
| 23.2 Corollary | 40 |
| 23.1 Theorem (nowhere differentiable functions are generic) | 40 |
| 24.1 Theorem | 41 |
| 24.1 Proposition | 42 |
| 24.2 Theorem (Stone) | 42 |
| 25.1 Corollary | 43 |
| 25.1 Theorem (Stone-Weierstrauss Theorem) | 44 |
| 26.1 Corollary (Stone-Weierstrauss without constant functions) | 45 |
| 26.2 Corollary | 45 |
| 27.1 Proposition (Properties of relatively compact subsets) | 46 |
| 27.1 Lemma | 47 |
| 28.1 Theorem (Arzela-Ascoli Theorem) | 47 |
| 29.1 Theorem (Peano's Theorem) | 48 |

Abstract

These notes are intended as a resource for myself; past, present, or future students of this course, and anyone interested in the material. The goal is to provide an end-to-end resource that covers all material discussed in the course displayed in an organized manner. If you spot any errors or would like to contribute, please contact me directly.

1 CHAINS AND ZORN'S LEMMA

Let (X, \leq) be a poset. A chain is any subset $C \subseteq X$ such that (C, \leq) is totally ordered.

Office hours:

1. Today 2:30 - 3:20
2. Wednesday next week 2:30 - 4:30

Or, email nspronk@uwaterloo.ca

2 CARDINAL ARITHMETIC

i. $:($

ii. $\mathbb{R} \underset{f}{\sim} (-1, 1), f(x) = x/|x| + 1$ (exercise: exhibit f^{-1})

iii. $a < b$ in $\mathbb{R}.(0, 1) \underset{g}{\sim} (a, b), g(x) = a + x(b - a)$

Notation: $\aleph_0 = |\mathbb{N}|$ ("aleph naught"), $c = |\mathbb{R}|$ ("continuous")

Arithmetic: Let A, B be sets.

$$|A| + |B| = |A \sqcup B|$$

$$|A||B| = |A \times B|$$

$$|A|^{|B|} = |A^B| (B \neq \emptyset, A^B = \{f : B \rightarrow A \mid \text{function}\})$$

$A \sqcup A$ is two copies of A , $\sim A \times \{1, 2\}$

Properties

- (commutativity) $|A| + |B| = |B| + |A|$, $|A||B| = |B||A|$
- (distributivity) $|A|(|B| + |C|) = |A||B| + |A||C|$

$$A \times (B \sqcup C) \sim (A \times B) \sqcup (A \times C)$$

- (Exponential laws)

$$|A|^{|B|+|C|} = |A|^{|B|}|A|^{|C|}, |A|^{|B||C|} = (|A|^{|B|})^{|C|}$$

$$(B \neq \emptyset \neq C)$$

$$A^{B \sqcup C} \sim A^B \times A^C \text{ via } \varphi \mapsto (\varphi|_B, \varphi|_C)$$

$$A^{B \times C} \sim (A^B)^C \text{ via } \varphi \mapsto (\varphi(b, \cdot) : C \rightarrow A)$$

Now, for sets A, B , define $A \preceq B$ if there is an injection $f : A \rightarrow B$.

Sometimes write $A \preceq B$. As above:

$$\text{(reflexivity)} \quad A \preceq A$$

$$\text{(transitivity)} \quad A \preceq B, B \preceq C \implies A \preceq C$$

Seems reasonable to write $|A| \leq |B|$, in this case.

Question: Is \leq in cardinal numbers anti-symmetric?

Theorem 2.1 (Cantor-Bernstein-Schroder Theorem). If, for non-empty set A, B we have $A \preceq B, B \preceq A$, then $A \sim B$. I.e. if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Proof. Our assumption is that we have injections $A \xrightarrow{\varphi} B, B \xrightarrow{\psi} A$.

To avoid triviality, let us suppose that neither φ nor ψ is surjective. Thus $\varphi(A) \subsetneq B, \psi(B) \subsetneq A$.

Let $A_0 = A, A_1 = \psi(B), A_2 = \psi \circ \varphi(A)$ and we inductively define $A_{n+2} = g(A_n), g = \psi \circ \varphi$.

Then $A_2 \subsetneq A_1 \subsetneq A_0$, so by applying injection g ,

$$\begin{aligned} A_2 &\subsetneq A_1 \subsetneq A_0 \\ &\vdots \\ A_{n+1} &\subsetneq A_n \subsetneq A_{n-1} \end{aligned}$$

Hence, we may decompose

$$\begin{aligned} A &= A_0 = (A_0 \setminus A_1) \cup A_1 \\ &= (A_0 \setminus A_1) \cup (A_1 \setminus A_2) \cup A_2 \\ &\vdots \\ &= \bigcup_{n=1}^{\infty} (A_{n-1} \setminus A_n) \cup A_{\infty} \end{aligned}$$

where $A_{\infty} = \bigcap_{n=1}^{\infty} A_n = \bigcap_{n=2}^{\infty} A_n$, we likewise observe

$$A_1 = \bigcup_{n=2}^{\infty} (A_{n-1} \setminus A_n) \cup A_{\infty}.$$

Picture:

$$\underbrace{\overbrace{A_0 \setminus A_1} \quad \overbrace{A_1 \setminus A_2} \quad \dots \quad \overbrace{A_{\infty}}} \quad A_0$$

Using definitions of the sets A_n ($n \geq 2$), we have $g(A_{n-1} \setminus A_n) = A_{n+1} \setminus A_{n+2}$. Define

$$h : A_0 \rightarrow A_1, h(x) = \begin{cases} g(x), & \text{if } x \in A_{n-1} \setminus A_n, n \text{ odd} \\ x, & \text{otherwise} \end{cases}$$

Then h is a bijection. Thus

$$A = A_0 \xrightarrow{h} A_1 = \psi(B), B \xrightarrow{\psi} \psi(B)$$

so we conclude that $A \sim B$. □

Examples:

1. Let $a < b$ in \mathbb{R} . Then $[a, b] \preceq \mathbb{R}$ (obvious)
 $\mathbb{R} \sim (-1, 1) \sim (0, 1) \sim (a, b) \preceq [a, b]$
 I.e. $[a, b] \preceq \mathbb{R}$ and $\mathbb{R} \preceq [a, b]$ so $\mathbb{R} \sim [a, b]$

3 2017-09-18

3.1 LAST CLASS: C.B.S THEOREM

If $A \preceq B$ and $B \preceq A$ then $A \sim B$.

Examples:

- (i) $\mathcal{P}(\mathbb{N}) \sim \mathbb{R}$, i.e. $|\mathcal{P}(\mathbb{N})| = c$.

$$\mathcal{P}(\mathbb{N}) \sim \{0, 1\}^{\mathbb{N}}, \text{ via } A \mapsto \chi_A \text{ where } \chi_A(n) \begin{cases} 1 & , n \in A \\ 0 & , n \notin A \end{cases} \text{ ("characteristic indicator")}$$

$$\{0, 1\}^{\mathbb{N}} \sim \mathcal{P}(\mathbb{N}), \text{ via } (x_k)_{k=1}^{\infty} \xrightarrow[\text{injective}]{\quad} \chi_A \text{ where } \sum_{k=1}^{\infty} \frac{x_k}{3^k} = 0.x_1x_2x_3\ldots \text{ (ternary representation)}$$

$$[0, 1] \sim \{0, 1\}^{\mathbb{N}}, 0.x_1x_2x_3\ldots = \sum_{k=1}^{\infty} \frac{x_k}{2^k} \text{ (binary representation) (never allow } 0.111\ldots = 1!) \mapsto (x_k)_{k=1}^{\infty}$$

$$\mathcal{P}(\mathbb{N}) \sim \{0, 1\}^{\mathbb{N}} \preceq [0, 1] \preceq \{0, 1\}^{\mathbb{N}} \sim \mathcal{P}(\mathbb{N})$$

so, by C.B.S. Theorem, we have $|\mathcal{P}(\mathbb{N})| = |[0, 1]| = c = |\mathbb{R}|$.

(ii)

2nd lecture:

- (iii) $\mathbb{N} \sim \mathbb{Q} \sim \mathbb{N}^2$

$$\mathbb{N} \preceq \mathbb{Q}$$

$$\mathbb{Q} \preceq \mathbb{Z} \times \mathbb{N}, \text{ via } \frac{m}{n} \mapsto (m, n) \text{ (gcd}(m, n) = 1)$$

$$\mathbb{Z} \times \mathbb{N} \sim \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}, \text{ as } \mathbb{Z} \sim \mathbb{N}$$

$$\mathbb{N}^2 \preceq \mathbb{N}, \text{ via } (m, n) \mapsto 2^m 3^n$$

Hence $\mathbb{N} \preceq \mathbb{Q} \preceq \mathbb{Z} \times \mathbb{N} \sim \mathbb{N}^2 \preceq \mathbb{N}$ so, by C.B.S. Theorem, $\mathbb{N} \sim \mathbb{Q} \sim \mathbb{N}^2$.

Notation: We say that a set A is

- countable if $A \preceq \mathbb{N}$, i.e. $|A| \leq \aleph_0$
- denumerable if $A \sim \mathbb{N}$, i.e. $|A| = \aleph_0$

Proposition 3.1 (surjectivity). Suppose X and Y are non-empty sets and there is a surjection $g : X \rightarrow Y$. Then $Y \preceq X$.

Proof. Let $f : \mathcal{P}(X) \setminus \{\emptyset\} \rightarrow X$ be a choice function (AC). For each $y \in Y$, we have $g^{-1}(\{y\}) = \{x \in X : g(x) = y\} \neq \emptyset$, as g is surjective. Define $h : Y \rightarrow X$ be given by $h(y) = f(g^{-1}(\{y\}))$ and h is injective, as if $y_1 \neq y_2$, $\{y_1\} \cap \{y_2\} = \emptyset$, so we see that $g^{-1}(\{y_1\}) \cap g^{-1}(\{y_2\}) = \emptyset$ too. \square

Theorem 3.1 (Comparison Theorem). Let X, Y be sets. Then either $X \preceq Y$ or $Y \preceq X$.

Proof. If $X \neq \emptyset$, then $X \preceq Y$; likewise if $Y = \emptyset$. Hence assume $X \neq \emptyset \neq Y$. We let

$$\Delta = \{(A, f) : A \in \mathcal{P}(X) \setminus \{\emptyset\}, f \in Y^A \text{ is an injection mapping from } A \text{ to } Y\}$$

We observe that $\Delta \neq \emptyset$. If $x \in A, y \in Y$, then $(\{x\}, x \mapsto y) \in \Delta$. On Δ let

$$(A, f) \preceq (B, g) \iff A \subseteq B \subseteq X, g|_A = f$$

Notice that \preceq is reflexive, anti-symmetric, and transitive, hence is a partial order on Δ . Let $\Gamma\{(A_i, f_i)\}_{i \in I}$ be a chain in (Δ, \preceq) . We let $A = \bigcup_{i \in I} A_i$ and $f \in Y^A$ be given by $f(x) = f_i(x)$ provided $x \in A_i$.

Notice that f is well-defined. Say $x \in A_i$ and $x \in A_j$, then, since Γ is a chain, $A_i \subseteq A_j$, say, and $f_j|_{A_i} = f_i$.

Furthermore, if $x_1 \neq x_2$ in A , then $x_1 \in A_{i_1}, x_2 \in A_{i_2}$, and we may suppose $A_{i_1} \subseteq A_{i_2}$. Then $f(x_1) = f_{i_1}(x_1) = f_{i_2}(x_1) \neq f_{i_2}(x_2) = f(x_2)$, so f is an injection. Thus $(A, f) \in \Delta$, and is an upper bound of Γ .

Thus, there is a maximal element $(M, g) \in \Delta$, by Zorn's Lemma.

Case #1: $M = X$. Then $X = M \preceq_g Y$.

Case #2: $M \subsetneq X$. We wish to see that g must be surjective. Suppose not, i.e., there is $y_0 \in Y \setminus g(M)$. Since $M \subsetneq X$, there is $x_0 \in X \setminus M$. Define $h : M \cup \{x_0\} \rightarrow Y$ by

$$h(x) = \begin{cases} g(x) & x \in M \\ y_0 & x = x_0 \end{cases} \text{ injective!}$$

Then $(M \cup \{x_0\}, h) \in \Delta$, and $(M, g) \not\preceq (M \cup \{x_0\}, h)$, contradicting maximality of (M, g) . Thus, we have that g is surjective. Thus $Y \underbrace{\preceq}_{g^{-1}} X$.

□

Proposition 3.2. Let A be a set. Then TFAE:

- (i) $n \leq |A|$ for all $n \in \mathbb{N}$
- (ii) $\aleph_0 \leq |A|$ (A is infinite)
- (iii) there is $B \subsetneq A$ s.t. $|B| = |A|$
- (iv) $1 + |A| = |A|$ (Hilbert hotel)
- (v) $\aleph_0 + |A| = |A|$

Proof. (i) \Rightarrow (ii) We have that for each n in \mathbb{N} there is an injection $\varphi_n : \{1, \dots, n\} \rightarrow A$. Inductively, define $f : \mathbb{N} \rightarrow A$ by

$$f(1) = \varphi_1(1)$$

$$f(n+1) = \varphi_{n+1}(k)$$

where $k = \min j \in \{1, \dots, n+1\} : \varphi_{n+1}(j) \notin \{f(1), \dots, f(n)\}$.

Then f is injective by construction.

(ii) \Rightarrow (iii) We have $\mathbb{N} \preceq_f A$. Let $B = A \setminus \{f(1)\}$. Define $g : A \rightarrow B$ by

$$g(x) = \begin{cases} f(n+1) & \text{if } x = f(n), n \in \mathbb{N} \\ x & \text{otherwise} \end{cases}$$

Then $A \sim_g B$, i.e., $|A| = |B|$.

(iii) \Rightarrow (iv) We suppose there is $x_0 \in A \setminus B$ and $B \sim A$. Thus $A \sim B \preceq B \cup \{x_0\} \preceq A$ so by C.B.S. Theorem $A \sim B$ and

$A \sim B \cup \{x_0\} \sim A \sqcup \{1\}$, i.e. $|A| = |A| + 1$.

(iv) \Rightarrow (i) We have $\{1\} \sqcup A \sim_\varphi A$. Then $\varphi(A) \subsetneq A$. Thus $\varphi \circ \varphi(A) \subsetneq \varphi(A) \subsetneq A$, and, by induction,

$$\varphi^{\circ n}(A) \subsetneq \varphi^{\circ n-1}(A) \subsetneq \cdots \subsetneq A$$

$$\underbrace{\varphi \circ \cdots \circ \varphi}_{n \text{ times}}$$

Hence $|A| \geq |A \setminus \varphi^{\circ n}(A)| \geq n$ (at each stage above, we gain at least one point).

(ii) \Rightarrow (v) We have $\mathbb{N} \preceq_f A$. Let $g : \mathbb{N} \sqcup A \rightarrow A$,

$$g(x) = \begin{cases} f(2n) & \text{if } x = n, n \in \mathbb{N} \\ f(2n+1) & \text{if } x = f(n) \in A, n \in \mathbb{N} \\ x & \text{otherwise} \end{cases}$$

(v) \Rightarrow (ii) $\aleph_0 \leq \aleph_0 + |A| = |A|$ by assumption. □

Corollary 3.1. If $A \in \mathcal{P}(\mathbb{N})$, then either A is finite or denumerable.

Proof. Either $n \leq |A|$ for all n , or $|A| < n$ (Comparison lemma). □

Theorem 3.2 (Cantor). For any set X , $|X| < |\mathcal{P}(X)|$.

Proof. : (□

Cantor's paradox: There is no “set” of all sets.

4 2017-09-22

4.1 METRIC SPACES

Example (French railroad / metro metric): Suppose we have a set $X \neq \emptyset$, and a function $f : X \rightarrow [0, \infty)$ which satisfies $f^{-1}(\{0\}) = \{p_0\}$. Notice, then, that $f(x) > 0$ if $x \in X \setminus \{p_0\}$.

$$d_f : X \times X \rightarrow [0, \infty), \quad d_f(x, y) = f(x) + f(y)$$

if $x \neq y$, 0 if $x = y$.

Easy exercise: this is a metric.

(Belongs to family of weighted graph metrics.)

$$\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$$

$$x^p = \begin{cases} e^{p \log x} & x > 0 \\ 0 & x = 0 \end{cases}$$

Lemma 4.1. Let $\alpha, \beta \geq 0$ in \mathbb{R} , $1 < p < \infty$ and q is chosen so that $\frac{1}{p} + \frac{1}{q} = 1$ (ie $q = \frac{p}{p-1}$) then

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$$

with equality when $\alpha^p = \beta^q$.

Proof. Consider the graph of $y = x^{p-1}$ (assume $p \geq 2$).

$$x = y^1 p - 1 = y^q p = y^{q-1}$$

Then

$$\alpha\beta \leq \underbrace{\int_0^\alpha x^{p-1} dx}_{A_1} + \underbrace{\int_0^\beta y^{q-1} dy}_{A_2}$$

(Equality holds only if $\beta = \alpha^{p-1} \Rightarrow \beta^1 q - 1 \Rightarrow \beta^q = \alpha^p$)

$$= \frac{\alpha^p}{p} + \frac{\beta^q}{q}$$

Holder's Inequality

□

5 2017-09-25

Lemma: $\alpha, \beta \geq 0$ in \mathbb{R} , $1 < p < \infty$ with q satisfying $\frac{1}{p} + \frac{1}{q} \Rightarrow \alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$

Holder's Inequality: If $x, y \in \mathbb{R}^n$, $1 < p < \infty$ and q satisfies $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\left| \sum_{j=1}^n x_j y_j \right| \underbrace{\leq}_{1\text{-ineq. of } |\cdot|} \sum_{j=1}^n |x_j| |y_j| \leq \left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^n |y_j|^q \right)^{\frac{1}{q}} := \|x\|_p \|y\|_q$$

Proof. If $\|x\|_p \|y\|_q = 0$, then $x = 0$ or $y = 0$ and the inequality is trivial. Assume $\|x\|_p \|y\|_q \neq 0$. For $j = 1, \dots, n$, let

$$\alpha_j = \frac{|x_j|}{\|x\|_p}, \quad \beta_j = \frac{|y_j|}{\|y\|_q}.$$

Then

$$\begin{aligned} \frac{1}{\|x\|_p \|y\|_q} \sum_{j=1}^n |x_j| |y_j| &= \sum_{j=1}^n \alpha_j \beta_j \\ &\leq \sum_{j=1}^n \left[\frac{\alpha_j^p}{p} + \frac{\beta_j^q}{q} \right] \text{ by lemma} \\ &= \frac{1}{p} \sum_{j=1}^n \alpha_j^p + \frac{1}{q} \sum_{j=1}^n \beta_j^q \\ &= \frac{1}{p \|x\|_p^p} \sum_{j=1}^n |x_j|^p + \frac{1}{q \|y\|_q^q} \sum_{j=1}^n |y_j|^q \\ &= \frac{1}{p} + \frac{1}{q} \\ &= 1 \end{aligned}$$

□

Theorem 5.1 (Minkowski's Inequality). Let $x, y \in \mathbb{R}^n$ and $1 < p < \infty$. Then

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

Proof. If $x + y = 0$ then this is trivial, so suppose $x + y \neq 0$.

$$\begin{aligned}
\|x + y\|_p^p &= \sum_{j=1}^n |x_j + y_j|^p \\
&= \sum_{j=1}^n |x_j + y_j| |x_j + y_j|^{p-1} \\
&\leq \sum_{j=1}^n (|x_j| + |y_j|) |x_j + y_j|^{p-1} \\
&= \sum_{j=1}^n |x_j| |x_j + y_j|^{p-1} + \sum_{j=1}^n |y_j| |x_j + y_j|^{p-1} \\
&\leq \left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^n |x_j + y_j|^{(p-1)q} \right)^{\frac{1}{q}} + \left(\sum_{j=1}^n |y_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^n |x_j + y_j|^{(p-1)q} \right)^{\frac{1}{q}} \\
&= (\|x\|_p + \|y\|_p) \left(\sum_{j=1}^n |x_j + y_j|^{(p-1)q} \right)^{\frac{1}{q}}
\end{aligned}$$

We have

$$\frac{1}{p} + \frac{1}{q} = 1 \implies \frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p} \implies p = q(p-1)$$

and thus

$$\begin{aligned}
\|x + y\|_p^p &\leq (\|x\|_p + \|y\|_p) \left(\sum_{j=1}^n |x_j + y_j|^p \right)^{\frac{1}{q}} \\
&= (\|x\|_p + \|y\|_p) \|x + y\|_p^{\frac{p}{q}}
\end{aligned}$$

Now, divide $\|x + y\|_p^{\frac{p}{q}} \neq 0$ to get

$$\begin{aligned}
\|x + y\|_p &= \|x + y\|_p^{p - \frac{p}{q}} \\
&\leq \|x\|_p + \|y\|_p
\end{aligned}$$

(since $p - \frac{p}{q} = p(1 - \frac{1}{q}) = 1$). □

Corollary 5.1. Given $1 < p < \infty$, $\|\cdot\|_p$ is a norm on \mathbb{R}^n .

Proof. Clearly $\|\cdot\|_p$ is non-negative and non-degenerate. If $\alpha \in \mathbb{R}, x \in \mathbb{R}^n$ then

$$\begin{aligned}
\|\alpha x\|_p &= \left(\sum_{j=1}^n |\alpha x_j|^p \right)^{\frac{1}{p}} \\
&= |\alpha| \left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} \\
&= |\alpha| \|x\|_p
\end{aligned}$$

Finally, subadditivity is provided by Minkowski's inequality. □

$$|x|^p = e^{p \log |x|}$$

5.1 THE ℓ_p -SPACES

Consider $\mathbb{R}^N = \{x = (x_k)_{k=1}^\infty : x_k \in \mathbb{R}\}$ which is a \mathbb{R} -vector space:

$$(x_k)_{k=1}^\infty + (y_k)_{k=1}^\infty = (x_k + y_k)_{k=1}^\infty, \alpha(x_k)_{k=1}^\infty = (\alpha x_k)_{k=1}^\infty.$$

We let for $1 \leq p < \infty$

$$\ell_p = \{x = (x_k)_{k=1}^\infty \in \mathbb{R}^N : \sum_{k=1}^\infty |x_k|^p = \lim_{n \rightarrow \infty} \sum_{k=1}^n |x_k|^p < \infty\}$$

and

$$\ell_\infty = \{x = (x_k)_{k=1}^\infty : \sup_{k \in \mathbb{N}} |x_k| < \infty\}.$$

On ℓ_p we define

$$\|x\|_p = \begin{cases} (\sum_{j=1}^n |x_j|^p)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty \\ \sum_{k \in \mathbb{N}} |x_k|, & \text{if } p = \infty \end{cases}$$

Theorem 5.2. Let $1 \leq p < \infty$. Then ℓ_p is a \mathbb{R} -subspace of \mathbb{R}^N and $\|\cdot\|_p$ is a norm.

Proof. We prove these together. Suppose that $x, y \in \ell_p$. Then

$$\begin{aligned} \|x + y\|_p &= \left(\sum_{k=1}^\infty |x_k + y_k|^p \right)^{\frac{1}{p}} \quad \text{if } \infty, \text{ treat } \infty^{\frac{1}{p}} = \infty \\ &= \left(\lim_{n \rightarrow \infty} \sum_{k=1}^n |x_k + y_k|^p \right)^{\frac{1}{p}} \\ &= \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n |x_k + y_k|^p \right)^{\frac{1}{p}} \quad x \mapsto x^{\frac{1}{p}} \text{ is continuous on } [0, \infty), \text{ if } x \rightarrow \infty, x^{\frac{1}{p}} \rightarrow \infty \\ &\leq \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n |x_k|^p \right)^{\frac{1}{p}} + \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n |y_k|^p \right)^{\frac{1}{p}} \quad \text{Minkowski applied on each } n \\ &= \left(\lim_{n \rightarrow \infty} \sum_{k=1}^n |x_k|^p \right)^{\frac{1}{p}} + \left(\lim_{n \rightarrow \infty} \sum_{k=1}^n |y_k|^p \right)^{\frac{1}{p}} \quad \text{continuity again} \\ &= \left(\sum_{k=1}^\infty |x_k|^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^\infty |y_k|^p \right)^{\frac{1}{p}} \\ &= \|x\|_p + \|y\|_p \\ &< \infty \end{aligned}$$

Thus $x + y \in \ell_p$, and we get subadditivity of $\|\cdot\|_p$.

We note that non-negativity and non-degeneracy of $\|\cdot\|_p$ are obvious. Likewise, the $|\cdot|$ -homogeneity is straightforward. \square

Theorem 5.3. $(\ell_\infty, \|\cdot\|_\infty)$ is a normed vector space.

Proof. If $x, y \in \ell_\infty$ then

$$\begin{aligned}
 \|x + y\|_\infty &= \sup_{k \in \mathbb{N}} |x_k + y_k| \\
 &\leq \sup_{k \in \mathbb{N}} (|x_k| + |y_k|) \\
 &\leq \sup_{j, k \in \mathbb{N}} (|x_j| + |y_k|) \\
 &= \sup_{j \in \mathbb{N}} |x_j| + \sup_{k \in \mathbb{N}} |y_k| \\
 &= \|x\|_\infty + \|y\|_\infty
 \end{aligned}$$

Other properties are very easy. □

6 2017-09-29

i) $X \neq \emptyset$ s.t. $|X| \geq 2$

$$\text{discrete metric } d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

For $x_0 \in X$,

$$\begin{aligned}
 B(x, \varepsilon) &= \begin{cases} \{x_0\} & 0 < \varepsilon \leq 1 \\ x & \varepsilon > 1 \end{cases} \\
 B[x, \varepsilon] &= \begin{cases} \{x_0\} & 0 < \varepsilon < 1 \\ x & \varepsilon \geq 1 \end{cases}
 \end{aligned}$$

ii) (geometry of balls in \mathbb{R}^2)

$$1 \leq p \leq \infty, B_p(0, 1) = \{x \in \mathbb{R}^2 : d_p(0, x) = \|x\|_p < 1\}$$

Proposition 6.1. (X, d) a metric space.

i) X, \emptyset are both open and closed.

ii) If $\{U_i\}_{i \in I}$ is a family of open sets, then $\bigcup_{i \in I} U_i$ is open.

iii) If $\{U_1, \dots, U_n\}$ is a finite family of open sets, then $\bigcap_{i=1}^n U_i$ is open.

iv) If $\{F_i\}_{i \in I}$ is a family of closed sets, then $\bigcap_{i \in I} F_i$ is closed.

v) If $\{U_1, \dots, U_n\}$ is a finite family of closed sets, then $\bigcup_{i=1}^n U_i$ is closed.

Proof. i) Let $x \in X$, then $x \in B(x, 1) \subseteq X$, so X is open. So $\emptyset = X \setminus X$, $X = X \setminus \emptyset$ are closed.

ii) Let $x \in U = \bigcup_{i \in I} U_i$. Then there is some i_0 in I s.t. $x \in U_{i_0}$, which is open, so there is $\varepsilon_x > 0$ s.t. $x \in B(x, \varepsilon_x) \subseteq U_{i_0} \subseteq U$.

iii) Let $x \in V = \bigcap_{i=1}^n U_i$. Then for each $i = 1, \dots, n$, there is $\varepsilon_i > 0$ s.t. $B(x, \varepsilon_i) \subseteq U_i$. Let $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\} \implies B(x, \varepsilon) \subseteq \bigcap_{i=1}^n B(x, \varepsilon_i) \subseteq V$.

iv), v) De Morgan's Laws. □

Given a metric space (X, d) , $A \subseteq X$, we define the boundary of A :

$$\partial A = \{x \in X : \forall \varepsilon > 0, B(x, \varepsilon) \cap A \neq \emptyset, B(x, \varepsilon) \setminus A \neq \emptyset\}.$$

Remark: $\partial A = \partial(X \setminus A)$.

Interior of A :

$$A^\circ = \bigcup \{U \subseteq X : U \subseteq A \text{ and } U \text{ is open}\}.$$

Proposition 6.2 (characterizations of interior). If $(X, d), A$ are as above then

$$\begin{aligned} A^\circ &= \{x \in X : \exists \varepsilon_x > 0 \text{ s.t. } B(x, \varepsilon_x) \subseteq A\} \\ &= A \setminus \partial A. \end{aligned}$$

Proof. Let $x \in A$. Then either:

- for some $\varepsilon_x > 0$, $B(x, \varepsilon_x) \subseteq A \implies x \in A^\circ$, or
- $\forall \varepsilon > 0, B(x, \varepsilon) \setminus A \neq \emptyset \implies$ since $x \in A \cap B(x, \varepsilon)$, $x \in \partial A$.

Since $A^\circ \subseteq A$, the proposition holds. □

Def: (X, d) a metric space, $(x_n)_{n=1}^\infty \subseteq X$ and $x_0 \in X$. Say $(x_n)_{n=1}^\infty$ converges to x_0 , i.e. $\lim_{n \rightarrow \infty} x_n = x_0$ or $x_n \xrightarrow{n \rightarrow \infty} x_0$ if $\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$ s.t. $n \geq n_\varepsilon \implies d(x_0, x_n) < \varepsilon$.

Remark: The limit, if it exists, is unique. Suppose $x_0 = \lim_{n \rightarrow \infty} x_n, y_0 = \lim_{n \rightarrow \infty} x_n$, then given $\varepsilon > 0, \exists n_\varepsilon, n_{\varepsilon'}$ in \mathbb{N} s.t.

$$\begin{aligned} n \geq n_\varepsilon &\implies d(x_0, x_n) < \varepsilon \\ n \geq n_{\varepsilon'} &\implies d(y_0, x_n) < \varepsilon. \end{aligned}$$

Now if $n \geq \max\{n_\varepsilon, n_{\varepsilon'}\}$, then

$$\begin{aligned} d(x_0, y_0) &\leq d(x_0, x_n) + d(x_n, y_0) < \varepsilon \\ &\implies d(x_0, y_0) = 0, \text{ so } x_0 = y_0. \end{aligned}$$

Example: Let $(V, \|\cdot\|)$ be a normed vector space. A subset $\{e_n\}_{n=1}^\infty \subseteq V$ is a Schauder basis if for each $x \in V, \exists$ a unique sequence $\{x_n\}_{n=1}^\infty$ s.t. $x = \lim_{n \rightarrow \infty} \sum_{k=1}^n x_k e_k$ in V .

In $\ell_p, 1 \leq p < \infty$, let $e_n = (0, \dots, 0, \underbrace{1}_{n\text{-th place}}, 0, \dots)$.

Let, for $(X, d), A$ as above, the set of accumulation points (cluster points) be given as

$$A' = \{x \in X : \forall \varepsilon > 0, \underbrace{B(x, \varepsilon) \setminus \{x\}}_{\text{punctured ball}} \cap A \neq \emptyset\}$$

Call elements of $A \setminus A'$ isolated points.

Proposition 6.3. Given $(X, d), A$ as above, we have

$$A' = \{x \in X : x = \lim_{n \rightarrow \infty} x_n, (x_n)_{n=1}^\infty \subseteq A \setminus \{x\}\}$$

Proof. If $x \in A'$, let $x_1 \in (B(x, 1) \setminus \{x\}) \cap A$, and $x_{n+1} \in (B(x, \varepsilon_n) \setminus \{x\}) \cap A$, where $\varepsilon_n = \min\{\frac{1}{n}, d(x, x_n)\}$.

Then $x = \lim_{n \rightarrow \infty} x_n$ while $(x_n)_{n=1}^\infty \subseteq A \setminus \{x\}$. Note x_1, x_2, \dots are distinct.

Converse direction: definition of limits. □

7 2017-10-02

Def: Given a metric space (X, d) and $A \subseteq X$, define the closure of A by

$$\bar{A} = \bigcap \{F \subseteq X : A \subseteq F, F \text{ is closed in } X\}$$

Of course $A^\circ \subseteq A \subseteq \bar{A}$.

Theorem 7.1 (characterization of the closure). Given a metric space (X, d) , $A \subseteq X$, the following sets are the same:

$$\bar{A}, A \cup \partial A, A \cup A'$$

("meet" set) $A_M = \{x \in X : \text{for any } \varepsilon > 0, B(x, \varepsilon) \cap A \neq \emptyset\}$

("limit" set) $A_L = \{x \in X : x = \lim_{n \rightarrow \infty} x_n, \text{ where } (x_n)_{n=1}^\infty \subseteq A\}$

(The notations A_L, A_M will not be used afterwards; we shall use \bar{A} .)

Proof. We have

$$\begin{aligned} \bar{A} &= \cap \{F \subseteq X : A \subseteq F, F \text{ closed}\} \\ &= \cap \{X \subseteq U : U \subseteq X \setminus A, U \text{ open in } X\} \\ &= X \setminus \cup \{U : U \subseteq X \setminus A, U \text{ open in } X\} \\ &= X \setminus [(X \setminus A)^\circ] \text{ complement of interior} \\ &= X \setminus [(X \setminus A) \setminus \partial(X \setminus A)] \text{ characterization of } (X \setminus A)^\circ \\ &= X \setminus [(X \setminus A) \setminus \partial A] \\ &= A \cup \partial A \end{aligned}$$

$$(\cap_{i \in I} (X \setminus U_i) = X \setminus \cup_{i \in I} U_i)$$

We thus have $\bar{A} = A \cup \partial A$.

Now if $x \in A \cup \partial A$, then for each $\varepsilon > 0$, we have that $B(x, \varepsilon) \cap A \neq \emptyset$ [i.e. either $x \in A$ so $x \in A \cap B(x, \varepsilon)$, or $x \in \partial A$, so $B(x, \varepsilon) \cap A \neq \emptyset$]. Thus $A \cup \partial A \subseteq A_M$. Conversely, if $x \in A_M$, then, either

- there is $\varepsilon > 0$ so $B(x, \varepsilon) \subset A \implies x \in A^\circ \subseteq A$, or
- for every $\varepsilon > 0$ we have $B(x, \varepsilon) \setminus A \neq \emptyset$ in which case $x \in \partial A$.

Hence, $x \in A_M \implies x \in A \cup \partial A$ so $A_M \subseteq A \cup \partial A$.

If $x \in A \cup A'$, then for each $\varepsilon > 0$, we have $B(x, \varepsilon) \cap A \neq \emptyset$. Indeed, as above, either $x \in A$, so for any $\varepsilon > 0$, $x \in B(x, \varepsilon) \cap A$, or $x \in A'$, so $B(x, \varepsilon) \cap A \supseteq (B(x, \varepsilon) \setminus \{x\}) \cap A \neq \emptyset$. Hence $A \cup A' \subseteq A_M$.

The definition of the limit of a sequence shows that $A_M = A_L$.

Finally, consider

$$\begin{aligned} X \setminus (A \cup A') &\subseteq \{x \in X : \text{there exists } \varepsilon_x > 0 \text{ s.t. } B(x, \varepsilon_x) \cap A = \emptyset, B(x, \varepsilon_x) \subseteq X \setminus A\} \\ &= (X \setminus A)^\circ \implies X \setminus [(X \setminus A)^\circ] \subseteq X \setminus [X \setminus (A \cup A')]. \end{aligned}$$

Hence

$$\begin{aligned} \bar{A} &= X \setminus [(X \setminus A)^\circ] \subseteq X \setminus [X \setminus (A \cup A')] \\ &= A \cup A'. \end{aligned}$$

Hence $\bar{A} \subseteq A \cup A' \subseteq A_M = \bar{A}$, so $\bar{A} = A \cup A'$. □

7.1 CONTINUITY

Def: Let (X, d_X) and (Y, d_Y) be metric spaces $f : X \rightarrow Y$ and $x_0 \in X$. We say that f is continuous at x_0 if given $\varepsilon > 0$, there is $\delta > 0$ s.t. $d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \varepsilon$. (★)

We say that f is continuous on X if it is continuous at each point.

Note:

$$\begin{aligned} (\star) &\iff f(B(x_0, \delta)) \subseteq B(f(x_0), \varepsilon) \\ &\iff B(x, \delta) \subseteq f^{-1}(B(f(x_0), \varepsilon)) \end{aligned}$$

Notation: In a metric space, a set N is a neighbourhood of a point x_0 if $x_0 \in N^\circ$ (interior).

Theorem 7.2 (characterization of continuity at a point). If $(X, d_X), (Y, d_Y), f : X \rightarrow Y, x \in X$ are as above, then TFAE:

- (i) f is continuous at x_0
- (ii) for any neighbourhood N of $f(x_0)$ in (Y, d_Y) , we have $f^{-1}(N)$ is a neighbourhood of x_0 in (X, d_X)
- (iii) if $x_0 = \lim_{n \rightarrow \infty} x_n$ in $(X, d_X) \implies f(x_0) = \lim_{n \rightarrow \infty} f(x_n)$ in (Y, d_Y) .

Proof. (i) \implies (ii) Given a neighbourhood of $f(x_0)$, there exists $\varepsilon > 0$ for which $B(f(x_0), \varepsilon) \subseteq N$. By assumption of continuity, there is $\delta > 0$ s.t.

$$\begin{aligned} B(x_0, \delta) &\subseteq f^{-1}(B(f(x_0), \varepsilon)) \\ &\subseteq f^{-1}(N), \text{ from above.} \end{aligned}$$

Thus $f^{-1}(N)$ is a neighbourhood of x_0 .

(ii) \implies (i) \implies (iii) Given $\varepsilon > 0$, $B(f(x_0), \varepsilon)$ is a neighbourhood of $f(x_0)$, so $f^{-1}(B(f(x_0), \varepsilon))$ is a neighbourhood of x_0 and hence there is $\delta > 0$ s.t. $B(x_0, \delta) \subseteq f^{-1}(B(f(x_0), \varepsilon))$, which gives (i).

Now, if $x_0 = \lim_{n \rightarrow \infty} x_n$ in (X, d_X) then there is n_δ in \mathbb{N} s.t. if $n \leq n_\delta, x_n \in B(x_0, \delta)$. But then for $n \leq n_\delta$, we have

$$f(x_n) \in f(B(x_0, \delta)) \subseteq B(f(x_0), \varepsilon)$$

and hence $f(x_0) = \lim_{n \rightarrow \infty} f(x_n)$.

(iii) \implies (i) (contrapositive) If (i) fails, then there exists $\varepsilon > 0$ s.t. for any $\delta > 0$, $B(x_0, \delta) \not\subseteq f^{-1}(B(f(x_0), \varepsilon))$.

Hence for each $n \in \mathbb{N}$ we may find $x_n \in B(x_0, \frac{1}{n}) \setminus f^{-1}(B(f(x_0), \varepsilon))$. Given $\varepsilon' > 0$, let $n_{\varepsilon'}$ satisfy $n_{\varepsilon'} \leq \frac{1}{\varepsilon'}$, thus $\lim_{n \rightarrow \infty} x_n = x_0$. However, each $f(x_n) \notin B(f(x_0), \varepsilon)$, so $f(x)$ does not go to. \square

8 2017-10-06

Corollary 8.1. A metric space is complete if whenever for any Cauchy sequence, we may find a converging subsequence.

Nested Intervals Theorem, Bolzano-Weierstrauss Theorem

Theorem 8.1. $(\ell_p, \|\cdot\|_p)$ ($1 \leq p < \infty$) is complete as a metric space.

Def: A normed space $(V, \|\cdot\|)$ is called a Banach space provided that V is complete w.r.t. metric $d(x, y) = \|x - y\|$. $(\ell_p, \|\cdot\|_p)$ is a Banach space.

9 2017-10-16

Theorem 9.1. The space of continuous bounded functions under the uniform metric, $(C_b(f), \|\cdot\|_\infty)$, is a Banach space.

Proof. (I) For $x \in X$, $(f_n(x))_{n=1}^\infty$ is Cauchy and admits a limit, so this defines $f : X \rightarrow \mathbb{R}$. The hard part is showing that f is continuous.

Next, show f is bounded, so $f \in C_b(X)$.

(II) $\lim_{n \rightarrow \infty} \|f - f_n\|_\infty = 0$, ie. $\lim_{n \rightarrow \infty} f_n = f$ uniformly in $C_b(X)$. \square

9.1 CHARACTERIZATIONS OF COMPLETENESS

Def: If (X, d) is a metric space, $\emptyset \neq A \subseteq X$, we let the diameter of A be given by

$$\text{diam}(A) = \sum_{x, y \in A} d(x, y) \text{ (may be } \infty)$$

Proposition 9.1. If (X, d) , A are as above then $\text{diam}(\bar{A}) = \text{diam}(A)$.

Proof. If $x, y \in \bar{A}$, $\varepsilon > 0$, then there are $x', y' \in A$ s.t. $d(x, x') < \frac{\varepsilon}{2}$, $d(y, y') < \frac{\varepsilon}{2}$ (using meet set characterization of \bar{A}). Then

$$\begin{aligned} d(x, y) &\leq d(x, x') + d(x', y') + d(y', y) \\ &\leq \frac{\varepsilon}{2} + \text{diam}(A) + \frac{\varepsilon}{2} \\ &= \text{diam}(A) + \varepsilon. \quad (\text{Assume } \text{diam}(A) < \infty). \end{aligned}$$

Thus, since $\varepsilon > 0$ is arbitrary, $d(x, y) \leq \text{diam}(A) \implies \text{diam}(\bar{A}) = \sup_{x, y \in \bar{A}} d(x, y) \leq \text{diam}(A)$. Since $A \subseteq \bar{A}$, $\text{diam}(A) \leq \text{diam}(\bar{A})$. \square

Theorem 9.2 (Nested set characterization of completeness). Let (X, d) be a metric space. Then (X, d) is complete \iff whenever we have closed sets,

- $F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$
- $\text{diam } F_n \xrightarrow{n \rightarrow \infty} 0$

then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

Proof. (\implies) For each n , choose $x_n \in F_n$. Given $\varepsilon > 0$, choose n_ε in \mathbb{N} s.t. $n \geq n_\varepsilon \implies \text{diam}(F_n) < \varepsilon$. Now, if $n, m \geq n_\varepsilon$ we have

$$x_n \in F_n \subseteq F_{n_\varepsilon}, x_m \in F_m \subseteq F_{n_\varepsilon} \implies d(x_n, x_m) \leq \text{diam}(F_{n_\varepsilon}) < \varepsilon$$

so $(x_n)_{n=1}^{\infty}$ is Cauchy, and has limit $x = \lim_{n \rightarrow \infty} x_n$. Since each $F_m = \bar{F}_m$ (closed), and we have for $n \geq m$, $x_n \in F_m$, $x = \lim_{n \rightarrow \infty} x_n \in F_m$ for all m . Hence $x \in \bigcap_{m=1}^{\infty} F_m$ (ie. $\neq \emptyset$).

(\impliedby) Let $(x_n)_{n=1}^{\infty} \subset X$ be Cauchy, let for n in \mathbb{N} , $F_n = \{x_k\}_{k \geq n}$. Then each F_n is closed and $F_n \supseteq F_{n+1}$ for each n . Further, $\text{diam } F_n = \text{diam}\{x_k\}_{k \geq n}$ (last proposition). Given $\varepsilon > 0$, there is n_ε in \mathbb{N} so $n, m \geq n_\varepsilon \implies d(x_n, x_m) < \varepsilon$. So for $n \geq n_\varepsilon$, we have $\text{diam}\{x_k\}_{k \geq n} = \sup_{k, l \geq n} d(x_k, x_l) < \varepsilon$. \square

10 2017-10-18

Continuing the proof that $(C_b(f), \|\cdot\|_\infty)$ is a Banach space from last time:

Theorem 10.1. The space of continuous bounded functions under the uniform metric, $(C_b(f), \|\cdot\|_\infty)$, is a Banach space.

Proof. (I) For $x \in X$, $(f_n(x))_{n=1}^{\infty}$ is Cauchy and admits a limit, so this defines $f : X \rightarrow \mathbb{R}$. f is continuous: let $x \in X$, and let $\varepsilon > 0$. Choose $n_\varepsilon \in \mathbb{N}$ so that

$$n, m \geq n_\varepsilon \implies |f_n(x) - f(x)| < \frac{\varepsilon}{4} \text{ and } \|f_n - f_m\|_\infty < \frac{\varepsilon}{4}.$$

Choose $\delta > 0$ so that for $x, y \in X$,

$$d(x, y) < \delta \implies |f_{n_\varepsilon}(x) - f_{n_\varepsilon}(y)| < \frac{\varepsilon}{4}.$$

Then, given $y \in B(x, \delta)$, let $n_y \in \mathbb{N}$ so that $n_y \geq n_\varepsilon$ and

$$n \geq n_y \implies |f_n(y) - f(y)| < \frac{\varepsilon}{4}.$$

Then for $n \geq n_y \geq n_\varepsilon$ we have

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_{n_\varepsilon}(x)| + |f_{n_\varepsilon}(x) - f_{n_\varepsilon}(y)| + |f_{n_\varepsilon}(y) - f_n(y)| + |f_n(y) - f(y)| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ &= \varepsilon. \end{aligned}$$

Also, f is bounded because

$$\begin{aligned} |f(x)| &\leq |f(x) - f_n(x)| + |f_n(x)| \\ &\leq |f(x) - f_n(x)| + \|f_n\|_\infty \\ &= o(1) + M. \end{aligned}$$

(II) Show that this is actually the limit (i.e. $\lim_{n \rightarrow \infty} \|f - f_n\|_\infty = 0$).

Let $\varepsilon > 0$. Choose $n_\varepsilon \in \mathbb{N}$ so that $m, n \geq n_\varepsilon \implies \|f_m - f_n\|_\infty < \frac{\varepsilon}{2}$. Also, given $x \in X$, choose $n_x \geq n_\varepsilon$ so that $n \geq n_x \implies |f_n(x) - f(x)| < \frac{\varepsilon}{2}$. Then, for $n \geq n_\varepsilon$, find $m \geq n_x \geq n_\varepsilon$ and observe that

$$\begin{aligned} |f(x) - f_n(x)| &\leq |f(x) - f_m(x)| + |f_m(x) - f_n(x)| \\ &< \frac{\varepsilon}{2} + \|f_m - f_n\|_\infty \\ &= \varepsilon. \end{aligned}$$

□

Example: Consider $(\ell_p, \|\cdot\|_p)$, $1 \leq p < \infty$. Let $e_n = (0, \dots, 0, \underbrace{1}_{n\text{-th place}}, 0, \dots)$ and let $F_n = \{e_k\}_{k \geq n} \subseteq \ell_p$.

- Each F_n is closed (easy exercise)
- $F_1 \supseteq F_2 \supseteq \dots$
- $\text{diam } F_n = 2^{\frac{1}{p}}$ (easy computation) (Finite diameter is not sufficient for Nested set characterization)

Notice that $\bigcap_{n=1}^\infty F_n = \emptyset$.

Theorem 10.2 (abstract M -test). Let $(V, \|\cdot\|)$ be a normed vector space. Then $(V, \|\cdot\|)$ is a Banach space \iff for every $(x_k)_{k=1}^\infty \subset V$ with $\sum_{k=1}^\infty \|x_k\| = \lim_{n \rightarrow \infty} \sum_{k=1}^n \|x_k\|$ converging, has that $\sum_{k=1}^\infty x_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n x_k$ converges in $(V, \|\cdot\|)$ [ie. V satisfies that “absolute convergence” \implies convergence.]

Proof. (\implies) Suppose $\sum_{k=1}^\infty \|x_k\|$ converges. Consider $(\sum_{k=1}^n x_k)_{n=1}^\infty \subset V$. We have for $m < n$ that

$$\left\| \sum_{k=1}^n x_k - \sum_{k=1}^m x_k \right\| \leq \underbrace{\sum_{k=m+1}^n \|x_k\|}_{\text{partial tail of converging series in } \mathbb{R}}$$

and hence $(\sum_{k=1}^n x_k)_{n=1}^\infty$ is Cauchy in $(V, \|\cdot\|)$, and thus converges.

(\Leftarrow) Suppose $(x_n)_{n=1}^\infty$ is a Cauchy seq in $(V, \|\cdot\|)$. Let n_1 in \mathbb{N} be so $m, n \geq n_1 \implies \|x_m - x_n\| < 1$, and, inductively, choose n_{k+1} in \mathbb{N} s.t. $n_{k+1} \geq n_k$ and $m, n \geq n_{k+1} \implies \|x_n - x_m\| < \frac{1}{2^k}$.

Let $y_0 = x_{n_1}$, $y_j = x_{n_{j+1}} - x_{n_j}$, $j \in \mathbb{N}$.

Then, each $\|y_j\| = \|x_{n_{j+1}} - x_{n_j}\| < \frac{1}{2^{j-1}}$, as $n_{j+1} > n_j \geq n_1$, so

$$\sum_{i=0}^\infty \|y_j\| = \|y_0\| + \sum_{j=1}^\infty \frac{1}{2^{j-1}},$$

which converges. (\star)

Now

$$\begin{aligned}
 x_{n_k} &= x_{n_1} + \sum_{j=1}^{k-1} (x_{n_{j+1}} - x_{n_j}) \\
 &= y_0 + \sum_{j=1}^{k-1} y_j \\
 &\xrightarrow{k \rightarrow \infty} y_0 + \sum_{j=1}^{\infty} y_j \text{ (by assumption and } (\star))
 \end{aligned}$$

In other words, $(x_{n_k})_{k=1}^{\infty}$ converges, hence $(x_n)_{n=1}^{\infty}$ converges as well. □

Application: a continuous nowhere differentiable function on \mathbb{R} .

Facts: $C_b(\mathbb{R})$ is complete; M -test.

Construction: Let $\varphi : \mathbb{R} \rightarrow [0, 1]$

$$\varphi(t) = \begin{cases} t - 2k & 2k \leq t < 2k + 1 \\ 2k + 2 - t & 2k + 1 \leq t < 2k + 2 \end{cases}$$

Picture: sawtooth function with zeros at $\dots, -4, -2, 0, 2, 4, \dots$

Then

- (i) φ is continuous and bounded
- (ii) φ is 2-periodic, ie. $\varphi(t + 2) = \varphi(t)$ for $t \in \mathbb{R}$
- (iii) $\varphi(2k) = 0, \varphi(2k + 1) = 1$ for $k \in \mathbb{Z}$
- (iv) if $k \leq s, t \leq k + 1$ ($k \in \mathbb{Z}$), then

$$|\varphi(s) - \varphi(t)| = |s - t|$$

Let for $t \in \mathbb{R}$

$$f(t) = \sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^k \underbrace{\varphi(4^k t)}_{\in [0,1]}$$

However, note that each $\varphi(4^k) \in C_b(\mathbb{R})$, $\|\varphi(4^k)\|_{\infty} = 1$, so by the M -test, $f \in C_b(\mathbb{R})$. Fix $t \in \mathbb{R}$. We show that f cannot be differentiable at t . Let $\ell_m = \lfloor 4^m t \rfloor$ ($m \in \mathbb{N}$) so

$$\begin{aligned}
 \ell_m &\leq 4^m t < \ell_m + 1 \\
 \implies p_m = \frac{\ell_m}{4^m} &\leq t < \frac{\ell_m + 1}{4^m} = q_m
 \end{aligned}$$

We compute

$$\begin{aligned}
& |f(p_m) - f(q_m)| \\
&= \left| \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^k [\varphi(4^k p_m) - \varphi(4^k q_m)] \right| \\
&= \left| \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^k [\varphi(4^{k-m} \ell_m) - \varphi(4^{k-m}(\ell_m + 1))] \right| \\
&= \left| \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^k [\varphi(4^{k-m} \ell_m) - \varphi(4^{k-m}(\ell_m + 1))] \right|, \text{ by (ii) (2-periodicity)} \\
&\text{(key step)} \geq \frac{3^m}{4} - \sum_{k=1}^{m-1} \frac{3^k}{4^k} \underbrace{|\varphi(4^{k-m} \ell_m) - \varphi(4^{k-m}(\ell_m + 1))|}_{=4^{k-m}, \text{ by (iv)}} \\
&= \frac{3^k}{4^k} - \frac{1}{4^m} \sum_{k=1}^{m-1} 3^k \\
&= \frac{1}{4^m} [3^m - \sum_{k=1}^{m-1} 3^k] \\
&= \frac{1}{4^m} \left[\frac{2 \cdot 3^m - 3^m + 1}{2} \right] \\
&= \frac{1}{4^m} \left(\frac{3^m + 1}{2} \right)
\end{aligned}$$

Since $|p_m - q_m| = \frac{1}{4^m}$, we have

$$\begin{aligned}
\frac{f(p_m) - f(q_m)}{p_m - q_m} &\geq \frac{3^m + 1}{2} \\
\left(p_m = \frac{\lfloor 4^m t \rfloor}{4^m} \right)
\end{aligned}$$

If $t = \frac{\ell}{4^{m_0}}$ ($\ell \in \mathbb{Z}$), then $t = p_m$ for $m \geq m_0$ and hence for $m \geq m_0$,

$$\left| \frac{f(t) - f(q_m)}{t - q_m} \right| \geq \frac{3^m + 1}{2}$$

while $\lim_{m \rightarrow \infty} q_m = t$, so $f'(t)$ does not exist.

$$\begin{aligned}
\frac{f(p_m) - f(q_m)}{p_m - q_m} &\leq \frac{|f(p_m) - f(t)| + |f(t) - f(q_m)|}{|p_m - q_m|} \\
&\leq \frac{|f(p_m) - f(t)|}{|p_m - t|} + \frac{|f(t) - f(q_m)|}{|t - q_m|}
\end{aligned}$$

Hence, for some $r_m \in \{p_m, q_m\}$, $\frac{|f(t) - f(r_m)|}{|t - r_m|} \geq \frac{3^m + 1}{2 \cdot 2}$.

We have $\left| \frac{f(t) - f(r_m)}{t - r_m} \right| \geq \frac{3^m + 1}{4}$ while $r_m \rightarrow t$.

11 2017-10-20

Corollary 11.1. $(\ell_\infty, \|\cdot\|_\infty)$ is a Banach space.

Proof. $\ell_\infty = C_b(\mathbb{N})$ with usual $|\cdot|$ metric on \mathbb{N} . If $f : \mathbb{N} \rightarrow \mathbb{R}$ is bounded, $U \subseteq \mathbb{R}$ open, then $f^{-1}(U) \in \mathcal{P}(\mathbb{N})$ is open (all subsets of \mathbb{N} are open) $\implies f$ is continuous.

If $(x_n)_{n=1}^\infty \in \ell_\infty$, define $f : \mathbb{N} \rightarrow \mathbb{R}$, $f(n) = x_n$, $f \in C_b(\mathbb{N})$, $\|f\|_\infty = \|(x_n)_{n=1}^\infty\|_\infty$. □

Eg. $(C[0, 2], \|\cdot\|_p)$, $\|f\|_p = (\int_0^2 |f|^p)^{\frac{1}{p}}$, $1 \leq p < \infty$.
 NOT a Banach space!

Let

$$f_n(t) = \begin{cases} 1 & 0 \leq t \leq \frac{1}{2} \\ n(\frac{1}{2} + \frac{1}{n} - t) & \frac{1}{2} < t \leq \frac{1}{2} + \frac{1}{n} \\ 0 & \frac{1}{2} + \frac{1}{n} < t \end{cases}.$$

Then for $m < n \in \mathbb{N}$,

$$\begin{aligned} \|f_n - f_m\|_p &= \left(\int_0^2 |f_n - f_m|^p \right)^{\frac{1}{p}} \\ &= \left(\underbrace{\int_0^{\frac{1}{2}} |f_n - f_m|^p}_0 + \underbrace{\int_{\frac{1}{2}}^{\frac{1}{2} + \frac{1}{m}} |f_n - f_m|^p}_{\leq \frac{1}{m}} + \underbrace{\int_{\frac{1}{2} + \frac{1}{m}}^2 |f_n - f_m|^p}_0 \right)^{\frac{1}{p}} \\ &\leq \frac{1}{m^{\frac{1}{p}}}. \end{aligned}$$

Hence $(f_n)_{n=1}^\infty$ is Cauchy in $(C[0, 2], \|\cdot\|_p)$.

Consider

$$\chi_{[0, \frac{1}{2}]}(t) = \begin{cases} 1 & 0 \leq t \leq \frac{1}{2} \\ 0 & \frac{1}{2} < t \end{cases}.$$

$\chi_{[0, \frac{1}{2}]}$ is bounded, piecewise continuous, so Riemann integrable.

$$\begin{aligned} \|f_n - \chi_{[0, \frac{1}{2}]} \|_p &= \left(\int_0^2 |f_n - \chi_{[0, \frac{1}{2}]}|^p \right)^{\frac{1}{p}} \leq \frac{1}{n^{\frac{1}{p}}} \\ \implies \lim_{n \rightarrow \infty} \|f_n - \chi_{[0, \frac{1}{2}]} \|_p &= 0. \end{aligned}$$

If $g \in C[0, 1]$ s.t. $\lim_{n \rightarrow \infty} \|f_n - g\|_p = 0$, then $\|g - \chi_{[0, \frac{1}{2}]} \|_p = 0$.

Using Riemann integration theory,

$$g(t) = \begin{cases} 1 & 0 \leq t \leq \frac{1}{2} \\ 0 & \frac{1}{2} < t \end{cases}.$$

Then $\lim_{t \rightarrow \frac{1}{2}} g$ does not exist!

11.1 COMPLETENESS OF METRIC SPACES

(X, d) metric space.

Remark: $|d(x, z) - d(y, z)| \leq d(x, y)$.

If $x = \lim_{n \rightarrow \infty} x_n$, $y = \lim_{n \rightarrow \infty} y_n$ in (X, d) , then $\lim_{n \rightarrow \infty} d(x_n, y_n) = d(x, y)$. (See solution to A3Q2).

Def: $(X, d_X), (Y, d_Y)$ metric spaces. $i : X \rightarrow Y$ is an isometry if $d_Y(i(x), i(y)) = d_X(x, y) \forall x, y \in X$.

Notes: An isometry is injective. Consider $i : X \rightarrow i(X) \subseteq Y \implies i^{-1} : i(X) \rightarrow X$ isometry.

Theorem 11.1. (X, d) metric space.

i) Existence of completion: there exists a metric space $(\overline{X}, \overline{d})$ s.t.

- a) $(\overline{X}, \overline{d})$ is complete
- b) \exists isometry $\bar{i} : X \rightarrow \overline{X}$
- c) $\overline{i(X)} = \overline{X}$; i.e. $i(X)$ is dense in \overline{X}

- ii) Uniqueness up to isometry: if (\tilde{X}, \tilde{d}) is a metric space with map $\tilde{i} : X \rightarrow \tilde{X}$ s.t. $(\tilde{X}, \tilde{d}), \tilde{i}$ satisfy (a),(b),(c), then \exists a surjective isometry $\varphi : \tilde{X} \rightarrow \bar{X}$ s.t. $\varphi \circ \tilde{i} = \bar{i}$.

Proof. 1. Fix $x_0 \in X$. For $u \in X$, let $f_u : X \rightarrow \mathbb{R}$, $f_u(x) = d(x, u) - d(x, x_0)$

$\implies f_u$ is continuous and $|f_u(x)| \leq d(u, x_0)$

$\implies \|f_u\|_\infty = \sup_{x \in X} |f_u(x)| \leq d(u, x_0) < \infty \implies f_u$ is bounded

$\implies f_u \in C_b(X)$.

For $u, v \in X, x \in X$,

$$|f_u(x) - f_v(x)| = |d(x, u) - d(x, v)| \leq d(u, v).$$

Thus $\|f_u - f_v\|_\infty \leq d(u, v)$. Finally,

$$\begin{aligned} |f_u(u) - f_v(u)| &= |d(u, u) - d(u, x_0) - d(u, v) + d(u, x_0)| \\ &= d(u, v). \end{aligned}$$

Thus $\|f_u - f_v\|_\infty \geq d(u, v) \implies \|f_u - f_v\|_\infty = d(u, v)$.

Define $\tau : X \rightarrow C_b(X)$, $\tau(u) = f_u$, τ isometry.

Let $\bar{X} = \overline{\tau(X)} = \{\overline{f_u} : u \in X\} \subseteq C_b(X)$.

By A3Q2(a), (\bar{X}, \bar{d}) is complete, where \bar{d} is relativized from the metric on $C_b(X)$.

2. Let $\varphi_0 = \tau \circ \tau^{-1} : \tau(X) \rightarrow \tau(X)$. φ_0 an isometry \implies uniformly continuous. Hence it admits an extension $\varphi = \overline{\varphi_0} : \bar{X} = \overline{\tau(X)} \rightarrow \bar{X} = \overline{\tau(X)}$.

Verify φ is an isometry:

If $\tilde{x}, \tilde{y} \in \bar{X}$, let $\tilde{x} = \lim_{n \rightarrow \infty} \tau(x_n), \tilde{y} = \lim_{n \rightarrow \infty} \tau(y_n), x_n, y_n \in X$. Then

$$\varphi(\tilde{x}) = \lim_{n \rightarrow \infty} \varphi_0(\tau(x_n)) = \lim_{n \rightarrow \infty} \tau(x_n).$$

Hence

$$\begin{aligned} \bar{d}(\varphi(\tilde{x}), \varphi(\tilde{y})) &= \lim_{n \rightarrow \infty} \bar{d}(\tau(x_n), \tau(y_n)) \\ &= \lim_{n \rightarrow \infty} d(x_n, y_n) \\ &= \lim_{n \rightarrow \infty} \tilde{d}(\tau(x_n), \tau(y_n)) = \tilde{d}(\tilde{x}, \tilde{y}). \end{aligned}$$

$\implies \varphi$ is an isometry. $\varphi \circ \tau = \tau$ comes for free.

□

12 2017-10-23

Assignment discussion – the completion vs A4,Q1:

Suppose $(V, \|\cdot\|)$ is a non-complete normed vector space, eg. $(C[0, 2], \|\cdot\|_p)$ ($1 \leq p < \infty$). Consider the map

$$\tau : V \rightarrow C_b(V)$$

$$\tau(v) \in C_b(V), \tau(v)(x) = \|x - v\| - \|x\|$$

We saw that τ is an isometry, hence we let

$$\bar{V} = \overline{\tau(V)}_{\text{complete}} \subseteq C_b(V)$$

Problem: τ is not linear, $\overline{\tau(V)}$ not evidently a subspace of $C_b(V)$.

A4, Q1 shows that an addition and a scalar multiplication may be imposed on $\bar{V} = \overline{\tau(V)}$ which makes it a Banach (complete normed vector) space. These two operations are not the same as addition and scalar multiplication in $C_b(V)$. (The only linear property that τ enjoys seems to be that it takes 0 to 0.)

12.1 COMPACTNESS

Let (X, d) be a metric space, and $K \subseteq X$. We say that K is compact if given a family of open sets $\{U_i\}_{i \in I}$ for which

$$K \subseteq \bigcup_{i \in I} U_i \text{ -- we say } \{U_i\}_{i \in I} \text{ is an "open cover"}$$

there is a finite subfamily $\{U_{i_1}, \dots, U_{i_n}\}$ such that

$$K \subseteq \bigcup_{k=1}^n U_{i_k} \text{ -- we say } \{U_i\}_{i \in I} \text{ admits a "finite subcover" .}$$

If $X = K$ itself is compact, we will call (X, d) a compact metric space.

Remark: If $K \subseteq X$ is compact, the relativized metric space (K, d_K) is a compact metric space.

Proposition 12.1. Let (X, d) be a metric space and $K \subseteq X$. If K is compact, then it must be closed.

Proof. Let us suppose, for sake of contradiction that there is $x \in \overline{K} \setminus K$. Then for n in \mathbb{N} ,

$$B(x, \frac{1}{n}) \cap K \neq \emptyset \implies B[x, \frac{1}{n}] \cap K \neq \emptyset. \quad (\star)$$

Further, $\bigcap_{n=1}^{\infty} B[x, \frac{1}{n}] = \{x\}$. Let $U_n = X \setminus B[x, \frac{1}{n}]$, which is open.

We have that

$$\bigcup_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} (X \setminus B[x, \frac{1}{n}]) = X \setminus \bigcap_{n=1}^{\infty} B[x, \frac{1}{n}] = X \setminus \{x\} \supseteq K.$$

But, for any finite m we have

$$\bigcup_{n=1}^m U_n = X \setminus \bigcap_{n=1}^m B[x, \frac{1}{n}] = X \setminus B[x, \frac{1}{m}] \not\supseteq K$$

by (\star) . Hence if $\overline{K} \setminus K \neq \emptyset$, K cannot be compact. So we are done. \square

Proposition 12.2. Let (X, d) be a compact metric space and $C \subseteq X$ is closed. Then C is compact.

Proof. Suppose $\{U_i\}_{i \in I}$ is an open cover of C . Then $\{U_i\}_{i \in I} \cup \{X \setminus C\}$ is an open cover of X . Hence X admits finite subcover $\{U_{i_1}, \dots, U_{i_n}\} \cup \{X \setminus C\}$, hence, $\{U_{i_1}, \dots, U_{i_n}\}$ is a finite subcover of C . \square

Theorem 12.1 (continuous image of compact is compact). Let (X, d_X) be a compact metric space, (Y, d_Y) be a metric space, and $f : X \rightarrow Y$ be continuous. Then $f(X) = \{f(x) : x \in X\}$ is compact.

Proof. Let $\{V_i\}_{i \in I}$ be an open cover of $f(X)$. Then $U_i = f^{-1}(V_i)$ is open, and $\{U_i\}_{i \in I}$ is an open cover of X . Hence there is a finite subcover, $X \subseteq \bigcup_{k=1}^n U_{i_k}$ so $f(X) \subseteq \bigcup_{k=1}^n f(U_{i_k}) = \bigcup_{k=1}^n V_{i_k}$, so $\{V_{i_1}, \dots, V_{i_n}\}$ is a finite subcover of $f(X)$. \square

Corollary 12.1 (Extreme Value Theorem). If (X, d) is a compact metric space, $f : X \rightarrow \mathbb{R}$ is continuous, then there are $x_{\min}, x_{\max} \in X$ for which

$$f(x_{\min}) \leq f(x) \leq f(x_{\max}) \quad \forall x \in X.$$

Proof. We have $f(X) \subseteq \mathbb{R}$ is compact. Hence $f(X)$ is closed. Also $\{(-n, n)\}_{n=1}^{\infty}$ (open intervals), then $f(X) \subseteq \mathbb{R} = \bigcup_{n=1}^{\infty} (-n, n)$ admits a finite subcover, $\{(-1, 1), \dots, (-n, n)\}$ and hence $f(X) \subseteq (-n, n)$. Thus we have $\inf(f(X)), \sup(f(X))$ exist.

Since $f(X)$ is closed we have

$$\inf(f(X)), \sup(f(X)) \in f(X)$$

(use meet-set of closure). Let x_{\min}, x_{\max} be so $f(x_{\min}) = \inf(f(X)), f(x_{\max}) = \sup(f(X))$. \square

– Assignment line –

Theorem 12.2 (finite intersection property). Let (X, d) be a metric space. Then (X, d) is compact \iff for any family $\{F_i\}_{i \in I}$ of closed subsets of X for which $\bigcap_{k=1}^n F_{i_k} \neq \emptyset$, $\{i_1, \dots, i_n\}$ finite in I , we must have $\bigcap_{i \in I} F_i \neq \emptyset$.

Proof. (\implies) (contrapositive) Let us suppose that $\{F_i\}_{i \in I}$ is a family of closed subsets with $\bigcap_{i \in I} F_i = \emptyset$. Then if $U_i = X \setminus F_i$, we have that $\{U_i\}_{i \in I}$ is an open cover (De Morgan's law) and hence admits finite subcover $\{U_{i_1}, \dots, U_{i_n}\}$. Again, by DeMorgan's law, $\bigcap_{k=1}^n F_{i_k} = \emptyset$. Hence we are done.

(\impliedby) Very similar, interchange roles of U_i s and $F_i = X \setminus U_i$. \square

Example: Let $X = B[0, 1]$ in ℓ_p ($1 \leq p \leq \infty$).

Let $e_n = (0, \dots, 0, \underbrace{1}_{n\text{-th place}}, 0, \dots)$ and let $F_n = \{e_k\}_{k \geq n}$ (seen before on Oct 18).

Each F_n is closed. Also

$$\bigcap_{n=1}^{\infty} F_n = \emptyset$$

$$\bigcap_{n=1}^m F_n = F_m \neq \emptyset$$

Conclusion: $(B[0, 1], d_p)$ ($d_p(x, y) = \|x - y\|_p$) is not compact.

13 2017-10-25

Def: Let (X, d) be a metric space. Then we say it is

- bounded if there are x_0 in X , and $R > 0$ such that $X \subseteq B[x_0, R]$ (of course “=” holds) (equivalently, for any $x \in X$, there is $R_x > 0$ such that $X \subseteq B[x, R_x]$; or, equivalently, $\text{diam}(X) < \infty$)
- totally bounded if, for any $\varepsilon > 0$, there are $x_1, \dots, x_n \in X$ such that $X \subseteq \bigcup_{k=1}^n B[x_k, \varepsilon]$

Totally bounded \implies bounded. [with $\varepsilon > 0$, x_1, \dots, x_n in defn, check that $\bigcup_{k=1}^n B[x_k, \varepsilon] \subseteq B[x_1, \varepsilon + \max_{k=2, \dots, n} d(x_1, x_k)]$]

Example: (bounded $\not\Rightarrow$ totally bounded)

In ℓ_p ($1 \leq p \leq \infty$), $e_n = (0, \dots, 0, \underbrace{1}_{n\text{-th place}}, 0, \dots)$, $F_n = \{e_k\}_{k \geq n} \subseteq \ell_p$,

F_n int, $F_n \subseteq B[0, 1] \subseteq B[e, 2]$ so F_n is bounded. But $n \neq m$, $d(e_n, e_m) = \begin{cases} 2^{\frac{1}{p}} & 1 \leq p < \infty \\ 1 & \text{otherwise} \end{cases} =: R$.

If $0 < \varepsilon < \frac{1}{2}R$, we see that $F_n \not\subseteq \bigcup_{k=1}^n B[e_k, \varepsilon]$ for any n .

Theorem 13.1 (Characterizations of compact metric spaces). Let (X, d) be a metric space. TFAE:

- (X, d) is compact,
- any sequence $(x_n)_{n=1}^{\infty} \subseteq X$ admits a subsequence which converges in X
- (X, d) is complete and totally bounded

Proof. (i) \implies (ii): Let $F_n = \overline{\{x_k\}_{k=n}^{\infty}}$. Then each F_n is closed, and $F_1 \supseteq F_2 \supseteq \dots$, so if $n_1 < n_2 < \dots < n_m$, then $\bigcap_{j=1}^m F_{n_j} = F_{n_m} \neq \emptyset$. Thus, by finite intersection property, we have that $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$. Let $x \in \bigcap_{n=1}^{\infty} F_n$. Now let

$$n_1 = \min\{k : x_k \in B(x, 1)\} \text{ (exists by meet-set closure definition)}$$

and, inductively,

$$n_{m+1} = \min\{k : k > n_m \text{ and } x_k \in B(x, \frac{1}{m+1})\}.$$

Then, as is easy to check, $\lim_{m \rightarrow \infty} x_{n_m} = x$.

(ii) \implies (iii): If $(x_n)_{n=1}^{\infty} \subseteq X$ is Cauchy, it admits a converging subsequence (by assumption), and hence itself converges

(earlier proposition). Thus (X, d) is complete.

Let us suppose that (X, d) is not totally bounded.

Thus, there exists $\varepsilon > 0$ so no finite collection of closed ε -balls covers X . Let

$$x_1 \in X \setminus B[x_1, \varepsilon], \dots, x_{n+1} \in X \setminus \bigcup_{k=1}^n B[x_k, \varepsilon] \text{ (always possible by assumption).}$$

Thus $d(x_n, x_m) > \varepsilon$ for $n \neq m$. Thus, this sequence $(x_n)_{n=1}^\infty$ admits no Cauchy subsequences, hence no subsequences which converge, violating assumption (ii). Thus (ii) $\implies (X, d)$ is totally bounded.

(iii) \implies (ii): We first use total boundedness. Given n in \mathbb{N} , there exist $y_{n1}, \dots, y_{nm_n} \in X$ such that the closed balls

$$B_{n1} = B[y_{n1}, \frac{1}{n}], \dots, B_{nm_n} = B[y_{nm_n}, \frac{1}{n}]$$

satisfy that $X \subseteq \bigcup_{k=1}^{m_n} B_{nk}$. Let

- B_1 be a ball from B_{11}, \dots, B_{1m_1} such that

$$|\{n \in \mathbb{N} : x_n \in B_1\}| = \aleph_0 \text{ (pigeonhole principle)}$$

• \vdots

- B_k be a ball from B_{k1}, \dots, B_{km_k} such that

$$|\{n \in \mathbb{N} : x_n \in \bigcap_{j=1}^k B_j\}| = \aleph_0$$

(we've covered X by 1-balls, B_1 by $\frac{1}{2}$ -balls, then $B_2 \cap B_1$ covered by $\frac{1}{3}$ -balls, ...)

Now we use completeness. Let $F_n = \bigcap_{k=1}^n B_k$ so each F_n is closed.

- $F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$
- $\text{diam}(F_n) \leq \text{diam}(B_n) = \frac{2}{n} \xrightarrow{n \rightarrow \infty} 0$

Thus, by nested sets theorem, $\bigcap_{n=1}^\infty F_n \neq \emptyset$.

Let $n_1 = \min\{k \in \mathbb{N} : x_k \in F_1\}$, inductively, $n_{m+1} = \min\{k \in \mathbb{N} : k > n_m \text{ and } x_k \in F_k\}$.

Then, if $x \in \bigcap_{n=1}^\infty F_n$, $d(x, x_m) \leq \text{diam}(F_m) \leq \text{diam}(B_m) = \frac{2}{m} \xrightarrow{m \rightarrow \infty} 0$ so $x = \lim_{n \rightarrow \infty} x_{n_m}$. □

14 2017-10-27

Office hours:

Mon 2:30 – 4:30

Tue 2 – 3:30

Proof. Continuing theorem from last time:

So far we did (i) $\xRightarrow{\text{F.I.P.}}$ (ii) $\xRightarrow{\text{routine}}$ (iii) $\xRightarrow{\text{harder, nested sets thm}}$ (ii)

(ii) \implies (i): Let $\{U_i\}_{i \in I}$ be an open cover of X .

(LN) There exists $r > 0$ s.t. for any x in X there exists i in I so $B(x, r) \subseteq U_i$.

(This number r is sometimes called the “Lebesgue number” of the covering; its existence is based on (ii).)

Suppose (LN) fails. Then for choice of $r = \frac{1}{n}$, there exists x_n in X s.t. $B(x_n, \frac{1}{n}) \not\subseteq U_i$ for all i in I .

Our assumption is that $(x_n)_{n=1}^\infty \subseteq X$ admits a subsequence $(x_{n_k})_{k=1}^\infty$ such that $x_0 = \lim_{k \rightarrow \infty} x_{n_k}$ exists.

Then $x_0 \in U_{i_0}$ for some i_0 , so there is $\varepsilon > 0$ such that $B(x_0, \varepsilon) \subseteq U_{i_0}$. Now, there is k_ε in \mathbb{N} so $k \geq k_\varepsilon \implies x_{n_k} \in B(x_0, \frac{\varepsilon}{2})$. Hence, let us choose $k \geq k_\varepsilon$ and $\frac{1}{n_k} < \frac{\varepsilon}{2}$. Thus, if $x \in B(x_{n_k}, \frac{1}{n_k})$, we have

$$d(x, x_0) \leq d(x, x_{n_k}) + d(x_{n_k}, x_0) < \frac{1}{n_k} + \frac{\varepsilon}{2} < \varepsilon$$

and hence $B(x_{n_k}, \frac{1}{n_k}) \subseteq B(x_0, \varepsilon) \subseteq U_{i_0}$, contradicting the choice of the elements x_n .

Hence, we must conclude that (LN) holds.

We saw in (ii) \implies (iii) above, that our assumption gives total boundedness of (X, d) . Hence there are y_1, \dots, y_m such that $X \subseteq \bigcup_{j=1}^m B[y_j, \frac{r}{2}] \subseteq \bigcup_{j=1}^m B(y_j, r)$. Now, for each $j = 1, \dots, m$, (LN) tells us that there is $i_j \in I$ so $B(y_j, r) \subseteq U_{i_j}$.

Thus $X \subseteq \bigcup_{j=1}^m B(y_j, r) \subseteq \bigcup_{j=1}^m U_{i_j}$, so $\{U_{i_1}, \dots, U_{i_m}\}$ is a finite subcover.

Remark: On \mathbb{R}^n , norms $\|\cdot\|_p$ ($1 \leq p \leq \infty$) are equivalent, and from A2, each gives the same open sets, and hence the same compact sets.

Corollary 14.1.

(i) (Bolzano-Weierstrauss Theorem for \mathbb{R}^n)

If $(x^{(n)})_{n=1}^\infty \subseteq [-R, R]^n = B_\infty[0, R]$, then it admits a converging subsequence.

(ii) (Heine-Borel Theorem)

A subset $K \subseteq \mathbb{R}^n$ is compact $\iff K$ is closed & K is bounded (with respect to any $\|\cdot\|_\infty$).

Proof. (i) We consider, first $(x_1^{(n)})_{n=1}^\infty \subseteq [-R, R] \subseteq \mathbb{R}$. By Bolzano-Weierstrauss for \mathbb{R} , this admits converging subsequence $(x_1^{(n_k)})_{n=1}^\infty$. Then $(x_2^{(n)})_{n=1}^\infty \subseteq [-R, R] \subseteq \mathbb{R}$ admits a converging subsequence $(x_2^{(n_k)})_{n=1}^\infty$. Etc. Hence, after finitely many (n) iterations, we get a subsequence of $(x^{(n)})_{n=1}^\infty$ which converges ($\mathbb{R}^n, \|\cdot\|_\infty$).

(ii) If K is compact, then K is closed by a result at the beginning of the section, and totally bounded by last theorem, hence bounded. Conversely, if K is closed and bounded, $K \subseteq [-R, R]^n$ for some $R > 0$. Let us consider a sequence $(x^{(n)})_{n=1}^\infty \subseteq K$. First, $(x^{(n)})_{n=1}^\infty$ admits a converging subsequence, by (i). Since K is closed, the limit of the subsequence is in K . □

Example: $P = \prod_{k=1}^\infty \{0, \frac{1}{2^k}\} \subseteq \ell_1$ is compact in $(\ell_1, \|\cdot\|_1)$.

First soln: The Cantor set C is closed and bounded in \mathbb{R} , so thus compact. And there is a continuous function $f : C \rightarrow \ell_1$ with $f(C) = P$ (A4, Q3), so P is compact. [In fact f is a bijection from C to P so $f^{-1} : P \rightarrow C$ is also continuous.]

Second soln: P is closed (A3). Hence the relativised metric space (P, d_P) is complete. Let us show total boundedness.

Let $\varepsilon > 0$, and n be so $\frac{1}{2^n} < \varepsilon$. For $(b_1, \dots, b_n) \in \{0, 1\}^n$, let $x_{b_1 \dots b_n} = \sum_{k=1}^\infty \frac{b_k}{2^k} e_k \in P$. If $b = (b_1, b_2, \dots) \in \{0, 1\}^\mathbb{N}$, then $x_b = \sum_{k=1}^\infty \frac{b_k}{2^k} e_k \in P$ (generic element of P).

Then for $b = (b_1, b_2, \dots)$ as above,

$$\|x_b - x_{b_1 \dots b_n}\|_1 = \sum_{k=n+1}^\infty \frac{1}{2^k} b_k \leq \sum_{k=n+1}^\infty \frac{1}{2^k} = \frac{1}{2^n} \leq \varepsilon.$$

Thus, $P \subseteq \bigcup_{(b_1, \dots, b_n) \in \{0, 1\}^n} B[x_{b_1 \dots b_n}, \varepsilon]$. □

– MIDTERM CUTOFF –

15 2017-10-30

Midterm: Wed evening

See info sheet on website

Office hours:

– 2:30 - 4:30

– 1:30 - 3:30

A5 - will be posted Friday

Theorem 15.1 (sequential characterization of uniform continuity). Let (X, d_X) and (Y, d_Y) be metric spaces, $f : X \rightarrow Y$. Then

$$f \text{ is uniformly continuous} \iff \text{whenever } d_X(x_n, y_n) \xrightarrow{n \rightarrow \infty} 0, \ x_n, y_n \in X, \\ \text{we must have } d_Y(f(x_n), f(y_n)) \xrightarrow{n \rightarrow \infty} 0.$$

Proof. (\implies) Given $\varepsilon > 0$, there is $\delta > 0$ such that $d_X(x, y) < \delta \ (x, y \text{ in } X) \implies d_Y(f(x), f(y)) < \varepsilon$. Now suppose $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty \subseteq X$ such that $\lim_{n \rightarrow \infty} d_X(x_n, y_n) = 0$. Then there is n_ε in \mathbb{N} such that

$$n \geq n_\varepsilon \implies d_X(x_n, y_n) < \delta \\ \implies d_Y(f(x_n), f(y_n)) < \varepsilon.$$

I.e. $\lim_{n \rightarrow \infty} d_Y(f(x_n), f(y_n)) = 0$.

(\impliedby) (contrapositive) Suppose f is not uniformly continuous, so there exists $\varepsilon > 0$ such that for all $\delta > 0$ there are x, y in X with $d_X(x, y) < \delta$ but $d_Y(f(x), f(y)) \geq \varepsilon$. For each choice $\delta = \frac{1}{n}$, let x_n, y_n in X so $d_X(x_n, y_n) < \frac{1}{n}$ for which $d_Y(f(x_n), f(y_n)) \geq \varepsilon$.

Plainly, $\lim_{n \rightarrow \infty} d_X(x_n, y_n) = 0$ while $\lim_{n \rightarrow \infty} d_Y(f(x_n), f(y_n)) \neq 0$ (if the limit exists).

Ex: Let $f(x) = x^2$ on \mathbb{R} . Let $x_n = n$, $y_n = n + \frac{1}{n}$. Then $|x_n - y_n| = \frac{1}{n} \xrightarrow{n \rightarrow \infty} 0$, while $|f(x_n) - f(y_n)| = 2 + \frac{1}{n^2} \not\xrightarrow{n \rightarrow \infty} 0$. Hence f is not uniformly continuous. \square

Theorem 15.2 (continuous on compact is uniformly continuous). Let (X, d_X) , (Y, d_Y) be metric spaces, with (X, d_X) compact, and $f : X \rightarrow Y$ continuous. Then f is uniformly continuous.

Proof. Let us suppose not. Then there is $\varepsilon > 0$ and $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty \subseteq X$ such that $d_X(x_n, y_n) \xrightarrow{n \rightarrow \infty} 0$ while $d_Y(f(x_n), f(y_n)) \geq \varepsilon$. Let $(x_{n_k})_{k=1}^\infty$ be a converging subsequence. Then let $(y_{n_k})_{k=1}^\infty$ be a sequence in X , hence admits converging subsequence $(y_{n_{k_\ell}})_{\ell=1}^\infty$. Then if $x = \lim_{k \rightarrow \infty} x_{n_k} = \lim_{\ell \rightarrow \infty} x_{n_{k_\ell}}$ then

$$d_X(x, y_{n_{k_\ell}}) \leq d_X(x, x_{n_{k_\ell}}) + d_X(x_{n_{k_\ell}}, y_{n_{k_\ell}}) \\ \xrightarrow{\ell \rightarrow \infty} 0$$

so $x = \lim_{\ell \rightarrow \infty} y_{n_{k_\ell}}$. Then we have $f(x) = \lim_{\ell \rightarrow \infty} f(y_{n_{k_\ell}})$, by continuity, so

$$0 = d_Y(f(x), f(x)) = \lim_{\ell \rightarrow \infty} d_Y(f(x_{n_{k_\ell}}), f(y_{n_{k_\ell}}))$$

contradicts (\star) . Thus, we conclude that f is uniformly continuous. \square

Definition: A map $f : X \rightarrow Y$ ($(X, d_X), (Y, d_Y)$) is called Lipschitz if there is $L \geq 0$ such that

$$d_Y(f(x), f(y)) \leq L d_X(x, y) \text{ for all } x, y \in X.$$

Notice that

$$\sup_{x, y \in X, x \neq y} \frac{d_Y(f(x), f(y))}{d_X(x, y)} = \inf \{ L \geq 0 : (\text{Lip}) \text{ is satisfied} \}$$

so there exists a minimum L satisfying (Lip). We call this the “Lipschitz constant”.

Remark: Lipschitz $\xrightarrow{\text{exercise}}$ uniform continuity \implies continuity
 Lipschitz $\not\xrightarrow{\text{assignment}}$ uniform continuity $\not\implies$ continuity

Theorem 15.3. Any two norms on \mathbb{R}^n are equivalent, i.e. if $\|\cdot\|, \|\cdot\|$ on \mathbb{R}^n satisfy $\|\cdot\| \approx \|\cdot\|$, i.e., there are $m, M > 0$ for which $m\|x\| \leq \|x\| \leq M\|x\|$.

Proof. Let $\|\cdot\|$ be a norm on \mathbb{R}^n . We will see that $\|\cdot\| \approx \|\cdot\|_1$ ($\|x\|_1 = \sum_{j=1}^n |x_j|$). Since \approx is an equivalence relation, we get $\|\cdot\| \approx \|\cdot\|_1$ so $\|\cdot\| \approx \|\cdot\|$.

Let $\{e_1, \dots, e_n\}$ be the standard basis, so if $x \in \mathbb{R}^n$, $x = \sum_{j=1}^n x_j e_j$. Then

$$\|x\| = \left\| \sum_{j=1}^n x_j e_j \right\| \underbrace{\leq}_{\text{properties of norm}} \sum_{j=1}^n |x_j| \|e_j\| \leq M \|x\|_1 \text{ where } M = \max_{j=1, \dots, n} \|e_j\|.$$

Notice, then, for x, y in \mathbb{R}^n we have

$$\| \|x\| - \|y\| \| \underbrace{\leq}_{\text{standard } \leq \text{ (shown before completeness of } C_b(X))} \|x - y\| \leq M \|x - y\|_1$$

so $\|\cdot\| : \mathbb{R}^n \rightarrow \mathbb{R}$ is Lipschitz with respect to $d_1(x, y) = \|x - y\|_1$ and thus continuous.

Let $S_1 = \{x \in \mathbb{R}^n : \|x\|_1 = 1\} = B_1[0, 1] \setminus \underbrace{B_1(0, 1)}_{\subseteq B_1[0, 1]}$ so S_1 is closed in $B_1[0, 1]$. Hence by Heine-Borel Theorem, it is compact.

Hence, by Extreme Value Theorem, there is x_{\min} in S_1 such that

$$\|x_{\min}\| = \inf\{\|x\| : x \in S_1\}.$$

Let $m = \|x_{\min}\| > 0$ (as $x_{\min} \neq 0$, since $\|x_{\min}\|_1 = 1 \neq 0$).

Now, if $x \in \mathbb{R}^n \setminus \{0\}$, then

$$m \leq \left\| \frac{1}{\underbrace{\|x\|_1}_{\in S_1}} x \right\| \implies m \|x\|_1 \leq \|x\| \quad (\dagger)$$

Then (\dagger) and (\ddagger) show that $\|\cdot\| \approx \|\cdot\|_1$. □

Corollary 15.1. If $\|\cdot\|$ is a norm on \mathbb{R}^n , $\|\cdot\|$ on \mathbb{R}^m and $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear. Then A is Lipschitz from $(\mathbb{R}^n, \|\cdot\|)$ to $(\mathbb{R}^m, \|\cdot\|)$, and hence continuous.

Proof. Let $\{e_1, \dots, e_n\}$ be the standard basis of \mathbb{R}^n , $\{e_1, \dots, e_m\}$ be the standard basis of \mathbb{R}^m . Then there is a matrix $[a_{ij}]$ such that $Ae_j = \sum_{i=1}^m a_{ij} e_i$.

Then for $x = \sum_{j=1}^n x_j e_j$ in \mathbb{R}^n we have

$$\begin{aligned} Ax &= \sum_{j=1}^n x_j Ae_j \\ &= \sum_{j=1}^n x_j \sum_{i=1}^m a_{ij} e_i \\ &= \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} x_j \right) e_i \in \mathbb{R}^m \end{aligned}$$

so

$$\begin{aligned}
\|Ax\| &\leq \sum_{j=1}^n \left| \sum_{i=1}^n a_{ij}x_j \right| \|e_i\|, & M &= \max_{j=1,\dots,n} \|e_j\| \\
&\leq M \sum_{j=1}^n \sum_{i=1}^m |a_{ij}| |x_j|, & \|A\|_\infty &= \max_{i=1,\dots,m, j=1,\dots,n} |a_{ij}| \\
&= M \sum_{i=1}^m \sum_{j=1}^n |a_{ij}| |x_j| \\
&\leq M \sum_{i=1}^m |A|_\infty |x|_1 \\
&= M \|x\|_1 \leq M
\end{aligned}$$

$$\|x\|_1 \leq M \|x\|$$

□

16 2017-11-01

Proposition 16.1. Let $(V, \|\cdot\|_V)$, $(W, \|\cdot\|_W)$ be normed linear spaces, $A : V \rightarrow W$ be linear. Then TFAE:

1. A is continuous
2. $\|A\| := \sup\{\|Ax\|_W : x \in \underbrace{B_V[0,1]}_{\text{closed ball, center 0 in } V}\} < \infty$
3. A is Lipschitz map with Lipschitz constant $\|A\|$

Moreover, in the case of (ii) (hence (iii)), above, $\|Ax\|_W \leq \|A\| \|x\|_V$ for any x in V .

Proof. (i) \implies (ii) A is continuous at 0 in V . Thus, letting $\varepsilon = 1$, there is $\delta > 0$ s.t. $A(B_V(0, \delta)) \subseteq B_W(0, 1)$. Now, if $x \in B_V[0, 1]$, then $\frac{\delta}{2}x \in B_V(0, \delta)$, so

$$\|Ax\|_W = \frac{2}{\delta} \left\| \underbrace{A\left(\frac{\delta}{2}x\right)}_{\in B_W(0,1)} \right\|_W < \frac{2}{\delta} 1 = \frac{2}{\delta} < \infty$$

so $\|A\| = \sup_{x \in B_V[0,1]} \|Ax\|_W \leq \frac{2}{\delta} < \infty$.

(ii) \implies (iii) If $x \in V \setminus \{0\}$, so $\frac{1}{\|x\|_V}x \in B_V[0, 1]$ and

$$(\star) \quad \|Ax\|_W = \|x\|_V \underbrace{\left\| A\left(\frac{1}{\|x\|_V}x\right) \right\|_W}_{\leq \|A\|} \leq \|A\| \|x\|_V.$$

Clearly, (\star) holds for $x = 0$ in V . Hence if $x, y \in V$,

$$\|Ax - Ay\|_W = \|A(x - y)\|_W \leq \|A\| \|x - y\|_V.$$

Thus A is Lipschitz and “Moreover...” holds. Furthermore, by (\star) ,

$$\|A\| = \sup_{x \in V \setminus \{0\}} \frac{\|Ax\|_W}{\|x\|_V} = \sup_{x \neq y \text{ in } V} \frac{\|Ax - Ay\|_W}{\|x - y\|_V}$$

which is the definition of the Lipschitz constant.

(iii) \implies (i) Obvious.

□

Remark: Let $B(V, W) = \{A : V \rightarrow W \mid A \text{ is linear and continuous}\}$. Notice that (ii) above shows that A must be bounded on $B_V[0, 1]$ and we call A a “bounded linear operator”.

$B(V, W)$ is a \mathbb{R} -vector space (pointwise addition and scalar multiplication) and $\|\cdot\|$ is a norm on $B(V, W)$, called “bounded operator norm”. (Exercise.)

Question: Is continuity automatic for linear operators?

Example: Consider the vector space $C[0, 1]$ of continuous \mathbb{R} -valued functions on $[0, 1]$. Let

$$\varphi : C[0, 1] \rightarrow \mathbb{R}, \quad \varphi(f) = f\left(\frac{1}{2}\right) \text{ (evaluation at } \frac{1}{2}\text{)}.$$

Then φ is linear: let $f, g \in C[0, 1]$, $\alpha \in \mathbb{R}$, then

$$\begin{aligned} \varphi(f + \alpha g) &= f\left(\frac{1}{2}\right) + \alpha g\left(\frac{1}{2}\right) \\ &= \varphi(f) + \alpha \varphi(g) \end{aligned}$$

(i) Consider $(C[0, 1], \|\cdot\|_\infty)$. Then

$$|\varphi(f)| = |f\left(\frac{1}{2}\right)| \leq \max_{t \in [0, 1]} |f(t)| = \|f\|_\infty.$$

Thus $\|\varphi\| \leq 1$ (easy to show that $\|\varphi\| = 1$), i.e., $\varphi \in B((C[0, 1], \|\cdot\|_\infty), \mathbb{R})$.

(ii) Now consider $(C[0, 1], \|\cdot\|_p)$ ($1 \leq p < \infty$). Let

$$f_n(t) = \begin{cases} 0 & \text{if } t \leq \frac{1}{2} - \frac{1}{n^{2p}} \\ n^{2p+1}(t - \frac{1}{2} + \frac{1}{n^{2p}}) & \text{if } \frac{1}{2} - \frac{1}{n^{2p}} < t \leq \frac{1}{2} \\ n^{2p+1}(\frac{1}{2} + \frac{1}{n^{2p}} - t) & \text{if } \frac{1}{2} < t \leq \frac{1}{2} + \frac{1}{n^{2p}} \\ 0 & \text{if } t > \frac{1}{2} + \frac{1}{n^{2p}} \end{cases}$$

[triangular spike at $[\frac{1}{2} - \frac{1}{n^{2p}}, \frac{1}{2} + \frac{1}{n^{2p}}]$ with peak at $\frac{1}{2}$ having value n .] Notice

$$\varphi(f_n) = f_n\left(\frac{1}{2}\right) = n$$

while

$$\begin{aligned} \|f_n\|_p &= \left(\int_0^1 f_n^p \right)^{\frac{1}{p}} \\ &= \left(\int_{\frac{1}{2} - \frac{1}{n^{2p}}}^{\frac{1}{2} + \frac{1}{n^{2p}}} \underbrace{f_n^p}_{0 \leq f_n^p \leq n^p} \right)^{\frac{1}{p}} \\ &\leq \left(\int_{\frac{1}{2} - \frac{1}{n^{2p}}}^{\frac{1}{2} + \frac{1}{n^{2p}}} \underbrace{n^p}_{\text{constant}} \right)^{\frac{1}{p}} \\ &= \left(n^p \frac{2}{n^{2p}} \right)^{\frac{1}{p}} = \frac{2^{\frac{1}{p}}}{n}. \end{aligned}$$

Thus

$$\frac{|\varphi(f_n)|}{\|f_n\|_p} = \frac{n}{\frac{2^{\frac{1}{p}}}{n}} = \frac{n^2}{2^{\frac{1}{p}}} \xrightarrow{n \rightarrow \infty} \infty.$$

Hence

$$\varphi \notin B((C[0, 1], \|\cdot\|_p), \mathbb{R}).$$

Example: (Axiom of choice) If $(V, \|\cdot\|)$ is an infinite dimensional normed vector space, then it admits an infinite linearly independent family $\{v_n\}_{n=1}^\infty$. There exists a basis $\{w_i\}_{i \in I}$ s.t. $\{v_n\}_{n=1}^\infty \subseteq \{w_i\}_{i \in I}$.

Define $f : V \rightarrow \mathbb{R}$

$$f(w_i) = \begin{cases} \frac{n}{\|v_n\|} & \text{if } w_i = v_n \\ 0 & \text{otherwise} \end{cases}$$

and extend uniquely to a linear operator on V .

Check that $f \notin B(V, \mathbb{R})$.

Why isn't $B[0, 1]$ in $(C[0, 1], \|\cdot\|_\infty)$ compact?

Reason: existence of subsequence with no converging subsequence [similar holds on $(\ell_p, \|\cdot\|_p)$].

Picture: [triangle spike to height $f_n(t) = 1$ on $[\frac{1}{n+1}, \frac{1}{n}]$, 0 elsewhere.]

Calculate that if $m \neq n$, $\|f_n - f_m\|_\infty = 1$. Conclude that $(f_n)_{n=1}^\infty \subset B[0, 1]$ admits no converging subsequence.

17 2017-11-03

Theorem 17.1 (Banach's Contraction Mapping Theorem). Let (X, d) be a complete metric space and let $\Gamma : X \rightarrow X$ be a strict contraction, i.e., there is $0 < c < 1$ s.t. $d(\Gamma(x), \Gamma(y)) < cd(x, y)$ for x, y in X (Γ is c -Lipschitz). Then

- (i) there is a unique fixed point x_{fix} for Γ , i.e. $\Gamma(x_{\text{fix}}) = x_{\text{fix}}$,
- (ii) given any x_0 in X , if we define a sequence by $x_n = \Gamma(x_{n-1}), n \in \mathbb{N}$, then it satisfies

$$d(x_n, x_{\text{fix}}) \leq \frac{c^n}{1-c} d(x_0, \Gamma(x_0))$$

and hence $\lim_{n \rightarrow \infty} x_n = x_{\text{fix}}$.

Proof. Let $x_0 \in X$. We define $(x_n)_{n=1}^\infty \subseteq X$ as in (ii), above. We note that $d(x_1, x_2) = d(\Gamma(x_0), \Gamma(x_1)) \leq cd(x_0, x_1) = cd(x_0, \Gamma(x_0))$.

Now, if

$$(\star) \quad d(x_n, x_{n+1}) \leq c^n d(x_0, \Gamma(x_0)),$$

then

$$d(x_{n+1}, x_{n+2}) = d(\Gamma(x_n), \Gamma(x_{n+1})) \leq cd(x_n, x_{n+1}) \leq c^{n+1} d(x_0, \Gamma(x_0))$$

so (\star) holds generally. Thus, if $m < n$ in \mathbb{N} we have

$$\begin{aligned} d(x_m, x_n) &\leq \sum_{j=m}^{n-1} d(x_j, x_{j+1}) \\ &\leq \sum_{j=m}^{n-1} c^j d(x_0, \Gamma(x_0)), \text{ by } (\star) \\ &\leq \sum_{j=m}^{\infty} c^j d(x_0, \Gamma(x_0)), \text{ by } (\star) = \frac{c^m}{1-c} d(x_0, \Gamma(x_0)). \end{aligned}$$

It follows that $(x_n)_{n=1}^\infty$ is Cauchy, and hence $x_{\text{fix}} = \lim_{n \rightarrow \infty} x_n$ exists. Then

$$x_{\text{fix}} = \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \Gamma(x_n) \underset{\substack{\Gamma \text{ Lipschitz} \implies \text{continuous}}}{=} \Gamma(\lim_{n \rightarrow \infty} x_n) = \Gamma(x_{\text{fix}}).$$

Hence x_{fix} is a fixed point. If y_{fix} is any other fixed point then

$$\begin{aligned} d(x_{\text{fix}}, y_{\text{fix}}) &= d(\Gamma(x_{\text{fix}}), \Gamma(y_{\text{fix}})) \\ &\leq cd(x_{\text{fix}}, y_{\text{fix}}) \\ &< d(x_{\text{fix}}, y_{\text{fix}}), \text{ if } d(x_{\text{fix}}, y_{\text{fix}}) > 0 \end{aligned}$$

so we must have $d(x_{\text{fix}}, y_{\text{fix}}) = 0$, i.e. $x_{\text{fix}} = y_{\text{fix}}$. Thus (i) holds.
Also we have for m, n , as above,

$$d(x_m, x_n) \leq \frac{c^m}{1-c} d(x_0, \Gamma(x_0)) \implies d(x_n, x_{\text{fix}}) = \lim_{n \rightarrow \infty} d(x_m, x_n) \leq \frac{c^m}{1-c} d(x_0, \Gamma(x_0))$$

so (ii) holds. □

Application: Some differentiable equations

Let $F : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous, and $y_0 \in \mathbb{R}$. We consider the following initial value problem:
Want $f \in C[a, b]$, with $\underbrace{f(a) = y_0}_{\text{initial value}}$ and $\underbrace{f'(t) = F(t, f(t))}_{\text{differential equation}}$ (IVP).

We use the Fundamental Theorem of Calculus to convert this to an integral equation:

Want $f \in C[a, b]$, $f(t) = y_0 + \int_a^t F(s, f(s)) ds$ (IE).

Theorem 17.2 (Picard-Lindelof Theorem). Let F, y_0 be as above and suppose that F is Lipschitz in the second variable: for all $t \in [a, b]$, $y, z \in \mathbb{R}$,

$$|F(t, y) - F(t, z)| \leq L|y - z|, \text{ for some } L > 0.$$

Then (IVP) admits a unique solution, f_{sol} in $C[a, b]$.

Proof. (I) Let us assume that $(b-a)L < 1$. Define $\Gamma : C[a, b] \rightarrow C[a, b]$ by, for $t \in [a, b]$,

$$\Gamma(F)(t) = y_0 + \int_a^t F(s, f(s)) ds.$$

Then for $f, g \in C[a, b]$, and $t \in [a, b]$, then

$$\begin{aligned} |\Gamma(f)(t) - \Gamma(g)(t)| &= \left| \int_a^t [F(s, f(s)) - F(s, g(s))] ds \right| \\ &\leq \int_a^t \underbrace{|F(s, f(s)) - F(s, g(s))|}_{\leq L|f(s) - g(s)|} ds \\ &\leq L \int_a^t \underbrace{|f(s) - g(s)|}_{\leq \|f - g\|_\infty} ds \\ &\leq L \|f - g\|_\infty \int_a^t 1 ds \\ &= L \|f - g\|_\infty (t - a) \leq (b - a)L \|f - g\|_\infty. \end{aligned}$$

In summary,

$$\begin{aligned} \|\Gamma(f) - \Gamma(g)\|_\infty &= \sup_{t \in [a, b]} \|\Gamma(f)(t) - \Gamma(g)(t)\| \\ &\leq \underbrace{(b-a)L}_{< 1} \|f - g\|_\infty. \end{aligned}$$

Hence, by the Contraction Mapping Theorem, applied to Γ on $(C[a, b], \|\cdot\|_\infty)$, there is a unique f_{sol} such that $\Gamma(f_{\text{sol}}) = f_{\text{sol}}$.

(II) Let

$$a = a_1 < a_2 < b_1 < b_3 < b_2 < \cdots < a_n < b_{n-1} < b_n = b$$

so that $(b_j - a_j)L < 1$ for $j = 1, \dots, n$.

Notice that $[a_j, b_j] \cap [a_{j+1}, b_{j+1}] = [a_j, b_{j+1}]$ has non-empty interior.

Let $f_1 \in C[a_1, b_1]$ be the unique solution to (IVP) with $f_1(a) = y_0$, by (I).

Then, let f_2 in $C[a_2, b_2]$ satisfy (IVP) with $f_2(a_2) = f_1(a_2)$. Then, let f_3 in $C[a_3, b_3]$ satisfy (IVP) with $f_3(a_3) = f_2(a_3)$. Etc. Let $f : [a, b] \rightarrow \mathbb{R}$ be given by

$$f(t) = f_j(t) \text{ for } t \in [a_j, b_j], j = 1, \dots, n.$$

Check that this is well-defined. Its value is uniquely determined on each $[a_{j+1}, b_j]$, thanks to uniqueness in (I). \square

18 2017-11-06

Example: (IVP) Want $f \in C[0, 1]$ s.t.

$$f(0) = 1, \quad f'(t) = tf(t).$$

We convert to

$$(IE) \quad f(t) = 1 + \int_0^t sf(s)ds.$$

This fits into Picard-Lindelof Theorem. Let $F(t, y) = ty$, so $f(t) = 1 + \int_0^t F(s, f(s))ds$ with $|F(t, y) - F(t, z)| = \underbrace{|t|}_{\leq 1} |y - z| \leq$

$|y - z|$. (Case (II) of Picard-Lindelof.)

However, let $\Gamma : C[0, 1] \rightarrow C[0, 1]$ by, for $t \in [0, 1]$,

$$\Gamma(f)(t) = 1 + \int_0^t sf(s)ds.$$

Let us see that Γ , itself, is a strict contraction. Let $f, g \in C[0, 1], t \in [0, 1]$,

$$\begin{aligned} |\Gamma(f)(t) - \Gamma(g)(t)| &\leq \int_0^t s \underbrace{|f(s) - g(s)|}_{\leq \|f - g\|_\infty} ds \\ &\leq \int_0^t s ds \|f - g\|_\infty \\ &= \underbrace{\frac{t^2}{2}}_{\leq \frac{1}{2}} \|f - g\|_\infty \\ &\leq \frac{1}{2} \|f - g\|_\infty. \end{aligned}$$

$$(\|\Gamma(f) - \Gamma(g)\|_\infty \leq \frac{1}{2} \|f - g\|_\infty)$$

Hence, contraction mapping theorem tells us that Γ has a unique fixed point, ie (IE) and (IVP) have a unique solution, f_{sol} . Furthermore, if we choose $f_0 \in C[0, 1]$ and let $f_n = \Gamma(f_{n-1})$ ($n \in \mathbb{N}$) then

$$\|f_{\text{sol}} - f_n\|_\infty \leq \underbrace{\frac{(\frac{1}{2})^n}{1 - \frac{1}{2}}}_{=\frac{1}{2^{n-1}}} \|f_0 - \Gamma(f_0)\|_\infty.$$

We can compute f_{sol} .

Let $f_0(t) = 0$ (constant zero).

$$\begin{aligned} f_1(t) &= \Gamma(f_0)(t) = 1 + \int_0^t s \cdot 0 \, ds = 1 \\ f_2(t) &= \Gamma(f_1)(t) = 1 + \int_0^t s \cdot 1 \, ds = 1 + \frac{t^2}{2} \\ f_3(t) &= \Gamma(f_2)(t) = 1 + \int_0^t s \left(1 + \frac{t^2}{2}\right) ds = 1 + \frac{t^2}{2} + \frac{t^4}{4 \cdot 2} \end{aligned}$$

(Use induction to check)

$$f_n(t) = 1 + \frac{t^2}{2} + \frac{t^4}{4 \cdot 2} + \cdots + \frac{t^{2(n-1)}}{[2(n-1)][2(n-2)] \cdots 2} = \sum_{k=1}^n \frac{t^{2(k-1)}}{2^{k-1}(k-1)!}.$$

Thus, at any t in $[0, 1]$,

$$f_{\text{sol}} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{t^{2(k-1)}}{2^{k-1}(k-1)!} = \sum_{k=1}^{\infty} \frac{t^{2(k-1)}}{2^{k-1}(k-1)!}.$$

Furthermore, for each n ,

$$\begin{aligned} \|f_{\text{sol}} - f_n\|_{\infty} &= \max_{t \in [0,1]} |f_{\text{sol}}(t) - f_n(t)| \\ &\leq \frac{1}{2^{n-1}} \left\| 0 - \underbrace{\Gamma(0)}_{=1} \right\|_{\infty} = \frac{1}{2^{n-1}}. \end{aligned}$$

Question: Suppose we only knew that

$$d(\Gamma(x), \Gamma(y)) < d(x, y) \text{ for } x \neq y \text{ in } X.$$

(“proper contraction” instead of “strict contraction”)

Does Γ necessarily admit a fixed point?

Answer #1: No.

Example: On $X = [1, \infty) \subset \mathbb{R}$, let $\Gamma(x) = x + \frac{1}{x}$. If $x < y$, we have there is $x < c_{x,y} < y$ such that

$$|\Gamma(x) - \Gamma(y)| = |\Gamma'(c_{x,y})| |x - y| = \left| 1 - \frac{1}{c_{x,y}^2} \right| |x - y| < |x - y|.$$

Notice: if $\Gamma(x) = x$ we'd have $x = x + \frac{1}{x} \implies 0 = \frac{1}{x}$. Hence Γ admits no fixed point in $[1, \infty)$.

Answer #2: Yes, provided we limit (X, d) .

Theorem 18.1 (Edelstein). Let (X, d) be compact, and $\Gamma : X \rightarrow X$ satisfy $d(\Gamma(x), \Gamma(y)) < d(x, y)$ for $x \neq y$ in X . Then

- (i) Γ admits a unique fixed point x_{fix} , and
- (ii) if $x_0 \in X$, and $x_n = \Gamma(x_{n-1})$ ($n \in \mathbb{N}$), then $x_{\text{fix}} = \lim_{n \rightarrow \infty} x_n$.

Proof. (i) Let $f : X \rightarrow \mathbb{R}, f(x) = d(x, \Gamma(x))$. Since Γ is continuous, f is continuous. [Check that f is 2-Lipschitz.] Hence, by EVT, there is x_{min} in X so $f(x_{\text{min}}) = \min f(X)$. Suppose $x_{\text{min}} \neq \Gamma(x_{\text{min}})$, then

$$\begin{aligned} f(\Gamma(x_{\text{min}})) &= d(\Gamma(x_{\text{min}}), \Gamma \circ \Gamma(x_{\text{min}})) \\ &< d(x_{\text{min}}, \Gamma(x_{\text{min}})) = f(x_{\text{min}}) \end{aligned}$$

violating choice of x_{\min} . Hence $x_{\min} = \Gamma(x_{\min})$, so write $x_{\min} = x_{\text{fix}}$.

If, also, $y = \Gamma(y)$ in X , with $y \neq x_{\text{fix}}$, then

$$d(y, x_{\text{fix}}) = d(\Gamma(y), \Gamma(x_{\text{fix}})) < d(y, x_{\text{fix}})$$

which is absurd.

(ii) Let $x_0 \in X, (x_n)_{n=1}^\infty$ be as above. Notice that

$$0 \leq d(x_{\text{fix}}, x_{n+1}) = d(\Gamma(x_{\text{fix}}), \Gamma(x_0)) < d(x_{\text{fix}}, x_0)$$

so $L = \lim_{n \rightarrow \infty} d(x_{\text{fix}}, x_n)$ exists (decreasing, bounded sequence in \mathbb{R}).

Consider any converging subsequence $(x_{n_k})_{k=1}^\infty$ of $(x_n)_{n=1}^\infty$, with $x = \lim_{k \rightarrow \infty} x_{n_k}$. Then $d(x_{\text{fix}}, x) = \lim_{k \rightarrow \infty} d(x_{\text{fix}}, x_{n_k}) = L$.

If $x \neq x_{\text{fix}}$, then

$$\begin{aligned} L &= \lim_{k \rightarrow \infty} d(x_{\text{fix}}, x_{n_k+1}) = \lim_{k \rightarrow \infty} d(x_{\text{fix}}, \Gamma(x_{n_k})) \\ &= d(x_{\text{fix}}, \Gamma(x)) < d(x_{\text{fix}}, x) = L \end{aligned}$$

which is absurd. Hence the sequence $(x_n)_{n=1}^\infty$ has that x_{fix} is the only possible limit of a subsequence. Thus $\lim_{n \rightarrow \infty} x_n = x_{\text{fix}}$ (check!). \square

19 2017-11-08

Office hours:

| | |
|----------|-----------|
| Today | 2:30-3:30 |
| Tomorrow | 2:30-4 |
| Friday | 2:30-3:30 |

19.1 BAIRE CATEGORY THEOREM

Definition: Let (X, d) be a metric space.

- (i) A subset $N \subset X$ is called nowhere dense if $(\overline{N})^\circ = \emptyset$ (ie. the interior of the closure of N is the empty set). [Equivalently, for any $x \in N, \varepsilon > 0, B(x, \varepsilon) \setminus \overline{N} \neq \emptyset$].
- (ii) A set $S \subseteq X$ will be called meager (or is 1st category) if S is a countable union of nowhere dense sets: i.e.

$$S = \bigcup_{n=1}^{\infty} N_n, \text{ each } (\overline{N}_n)^\circ = \emptyset.$$

(ii') $S \subseteq X$ is non-meager (or is 2nd category) provided that it is not meager.

- (iii) A set $R \subseteq X$ is residual if $X \setminus R$ is meager.

Remarks:

nowhere dense \implies meager

residual \implies non-meager (provided (X, d) is complete;

consequence of B.C.T, Baire Category Theorem)

If (X, d) is complete, we think of meager = “small”, non-meager = “not small” \iff residual.

Examples:

- (i) If $x_0 \in X, \{x_0\}$ is nowhere dense $\iff x_0$ is an accumulation point.

- (ii) In $(\mathbb{R}^2, \|\cdot\|_2)$, $\mathbb{R} \times \{0\}$ is meager (exercise).
- (iii) In $(\mathbb{R}, |\cdot|)$, the Cantor set C is nowhere dense.
Indeed, C is closed. If $t = 0.t_1t_2\cdots \in C$ (ternary representation), then given $\varepsilon > 0$, find k so $\frac{1}{3^k} < \varepsilon$ and then

$$t' = 0.t_1t_2\cdots t_{k-1}1t_{k+1}\cdots \in B(t, \varepsilon) \setminus C.$$

- (iv) $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$ is meager in $(\mathbb{R}, |\cdot|)$ (using (i)).
- (v) $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$ is meager in $(\mathbb{Q}, |\cdot|)$ (using (i)).

Note: if (X, d) is not complete, it may be meager in itself. [meager sets are interesting in complete settings.]

Remark: If (X, d) is a metric space, $U \subseteq X$ is open and $x_0 \in U$, then there is $\varepsilon > 0$, s.t. $B[x, \varepsilon] \subseteq U$ (Indeed, let $\varepsilon' > 0$ be so $B(x, \varepsilon') \subseteq U$, and $\varepsilon \in (0, \varepsilon')$).

Lemma 19.1. Let (X, d) be a metric space, $N \subset X$. Then N is nowhere dense $\iff \overline{X \setminus \overline{N}} = X$.

Proof.

$$\begin{aligned} N \text{ is nowhere dense} &\iff \text{for any } x \in \overline{N}, \varepsilon > 0, B(x, \varepsilon) \setminus \overline{N} \neq \emptyset \\ &\iff x \in \overline{X \setminus \overline{N}} \text{ for any } x \in \overline{N} \cup (X \setminus \overline{N}). \end{aligned}$$

□

Theorem 19.1 (Baire Category Theorem). Let (X, d) be a complete metric space.

- (i) Suppose $\{U_n\}_{n=1}^\infty$ is a sequence of open sets, each dense in X . Then $\bigcap_{n=1}^\infty U_n$ is dense in X .
- (ii) If $M \subset X$ is meager, then $M^\circ = \emptyset$.

Proof. (i) Let $x_0 \in X$ and $\varepsilon_0 > 0$. We wish to show that $B(x_0, \varepsilon_0) \cap \bigcap_{n=1}^\infty U_n \neq \emptyset$.

Since $\overline{U_1} = X$, there is $x_1 \in B(x_0, \varepsilon_0) \cap U_1$ (using meet set characterization of closure). Let $\varepsilon_1 > 0$ be chosen so $B[x_1, \varepsilon_1] \subseteq B(x_0, \varepsilon_0) \cap U_1$.

Since $\overline{U_2} = X$, there is $x_2 \in B(x_1, \varepsilon_1) \cap U_2$.

Let $\varepsilon_2 \in (0, \frac{\varepsilon_1}{2}]$ be so $B[x_2, \varepsilon_2] \subseteq B(x_1, \varepsilon_1) \cap U_2$.

Inductively, having chosen x_n, ε_n , we appeal to the fact that $\overline{U_{n+1}} = X$ to find $x_{n+1} \in B(x_n, \varepsilon_n) \cap U_{n+1}$, then choose $\varepsilon_{n+1} \in (0, \frac{\varepsilon_n}{2}]$ and $B[x_{n+1}, \varepsilon_{n+1}] \subseteq B(x_n, \varepsilon_n) \cap U_{n+1}$.

Thus, we have $(x_n)_{n=1}^\infty \subseteq X$, $(\varepsilon_n)_{n=1}^\infty \subset (0, \infty)$ s.t.

- (a) $B[x_{n+1}, \varepsilon_{n+1}] \subseteq B(x_n, \varepsilon_n) \subseteq B[x_n, \varepsilon_n]$
(b) $\text{diam } B[x_n, \varepsilon_n] = 2\varepsilon_n \leq \varepsilon_{n-1} \leq \frac{\varepsilon_{n-2}}{2} \leq \cdots \leq \frac{\varepsilon_1}{2^{n-1}}$.
(c) $B[x_n, \varepsilon_n] \subseteq U_n \cap B(x_0, \varepsilon_0)$.

Then (a) & (b), with the Nested Sets Theorem, show that $\bigcap_{n=1}^\infty B[x_n, \varepsilon_n] \neq \emptyset$.

Further, (c) shows that $\emptyset \neq \bigcap_{n=1}^\infty B[x_n, \varepsilon_n] \subseteq \bigcap_{n=1}^\infty U_n \cap B(x_0, \varepsilon_0)$.

Hence, for any $x_0 \in X$, $\varepsilon_0 > 0$, $B(x_0, \varepsilon_0) \cap \bigcap_{n=1}^\infty U_n \neq \emptyset$, so $\bigcap_{n=1}^\infty \overline{U_n} = X$.

- (ii) Write $M = \bigcup_{n=1}^\infty N_n$, each $(\overline{N_n})^\circ = \emptyset$. Then $U_n = X \setminus \overline{N_n}$ is open, and dense in X , by Lemma. We have

$$\begin{aligned} X \setminus M &= X \setminus \bigcup_{n=1}^\infty N_n \supseteq X \setminus \bigcup_{n=1}^\infty \overline{N_n} \text{ (as each } N_n \subseteq \overline{N_n}) \\ &= \bigcap_{n=1}^\infty (X \setminus \overline{N_n}) = \bigcap_{n=1}^\infty U_n \end{aligned}$$

so $\overline{X \setminus M} = X$. Thus if $x \in M, \varepsilon > 0$, we have $B(x, \varepsilon) \setminus M = B(x, \varepsilon) \cap (X \setminus M) \neq \emptyset$. Thus $x \notin M^\circ$, i.e. $M^\circ = \emptyset$. \square

Question: Let $\{q_k\}_{k=1}^\infty = \mathbb{Q}$. Let for n in \mathbb{N}

$$U_n = \underbrace{\bigcup_{k=1}^\infty \underbrace{\left(q_k - \frac{1}{2^{kn+1}}, q_k + \frac{1}{2^{kn+1}}\right)}_{\text{length is } \frac{1}{2^{nk}}}}_{U_n \text{ is a union of intervals, sum of lengths is } \sum_{k=1}^\infty \frac{1}{(2^n)^k} = \frac{1}{1 - \frac{1}{2^n}}}$$

Is $\mathbb{Q} = \bigcap_{n=1}^\infty U_n$?

20 2017-11-10

Remark: In particular, a nonempty open subset in a complete metric space is nonmeager. The whole of X is a nonempty open set.

Corollary 20.1. A residual set in a complete metric space is nonmeager.

Proof. Let $R \subset X$ be residual, so $M = X \setminus R$ is meager, so $X \setminus R = \bigcup_{n=1}^\infty N_n$, each $(\overline{N_n})^\circ = \emptyset$. If we had that R was meager, i.e. $R = \bigcup_{n=1}^\infty N'_n$, $(\overline{N'_n})^\circ = \emptyset$, then

$$X = R \cup (X \setminus R) = \underbrace{\bigcup_{n=1}^\infty N'_n}_{\text{countable union of nowhere dense sets}} \cup \bigcup_{n=1}^\infty N_n.$$

But $X^\circ = X$, so this contradicts B.C.T. \square

meager = “small”, residual = “bigness”, “typical elements”

Definition: Let (X, d) be a metric space.

1. $G \subseteq X$ is a G_δ -set if $G = \bigcap_{n=1}^\infty U_n$, each U_n open
2. $F \subseteq X$ is an F_σ -set if $F = \bigcup_{n=1}^\infty F_n$, each F_n closed

Examples:

1. In A4, Q2, we saw that any closed set is G_δ
(i') Any open set $U \subseteq X$ is F_σ (De Morgan's law)

2. $\mathbb{R} \setminus \mathbb{Q}$ is not F_σ .

First, $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$ is F_σ . Second, if $F \subset \mathbb{R} \setminus \mathbb{Q}$ is closed, then F is nowhere dense (this just follows density of \mathbb{Q}). Thus if we had an F_σ realization $\mathbb{R} \setminus \mathbb{Q} = \bigcup_{n=1}^\infty F_n$, $F_n \subset \mathbb{R} \setminus \mathbb{Q}$ closed, then $\mathbb{R} \setminus \mathbb{Q}$ is meager. Thus,

$$\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q}) = \bigcup_{q \in \mathbb{Q}} \{q\} \cup \bigcup_{n=1}^\infty F_n$$

would be meager which violates B.C.T. (Corollary just stated).

(ii') \mathbb{Q} is not G_δ (De Morgan, from (ii)).

In particular

$$\mathbb{Q} \not\subseteq \bigcap_{n=1}^\infty \bigcup_{k=1}^\infty \underbrace{\left(q_k - \frac{1}{2^{kn+1}}, q_k + \frac{1}{2^{kn+1}}\right)}_{U_n}.$$

$$\{q_k\}_{k=1}^{\infty} = \mathbb{Q}.$$

Corollary 20.2. In a complete metric space, a dense G_δ -subset is residual.

Proof. In complete (X, d) , if $G = \bigcap_{n=1}^{\infty} U_n$, each U_n open, and $\overline{G} = X$, then each $\overline{U_n} = X$. Thus, by lemma before B.C.T., each $X \setminus U_n$ is nowhere dense hence $X \setminus G = X \setminus \bigcap_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} (X \setminus U_n)$ is meager. \square

Theorem 20.1 (Uniform Boundedness Principle). Let (X, d) be a complete metric space and $\{f_i\}_{i \in I} \subset C(X)$ (continuous \mathbb{R} -valued functions) which satisfies for each x

$$\sup_{i \in I} |f_i(x)| < \infty \text{ (pointwise boundedness).}$$

Then there exists an open $\emptyset \neq U \subseteq X$ s.t.

$$\sup_{i \in I} \sup_{x \in U} |f_i(x)| < \infty \text{ (uniform boundedness on } U\text{).}$$

Proof. For n in \mathbb{N} , let

$$F_n = \{x \in X : |f_i(x)| \leq n \text{ for all } i \in I\}.$$

By our pointwise boundedness assumption,

$$X = \bigcup_{n=1}^{\infty} F_n \quad (\star).$$

Each F_n is closed:

$$F_n = \bigcap_{i \in I} |f_i|^{-1}((-\infty, n]) = \bigcap_{i \in I} (X \setminus \underbrace{|f_i|^{-1}(n, \infty)}_{\substack{\text{open, as } |f_i(\cdot)| \text{ is continuous} \\ \text{closed}}})$$

But B.C.T. tells us that our complete X is non-meager, so for some n_0 , $F_{n_0}^\circ \neq \emptyset$. Let $U = F_{n_0}^\circ$, and for all $x \in U \subseteq F_n$

$$\begin{aligned} & |f_i(x)| \leq n_0 \text{ for all } i \in I \\ \implies & \sup_{x \in U} |f_i(x)| \leq n_0 \text{ for all } i \in I \\ \implies & \sup_{i \in I} \sup_{x \in U} |f_i(x)| \leq n_0 < \infty. \end{aligned}$$

\square

Corollary 20.3 (Banach-Stenhaus Theorem). Let $(V, \|\cdot\|_V)$ be a Banach space, $(W, \|\cdot\|_W)$ a normed vector space, and $\{T_i\}_{i \in I} \subset B(V, W)$ satisfies

$$\sup_{i \in I} \|T_i x\|_W < \infty \text{ for each } x \in V.$$

Then

$$\sup_{i \in I} \|T_i\| < \infty. \text{ [Recall } \|T_i\| = \sup_{x \in B_V[0,1]} \|T_i x\|_W.]$$

Proof. Let $f_i(x) = \|T_i x\|_W$, for $i \in I, x \in V$, so $\{f_i\}_{i \in I} \subset C(V)$. Our assumption on $\{T_i\}_{i \in I}$, gives pointwise boundedness of $\{f_i\}_{i \in I}$, so U.B.P provides $\emptyset \neq U \subset V$ for which

$$M = \sup_{i \in I} \sup_{x \in U} \|T_i x\| < \infty.$$

As U is open, if $x_0 \in U$, there is $\varepsilon > 0$, $B[x_0, \varepsilon] \subset U$.

Now if $z \in B_V[0, 1]$, then we may write

$$z = \frac{1}{2\varepsilon}(-x_0 + \varepsilon z) + \frac{1}{2\varepsilon}(x_0 + \varepsilon z)$$

and, for i in I , we have

$$\begin{aligned}\|T_i z\|_W &\leq \frac{1}{2\varepsilon} \left\| T_i \left(\underbrace{x_0 - \varepsilon z}_{\in B[x, \varepsilon] \subset U} \right) \right\|_W + \frac{1}{2\varepsilon} \left\| T_i \left(\underbrace{x_0 + \varepsilon z}_{\in B[x, \varepsilon] \subset U} \right) \right\|_W \\ &\leq \frac{1}{2\varepsilon} M + \frac{1}{2\varepsilon} M = \frac{M}{\varepsilon}. \\ \Rightarrow \|T_i\| &= \sup_{z \in B_V[0,1]} \|T_i z\|_W \leq \frac{M}{\varepsilon} < \infty.\end{aligned}$$

□

21 2017-11-13

21.1 BAIRE-1 FUNCTIONS

Def: Let $\emptyset \neq X \subseteq \mathbb{R}$, so (X, d) is a metric space with relativized metric from \mathbb{R} .

A function $f : X \rightarrow \mathbb{R}$ is called Baire-1 if there is a sequence $(f_n)_{n=1}^\infty \subset C(X)$ such that for $t \in X$,

$$f(t) = \lim_{n \rightarrow \infty} f_n(t) \text{ (pointwise limit).}$$

Remark: Unlike uniform limits, pointwise limits of continuous functions need not be continuous.

Example: Let $X = [0, 1]$, $f_n(t) = t^n$. Then

$$\lim_{n \rightarrow \infty} f_n(t) = \begin{cases} 0 & t \in [0, 1) \\ 1 & t = 1. \end{cases}$$

Question: Let for t in $[0, 1]$,

$$f_n(t) = \cos(n! \pi t)^{n!}.$$

If $t = \frac{k}{\ell} \in \mathbb{Q}$, $\ell \in \mathbb{N}$, then $f_n(t) = 1$, if $t \geq \ell + 1$.

Does $\lim_{n \rightarrow \infty} f_n(t) = \chi_{\mathbb{Q} \cap [0, 1]}(t)$ for t in $[0, 1]$?

Answer: No. (Probably the limit does not exist.)

The answer will follow from (corollary to) the next theorem and B.C.T.

Theorem 21.1 (Baire). Let $a < b$, and $f : (a, b) \rightarrow \mathbb{R}$ be a Baire-1 function, then there is t_0 in (a, b) such that f is continuous at t_0 .

$$\chi_{\mathbb{Q}}(t) = \lim_{n \rightarrow \infty} \underbrace{\lim_{m \rightarrow \infty} |\cos(n! \pi t)^m|}_{\chi_{\{\frac{k}{n!}, k \in \mathbb{Z}\}}(t)}$$

Baire-2 = pointwise limit of Baire-1 functions.

At no t_0 is $\chi_{\mathbb{Q}}$ continuous, thus not Baire-1.

Proof. Let $f(t) = \lim_{n \rightarrow \infty} f_n(t)$, $t \in (a, b)$, $(f_n)_{n=1}^\infty \subset C(a, b)$.

(I) Given $\varepsilon > 0$, we will show that there are $\alpha < \beta$ in (a, b) , and N_ε in \mathbb{N} such that for all $n, m \geq N_\varepsilon$,

$$|f_n(t) - f_m(t)| \leq \varepsilon \text{ for } t \in [\alpha, \beta].$$

Let us proceed by contradiction. Hence, there exists t_1 in (a, b) , and $n_1, m_1 \in \mathbb{N}$ such that

$$|f_{n_1}(t_1) - f_{m_1}(t_1)| > \varepsilon.$$

Since each f_{n_1}, f_{m_1} is continuous, there is an open interval $I_1 \subset \overline{I_1} \subset (a, b)$ such that

$$|f_{n_1}(t) - f_{m_1}(t)| > \varepsilon \text{ for } t \in I_1.$$

$[t \mapsto |f_{n_1}(t) - f_{m_1}(t)| \text{ is continuous.}]$

Next, by assumption, there is $t_2 \in I_1$ such that there exist $n_2, m_2 > \max\{n_1, m_1\}$ such that

$$|f_{n_2}(t_2) - f_{m_2}(t_2)| > \varepsilon.$$

Again, as f_{n_2}, f_{m_2} are continuous, there is an open interval $I_2 \subset \overline{I_2} \subset I_1$ such that

$$|f_{n_2}(t) - f_{m_2}(t)| > \varepsilon \text{ for } t \in I_2.$$

Inductively, we obtain

- a sequence of intervals

$$\overline{I_1} \supset I_1 \supset \overline{I_2} \supset I_2 \supset \cdots \supset \overline{I_n} \supset I_n \supset \cdots, \text{ and}$$

- two increasing sequences $(n_k)_{k=1}^\infty, (m_k)_{k=1}^\infty \subseteq \mathbb{N}$ such that

$$|f_{n_k}(t) - f_{m_k}(t)| > \varepsilon \text{ for } t \in I_k.$$

Thus, by Nested Intervals Theorem, there exists

$$t_0 \in \bigcap_{k=1}^\infty \overline{I_k} = \bigcap_{k=2}^\infty \overline{I_k} \subseteq \bigcap_{k=1}^\infty I_k$$

so $t_0 \in I_k$ for each k , so

$$|f_{n_k}(t_0) - f_{m_k}(t_0)| > \varepsilon. \quad (\dagger)$$

But, by pointwise convergence, $f(t_0) = \lim_{k \rightarrow \infty} f_k(t_0)$ so $(f_n(t_0))_{n=1}^\infty \subset \mathbb{R}$ is Cauchy. This violates (\dagger) . Hence (I) holds.

(II) We use (I), with $\varepsilon = 1$, to find $\alpha_1 < \beta_1$ in (a, b) and N_1 in \mathbb{N} so

$$|f_n(t) - f_m(t)| \leq 1 \text{ for } t \in [\alpha_1, \beta_1], \text{ if } n, m \geq N_1.$$

We again use (I), with $\varepsilon = \frac{1}{2}$, to find $\alpha_2 < \beta_2$ in (a, b) and N_2 in \mathbb{N} so

$$|f_n(t) - f_m(t)| \leq \frac{1}{2} \text{ for } t \in [\alpha_2, \beta_2], \text{ if } n, m \geq N_2.$$

Inductively, we obtain

- intervals

$$(a, b) \supset [\alpha_1, \beta_1] \supset (\alpha_1, \beta_1) \supset [\alpha_2, \beta_2] \supset (\alpha_2, \beta_2) \supset \cdots \supset [\alpha_n, \beta_n] \supset (\alpha_n, \beta_n) \supset \cdots, \text{ and}$$

- an increasing sequence $(N_k)_{k=1}^\infty \subset \mathbb{N}$ such that

$$|f_n(t) - f_m(t)| \leq \frac{1}{k} \text{ for } t \in [\alpha_k, \beta_k], \text{ if } n, m \geq N_k. \quad (\ddagger)$$

By N.I.T. (Nested Intervals Theorem), there exists

$$t_0 \in \bigcap_{k=1}^\infty [\alpha_k, \beta_k] \subseteq \bigcap_{k=1}^\infty (\alpha_k, \beta_k).$$

Now, given $\varepsilon > 0$, let k in \mathbb{N} so $\frac{1}{k} < \varepsilon$, and then let $\delta = \min\{t_0 - \alpha_k, \beta_k - t_0\} > 0$ so $(t_0 - \delta, t_0 + \delta) \subset (\alpha_k, \beta_k) \subset [\alpha_k, \beta_k]$. Hence by (\ddagger) , we have that

$$|f_n(t) - f_m(t)| \leq \frac{1}{k} < \varepsilon \text{ whenever } t \in (t_0 - \delta, t_0 + \delta), n, m \geq N_k.$$

Hence $(f_n)_{n=1}^\infty$ converges “uniformly at t_0 ” (see Assignment 6), so f is continuous at t_0 (Assignment 6). \square

Corollary 21.1. Let $a < b$ in \mathbb{R} , $f : (a, b) \rightarrow \mathbb{R}$ be a Baire-1 function. The set $G = \{t \in (a, b) : f \text{ is continuous at } t\}$ is a dense G_δ -subset of (a, b) . [By B.C.T., $G \subset [a, b]$ is residual.]

Proof. If $t_0 \in (a, b)$ and $\varepsilon > 0$, then there exists $t \in (t_0 - \varepsilon, t_0 + \varepsilon) \cap (a, b) \cap G$. I.e. $G \cap (t_0 - \varepsilon, t_0 + \varepsilon) \neq \emptyset$, so $\overline{G} = (a, b)$ (relativized topology). Furthermore, the set G is always G_δ (Assignment 6). \square

Example: $\underbrace{\chi_{\mathbb{Q}}}_{\text{nowhere continuous}}$ is not Baire-1 on any interval.

22 2017-11-15

Corollary 22.1. Let $f \in C(a, b)$ ($a < b$ in \mathbb{R}) be right differentiable on (a, b) . Then f'_+ (right derivative) is continuous on a dense G_δ -subset of (a, b) . [In particular, if f is differentiable, f' is continuous on a dense G_δ -subset.]

Proof. Let $h_n(t) = \min\{b - t, \frac{1}{n}\}$ for n in \mathbb{N}, t in (a, b) . Then

$$f_n(t) = \frac{f(t + h_n(t)) - f(t)}{h_n(t)} \quad \left(= \frac{f(t + \frac{1}{n}) - f(t)}{\frac{1}{n}}, n \text{ large} \right)$$

satisfies that each $f_n \in C(a, b)$ and

$$f'_+(t) = \lim_{n \rightarrow \infty} f_n(t) \text{ for each } t \in (a, b).$$

\square

22.1 ON THE BANACH SPACES $C(X)$, X COMPACT

First case $X = [a, b]$, compact interval in \mathbb{R} .

Lemma 22.1. For n in \mathbb{N} let $q_n(t) = c_n(1 - t^2)^n$ where c_n satisfies

$$1 = c_n \int_{-1}^1 (1 - t^2)^n dt.$$

Then

(q1) $q_n(t) \geq 0$ for $t \in [-1, 1], n$ in \mathbb{N} (non-negative)

(q2) $\int_{-1}^1 q_n(t) dt = 1, n$ in \mathbb{N} (total mass 1)

(q3) if $\delta \in (0, 1)$, then $\left(\int_{-1}^{-\delta} + \int_{\delta}^1 \right) q_n(t) dt \xrightarrow{n \rightarrow \infty} 0$ (concentration of mass near 0)

Proof. (q1) and (q2) are obvious. Now for $t \in [0, 1]$,

$$\begin{aligned} t^2 \leq t &\implies 1 - t \leq 1 - t^2 \\ &\implies (1 - t)^n \leq (1 - t^2)^n \end{aligned}$$

and hence

$$\begin{aligned} \frac{1}{c_n} &= \int_{-1}^1 (1 - t^2)^n dt = 2 \int_0^1 (1 - t^2)^n dt \\ &\leq 2 \int_0^1 (1 - t)^n dt = \frac{-2}{n+1} (1 - t)^{n+1} \Big|_0^1 = \frac{2}{n+1} \end{aligned}$$

so $c_n \leq \frac{n+1}{2}$. Hence, for $|t| \in (\delta, 1)$, we have

$$\begin{aligned} q_n(t) &= c_n(1-t^2)^n \leq c_n(1-t^2)^n \\ &\leq \frac{n+1}{2} \underbrace{(1-t^2)^n}_{<1} \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Thus

$$\begin{aligned} \left(\int_{-1}^{-\delta} + \int_{\delta}^1 \right) q_n(t) dt &\leq \left(\int_{-1}^{-\delta} + \int_{\delta}^1 \right) \frac{n+1}{2} (1-t^2)^n dt \\ &= (1-\delta)(n+1)(1-\delta^2)^n \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

□

Theorem 22.1 (Weierstrauss approximation theorem). Given $a < b$ in \mathbb{R} , $f \in C[a, b]$, there exists a sequence $(p_n)_{n=1}^{\infty}$ of polynomial functions such that

$$(WA) \|p_n - f\|_{\infty} = \max_{t \in [a, b]} |p_n(t) - f(t)| \xrightarrow{n \rightarrow \infty} 0.$$

Proof. (I) We condition f . Let $\tilde{f} \in C[0, 1]$ be given by

$$\tilde{f}(t) = f(a + t(b-a)) - [f(b) - f(a)]t - f(a).$$

So

- $\tilde{f}(0) = f(b) - f(a) = 0$
- $\tilde{f}(1) = f(b) - [f(b) - f(a)]1 - f(a) = 0.$

If we can find a sequence $(\tilde{p}_n)_{n=1}^{\infty}$ of polynomials,

$$\|\tilde{p}_n - \tilde{f}\|_{\infty} = \sup_{t \in [0, 1]} |\tilde{p}_n(t) - \tilde{f}(t)| \xrightarrow{n \rightarrow \infty} 0$$

we are done. Indeed, if $s \in [a, b]$, then define each $p_n(s) = \tilde{p}_n(\frac{1}{b-a}(s-a)) + \frac{f(b)-f(a)}{b-a}(s-a) + f(a)$; may be easily shown to satisfy (WA).

(II) Let us assume that

$$f \in C[0, 1], f(0) = 0 = f(1).$$

We can extend f to \mathbb{R} by letting $f(t) = 0$ for $t \in (-\infty, 0) \cup (1, \infty)$, so $f \in C_b(\mathbb{R})$, but $f(t) \neq 0$ only possibly for $t \in [0, 1]$, and f is uniformly continuous [any function in $C[0, 1]$ is uniformly continuous].

Let $(q_n)_{n=1}^{\infty}$ be as in the last lemma, and let for each n in \mathbb{N} and each t in $[0, 1]$,

$$p_n(t) = \int_0^1 q_n(s-t)f(s)ds.$$

Let us compute, for each n, t ,

$$\begin{aligned} \frac{d^{2n+1}}{dt^{2n+1}} p_n(t) &= \int_0^1 \frac{\partial^{2n+1}}{\partial t^{2n+1}} \underbrace{q_n(s-t)}_{\text{function is } 2n+2\text{-times continuously differentiable}} f(s) ds \\ &= 0, \text{ since } \deg q_n(t) = \deg(1-t^2)^n = 2n. \end{aligned}$$

$\implies p_n$ is a polynomial, $\deg p_n(t) \leq 2n$.

By change of variable $u = s - t$,

$$\begin{aligned} p_n(t) &= \int_0^1 q_n(s-t)f(s)ds \\ &= \int_{-t}^{1-t} q_n(u)f(u+t)du \\ &= \int_{-1}^1 q_n(u)f(u+t)du, \text{ since } f(u+t) \geq 0 \text{ possibly only on } [-t, 1-t]. \end{aligned}$$

Hence for t in $[0, 1]$,

$$\begin{aligned} |p_n(t) - f(t)| &= \left| \int_{-1}^1 q_n(u)f(u+t)du - \underbrace{\int_{-1}^1 q_n(u)f(t)du}_{\text{property (q2)}} \right| \\ &\leq \int_{-1}^1 q_n(u)|f(u+t) - f(t)|du. \end{aligned}$$

Given $\varepsilon > 0$, let $\delta > 0$ be so $|x - y| < \delta(x, y \in \mathbb{R}) \implies |f(x) - f(y)| < \frac{\varepsilon}{2}$ and then

$$\begin{aligned} |p_n(t) - f(t)| &\leq \int_{-\delta}^{\delta} q_n(u) \underbrace{|f(u+t) - f(t)|}_{< \frac{\varepsilon}{2}, \text{ by choice of } \delta} du + \left(\int_{-1}^{-\delta} + \int_{\delta}^1 \right) q_n(u) \underbrace{|f(u+t) - f(t)|}_{\leq 2\|f\|_{\infty}} du \\ &\leq \frac{\varepsilon}{2} \int_{-1}^1 q_n(u)du + \left(\int_{-1}^{-\delta} + \int_{\delta}^1 \right) q_n(u)2\|f\|_{\infty} du \text{ by (q1)} \xrightarrow{n \rightarrow \infty} \frac{\varepsilon}{2} + 0. \end{aligned}$$

(Continued next lecture.)

23 2017-11-17

We saw p_n is polynomial, i.e. $d^{2n+1}/dt^{2n+1}p_n(t) = 0$. Need approx.

Using (q2) we saw for $t \in [0, 1]$

$$|p_n(t) - f(t)| \leq \int_{-1}^1 \underbrace{q_n(u)}_{(q1)} |f(u+t) - f(t)| du$$

Given $\varepsilon > 0$, use uniform continuity of f to find $\delta > 0$ s.t. $|x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{2}$.

$$\begin{aligned}
|p_n(t) - f(t)| &\leq \int_{-1}^1 q_n(u) |f(u+t) - f(t)| du \\
&= \int_{-\delta}^{\delta} q_n(u) |f(u+t) - f(t)| du + \left(\int_{-1}^{-\delta} + \int_{\delta}^1 \right) q_n(u) \underbrace{|f(u+t) - f(t)|}_{\leq 2\|f\|_{\infty}} du \\
&\leq \int_{-\delta}^{\delta} q_n(u) \frac{\varepsilon}{2} du + \left(\int_{-1}^{-\delta} + \int_{\delta}^1 \right) q_n(u) 2\|f\|_{\infty} du \\
&\leq \underbrace{\frac{\varepsilon}{2} \int_{-\delta}^{\delta} q_n(u) du}_{=1(q2)} + 2\|f\|_{\infty} \left(\int_{-1}^{-\delta} + \int_{\delta}^1 \right) q_n(u) du.
\end{aligned}$$

Hence, if n_{ε} is so $n \geq n_{\varepsilon} \implies \left(\int_{-1}^{-\delta} + \int_{\delta}^1 \right) q_n(u) du \leq \frac{\varepsilon}{2(2\|f\|_{\infty}+1)}$
we have for $n \geq n_{\varepsilon}$,

$$|p_n(t) - f(t)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and we thus have

$$\|p_n - f\|_{\infty} = \max_{t \in [0,1]} |p_n(t) - f(t)| < \varepsilon$$

and we thus see that $\lim_{n \rightarrow \infty} p_n = f$ in $(C[0,1], \|\cdot\|_{\infty})$. □

Corollary 23.1. If $f \in C^1[a, b]$ (differentiable on $[a, b]$, with continuous derivative). Then, given $\varepsilon > 0$, there is a polynomial p s.t.

$$\begin{aligned}
\|p' - f\|_{\infty} &< \varepsilon \\
\|p - f\|_{\infty} &< (b-a)\varepsilon.
\end{aligned}$$

Proof. By Weierstrauss approximation theorem, find a polynomial q s.t. $\|f' - q\|_{\infty} < \varepsilon$. Let $p(t) = f(a) + \int_a^t q(s) ds$. Check that this works. (Remember Fundamental Theorem of Calculus.) □

Corollary 23.2. $(C[a, b], \|\cdot\|_{\infty})$ is separable.

Proof. Let $f \in C[a, b], \varepsilon > 0$.

By Weierstrauss approximation theorem, find polynomial p s.t.

$$\|f - p\|_{\infty} < \frac{\varepsilon}{2}.$$

Write $p(t) = a_0 + a_1 t + \dots + a_n t^n$. For $j = 1, \dots, n$ let $q_j \in \mathbb{Q}$ be such that

$$|a_j - q_j| < \frac{\varepsilon}{2(n+1) \max\{|a|^j, |b|^j\}}$$

then let $r(t) = q_0 + q_1 t + \dots + q_n t^n$.

Check that for each t in $[a, b]$,

$$|p(t) - r(t)| < \frac{\varepsilon}{2}$$

so $\|p - r\|_{\infty} = \max_{t \in [a,b]} |p(t) - r(t)| < \frac{\varepsilon}{2}$,
and thus

$$\|f - r\|_{\infty} \leq \|f - p\|_{\infty} + \|p - r\|_{\infty} < \varepsilon.$$

□

Theorem 23.1 (nowhere differentiable functions are generic). Let $ND[0, 1]$ denote the set of f in $C[0, 1]$ which are nowhere differentiable. Then $ND[0, 1]$ is residual in $C[a, b]$.

Proof. Recall for $M, \delta > 0$,

$$F_{M,\delta} = \{f \in C[0,1] : \text{there is } x \text{ in } [0,1] \text{ so } \frac{|f(x) - f(t)|}{|x - t|} \leq M \\ \text{for all } t \in [0,1] \cap [(x - \delta, x) \cup (x, x + \delta)] \}$$

(A5,Q1).

(I) Let us see that each $F_{M,\delta}$ is nowhere dense in $(C[0,1], \|\cdot\|_\infty)$.

To this end, let $f \in F_{M,\delta}, \varepsilon > 0$.

First, use Weierstrauss approximation to get a polynomial p so $\|f - p\|_\infty < \frac{\varepsilon}{2}$. In particular, p' exists everywhere, let $M' = \sup_{t \in [0,1]} \|p'(t)\|$.

Let

$$\varphi : [0, \infty) \rightarrow [0, 1], \varphi(t) = \begin{cases} t - n & t \in [n, n+1], n \in \{0\} \cup \mathbb{N} \text{ is even} \\ n+1 - t & t \in [n, n+1], n \in \mathbb{N} \text{ is odd} \end{cases}$$

For each k in \mathbb{N} let $\varphi_k(t) = \frac{1}{k} \varphi(k^2 t)$.

For $s, t \in [\frac{n-1}{k^2}, \frac{n}{k^2}]$, $n \in \mathbb{N}$,

$$\frac{|\varphi_k(s) - \varphi_k(t)|}{|s - t|} = k \quad (\dagger).$$

Now let k be so $\frac{1}{k} < \frac{\varepsilon}{2}$ and $k - M' > M, \frac{1}{k^2} < \delta$.

Let $\psi_k = p + \varphi_k$ and we have for s, t satisfying (\dagger) ,

$$\begin{aligned} \frac{|\psi_k(s) - \psi_k(t)|}{|s - t|} &= \left| \frac{p(s) - p(t)}{s - t} - \frac{\varphi_k(s) - \varphi_k(t)}{s - t} \right| \\ &\geq \left| \underbrace{\frac{|\psi_k(s) - \psi_k(t)|}{|s - t|}}_k - \underbrace{\frac{|p(s) - p(t)|}{|s - t|}}_{\leq M', \text{ by Mean Value Theorem}} \right| \\ &\geq |k - M'| = k - M' > M. \end{aligned}$$

Hence $\psi_k \notin F_{M,\delta}$. And $\|f - \psi_k\|_\infty \leq \|f - p\|_\infty + \left\| \underbrace{p - \psi_k}_{-\varphi_k} \right\|_\infty < \frac{\varepsilon}{2} + \frac{1}{k} < \varepsilon$. □

24 2017-11-20

Theorem 24.1. $ND[0,1] = \{f \in C[0,1] : f \text{ is nowhere differentiable}\}$ is a residual set in $(C[0,1], \|\cdot\|_\infty)$.

Proof. We saw:

Each

$$F_{M,\delta} = \{f \in C[0,1] : \exists x \text{ in } [0,1], \frac{|f(x) - f(t)|}{|x - t|} \leq M \text{ for } t \in [0,1] \cap [(x - \delta, x) \cup (x, x + \delta)]\}$$

is closed (A5), nowhere dense (I).

(II) Let $SD[0,1] = C[0,1] \setminus ND[0,1]$ ("somewhere differentiable"). If $f \in SD[0,1]$, in A5, it was shown that $f \in F_{M,\delta}$ for some $M > 0, \delta > 0$. If $n \in \mathbb{N}$, with $n > \max\{M, \frac{1}{\delta}\}$, then $F_{M,\delta} \subseteq F_{n, \frac{1}{n}}$. Then

$$SD[0,1] = \bigcup_{n=1}^{\infty} F_{n, \frac{1}{n}}, \text{ each } F_{n, \frac{1}{n}} \text{ closed, } F_{n, \frac{1}{n}}^\circ = \emptyset.$$

Thus $SD[0,1]$ is meager, so $ND[0,1] = C[0,1] \setminus SD[0,1]$ is residual. □

Remark: Baire Category Theorem tells us that in the complete metric space $(C[0,1], \|\cdot\|_\infty)$.
residual = "large" = "generic"

24.1 TOWARDS STONE-WEIERSTRAUSS THEOREM

Notation: (lattice structure)

Let X be non-empty, $f, g : X \rightarrow \mathbb{R}$. Define

$$\begin{aligned} \text{("join")} \quad & f \vee g : X \rightarrow \mathbb{R}, f \vee g(x) = \max\{f(x), g(x)\} \\ \text{("meet", min)} \quad & f \wedge g : X \rightarrow \mathbb{R}, f \wedge g(x) = \min\{f(x), g(x)\}. \end{aligned}$$

Proposition 24.1. Let (X, d) be a (compact) metric space, $f, g \in C(X)$. Then $f \vee g, f \wedge g \in C(X)$.

Proof. If $a, b \in \mathbb{R}$, then $\max\{a, b\} = \frac{1}{2}(a + b) + \frac{1}{2}|a - b|$.

Hence

$$f \vee g = \frac{1}{2}(f + g) + \frac{1}{2} \underbrace{|f - g|}_{f-g \text{ compact with } |\cdot|} \in C(X).$$

Also $\min\{a, b\} = -\max\{-a, -b\}$, so

$$f \wedge g = -(-f) \vee (-g) \in C(X).$$

□

Notation: A family $\mathcal{L} \subseteq C(X)$ is called a lattice if for each $f, g \in \mathcal{L}$, $f \vee g, f \wedge g \in \mathcal{L}$. Notice if $f_1, \dots, f_n \in \mathcal{L}$,

$$\begin{aligned} f_1 \vee f_2 &\in \mathcal{L} \\ \implies f_1 \vee f_2 \vee f_3 &\in \mathcal{L} \\ &\vdots \text{ (obvious induction)} \\ \implies f_1 \vee \dots \vee f_n &\in \mathcal{L}. \end{aligned}$$

Likewise $f_1 \wedge \dots \wedge f_n \in \mathcal{L}$.

Theorem 24.2 (Stone). Let (X, d) be a compact metric space and let the lattice $\mathcal{L} \subseteq C(X)$ satisfy

- \mathcal{L} is a \mathbb{R} -space
- $1 \in \mathcal{L}$ (contains constant function)
- \mathcal{L} separates points: if $x \neq y$ in X , there exists $\varphi \in \mathcal{L}$, so $\varphi(x) \neq \varphi(y)$.

Then $\overline{\mathcal{L}} = C(X)$ (\mathcal{L} is uniformly dense in $C(X)$).

Proof. Suppose $x \neq y$ in X and $\alpha, \beta \in \mathbb{R}$. Since \mathcal{L} separates points, there is $\varphi \in \mathcal{L}$ with $\varphi(x) \neq \varphi(y)$. Then

$$g = \alpha 1 + \frac{\beta - \alpha}{\varphi(y) - \varphi(x)} [\varphi - \varphi(x)1] \in \mathcal{L} \text{ as } 1 \in \mathcal{L}, \mathcal{L} \text{ is a } \mathbb{R}\text{-subspace}$$

with $g(x) = \alpha, g(y) = \beta$.

Fix $f \in C(X), \varepsilon > 0$.

(I) Fix x in X . For each y in X , letting $\alpha = f(x), \beta = f(y)$, if $y \neq x$, we have that there is

$$g_{x,y} \in \mathcal{L} \text{ s.t. } g_{x,y}(x) = f(x), g_{x,y}(y) = f(y).$$

Since each $f, g_{x,y}$ are continuous (near y), there are $\delta_y > 0$ so that

$$d(z, y) < \delta_y \implies g_{x,y}(z) < f(z) + \varepsilon \text{ i.e. } g_{x,y} < f + \varepsilon \text{ on } B(y, \delta_y)$$

$$\text{(i.e. } g_{x,y} - f \text{ is 0 at } y \text{ so } < \varepsilon \text{ in a neighbourhood of } y)$$

Since $X = \bigcup_{y \in X} B(y, \delta_y)$, by compactness, there are y_1, \dots, y_m s.t. $X = \bigcup_{j=1}^m B(y_j, \delta_{y_j})$. Let

$$g_x = g_{x, y_1} \wedge \dots \wedge g_{x, y_m} \in \mathcal{L}$$

and we have $g_x \leq g_{x, y} < f + \varepsilon 1$.

Notice that $g_x(x) = \min\{f_{x, y_1}(x), \dots, f_{x, y_m}(x)\} = f(x)$. □

25 2017-11-22

Small goof up:

Then we let $g_x = g_{x, y_1} \wedge \dots \wedge g_{x, y_m} \in \mathcal{L}$.

Now, if $z \in X$, then $z \in B(y_j, \delta_{y_j})$ for some $j = 1, \dots, m$ and then

$$g_x(z) = g_{x, y_1} \wedge \dots \wedge g_{x, y_m} \leq g_{x, y_j}(z) < f(z) + \varepsilon, \text{ property of } \delta_{y_j} \text{ w.r.t. } y_j$$

so we have

$$g_x < f + \varepsilon 1, \text{ and } g_x(x) = f(x).$$

(II) For each x in X , we found $g_x \in \mathcal{L}$ s.t. $g_x < f + \varepsilon 1, g_x(x) = f(x)$.

Hence $g_x(x) = f(x) < f(x) + \varepsilon$ at each x , so there is $\delta_x > 0$, s.t.

$$g_x(z) > f(z) - \varepsilon \text{ for } z \in B(x, \delta_x).$$

We have $X = \bigcup_{x \in X} B(x, \delta_x)$ so there are $x_1, \dots, x_n \in X$ so $X = \bigcup_{j=1}^n B(x_j, \delta_{x_j})$. We then let

$$g = g_{x_1} \vee \dots \vee g_{x_n} \in \mathcal{L}.$$

For $z \in X, z \in B(x_j, \delta_{x_j})$ for some $j = 1, \dots, n$, so

$$g(z) \geq g_{x_j}(z) > \dots > f(z) - \varepsilon$$

and thus

$$g > f - \varepsilon 1.$$

Furthermore, each $g_{x_j} < f + \varepsilon 1$, so if $z \in X$, then $g(z) = g_{x_j}(z)$ for some j , so

$$g(z) = g_{x_j}(z) < f(z) + \varepsilon \implies g < f + \varepsilon 1$$

i.e. $f - \varepsilon 1 < g < f + \varepsilon 1$, so $g \in B(f, \varepsilon)$ in $(C(X), \|\cdot\|_\infty)$.

In summary, given $f \in C(X), \varepsilon > 0, B(f, \varepsilon) \cap \mathcal{L} \neq \emptyset$. Hence, $\overline{\mathcal{L}} = C(X)$. □

Corollary 25.1. (i) Let $\mathcal{L} = \{f \in C[a, b] : f \text{ is piecewise affine (A5)}\}$. Then $\overline{\mathcal{L}} = C[a, b]$.

(ii) Let C be the Cantor set and $\mathcal{L} = \{f \in C(C) : |f(C)| < \aleph_0\}$. Then $\overline{\mathcal{L}} = C(C)$.

Definition: Let (X, d) be a (compact) metric space. A subset $A \subseteq C(X)$ is called an algebra if for $f, g \in A, \alpha \in \mathbb{R}$, we have

$$\begin{aligned} f + \alpha g &\in A && (A \text{ is a } \mathbb{R}\text{-subspace}) \\ fg &\in A && (A \text{ is closed under pointwise multiplication}) \end{aligned}$$

(If $f, g \in C(X)$, then $fg \in C(X)$, too.) If $f_1, \dots, f_n \in A, f_1 \cdots f_n \in A$ too.

If $1 \in A$, and $p(t) = \sum_{i=1}^n a_i t^i$, then for $f \in A$,

$$p \circ f = a_0 1 + a_1 f + a_2 f^2 + \dots + a_n f^n \in A.$$

$(f^k(x) = f(x)^k \text{ for } x \in X.)$

Theorem 25.1 (Stone-Weierstrauss Theorem). If (X, d) is a compact metric space, $A \subseteq C(X)$ satisfies

- A is an algebra
- $1 \in A$
- A separates points: for $x \neq y$ in X , there is $g \in A$ so $g(x) \neq g(y)$

Then $\overline{A} = C(X)$ (uniform closure).

Proof. (I) If $f \in A$, then $|f| \in \overline{A}$. First, since (X, d) is compact, f continuous, $f(X) \subset \mathbb{R}$ is compact, hence bounded, so there is $a > 0$ s.t. $f(X) \subseteq [-a, a]$. Now, the Weierstrauss approximation theorem provides $(p_n)_{n=1}^\infty$ of polynomials s.t. $\|p_n - |\cdot|\|_\infty = \max_{t \in [-a, a]} |p_n(t) - |t|| \rightarrow 0$. Hence $\|p_n \circ f - |f|\|_\infty = \max_{x \in X} |p_n(f(x)) - |f(x)|| \rightarrow 0$. Each $p_n \circ f \in A$.

(II) Since A is a \mathbb{R} -subspace, so is \overline{A} (A4 Q1). If $f, g \in \overline{A}$, let $f = \lim_{n \rightarrow \infty} f_n, g = \lim_{n \rightarrow \infty} g_n$ under uniform limits, each $f_n, g_n \in A$. Then

$$\begin{aligned} f \vee g &= \frac{1}{2}(f + g) + \frac{1}{2}|f - g| \\ &= \lim_{n \rightarrow \infty} \underbrace{\frac{1}{2}(f_n + g_n)}_{\in A \subseteq \overline{A}} + \underbrace{\frac{1}{2}|f_n - g_n|}_{\in A \text{ by (I)}} \in \overline{A} \end{aligned}$$

since \overline{A} is closed.

Also, $f \wedge g = -(-f) \vee (-g) \in \overline{A}$ as well.

$\implies \overline{A}$ is a \mathbb{R} -subspace and a lattice. Also, $1 \in A \subseteq \overline{A}$, and A separates points, hence \overline{A} separates points.

Thus \overline{A} is dense in $C(X)$, but is closed, so $\overline{A} = C(X)$. □

26 2017-11-24

Example: Let $I = [a_1, b_1] \times \cdots \times [a_n, b_n]$ be a non-empty compact interval in \mathbb{R}^n . A polynomial on I is any function

$$p(t_1, \dots, t_n) = \sum_{j_1, \dots, j_n=1}^N a_{j_1, \dots, j_n} t_1^{j_1} \cdots t_n^{j_n}$$

where each $a_{j_1, \dots, j_n} \in \mathbb{R}, N \in \mathbb{N}$. By Stone-Weierstrauss Theorem, the family $P(I)$ of polynomial functions is dense in $C(I)$.

Example: Let $(X, d_X), (Y, d_Y)$ be compact metric spaces. Let $\|\cdot\|$ be a norm on \mathbb{R}^2 . Define

$$\rho(X \times Y) \times (X \times Y) \rightarrow [0, \infty) \text{ by}$$

$$\rho((x_1, y_1), (x_2, y_2)) = \|(d_X(x_1, x_2), d_Y(y_1, y_2))\|.$$

It is “obvious” that ρ is a metric on $X \times Y$.

(Usually, $\|\cdot\| = \|\cdot\|_\infty, \|\cdot\|_1, \|\cdot\|_2$ on \mathbb{R}^2 .)

Furthermore, $(X \times Y, \rho)$ is compact. Indeed, let $((x_n, y_n))_{n=1}^\infty \subseteq X \times Y$ be a sequence. Then $(x_n)_{n=1}^\infty \subseteq X$ admits a converging subsequence: let $x = \lim_{k \rightarrow \infty} x_{n_k} \in X$. Then $(y_{n_k})_{k=1}^\infty \subseteq Y$ admits a converging subsequence: let $y = \lim_{\ell \rightarrow \infty} y_{n_{k_\ell}} \in Y$.

Notice that

$$\begin{aligned} &\rho((x, y), (x_{n_{k_\ell}}, y_{n_{k_\ell}})) \\ &= \left\| (d_X(x, x_{n_{k_\ell}}), d_Y(y, y_{n_{k_\ell}})) \right\| \\ &\leq d_X(x, x_{n_{k_\ell}}) \|(1, 0)\| + d_Y(y, y_{n_{k_\ell}}) \|(0, 1)\| \\ &\xrightarrow{\ell \rightarrow \infty} 0. \end{aligned}$$

Hence $((x_{n_{k_\ell}}, y_{n_{k_\ell}}))_{\ell=1}^\infty$ is a converging subsequence of $((x_n, y_n))_{n=1}^\infty$. Suppose that each $A_X \subseteq C(X)$ and $A_Y \subseteq C(Y)$, each satisfy assumptions of Stone-Weierstrauss Theorem. If $f \in A_X, g \in A_Y$,

$$f \otimes g : X \times Y \rightarrow \mathbb{R}, f \otimes g(x, y) = f(x)g(y).$$

Let $A_X \otimes A_Y = \text{span}_{\mathbb{R}}\{f \otimes g : f \in A_X, g \in A_Y\}$. Convince yourself that $A_X \otimes A_Y \subseteq C(X \times Y)$ and satisfies assumptions of Stone-Weierstrauss Theorem.

Hence $\overline{A_X \otimes A_Y} = C(X \times Y)$ (uniform closure).

Corollary 26.1 (Stone-Weierstrauss without constant functions). Let (X, d) be a compact metric space, and $A \subseteq C(X)$ satisfy

- A is an algebra
- A separates points
- there is x_0 in X s.t. $f(x_0) = 0$ for f in A .

Then $\overline{A} = C_{x_0}(X) := \{f \in C(X) : f(x_0) = 0\}$.

Proof. First, $C_{x_0}(X)$ is closed in $C(X)$. (Let $\varphi : C(X) \rightarrow \mathbb{R}, \varphi(f) = f(x_0)$, which is linear and continuous: $\|\varphi\| \leq 1$ (seen before). Then $C_{x_0}(X) = \varphi^{-1}(\{0\}) = C(X) \setminus \underbrace{\varphi^{-1}(\mathbb{R} \setminus \{0\})}_{\substack{\text{open} \\ \text{open}}} \underbrace{\phantom{C(X) \setminus \varphi^{-1}(\mathbb{R} \setminus \{0\})}}_{\text{closed}}$. Since $A \subseteq C_{x_0}(X) \implies \overline{A} \subseteq C_{x_0}(X)$.)

Second, note that $\mathbb{R}1 + A = \{\alpha 1 + f : \alpha \in \mathbb{R}, f \in A\}$ satisfies $\overline{\mathbb{R}1 + A} = C(X)$. If $g \in \mathbb{R}1 + A$, write $g = \alpha 1 + h, \alpha \in \mathbb{R}, h \in A$, and $g(x_0) = \alpha + h(x_0) = \alpha$ so $g = g(x_0)1 + h$.

Now, if $f \in C_{x_0}(X)$, there exists $(g_n)_{n=1}^\infty \subseteq \mathbb{R}1 + A$ s.t. $\|f - g\|_\infty \xrightarrow{n \rightarrow \infty} 0$ (Stone-Weierstrauss Theorem). Write each $g_n = g_n(x_0)1 + h_n$ where $h_n \in A$. Notice that $0 = f(x_0) = \lim_{n \rightarrow \infty} g_n(x_0)$. Hence

$$\begin{aligned} \|f - h_n\|_\infty &\leq \|f - (g_n(x_0)1 + h_n)\|_\infty + \|g_n(x_0)1\|_\infty \\ &= \|f - g_n\|_\infty + |g_n(x_0)| \quad (\|1\|_\infty = 1) \\ &\xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Thus $C_{x_0}(X) \subseteq \overline{A}$. □

Def: Let $C_\infty(\mathbb{R}) = \{\bar{f} \in C(\mathbb{R}) : \lim_{|t| \rightarrow \infty} f(t) = 0\}$. Then $C_\infty(\mathbb{R}) \subseteq C_b(\mathbb{R})$ and is a closed subspace. ($L_\pm : C_b(\mathbb{R}) \rightarrow \mathbb{R}, L_\pm(f) = \lim_{t \rightarrow \pm\infty} f(t)$, then L_+, L_- are linear and with $\|L_\pm\| \leq 1$. Then $C_\infty(\mathbb{R}) = L_+^{-1}(\{0\}) \cap L_-^{-1}(\{0\})$ is closed.)

Corollary 26.2. Let $A \subseteq C_\infty(\mathbb{R})$ satisfy that

- A is an algebra
- A separates points
- for each t of \mathbb{R} , there is $f \in A$ s.t. $f(t) \neq 0$.

Then $\overline{A} = C_\infty(\mathbb{R})$ (uniform closure).

Proof. (Sketch of proof) $\psi : \mathbb{R} \rightarrow (-1, 1), \psi(t) = \frac{t}{|t|+1}$, then ψ is continuous and bijective with $\psi^{-1}(-1, 1) \rightarrow \mathbb{R}$ continuous. Let $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

$$\begin{aligned} \varphi(-1, 1) &\rightarrow S \setminus \{(-1, 0)\} \\ \varphi(s) &= (\cos(\pi s), \sin(\pi s)) \end{aligned}$$

so φ is a continuous bijection with continuous inverse. Hence, $\varphi \circ \psi : \mathbb{R} \rightarrow S \setminus \{(-1, 0)\}$ is a homeomorphism, i.e. continuous bijection with continuous inverse.

Define

$$\begin{aligned}\Psi : C_\infty(\mathbb{R}) &\rightarrow C_{(-1,0)}(S) \\ \Psi(f)(x, y) &= f(\psi^{-1} \circ \varphi^{-1}(x, y)).\end{aligned}$$

Check that Ψ is a surjective isometry, between $(C_\infty(\mathbb{R}), \|\cdot\|_\infty)$ and $(C_{(-1,0)}(S), \|\cdot\|_\infty)$, and hence has isometric inverse.

We have $\Psi(A) \subseteq C_{(-1,0)}(S)$ satisfies assumptions of last corollary, so $\overline{\Psi(A)} = C_{(-1,0)}(S)$ but it follows that $\overline{A} = \Psi^{-1}(\overline{\Psi(A)}) = C_\infty(\mathbb{R})$. \square

27 2017-11-27

Today's subject: towards Arzela-Ascoli Theorem (by guest lecturer)

Def: Let (X, d) be a complete metric space. Let $F \subseteq X$ be a subset. We say F is relatively compact if \overline{F} is compact. (Here \overline{F} means the closure of F .)

Proposition 27.1 (Properties of relatively compact subsets). Let (X, d) be a metric space, $F \subseteq X$. TFAE:

1. F is relatively compact
2. Every sequence (x_n) admits a Cauchy subsequence (x_{n_k})
3. F is totally bounded

Proof. (i) \implies (ii) Let (x_n) be a sequence in F . Since (x_n) is in \overline{F} and \overline{F} is compact, (x_n) has a Cauchy subsequence (x_{n_k}) (that may converge to a point in $\overline{F} \setminus F$).

(ii) \implies (i) Let (x_n) be a sequence in \overline{F} . We want to show there is a subsequence (x_{n_k}) converging to a point in \overline{F} (note this is nonempty by characterization of the closure).

Now, by (ii), there is a Cauchy subsequence (y_{n_k}) .

Claim: (x_{n_k}) is Cauchy.

For $k, \ell \geq 1$,

$$\begin{aligned}d(x_{n_k}, x_{n_\ell}) &\leq d(x_{n_k}, y_{n_k}) + d(x_{n_k}, y_{n_\ell}) + d(x_{n_\ell}, y_{n_\ell}) \\ &\leq \frac{1}{n_k} + d(y_{n_k}, y_{n_\ell}) + \frac{1}{n_\ell} \xrightarrow{k, \ell \rightarrow \infty} 0.\end{aligned}$$

(i) \implies (iii) \overline{F} is totally bounded since it is compact. So for $\frac{\varepsilon}{2} > 0$, there are $x_1, \dots, x_n \in \overline{F}$ s.t. $B(x_i, \frac{\varepsilon}{2})$ covers \overline{F} (i.e. $\bigcup_{i=1}^n B(x_i, \frac{\varepsilon}{2}) \supseteq \overline{F}$).

For each i , choose $y_i \in B(x_i, \frac{\varepsilon}{2}) \cap F$. Then $B(y_i, \varepsilon) \supseteq B(x_i, \frac{\varepsilon}{2})$ so y_1, \dots, y_n is an ε -net for F .

(iii) \implies (i) Since F is totally bounded, there is an ε -net $y_1, \dots, y_n \in F$. So

$$\begin{aligned}F &\subseteq \bigcup_{i=1}^n B(y_i, \varepsilon) \\ \implies \overline{F} &\subseteq \bigcup_{i=1}^n \overline{B(y_i, \varepsilon)} \\ \implies \overline{F} &\subseteq \bigcup_{i=1}^n B(y_i, 2\varepsilon).\end{aligned}$$

So \overline{F} is totally bounded. \square

Def: [Equicontinuity] Let (X, d) be a (compact) metric space. A subset $F \subseteq C(X)$ is equicontinuous if for $\varepsilon > 0$ and $x \in X$ there is $\delta > 0$ s.t. if $d(x, y) < \delta$ then $|f(y) - f(x)| < \varepsilon \forall f \in F$ (holds for all f simultaneously).

Lemma 27.1. If (X, d) is compact and $F \subseteq C(X)$ then F is equicontinuous $\iff F$ is uniformly equicontinuous meaning for $\varepsilon > 0$ there is $\delta > 0$ s.t. if $x, y \in X$ and $d(x, y) < \delta$ then $|f(x) - f(y)| < \varepsilon \forall f \in F$.

Proof. If F is uniformly equicontinuous it is clearly equicontinuous.

For the other direction, fix $\varepsilon > 0$. For each x there is δ_x s.t. if $d(x, y) < \delta_x$ then $|f(y) - f(x)| < \varepsilon/2 \forall f \in F$. Then $(B(x, \delta_x))_{x \in X}$ is an open cover. Let $\delta > 0$ be the corresponding Lebesgue covering number. So for any $y \in X$, $B(y, \delta) \subseteq B(x, \delta_x)$ for some $x \in X$. So if $y, z \in X$ with $d(y, z) < \delta$, choose $x \in X$ s.t. $B(y, \delta) \subseteq B(x, \delta_x)$, then

$$\begin{aligned} |f(y) - f(z)| &\leq |f(y) - f(x)| + |f(x) - f(z)| \quad (z \in B(x, \delta_x)) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

□

Ex: Let F be a set of differentiable functions from $[0, 1]$ to \mathbb{R} s.t. $|f'(x)| \leq M \forall f \in F, x \in [0, 1]$ for some M . By the MVT, for $x, y \in [0, 1]$ there is $z \in [0, 1]$ s.t. $M \geq |f'(z)| = \frac{|f(y) - f(x)|}{|y - x|}$.

$$|f(y) - f(x)| \leq M|y - x| \forall y, x \in [0, 1], \forall f \in F.$$

Now take $\delta = \frac{\varepsilon}{M}$. Then if $|x - y| < \delta$ then

$$\begin{aligned} |f(x) - f(y)| &\leq M|x - y| \\ &< M \frac{\delta}{M} = \varepsilon. \end{aligned}$$

28 2017-11-29

Office Hours:

Today: 2:30-4:30

Tomorrow: 2-4 pm

Last time:

In complete (X, d) , TFAE:

- (i) relative compactness
- (ii) every sequence admits a Cauchy subsequence
- (iii) total boundedness

Discussed for $F \subset C(X)$:

- equicontinuity \implies uniform equicontinuity if (X, d) compact
- pointwise boundedness

Theorem 28.1 (Arzela-Ascoli Theorem). Let (X, d) be a compact metric space, $F \subset C(X)$. Then

F is relatively compact in $(C(X), \|\cdot\|_\infty) \iff F$ is both equicontinuous and pointwise bounded.

Proof. (\implies) F is totally bounded. In particular, F is bounded: $\sup_{f \in F} \|f\|_\infty < \infty$ (totally bounded \implies bounded). Hence for x in X , $\sup_{f \in F} |f(x)| < \sup_{f \in F} \sup_{x \in X} |f(x)| = \sup_{f \in F} \|f\|_\infty < \infty$.

Given $\varepsilon > 0$, let $f_1, \dots, f_n \in F$ s.t. $F \subseteq \bigcup_{j=1}^n B[f_j, \frac{\varepsilon}{3}]$. Let for $j = 1, \dots, n$, $\delta_j > 0$ be so for x, y in X , $d(x, y) < \delta_j \implies |f_j(x) - f_j(y)| < \frac{\varepsilon}{3}$ (uniform continuity of f_j). Then let $\delta = \min\{\delta_1, \dots, \delta_n\}$ and then for x, y in X , $d(x, y) < \delta$, we have for f in F , then $f \in B[f_j, \frac{\varepsilon}{3}]$ for some j . Then

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f(y)| \\ &< \|f - f_j\|_\infty + \frac{\varepsilon}{3} + \|f - f_j\|_\infty \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Hence, F is (uniformly) equicontinuous, thus equicontinuous.

(\Leftarrow) Let $(x_n)_{n=1}^\infty \subset X$ satisfy that there are $n_1 < n_2 < n_3 < \dots$ for which

$$X = \bigcap_{k=1}^\infty \bigcup_{j=1}^{n_k} B[x_j, \frac{1}{k}] \quad (\dagger)$$

(assignment 5, (X, d) compact $\implies (X, d)$ separable).

Now, let $(f_n)_{n=1}^\infty \subseteq F$. We wish to extract a uniformly Cauchy subsequence, hence showing F is relatively compact.

(I) Let us extract a candidate Cauchy subsequence. This technique is a variant of “Cantor’s diagonalization argument”. First, $(f_n(x_1))_{n=1}^\infty \subset \mathbb{R}$ is bounded (pointwise bounded assumption) so by Bolzano-Weierstrauss admits a Cauchy subsequence $(f_{n_k}(x_1))_{k=1}^\infty \subset \mathbb{R}$. Let $f_{1,k} = f_{n_k}$ for each k . Second, $(f_{1,n}(x_2))_{n=1}^\infty \subset \mathbb{R}$ is bounded, and again admits a Cauchy subsequence $(f_{1,n_k}(x_2))_{k=1}^\infty \subset \mathbb{R}$. Let $f_{2,k} = f_{1,n_k}$.

Inductively, we continue. We build sequences $(f_{1,k})_{k=1}^\infty, (f_{2,k})_{k=1}^\infty, \dots, (f_{n,k})_{k=1}^\infty, \dots \subseteq F$ which satisfy

- $m < n$, $(f_{n,k})_{k=1}^\infty$ is a subsequence of $(f_{m,k})_{k=1}^\infty$
- $(f_{n,k}(x_n))_{k=1}^\infty \subset \mathbb{R}$ is Cauchy.

We now let

$$g_n = f_{n,n}.$$

Then $(g_n)_{n=m}^\infty$ is a subsequence of $(f_{m,n})_{n=1}^\infty$ so $(g_n(x_m))_{n=1}^\infty$ is Cauchy in \mathbb{R} , (being a subsequence of $(f_{m,n}(x_m))_{n=1}^\infty$). Thus $(g_n(x_m))_{m=1}^\infty$ is Cauchy for each m in \mathbb{N} , and $(g_k)_{k=1}^\infty$ is a subsequence of $(f_n)_{n=1}^\infty$.

(II) Let us show that $(g_n)_{n=1}^\infty$ is Cauchy in $(C(X), \|\cdot\|_\infty)$, i.e., Cauchy in $\|\cdot\|_\infty$.

Given $\varepsilon > 0$, our set F , being equicontinuous on compact (X, d) , is uniformly equicontinuous (lemma Monday), so there is $\delta > 0$ s.t. $|f(x) - f(y)| < \frac{\varepsilon}{3}$ whenever $x, y \in X$, $d(x, y) < \delta$ and $f \in F$.

Now, let k in \mathbb{N} satisfy $\frac{1}{k} < \delta$, and we have from (\dagger) that $X = \bigcup_{j=1}^{n_k} B[x_j, \delta]$.

Now, for $j = 1, \dots, n_k$, let N_j in \mathbb{N} be s.t. $m, n \geq N_j \implies |g_m(x_j) - g_n(x_j)| < \frac{\varepsilon}{3}$ (i.e. $(g_n(x_j))_{n=1}^\infty$ is Cauchy). Let $N = \max\{N_1, \dots, N_{n_k}\}$. If $x \in X$, so $x \in B[x_j, \frac{\varepsilon}{3}]$ for some $j = 1, \dots, n_k$, and we have for $m, n \geq N$ that

$$\begin{aligned} |g_m(x) - g_n(x)| &\leq |g_m(x) - g_m(x_j)| + |g_m(x_j) - g_n(x_j)| + |g_n(x_j) - g_n(x)| \\ &< \underbrace{\frac{\varepsilon}{3}}_{\text{thanks to uniform equicontinuity of } F; g_n \in F} + \underbrace{\frac{\varepsilon}{3}}_{n, m \geq N \geq N_j \text{ Cauchy at } x_j} \\ &\quad + \underbrace{\frac{\varepsilon}{3}}_{\text{thanks to uniform equicontinuity of } F; g_n \in F} = \varepsilon. \end{aligned}$$

Hence $\|g_m - g_n\|_\infty = \max_{x \in X} |g_m(x) - g_n(x)| < \varepsilon$.

– END OF FINAL LINE (except Assignment 7) –

□

29 2017-12-01

Theorem 29.1 (Peano’s Theorem). Let $D \subset \mathbb{R}^2$ be open and $F : D \rightarrow \mathbb{R}$ be continuous, and $(t_0, y_0) \in D$. Then there are $a < b$ in \mathbb{R} so $t_0 \in (a, b)$ for which

$$(IVP) \quad f'(t) = F(t, f(t)), f(t_0) = y_0, t \in (a, b)$$

admits a solution.

(This is stronger than Picard-Lindelof, which required a Lipschitz condition on the second variable of a two variable function.) The solution here may not be unique.

Proof. (Most of proof):

(I) (Get $a < b$.) Let $R = [a_1, b_1] \times [a_2, b_2] \subset D$ (compact interval) so $(t_0, y_0) \in R^\circ$ (interior), and let $M = \max_{(t,y) \in R} |F(t, y)|$.

We let

$$W = \{(t, y) \in D : |y - y_0| \leq M|t - t_0|\}$$

and $a < b$ in \mathbb{R} so

$$([a, b] \times \mathbb{R}) \cap W \subset R.$$

(II) (Work on $[t_0, b]$, find a particular family of piecewise affine functions.) Given $\varepsilon > 0$, the uniform continuity of F on R provides $\delta > 0$ such that

$$\begin{aligned} (s, x), (t, y) \in R \text{ with } \max\{|s - t|, |x - y|\} = \|(s, x) - (t, y)\|_\infty < \delta \\ \implies |F(s, x) - F(t, y)| < \varepsilon. \end{aligned}$$

We partition $[t_0, b]$, $t_0 < t_1 < \dots < t_n = b$, so $\max_{j=1, \dots, n} (t_j - t_{j-1}) < \frac{\delta}{M+1}$ (let $M = 0$).

We define $f_\varepsilon : [t_0, b] \rightarrow \mathbb{R}$ inductively by

$$f_\varepsilon(t) = \begin{cases} y_0 + F(t_0, y_0)(t - t_0) & t \in [t_0, t_1] \\ f_\varepsilon(t_1) + F(t_1, f_\varepsilon(t_1))(t - t_1) & t \in (t_1, t_2] \\ \vdots & \\ f_\varepsilon(t_{n-1}) + F(t_{n-1}, f_\varepsilon(t_{n-1}))(t - t_{n-1}) & t \in (t_{n-1}, t_n] \end{cases}.$$

Two nice properties (exercise):

- graph of f_ε on $[t_0, b]$ is in R , so $\max_{t \in [t_0, b]} |f_\varepsilon(t)| \leq \max\{|a_2|, |b_2|\}$
- if $s < t$ in $[t_0, b]$, then $|f_\varepsilon(t) - f_\varepsilon(s)| \leq M|t - s|$ (\dagger).

These estimates are independent of ε . I.e. if we form $K = \{f_\varepsilon\}_{\varepsilon \in (0, \infty)}$ it is

- pointwise bounded & equi-Lipschitz \implies (uniformly) equicontinuous.

Hence K is relatively compact.

(III) (Relate $K = \{f_\varepsilon\}_{\varepsilon \in (0, \infty)}$ to the (IVP).) Fix $f_\varepsilon, \varepsilon$ and δ as in $(\varepsilon - \delta)$ above. If $t \in (t_j, t_{j+1})$, $j = 0, \dots, n-1$ then

$$f'_\varepsilon(t) = F(t_j, f_\varepsilon(t_j)). \quad (\star)$$

Also, for such t as above, then $|t - t_j| < \frac{\delta}{M+1}$ so by (\dagger)

$$|f_\varepsilon(t) - f_\varepsilon(t_j)| \leq M|t - t_j| \leq \delta \frac{M}{M+1} < \delta$$

so, by choice of δ ,

$$\begin{aligned} |F(t, f_\varepsilon(t)) - F(t_j, f_\varepsilon(t_j))| &< \varepsilon \\ (\text{using } (\star)) \implies |F(t, f_\varepsilon(t)) - f'_\varepsilon(t)| &< \varepsilon \quad (\star\star). \end{aligned}$$

Thus for $t \in [t_0, b]$ we have

$$\begin{aligned} f_\varepsilon(t) &= y_0 + \int_{t_0}^t f'_\varepsilon(s) ds \text{ (piecing together F.T. of C., as } f'_\varepsilon(t) \text{ exists except at } t_1, \dots, t_{n-1}) \\ &= y_0 + \int_{t_0}^t F(s, f_\varepsilon(s)) ds + \int_{t_0}^t [f'_\varepsilon(s) - F(s, f_\varepsilon(s))] ds \end{aligned}$$

Let $\tilde{f}_\varepsilon(t) = y_0 + \int_{t_0}^t F(s, f_\varepsilon(s))ds$, and we have for $t \in [t_0, b]$

$$|f_\varepsilon(t) - \tilde{f}_\varepsilon(t)| \leq \int_{t_0}^t \underbrace{|f'_\varepsilon(s) - F(s, f_\varepsilon(s))|}_{< \varepsilon} ds$$

$$(\star \star \star) \quad \leq (t - t_0)\varepsilon \leq (b - t_0)\varepsilon.$$

We now consider a sequence $(f_{\frac{1}{n}})_{n=1}^\infty \subseteq K$. By relative compactness, we get a uniformly Cauchy, hence uniformly converging subsequence $(f_{\frac{1}{n_k}})_{k=1}^\infty, f = \lim_{k \rightarrow \infty} f_{\frac{1}{n_k}}$ (uniform limit). Let $\tilde{f}(t) = y_0 + \int_{t_0}^t F(s, f(s))ds$.

We have

$$\|f - \tilde{f}\|_\infty \leq \|f - f_{\frac{1}{n_k}}\|_\infty + \|f_{\frac{1}{n_k}} - \tilde{f}_{\frac{1}{n_k}}\|_\infty + \|\tilde{f}_{\frac{1}{n_k}} - \tilde{f}\|_\infty$$

We have $\lim_{k \rightarrow \infty} f_{\frac{1}{n_k}}(s) = f(s)$ uniformly for $s \in [t_0, b]$, so, by uniform continuity $\lim_{k \rightarrow \infty} |F(s, f_{\frac{1}{n_k}}(s)) - F(s, f(s))| = 0$ uniformly for s in $[t_0, b]$, and thus $(\ddagger) \xrightarrow{k \rightarrow \infty} 0$. In conclusion

$$\|f - \tilde{f}\|_\infty \leq \|\tilde{f}_{\frac{1}{n_k}}\|_\infty + (b - t_0)\frac{1}{n_k} + (\ddagger)$$

$\implies f(t) = \tilde{f}(t) = y_0 + \int_{t_0}^t F(s, f(s))ds$, i.e. f satisfies (IE) \implies (IVP).

□