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# PMATH 351

## REAL ANALYSIS

PROF: NICO SPRONK • FALL 2017 • UNIVERSITY OF WATERLOO

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**Abstract**

These notes are intended as a resource for myself; past, present, or future students of this course, and anyone interested in the material. The goal is to provide an end-to-end resource that covers all material discussed in the course displayed in an organized manner. If you spot any errors or would like to contribute, please contact me directly.

**1 CHAINS AND ZORN'S LEMMA**

Let  $(X, \leq)$  be a poset. A chain is any subset  $C \subseteq X$  such that  $(C, \leq)$  is totally ordered.

Office hours:

1. Today 2:30 - 3:20
2. Wednesday next week 2:30 - 4:30

Or, email [nspronk@uwaterloo.ca](mailto:nspronk@uwaterloo.ca)

**2 CARDINAL ARITHMETIC**

i.  $:($

ii.  $\mathbb{R} \underset{f}{\sim} (-1, 1), f(x) = x/|x| + 1$  (exercise: exhibit  $f^{-1}$ )

iii.  $a < b$  in  $\mathbb{R}.(0, 1) \underset{g}{\sim} (a, b), g(x) = a + x(b - a)$

Notation:  $\aleph_0 = |\mathbb{N}|$  ("aleph naught"),  $c = |\mathbb{R}|$  ("continuous")

Arithmetic: Let  $A, B$  be sets.

$$|A| + |B| = |A \sqcup B|$$

$$|A||B| = |A \times B|$$

$$|A|^{|B|} = |A^B| (B \neq \emptyset, A^B = \{f : B \rightarrow A \mid \text{function}\})$$

$A \sqcup A$  is two copies of  $A$ ,  $\sim A \times \{1, 2\}$

Properties

- (commutativity)  $|A| + |B| = |B| + |A|$ ,  $|A||B| = |B||A|$
- (distributivity)  $|A|(|B| + |C|) = |A||B| + |A||C|$

$$A \times (B \sqcup C) \sim (A \times B) \sqcup (A \times C)$$

- (Exponential laws)

$$|A|^{|B|+|C|} = |A|^{|B|}|A|^{|C|}, |A|^{|B||C|} = (|A|^{|B|})^{|C|}$$

$$(B \neq \emptyset \neq C)$$

$$A^{B \sqcup C} \sim A^B \times A^C \text{ via } \varphi \mapsto (\varphi|_B, \varphi|_C)$$

$$A^{B \times C} \sim (A^B)^C \text{ via } \varphi \mapsto (\varphi(b, \cdot) : C \rightarrow A)$$

Now, for sets  $A, B$ , define  $A \preceq B$  if there is an injection  $f : A \rightarrow B$ .

Sometimes write  $A \preceq B$ . As above:

$$\text{(reflexivity)} \quad A \preceq A$$

$$\text{(transitivity)} \quad A \preceq B, B \preceq C \implies A \preceq C$$

Seems reasonable to write  $|A| \leq |B|$ , in this case.

Question: Is  $\leq$  in cardinal numbers anti-symmetric?

**Theorem 2.1** (Cantor-Bernstein-Schroder Theorem). If, for non-empty set  $A, B$  we have  $A \preceq B, B \preceq A$ , then  $A \sim B$ . I.e. if  $|A| \leq |B|$  and  $|B| \leq |A|$ , then  $|A| = |B|$ .

*Proof.* Our assumption is that we have injections  $A \xrightarrow{\varphi} B, B \xrightarrow{\psi} A$ .

To avoid triviality, let us suppose that neither  $\varphi$  nor  $\psi$  is surjective. Thus  $\varphi(A) \subsetneq B, \psi(B) \subsetneq A$ .

Let  $A_0 = A, A_1 = \psi(B), A_2 = \psi \circ \varphi(A)$  and we inductively define  $A_{n+2} = g(A_n), g = \psi \circ \varphi$ .

Then  $A_2 \subsetneq A_1 \subsetneq A_0$ , so by applying injection  $g$ ,

$$\begin{aligned} A_2 &\subsetneq A_1 \subsetneq A_0 \\ &\vdots \\ A_{n+1} &\subsetneq A_n \subsetneq A_{n-1} \end{aligned}$$

Hence, we may decompose

$$\begin{aligned} A &= A_0 = (A_0 \setminus A_1) \cup A_1 \\ &= (A_0 \setminus A_1) \cup (A_1 \setminus A_2) \cup A_2 \\ &\vdots \\ &= \bigcup_{n=1}^{\infty} (A_{n-1} \setminus A_n) \cup A_{\infty} \end{aligned}$$

where  $A_{\infty} = \bigcap_{n=1}^{\infty} A_n = \bigcap_{n=2}^{\infty} A_n$ , we likewise observe

$$A_1 = \bigcup_{n=2}^{\infty} (A_{n-1} \setminus A_n) \cup A_{\infty}.$$

Picture:

$$\underbrace{\overbrace{A_0 \setminus A_1} \quad \overbrace{A_1 \setminus A_2} \quad \dots \quad \overbrace{A_{\infty}}} \quad A_0$$

Using definitions of the sets  $A_n$  ( $n \geq 2$ ), we have  $g(A_{n-1} \setminus A_n) = A_{n+1} \setminus A_{n+2}$ . Define

$$h : A_0 \rightarrow A_1, h(x) = \begin{cases} g(x), & \text{if } x \in A_{n-1} \setminus A_n, n \text{ odd} \\ x, & \text{otherwise} \end{cases}$$

Then  $h$  is a bijection. Thus

$$A = A_0 \xrightarrow{h} A_1 = \psi(B), B \xrightarrow{\psi} \psi(B)$$

so we conclude that  $A \sim B$ . □

Examples:

1. Let  $a < b$  in  $\mathbb{R}$ . Then  $[a, b] \preceq \mathbb{R}$  (obvious)  
 $\mathbb{R} \sim (-1, 1) \sim (0, 1) \sim (a, b) \preceq [a, b]$   
 I.e.  $[a, b] \preceq \mathbb{R}$  and  $\mathbb{R} \preceq [a, b]$  so  $\mathbb{R} \sim [a, b]$

### 3 2017-09-18

#### 3.1 LAST CLASS: C.B.S THEOREM

If  $A \preceq B$  and  $B \preceq A$  then  $A \sim B$ .

Examples:

- (i)  $\mathcal{P}(\mathbb{N}) \sim \mathbb{R}$ , i.e.  $|\mathcal{P}(\mathbb{N})| = c$ .

$$\mathcal{P}(\mathbb{N}) \sim \{0, 1\}^{\mathbb{N}}, \text{ via } A \mapsto \chi_A \text{ where } \chi_A(n) \begin{cases} 1 & , n \in A \\ 0 & , n \notin A \end{cases} \text{ ("characteristic indicator")}$$

$$\{0, 1\}^{\mathbb{N}} \sim \mathcal{P}(\mathbb{N}), \text{ via } (x_k)_{k=1}^{\infty} \xrightarrow[\text{injective}]{\quad} \chi_A \text{ where } \sum_{k=1}^{\infty} \frac{x_k}{3^k} = 0.x_1x_2x_3\ldots \text{ (ternary representation)}$$

$$[0, 1] \sim \{0, 1\}^{\mathbb{N}}, 0.x_1x_2x_3\ldots = \sum_{k=1}^{\infty} \frac{x_k}{2^k} \text{ (binary representation) (never allow } 0.111\ldots = 1!) \mapsto (x_k)_{k=1}^{\infty}$$

$$\mathcal{P}(\mathbb{N}) \sim \{0, 1\}^{\mathbb{N}} \preceq [0, 1] \preceq \{0, 1\}^{\mathbb{N}} \sim \mathcal{P}(\mathbb{N})$$

so, by C.B.S. Theorem, we have  $|\mathcal{P}(\mathbb{N})| = |[0, 1]| = c = |\mathbb{R}|$ .

(ii)

2nd lecture:

- (iii)  $\mathbb{N} \sim \mathbb{Q} \sim \mathbb{N}^2$

$$\mathbb{N} \preceq \mathbb{Q}$$

$$\mathbb{Q} \preceq \mathbb{Z} \times \mathbb{N}, \text{ via } \frac{m}{n} \mapsto (m, n) \text{ (gcd}(m, n) = 1)$$

$$\mathbb{Z} \times \mathbb{N} \sim \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}, \text{ as } \mathbb{Z} \sim \mathbb{N}$$

$$\mathbb{N}^2 \preceq \mathbb{N}, \text{ via } (m, n) \mapsto 2^m 3^n$$

Hence  $\mathbb{N} \preceq \mathbb{Q} \preceq \mathbb{Z} \times \mathbb{N} \sim \mathbb{N}^2 \preceq \mathbb{N}$  so, by C.B.S. Theorem,  $\mathbb{N} \sim \mathbb{Q} \sim \mathbb{N}^2$ .

Notation: We say that a set  $A$  is

- countable if  $A \preceq \mathbb{N}$ , i.e.  $|A| \leq \aleph_0$
- denumerable if  $A \sim \mathbb{N}$ , i.e.  $|A| = \aleph_0$

**Proposition 3.1** (surjectivity). Suppose  $X$  and  $Y$  are non-empty sets and there is a surjection  $g : X \rightarrow Y$ . Then  $Y \preceq X$ .

*Proof.* Let  $f : \mathcal{P}(X) \setminus \{\emptyset\} \rightarrow X$  be a choice function (AC). For each  $y \in Y$ , we have  $g^{-1}(\{y\}) = \{x \in X : g(x) = y\} \neq \emptyset$ , as  $g$  is surjective. Define  $h : Y \rightarrow X$  be given by  $h(y) = f(g^{-1}(\{y\}))$  and  $h$  is injective, as if  $y_1 \neq y_2$ ,  $\{y_1\} \cap \{y_2\} = \emptyset$ , so we see that  $g^{-1}(\{y_1\}) \cap g^{-1}(\{y_2\}) = \emptyset$  too.  $\square$

**Theorem 3.1** (Comparison Theorem). Let  $X, Y$  be sets. Then either  $X \preceq Y$  or  $Y \preceq X$ .

*Proof.* If  $X \neq \emptyset$ , then  $X \preceq Y$ ; likewise if  $Y = \emptyset$ . Hence assume  $X \neq \emptyset \neq Y$ . We let

$$\Delta = \{(A, f) : A \in \mathcal{P}(X) \setminus \{\emptyset\}, f \in Y^A \text{ is an injection mapping from } A \text{ to } Y\}$$

We observe that  $\Delta \neq \emptyset$ . If  $x \in A, y \in Y$ , then  $(\{x\}, x \mapsto y) \in \Delta$ . On  $\Delta$  let

$$(A, f) \preceq (B, g) \iff A \subseteq B \subseteq X, g|_A = f$$

Notice that  $\preceq$  is reflexive, anti-symmetric, and transitive, hence is a partial order on  $\Delta$ . Let  $\Gamma\{(A_i, f_i)\}_{i \in I}$  be a chain in  $(\Delta, \preceq)$ . We let  $A = \bigcup_{i \in I} A_i$  and  $f \in Y^A$  be given by  $f(x) = f_i(x)$  provided  $x \in A_i$ .

Notice that  $f$  is well-defined. Say  $x \in A_i$  and  $x \in A_j$ , then, since  $\Gamma$  is a chain,  $A_i \subseteq A_j$ , say, and  $f_j|_{A_i} = f_i$ .

Furthermore, if  $x_1 \neq x_2$  in  $A$ , then  $x_1 \in A_{i_1}, x_2 \in A_{i_2}$ , and we may suppose  $A_{i_1} \subseteq A_{i_2}$ . Then  $f(x_1) = f_{i_1}(x_1) = f_{i_2}(x_1) \neq f_{i_2}(x_2) = f(x_2)$ , so  $f$  is an injection. Thus  $(A, f) \in \Delta$ , and is an upper bound of  $\Gamma$ .

Thus, there is a maximal element  $(M, g) \in \Delta$ , by Zorn's Lemma.

Case #1:  $M = X$ . Then  $X = M \preceq_g Y$ .

Case #2:  $M \subsetneq X$ . We wish to see that  $g$  must be surjective. Suppose not, i.e., there is  $y_0 \in Y \setminus g(M)$ . Since  $M \subsetneq X$ , there is  $x_0 \in X \setminus M$ . Define  $h : M \cup \{x_0\} \rightarrow Y$  by

$$h(x) = \begin{cases} g(x) & x \in M \\ y_0 & x = x_0 \end{cases} \text{ injective!}$$

Then  $(M \cup \{x_0\}, h) \in \Delta$ , and  $(M, g) \not\preceq (M \cup \{x_0\}, h)$ , contradicting maximality of  $(M, g)$ . Thus, we have that  $g$  is surjective. Thus  $Y \underbrace{\preceq}_{g^{-1}} X$ .

□

**Proposition 3.2.** Let  $A$  be a set. Then TFAE:

- (i)  $n \leq |A|$  for all  $n \in \mathbb{N}$
- (ii)  $\aleph_0 \leq |A|$  ( $A$  is infinite)
- (iii) there is  $B \subsetneq A$  s.t.  $|B| = |A|$
- (iv)  $1 + |A| = |A|$  (Hilbert hotel)
- (v)  $\aleph_0 + |A| = |A|$

*Proof.* (i)  $\Rightarrow$  (ii) We have that for each  $n$  in  $\mathbb{N}$  there is an injection  $\varphi_n : \{1, \dots, n\} \rightarrow A$ . Inductively, define  $f : \mathbb{N} \rightarrow A$  by

$$f(1) = \varphi_1(1)$$

$$f(n+1) = \varphi_{n+1}(k)$$

where  $k = \min j \in \{1, \dots, n+1\} : \varphi_{n+1}(j) \notin \{f(1), \dots, f(n)\}$ .

Then  $f$  is injective by construction.

(ii)  $\Rightarrow$  (iii) We have  $\mathbb{N} \preceq_f A$ . Let  $B = A \setminus \{f(1)\}$ . Define  $g : A \rightarrow B$  by

$$g(x) = \begin{cases} f(n+1) & \text{if } x = f(n), n \in \mathbb{N} \\ x & \text{otherwise} \end{cases}$$

Then  $A \sim_g B$ , i.e.,  $|A| = |B|$ .

(iii)  $\Rightarrow$  (iv) We suppose there is  $x_0 \in A \setminus B$  and  $B \sim A$ . Thus  $A \sim B \preceq B \cup \{x_0\} \preceq A$  so by C.B.S. Theorem  $A \sim B$  and



$A \sim B \cup \{x_0\} \sim A \sqcup \{1\}$ , i.e.  $|A| = |A| + 1$ .

(iv)  $\Rightarrow$  (i) We have  $\{1\} \sqcup A \sim_\varphi A$ . Then  $\varphi(A) \subsetneq A$ . Thus  $\varphi \circ \varphi(A) \subsetneq \varphi(A) \subsetneq A$ , and, by induction,

$$\varphi^{\circ n}(A) \subsetneq \varphi^{\circ n-1}(A) \subsetneq \cdots \subsetneq A$$

$$\underbrace{\varphi \circ \cdots \circ \varphi}_{n \text{ times}}$$

Hence  $|A| \geq |A \setminus \varphi^{\circ n}(A)| \geq n$  (at each stage above, we gain at least one point).

(ii)  $\Rightarrow$  (v) We have  $\mathbb{N} \preceq_f A$ . Let  $g : \mathbb{N} \sqcup A \rightarrow A$ ,

$$g(x) = \begin{cases} f(2n) & \text{if } x = n, n \in \mathbb{N} \\ f(2n+1) & \text{if } x = f(n) \in A, n \in \mathbb{N} \\ x & \text{otherwise} \end{cases}$$

(v)  $\Rightarrow$  (ii)  $\aleph_0 \leq \aleph_0 + |A| = |A|$  by assumption. □

**Corollary 3.1.** If  $A \in \mathcal{P}(\mathbb{N})$ , then either  $A$  is finite or denumerable.

*Proof.* Either  $n \leq |A|$  for all  $n$ , or  $|A| < n$  (Comparison lemma). □

**Theorem 3.2** (Cantor). For any set  $X$ ,  $|X| < |\mathcal{P}(X)|$ .

*Proof.* : ( □

Cantor's paradox: There is no "set" of all sets.

## 4 2017-09-22

### 4.1 METRIC SPACES

Example (French railroad / metro metric): Suppose we have a set  $X \neq \emptyset$ , and a function  $f : X \rightarrow [0, \infty)$  which satisfies  $f^{-1}(\{0\}) = \{p_0\}$ . Notice, then, that  $f(x) > 0$  if  $x \in X \setminus \{p_0\}$ .

$$d_f : X \times X \rightarrow [0, \infty), \quad d_f(x, y) = f(x) + f(y)$$

if  $x \neq y$ , 0 if  $x = y$ .

Easy exercise: this is a metric.

(Belongs to family of weighted graph metrics.)

$$\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$$

$$x^p = \begin{cases} e^{p \log x} & x > 0 \\ 0 & x = 0 \end{cases}$$

**Lemma 4.1.** Let  $\alpha, \beta \geq 0$  in  $\mathbb{R}$ ,  $1 < p < \infty$  and  $q$  is chosen so that  $\frac{1}{p} + \frac{1}{q} = 1$  (ie  $q = \frac{p}{p-1}$ ) then

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$$

with equality when  $\alpha^p = \beta^q$ .

*Proof.* Consider the graph of  $y = x^{p-1}$  (assume  $p \geq 2$ ).

$$x = y^1 p - 1 = y^q p = y^{q-1}$$

Then

$$\alpha\beta \leq \underbrace{\int_0^\alpha x^{p-1} dx}_{A_1} + \underbrace{\int_0^\beta y^{q-1} dy}_{A_2}$$

(Equality holds only if  $\beta = \alpha^{p-1} \Rightarrow \beta^1 q - 1 \Rightarrow \beta^q = \alpha^p$ )

$$= \frac{\alpha^p}{p} + \frac{\beta^q}{q}$$

Holder's Inequality

□

5 2017-09-25

Lemma:  $\alpha, \beta \geq 0$  in  $\mathbb{R}$ ,  $1 < p < \infty$  with  $q$  satisfying  $\frac{1}{p} + \frac{1}{q} \Rightarrow \alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$

Holder's Inequality: If  $x, y \in \mathbb{R}^n$ ,  $1 < p < \infty$  and  $q$  satisfies  $\frac{1}{p} + \frac{1}{q} = 1$ , then

$$\left| \sum_{j=1}^n x_j y_j \right| \underbrace{\leq}_{1\text{-ineq. of } |\cdot|} \sum_{j=1}^n |x_j| |y_j| \leq \left( \sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} \left( \sum_{j=1}^n |y_j|^q \right)^{\frac{1}{q}} := \|x\|_p \|y\|_q$$

*Proof.* If  $\|x\|_p \|y\|_q = 0$ , then  $x = 0$  or  $y = 0$  and the inequality is trivial. Assume  $\|x\|_p \|y\|_q \neq 0$ . For  $j = 1, \dots, n$ , let

$$\alpha_j = \frac{|x_j|}{\|x\|_p}, \quad \beta_j = \frac{|y_j|}{\|y\|_q}.$$

Then

$$\begin{aligned} \frac{1}{\|x\|_p \|y\|_q} \sum_{j=1}^n |x_j| |y_j| &= \sum_{j=1}^n \alpha_j \beta_j \\ &\leq \sum_{j=1}^n \left[ \frac{\alpha_j^p}{p} + \frac{\beta_j^q}{q} \right] \text{ by lemma} \\ &= \frac{1}{p} \sum_{j=1}^n \alpha_j^p + \frac{1}{q} \sum_{j=1}^n \beta_j^q \\ &= \frac{1}{p \|x\|_p^p} \sum_{j=1}^n |x_j|^p + \frac{1}{q \|y\|_q^q} \sum_{j=1}^n |y_j|^q \\ &= \frac{1}{p} + \frac{1}{q} \\ &= 1 \end{aligned}$$

□

**Theorem 5.1** (Minkowski's Inequality). Let  $x, y \in \mathbb{R}^n$  and  $1 < p < \infty$ . Then

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

*Proof.* If  $x + y = 0$  then this is trivial, so suppose  $x + y \neq 0$ .

$$\begin{aligned}
\|x + y\|_p^p &= \sum_{j=1}^n |x_j + y_j|^p \\
&= \sum_{j=1}^n |x_j + y_j| |x_j + y_j|^{p-1} \\
&\leq \sum_{j=1}^n (|x_j| + |y_j|) |x_j + y_j|^{p-1} \\
&= \sum_{j=1}^n |x_j| |x_j + y_j|^{p-1} + \sum_{j=1}^n |y_j| |x_j + y_j|^{p-1} \\
&\leq \left( \sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} \left( \sum_{j=1}^n |x_j + y_j|^{(p-1)q} \right)^{\frac{1}{q}} + \left( \sum_{j=1}^n |y_j|^p \right)^{\frac{1}{p}} \left( \sum_{j=1}^n |x_j + y_j|^{(p-1)q} \right)^{\frac{1}{q}} \\
&= (\|x\|_p + \|y\|_p) \left( \sum_{j=1}^n |x_j + y_j|^{(p-1)q} \right)^{\frac{1}{q}}
\end{aligned}$$

We have

$$\frac{1}{p} + \frac{1}{q} = 1 \implies \frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p} \implies p = q(p-1)$$

and thus

$$\begin{aligned}
\|x + y\|_p^p &\leq (\|x\|_p + \|y\|_p) \left( \sum_{j=1}^n |x_j + y_j|^p \right)^{\frac{1}{q}} \\
&= (\|x\|_p + \|y\|_p) \|x + y\|_p^{\frac{p}{q}}
\end{aligned}$$

Now, divide  $\|x + y\|_p^{\frac{p}{q}} \neq 0$  to get

$$\begin{aligned}
\|x + y\|_p &= \|x + y\|_p^{p - \frac{p}{q}} \\
&\leq \|x\|_p + \|y\|_p
\end{aligned}$$

(since  $p - \frac{p}{q} = p(1 - \frac{1}{q}) = 1$ ). □

**Corollary 5.1.** Given  $1 < p < \infty$ ,  $\|\cdot\|_p$  is a norm on  $\mathbb{R}^n$ .

*Proof.* Clearly  $\|\cdot\|_p$  is non-negative and non-degenerate. If  $\alpha \in \mathbb{R}, x \in \mathbb{R}^n$  then

$$\begin{aligned}
\|\alpha x\|_p &= \left( \sum_{j=1}^n |\alpha x_j|^p \right)^{\frac{1}{p}} \\
&= |\alpha| \left( \sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} \\
&= |\alpha| \|x\|_p
\end{aligned}$$

Finally, subadditivity is provided by Minkowski's inequality. □

$$|x|^p = e^{p \log |x|}$$

### 5.1 THE $\ell_p$ -SPACES

Consider  $\mathbb{R}^N = \{x = (x_k)_{k=1}^\infty : x_k \in \mathbb{R}\}$  which is a  $\mathbb{R}$ -vector space:

$$(x_k)_{k=1}^\infty + (y_k)_{k=1}^\infty = (x_k + y_k)_{k=1}^\infty, \alpha(x_k)_{k=1}^\infty = (\alpha x_k)_{k=1}^\infty.$$

We let for  $1 \leq p < \infty$

$$\ell_p = \{x = (x_k)_{k=1}^\infty \in \mathbb{R}^N : \sum_{k=1}^\infty |x_k|^p = \lim_{n \rightarrow \infty} \sum_{k=1}^n |x_k|^p < \infty\}$$

and

$$\ell_\infty = \{x = (x_k)_{k=1}^\infty : \sup_{k \in \mathbb{N}} |x_k| < \infty\}.$$

On  $\ell_p$  we define

$$\|x\|_p = \begin{cases} (\sum_{j=1}^n |x_j|^p)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty \\ \sum_{k \in \mathbb{N}} |x_k|, & \text{if } p = \infty \end{cases}$$

**Theorem 5.2.** Let  $1 \leq p < \infty$ . Then  $\ell_p$  is a  $\mathbb{R}$ -subspace of  $\mathbb{R}^N$  and  $\|\cdot\|_p$  is a norm.

*Proof.* We prove these together. Suppose that  $x, y \in \ell_p$ . Then

$$\begin{aligned} \|x + y\|_p &= \left( \sum_{k=1}^\infty |x_k + y_k|^p \right)^{\frac{1}{p}} \quad \text{if } \infty, \text{ treat } \infty^{\frac{1}{p}} = \infty \\ &= \left( \lim_{n \rightarrow \infty} \sum_{k=1}^n |x_k + y_k|^p \right)^{\frac{1}{p}} \\ &= \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n |x_k + y_k|^p \right)^{\frac{1}{p}} \quad x \mapsto x^{\frac{1}{p}} \text{ is continuous on } [0, \infty), \text{ if } x \rightarrow \infty, x^{\frac{1}{p}} \rightarrow \infty \\ &\leq \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n |x_k|^p \right)^{\frac{1}{p}} + \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n |y_k|^p \right)^{\frac{1}{p}} \quad \text{Minkowski applied on each } n \\ &= \left( \lim_{n \rightarrow \infty} \sum_{k=1}^n |x_k|^p \right)^{\frac{1}{p}} + \left( \lim_{n \rightarrow \infty} \sum_{k=1}^n |y_k|^p \right)^{\frac{1}{p}} \quad \text{continuity again} \\ &= \left( \sum_{k=1}^\infty |x_k|^p \right)^{\frac{1}{p}} + \left( \sum_{k=1}^\infty |y_k|^p \right)^{\frac{1}{p}} \\ &= \|x\|_p + \|y\|_p \\ &< \infty \end{aligned}$$

Thus  $x + y \in \ell_p$ , and we get subadditivity of  $\|\cdot\|_p$ .

We note that non-negativity and non-degeneracy of  $\|\cdot\|_p$  are obvious. Likewise, the  $|\cdot|$ -homogeneity is straightforward.  $\square$

**Theorem 5.3.**  $(\ell_\infty, \|\cdot\|_\infty)$  is a normed vector space.

*Proof.* If  $x, y \in \ell_\infty$  then

$$\begin{aligned}
 \|x + y\|_\infty &= \sup_{k \in \mathbb{N}} |x_k + y_k| \\
 &\leq \sup_{k \in \mathbb{N}} (|x_k| + |y_k|) \\
 &\leq \sup_{j, k \in \mathbb{N}} (|x_j| + |y_k|) \\
 &= \sup_{j \in \mathbb{N}} |x_j| + \sup_{k \in \mathbb{N}} |y_k| \\
 &= \|x\|_\infty + \|y\|_\infty
 \end{aligned}$$

Other properties are very easy. □

## 6 2017-09-29

i)  $X \neq \emptyset$  s.t.  $|X| \geq 2$

$$\text{discrete metric } d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

For  $x_0 \in X$ ,

$$\begin{aligned}
 B(x, \varepsilon) &= \begin{cases} \{x_0\} & 0 < \varepsilon \leq 1 \\ x & \varepsilon > 1 \end{cases} \\
 B[x, \varepsilon] &= \begin{cases} \{x_0\} & 0 < \varepsilon < 1 \\ x & \varepsilon \geq 1 \end{cases}
 \end{aligned}$$

ii) (geometry of balls in  $\mathbb{R}^2$ )

$$1 \leq p \leq \infty, B_p(0, 1) = \{x \in \mathbb{R}^2 : d_p(0, x) = \|x\|_p < 1\}$$

**Proposition 6.1.**  $(X, d)$  a metric space.

i)  $X, \emptyset$  are both open and closed.

ii) If  $\{U_i\}_{i \in I}$  is a family of open sets, then  $\bigcup_{i \in I} U_i$  is open.

iii) If  $\{U_1, \dots, U_n\}$  is a finite family of open sets, then  $\bigcap_{i=1}^n U_i$  is open.

iv) If  $\{F_i\}_{i \in I}$  is a family of closed sets, then  $\bigcap_{i \in I} F_i$  is closed.

v) If  $\{U_1, \dots, U_n\}$  is a finite family of closed sets, then  $\bigcup_{i=1}^n U_i$  is closed.

*Proof.* i) Let  $x \in X$ , then  $x \in B(x, 1) \subseteq X$ , so  $X$  is open. So  $\emptyset = X \setminus X$ ,  $X = X \setminus \emptyset$  are closed.

ii) Let  $x \in U = \bigcup_{i \in I} U_i$ . Then there is some  $i_0$  in  $I$  s.t.  $x \in U_{i_0}$ , which is open, so there is  $\varepsilon_x > 0$  s.t.  $x \in B(x, \varepsilon_x) \subseteq U_{i_0} \subseteq U$ .

iii) Let  $x \in V = \bigcap_{i=1}^n U_i$ . Then for each  $i = 1, \dots, n$ , there is  $\varepsilon_i > 0$  s.t.  $B(x, \varepsilon_i) \subseteq U_i$ . Let  $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\} \implies B(x, \varepsilon) \subseteq \bigcap_{i=1}^n B(x, \varepsilon_i) \subseteq V$ .

iv), v) De Morgan's Laws. □

Given a metric space  $(X, d)$ ,  $A \subseteq X$ , we define the boundary of  $A$ :

$$\partial A = \{x \in X : \forall \varepsilon > 0, B(x, \varepsilon) \cap A \neq \emptyset, B(x, \varepsilon) \setminus A \neq \emptyset\}.$$

Remark:  $\partial A = \partial(X \setminus A)$ .

Interior of  $A$  :

$$A^\circ = \bigcup \{U \subseteq X : U \subseteq A \text{ and } U \text{ is open}\}.$$

**Proposition 6.2** (characterizations of interior). If  $(X, d), A$  are as above then

$$\begin{aligned} A^\circ &= \{x \in X : \exists \varepsilon_x > 0 \text{ s.t. } B(x, \varepsilon_x) \subseteq A\} \\ &= A \setminus \partial A. \end{aligned}$$

*Proof.* Let  $x \in A$ . Then either:

- for some  $\varepsilon_x > 0$ ,  $B(x, \varepsilon_x) \subseteq A \implies x \in A^\circ$ , or
- $\forall \varepsilon > 0, B(x, \varepsilon) \setminus A \neq \emptyset \implies$  since  $x \in A \cap B(x, \varepsilon)$ ,  $x \in \partial A$ .

Since  $A^\circ \subseteq A$ , the proposition holds. □

Def:  $(X, d)$  a metric space,  $(x_n)_{n=1}^\infty \subseteq X$  and  $x_0 \in X$ . Say  $(x_n)_{n=1}^\infty$  converges to  $x_0$ , i.e.  $\lim_{n \rightarrow \infty} x_n = x_0$  or  $x_n \xrightarrow{n \rightarrow \infty} x_0$  if  $\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$  s.t.  $n \geq n_\varepsilon \implies d(x_0, x_n) < \varepsilon$ .

Remark: The limit, if it exists, is unique. Suppose  $x_0 = \lim_{n \rightarrow \infty} x_n, y_0 = \lim_{n \rightarrow \infty} x_n$ , then given  $\varepsilon > 0, \exists n_\varepsilon, n_{\varepsilon'}$  in  $\mathbb{N}$  s.t.

$$\begin{aligned} n \geq n_\varepsilon &\implies d(x_0, x_n) < \varepsilon \\ n \geq n_{\varepsilon'} &\implies d(y_0, x_n) < \varepsilon. \end{aligned}$$

Now if  $n \geq \max\{n_\varepsilon, n_{\varepsilon'}\}$ , then

$$\begin{aligned} d(x_0, y_0) &\leq d(x_0, x_n) + d(x_n, y_0) < \varepsilon \\ &\implies d(x_0, y_0) = 0, \text{ so } x_0 = y_0. \end{aligned}$$

Example: Let  $(V, \|\cdot\|)$  be a normed vector space. A subset  $\{e_n\}_{n=1}^\infty \subseteq V$  is a Schauder basis if for each  $x \in V, \exists$  a unique sequence  $\{x_n\}_{n=1}^\infty$  s.t.  $x = \lim_{n \rightarrow \infty} \sum_{k=1}^n x_k e_k$  in  $V$ .

In  $\ell_p, 1 \leq p < \infty$ , let  $e_n = (0, \dots, 0, \underbrace{1}_{n\text{-th place}}, 0, \dots)$ .

Let, for  $(X, d), A$  as above, the set of accumulation points (cluster points) be given as

$$A' = \{x \in X : \forall \varepsilon > 0, \underbrace{B(x, \varepsilon) \setminus \{x\}}_{\text{punctured ball}} \cap A \neq \emptyset\}$$

Call elements of  $A \setminus A'$  isolated points.

**Proposition 6.3.** Given  $(X, d), A$  as above, we have

$$A' = \{x \in X : x = \lim_{n \rightarrow \infty} x_n, (x_n)_{n=1}^\infty \subseteq A \setminus \{x\}\}$$

*Proof.* If  $x \in A'$ , let  $x_1 \in (B(x, 1) \setminus \{x\}) \setminus A$ , and  $x_{n+1} \in (B(x, \varepsilon_n) \setminus \{x\}) \setminus A$ , where  $\varepsilon_n = \min\{\frac{1}{n}, d(x, x_n)\}$ .

Then  $x = \lim_{n \rightarrow \infty} x_n$  while  $(x_n)_{n=1}^\infty \subseteq A \setminus \{x\}$ . Note  $x_1, x_2, \dots$  are distinct.

Converse direction: definition of limits. □

7 2017-10-02

Def: Given a metric space  $(X, d)$  and  $A \subseteq X$ , define the closure of  $A$  by

$$\bar{A} = \bigcap \{F \subseteq X : A \subseteq F, F \text{ is closed in } X\}$$

Of course  $A^\circ \subseteq A \subseteq \bar{A}$ .

**Theorem 7.1** (characterization of the closure). Given a metric space  $(X, d)$ ,  $A \subseteq X$ , the following sets are the same:

$$\bar{A}, A \cup \partial A, A \cup A'$$

("meet" set)  $A_M = \{x \in X : \text{for any } \varepsilon > 0, B(x, \varepsilon) \cap A \neq \emptyset\}$

("limit" set)  $A_L = \{x \in X : x = \lim_{n \rightarrow \infty} x_n, \text{ where } (x_n)_{n=1}^\infty \subseteq A\}$

(The notations  $A_L, A_M$  will not be used afterwards; we shall use  $\bar{A}$ .)

*Proof.* We have

$$\begin{aligned} \bar{A} &= \cap \{F \subseteq X : A \subseteq F, F \text{ closed}\} \\ &= \cap \{X \subseteq U : U \subseteq X \setminus A, U \text{ open in } X\} \\ &= X \setminus \cup \{U : U \subseteq X \setminus A, U \text{ open in } X\} \\ &= X \setminus [(X \setminus A)^\circ] \text{ complement of interior} \\ &= X \setminus [(X \setminus A) \setminus \partial(X \setminus A)] \text{ characterization of } (X \setminus A)^\circ \\ &= X \setminus [(X \setminus A) \setminus \partial A] \\ &= A \cup \partial A \end{aligned}$$

$$(\cap_{i \in I} (X \setminus U_i) = X \setminus \cup_{i \in I} U_i)$$

We thus have  $\bar{A} = A \cup \partial A$ .

Now if  $x \in A \cup \partial A$ , then for each  $\varepsilon > 0$ , we have that  $B(x, \varepsilon) \cap A \neq \emptyset$  [i.e. either  $x \in A$  so  $x \in A \cap B(x, \varepsilon)$ , or  $x \in \partial A$ , so  $B(x, \varepsilon) \cap A \neq \emptyset$ ]. Thus  $A \cup \partial A \subseteq A_M$ . Conversely, if  $x \in A_M$ , then, either

- there is  $\varepsilon > 0$  so  $B(x, \varepsilon) \subset A \implies x \in A^\circ \subseteq A$ , or
- for every  $\varepsilon > 0$  we have  $B(x, \varepsilon) \setminus A \neq \emptyset$  in which case  $x \in \partial A$ .

Hence,  $x \in A_M \implies x \in A \cup \partial A$  so  $A_M \subseteq A \cup \partial A$ .

If  $x \in A \cup A'$ , then for each  $\varepsilon > 0$ , we have  $B(x, \varepsilon) \cap A \neq \emptyset$ . Indeed, as above, either  $x \in A$ , so for any  $\varepsilon > 0$ ,  $x \in B(x, \varepsilon) \cap A$ , or  $x \in A'$ , so  $B(x, \varepsilon) \cap A \supseteq (B(x, \varepsilon) \setminus \{x\}) \cap A \neq \emptyset$ . Hence  $A \cup A' \subseteq A_M$ .

The definition of the limit of a sequence shows that  $A_M = A_L$ .

Finally, consider

$$\begin{aligned} X \setminus (A \cup A') &\subseteq \{x \in X : \text{there exists } \varepsilon_x > 0 \text{ s.t. } B(x, \varepsilon_x) \cap A = \emptyset, B(x, \varepsilon_x) \subseteq X \setminus A\} \\ &= (X \setminus A)^\circ \implies X \setminus [(X \setminus A)^\circ] \subseteq X \setminus [X \setminus (A \cup A')] \end{aligned}$$

Hence

$$\begin{aligned} \bar{A} &= X \setminus [(X \setminus A)^\circ] \subseteq X \setminus [X \setminus (A \cup A')] \\ &= A \cup A'. \end{aligned}$$

Hence  $\bar{A} \subseteq A \cup A' \subseteq A_M = \bar{A}$ , so  $\bar{A} = A \cup A'$ . □

## 7.1 CONTINUITY

Def: Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces  $f : X \rightarrow Y$  and  $x_0 \in X$ . We say that  $f$  is continuous at  $x_0$  if given  $\varepsilon > 0$ , there is  $\delta > 0$  s.t.  $d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \varepsilon$ . (★)

We say that  $f$  is continuous on  $X$  if it is continuous at each point.

Note:

$$\begin{aligned} (\star) &\iff f(B(x_0, \delta)) \subseteq B(f(x_0), \varepsilon) \\ &\iff B(x, \delta) \subseteq f^{-1}(B(f(x_0), \varepsilon)) \end{aligned}$$

**Notation:** In a metric space, a set  $N$  is a neighbourhood of a point  $x_0$  if  $x_0 \in N^\circ$  (interior).

**Theorem 7.2** (characterization of continuity at a point). If  $(X, d_X), (Y, d_Y), f : X \rightarrow Y, x \in X$  are as above, then TFAE:

- (i)  $f$  is continuous at  $x_0$
- (ii) for any neighbourhood  $N$  of  $f(x_0)$  in  $(Y, d_Y)$ , we have  $f^{-1}(N)$  is a neighbourhood of  $x_0$  in  $(X, d_X)$
- (iii) if  $x_0 = \lim_{n \rightarrow \infty} x_n$  in  $(X, d_X) \implies f(x_0) = \lim_{n \rightarrow \infty} f(x_n)$  in  $(Y, d_Y)$ .

*Proof.* (i)  $\implies$  (ii) Given a neighbourhood of  $f(x_0)$ , there exists  $\varepsilon > 0$  for which  $B(f(x_0), \varepsilon) \subseteq N$ . By assumption of continuity, there is  $\delta > 0$  s.t.

$$\begin{aligned} B(x_0, \delta) &\subseteq f^{-1}(B(f(x_0), \varepsilon)) \\ &\subseteq f^{-1}(N), \text{ from above.} \end{aligned}$$

Thus  $f^{-1}(N)$  is a neighbourhood of  $x_0$ .

(ii)  $\implies$  (i)  $\implies$  (iii) Given  $\varepsilon > 0$ ,  $B(f(x_0), \varepsilon)$  is a neighbourhood of  $f(x_0)$ , so  $f^{-1}(B(f(x_0), \varepsilon))$  is a neighbourhood of  $x_0$  and hence there is  $\delta > 0$  s.t.  $B(x_0, \delta) \subseteq f^{-1}(B(f(x_0), \varepsilon))$ , which gives (i).

Now, if  $x_0 = \lim_{n \rightarrow \infty} x_n$  in  $(X, d_X)$  then there is  $n_\delta$  in  $\mathbb{N}$  s.t. if  $n \leq n_\delta, x_n \in B(x_0, \delta)$ . But then for  $n \leq n_\delta$ , we have

$$f(x_n) \in f(B(x_0, \delta)) \subseteq B(f(x_0), \varepsilon)$$

and hence  $f(x_0) = \lim_{n \rightarrow \infty} f(x_n)$ .

(iii)  $\implies$  (i) (contrapositive) If (i) fails, then there exists  $\varepsilon > 0$  s.t. for any  $\delta > 0$ ,  $B(x_0, \delta) \not\subseteq f^{-1}(B(f(x_0), \varepsilon))$ .

Hence for each  $n \in \mathbb{N}$  we may find  $x_n \in B(x_0, \frac{1}{n}) \setminus f^{-1}(B(f(x_0), \varepsilon))$ . Given  $\varepsilon' > 0$ , let  $n_{\varepsilon'}$  satisfy  $n_{\varepsilon'} \leq \frac{1}{\varepsilon'}$ , thus  $\lim_{n \rightarrow \infty} x_n = x_0$ . However, each  $f(x_n) \notin B(f(x_0), \varepsilon)$ , so  $f(x)$  does not go to.  $\square$

## 8 2017-10-06

**Corollary 8.1.** A metric space is complete if whenever for any Cauchy sequence, we may find a converging subsequence.

Nested Intervals Theorem, Bolzano-Weierstrauss Theorem

**Theorem 8.1.**  $(\ell_p, \|\cdot\|_p)$  ( $1 \leq p < \infty$ ) is complete as a metric space.

**Def:** A normed space  $(V, \|\cdot\|)$  is called a Banach space provided that  $V$  is complete w.r.t. metric  $d(x, y) = \|x - y\|$ .  $(\ell_p, \|\cdot\|_p)$  is a Banach space.

## 9 2017-10-16

**Theorem 9.1.** The space of continuous bounded functions under the uniform metric,  $(C_b(f), \|\cdot\|_\infty)$ , is a Banach space.

*Proof.* (I) For  $x \in X$ ,  $(f_n(x))_{n=1}^\infty$  is Cauchy and admits a limit, so this defines  $f : X \rightarrow \mathbb{R}$ . The hard part is showing that  $f$  is continuous.

Next, show  $f$  is bounded, so  $f \in C_b(X)$ .

(II)  $\lim_{n \rightarrow \infty} \|f - f_n\|_\infty = 0$ , ie.  $\lim_{n \rightarrow \infty} f_n = f$  uniformly in  $C_b(X)$ .  $\square$

### 9.1 CHARACTERIZATIONS OF COMPLETENESS

**Def:** If  $(X, d)$  is a metric space,  $\emptyset \neq A \subseteq X$ , we let the diameter of  $A$  be given by

$$\text{diam}(A) = \sum_{x, y \in A} d(x, y) \text{ (may be } \infty)$$



**Proposition 9.1.** If  $(X, d)$ ,  $A$  are as above then  $\text{diam}(\bar{A}) = \text{diam}(A)$ .

*Proof.* If  $x, y \in \bar{A}$ ,  $\varepsilon > 0$ , then there are  $x', y' \in A$  s.t.  $d(x, x') < \frac{\varepsilon}{2}$ ,  $d(y, y') < \frac{\varepsilon}{2}$  (using meet set characterization of  $\bar{A}$ ). Then

$$\begin{aligned} d(x, y) &\leq d(x, x') + d(x', y') + d(y', y) \\ &\leq \frac{\varepsilon}{2} + \text{diam}(A) + \frac{\varepsilon}{2} \\ &= \text{diam}(A) + \varepsilon. \quad (\text{Assume } \text{diam}(A) < \infty). \end{aligned}$$

Thus, since  $\varepsilon > 0$  is arbitrary,  $d(x, y) \leq \text{diam}(A) \implies \text{diam}(\bar{A}) = \sup_{x, y \in \bar{A}} d(x, y) \leq \text{diam}(A)$ . Since  $A \subseteq \bar{A}$ ,  $\text{diam}(A) \leq \text{diam}(\bar{A})$ .  $\square$

**Theorem 9.2** (Nested set characterization of completeness). Let  $(X, d)$  be a metric space. Then  $(X, d)$  is complete  $\iff$  whenever we have closed sets,

- $F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$
- $\text{diam } F_n \xrightarrow{n \rightarrow \infty} 0$

then  $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$ .

*Proof.* ( $\implies$ ) For each  $n$ , choose  $x_n \in F_n$ . Given  $\varepsilon > 0$ , choose  $n_\varepsilon$  in  $\mathbb{N}$  s.t.  $n \geq n_\varepsilon \implies \text{diam}(F_n) < \varepsilon$ . Now, if  $n, m \geq n_\varepsilon$  we have

$$x_n \in F_n \subseteq F_{n_\varepsilon}, x_m \in F_m \subseteq F_{n_\varepsilon} \implies d(x_n, x_m) \leq \text{diam}(F_{n_\varepsilon}) < \varepsilon$$

so  $(x_n)_{n=1}^{\infty}$  is Cauchy, and has limit  $x = \lim_{n \rightarrow \infty} x_n$ . Since each  $F_m = \bar{F}_m$  (closed), and we have for  $n \geq m$ ,  $x_n \in F_m$ ,  $x = \lim_{n \rightarrow \infty} x_n \in F_m$  for all  $m$ . Hence  $x \in \bigcap_{m=1}^{\infty} F_m$  (ie.  $\neq \emptyset$ ).

( $\impliedby$ ) Let  $(x_n)_{n=1}^{\infty} \subset X$  be Cauchy, let for  $n$  in  $\mathbb{N}$ ,  $F_n = \{x_k\}_{k \geq n}$ . Then each  $F_n$  is closed and  $F_n \supseteq F_{n+1}$  for each  $n$ . Further,  $\text{diam } F_n = \text{diam}\{x_k\}_{k \geq n}$  (last proposition). Given  $\varepsilon > 0$ , there is  $n_\varepsilon$  in  $\mathbb{N}$  so  $n, m \geq n_\varepsilon \implies d(x_n, x_m) < \varepsilon$ . So for  $n \geq n_\varepsilon$ , we have  $\text{diam}\{x_k\}_{k \geq n} = \sup_{k, l \geq n} d(x_k, x_l) < \varepsilon$ .  $\square$

## 10 2017-10-18

Continuing the proof that  $(C_b(f), \|\cdot\|_\infty)$  is a Banach space from last time:

**Theorem 10.1.** The space of continuous bounded functions under the uniform metric,  $(C_b(f), \|\cdot\|_\infty)$ , is a Banach space.

*Proof.* (I) For  $x \in X$ ,  $(f_n(x))_{n=1}^{\infty}$  is Cauchy and admits a limit, so this defines  $f : X \rightarrow \mathbb{R}$ .  $f$  is continuous: let  $x \in X$ , and let  $\varepsilon > 0$ . Choose  $n_\varepsilon \in \mathbb{N}$  so that

$$n, m \geq n_\varepsilon \implies |f_n(x) - f(x)| < \frac{\varepsilon}{4} \text{ and } \|f_n - f_m\|_\infty < \frac{\varepsilon}{4}.$$

Choose  $\delta > 0$  so that for  $x, y \in X$ ,

$$d(x, y) < \delta \implies |f_{n_\varepsilon}(x) - f_{n_\varepsilon}(y)| < \frac{\varepsilon}{4}.$$

Then, given  $y \in B(x, \delta)$ , let  $n_y \in \mathbb{N}$  so that  $n_y \geq n_\varepsilon$  and

$$n \geq n_y \implies |f_n(y) - f(y)| < \frac{\varepsilon}{4}.$$

Then for  $n \geq n_y \geq n_\varepsilon$  we have

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_{n_\varepsilon}(x)| + |f_{n_\varepsilon}(x) - f_{n_\varepsilon}(y)| + |f_{n_\varepsilon}(y) - f_n(y)| + |f_n(y) - f(y)| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ &= \varepsilon. \end{aligned}$$

Also,  $f$  is bounded because

$$\begin{aligned} |f(x)| &\leq |f(x) - f_n(x)| + |f_n(x)| \\ &\leq |f(x) - f_n(x)| + \|f_n\|_\infty \\ &= o(1) + M. \end{aligned}$$

(II) Show that this is actually the limit (i.e.  $\lim_{n \rightarrow \infty} \|f - f_n\|_\infty = 0$ ).

Let  $\varepsilon > 0$ . Choose  $n_\varepsilon \in \mathbb{N}$  so that  $m, n \geq n_\varepsilon \implies \|f_m - f_n\|_\infty < \frac{\varepsilon}{2}$ . Also, given  $x \in X$ , choose  $n_x \geq n_\varepsilon$  so that  $n \geq n_x \implies |f_n(x) - f(x)| < \frac{\varepsilon}{2}$ . Then, for  $n \geq n_\varepsilon$ , find  $m \geq n_x \geq n_\varepsilon$  and observe that

$$\begin{aligned} |f(x) - f_n(x)| &\leq |f(x) - f_m(x)| + |f_m(x) - f_n(x)| \\ &< \frac{\varepsilon}{2} + \|f_m - f_n\|_\infty \\ &= \varepsilon. \end{aligned}$$

□

Example: Consider  $(\ell_p, \|\cdot\|_p)$ ,  $1 \leq p < \infty$ . Let  $e_n = (0, \dots, 0, \underbrace{1}_{n\text{-th place}}, 0, \dots)$  and let  $F_n = \{e_k\}_{k \geq n} \subseteq \ell_p$ .

- Each  $F_n$  is closed (easy exercise)
- $F_1 \supseteq F_2 \supseteq \dots$
- $\text{diam } F_n = 2^{\frac{1}{p}}$  (easy computation) (Finite diameter is not sufficient for Nested set characterization)

Notice that  $\bigcap_{n=1}^\infty F_n = \emptyset$ .

**Theorem 10.2** (abstract  $M$ -test). Let  $(V, \|\cdot\|)$  be a normed vector space. Then  $(V, \|\cdot\|)$  is a Banach space  $\iff$  for every  $(x_k)_{k=1}^\infty \subset V$  with  $\sum_{k=1}^\infty \|x_k\| = \lim_{n \rightarrow \infty} \sum_{k=1}^n \|x_k\|$  converging, has that  $\sum_{k=1}^\infty x_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n x_k$  converges in  $(V, \|\cdot\|)$  [ie.  $V$  satisfies that “absolute convergence”  $\implies$  convergence.]

*Proof.* ( $\implies$ ) Suppose  $\sum_{k=1}^\infty \|x_k\|$  converges. Consider  $(\sum_{k=1}^n x_k)_{n=1}^\infty \subset V$ . We have for  $m < n$  that

$$\left\| \sum_{k=1}^n x_k - \sum_{k=1}^m x_k \right\| \leq \underbrace{\sum_{k=m+1}^n \|x_k\|}_{\text{partial tail of converging series in } \mathbb{R}}$$

and hence  $(\sum_{k=1}^n x_k)_{n=1}^\infty$  is Cauchy in  $(V, \|\cdot\|)$ , and thus converges.

( $\Leftarrow$ ) Suppose  $(x_n)_{n=1}^\infty$  is a Cauchy seq in  $(V, \|\cdot\|)$ . Let  $n_1$  in  $\mathbb{N}$  be so  $m, n \geq n_1 \implies \|x_m - x_n\| < 1$ , and, inductively, choose  $n_{k+1}$  in  $\mathbb{N}$  s.t.  $n_{k+1} \geq n_k$  and  $m, n \geq n_{k+1} \implies \|x_n - x_m\| < \frac{1}{2^k}$ .

Let  $y_0 = x_{n_1}$ ,  $y_j = x_{n_{j+1}} - x_{n_j}$ ,  $j \in \mathbb{N}$ .

Then, each  $\|y_j\| = \|x_{n_{j+1}} - x_{n_j}\| < \frac{1}{2^{j-1}}$ , as  $n_{j+1} > n_j \geq n_1$ , so

$$\sum_{i=0}^\infty \|y_j\| = \|y_0\| + \sum_{j=1}^\infty \frac{1}{2^{j-1}},$$

which converges. ( $\star$ )

Now

$$\begin{aligned}
 x_{n_k} &= x_{n_1} + \sum_{j=1}^{k-1} (x_{n_{j+1}} - x_{n_j}) \\
 &= y_0 + \sum_{j=1}^{k-1} y_j \\
 &\xrightarrow{k \rightarrow \infty} y_0 + \sum_{j=1}^{\infty} y_j \text{ (by assumption and } (\star))
 \end{aligned}$$

In other words,  $(x_{n_k})_{k=1}^{\infty}$  converges, hence  $(x_n)_{n=1}^{\infty}$  converges as well. □

Application: a continuous nowhere differentiable function on  $\mathbb{R}$ .

Facts:  $C_b(\mathbb{R})$  is complete;  $M$ -test.

Construction: Let  $\varphi : \mathbb{R} \rightarrow [0, 1]$

$$\varphi(t) = \begin{cases} t - 2k & 2k \leq t < 2k + 1 \\ 2k + 2 - t & 2k + 1 \leq t < 2k + 2 \end{cases}$$

Picture: sawtooth function with zeros at  $\dots, -4, -2, 0, 2, 4, \dots$

Then

- (i)  $\varphi$  is continuous and bounded
- (ii)  $\varphi$  is 2-periodic, ie.  $\varphi(t + 2) = \varphi(t)$  for  $t \in \mathbb{R}$
- (iii)  $\varphi(2k) = 0, \varphi(2k + 1) = 1$  for  $k \in \mathbb{Z}$
- (iv) if  $k \leq s, t \leq k + 1$  ( $k \in \mathbb{Z}$ ), then

$$|\varphi(s) - \varphi(t)| = |s - t|$$

Let for  $t \in \mathbb{R}$

$$f(t) = \sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^k \underbrace{\varphi(4^k t)}_{\in [0,1]}$$

However, note that each  $\varphi(4^k) \in C_b(\mathbb{R})$ ,  $\|\varphi(4^k)\|_{\infty} = 1$ , so by the  $M$ -test,  $f \in C_b(\mathbb{R})$ . Fix  $t \in \mathbb{R}$ . We show that  $f$  cannot be differentiable at  $t$ . Let  $\ell_m = \lfloor 4^m t \rfloor$  ( $m \in \mathbb{N}$ ) so

$$\begin{aligned}
 \ell_m &\leq 4^m t < \ell_m + 1 \\
 \implies p_m = \frac{\ell_m}{4^m} &\leq t < \frac{\ell_m + 1}{4^m} = q_m
 \end{aligned}$$

We compute

$$\begin{aligned}
& |f(p_m) - f(q_m)| \\
&= \left| \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^k [\varphi(4^k p_m) - \varphi(4^k q_m)] \right| \\
&= \left| \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^k [\varphi(4^{k-m} \ell_m) - \varphi(4^{k-m}(\ell_m + 1))] \right| \\
&= \left| \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^k [\varphi(4^{k-m} \ell_m) - \varphi(4^{k-m}(\ell_m + 1))] \right|, \text{ by (ii) (2-periodicity)} \\
&\text{(key step)} \geq \frac{3^m}{4} - \sum_{k=1}^{m-1} \frac{3^k}{4^k} \underbrace{|\varphi(4^{k-m} \ell_m) - \varphi(4^{k-m}(\ell_m + 1))|}_{=4^{k-m}, \text{ by (iv)}} \\
&= \frac{3^k}{4^k} - \frac{1}{4^m} \sum_{k=1}^{m-1} 3^k \\
&= \frac{1}{4^m} [3^m - \sum_{k=1}^{m-1} 3^k] \\
&= \frac{1}{4^m} \left[ \frac{2 \cdot 3^m - 3^m + 1}{2} \right] \\
&= \frac{1}{4^m} \left( \frac{3^m + 1}{2} \right)
\end{aligned}$$

Since  $|p_m - q_m| = \frac{1}{4^m}$ , we have

$$\begin{aligned}
\frac{f(p_m) - f(q_m)}{p_m - q_m} &\geq \frac{3^m + 1}{2} \\
\left( p_m = \frac{\lfloor 4^m t \rfloor}{4^m} \right)
\end{aligned}$$

If  $t = \frac{\ell}{4^{m_0}}$  ( $\ell \in \mathbb{Z}$ ), then  $t = p_m$  for  $m \geq m_0$  and hence for  $m \geq m_0$ ,

$$\left| \frac{f(t) - f(q_m)}{t - q_m} \right| \geq \frac{3^m + 1}{2}$$

while  $\lim_{m \rightarrow \infty} q_m = t$ , so  $f'(t)$  does not exist.

$$\begin{aligned}
\frac{f(p_m) - f(q_m)}{p_m - q_m} &\leq \frac{|f(p_m) - f(t)| + |f(t) - f(q_m)|}{|p_m - q_m|} \\
&\leq \frac{|f(p_m) - f(t)|}{|p_m - t|} + \frac{|f(t) - f(q_m)|}{|t - q_m|}
\end{aligned}$$

Hence, for some  $r_m \in \{p_m, q_m\}$ ,  $\frac{|f(t) - f(r_m)|}{|t - r_m|} \geq \frac{3^m + 1}{2 \cdot 2}$ .

We have  $\left| \frac{f(t) - f(r_m)}{t - r_m} \right| \geq \frac{3^m + 1}{4}$  while  $r_m \rightarrow t$ .

## 11 2017-10-20

**Corollary 11.1.**  $(\ell_\infty, \|\cdot\|_\infty)$  is a Banach space.

*Proof.*  $\ell_\infty = C_b(\mathbb{N})$  with usual  $|\cdot|$  metric on  $\mathbb{N}$ . If  $f : \mathbb{N} \rightarrow \mathbb{R}$  is bounded,  $U \subseteq \mathbb{R}$  open, then  $f^{-1}(U) \in \mathcal{P}(\mathbb{N})$  is open (all subsets of  $\mathbb{N}$  are open)  $\implies f$  is continuous.

If  $(x_n)_{n=1}^\infty \in \ell_\infty$ , define  $f : \mathbb{N} \rightarrow \mathbb{R}$ ,  $f(n) = x_n$ ,  $f \in C_b(\mathbb{N})$ ,  $\|f\|_\infty = \|(x_n)_{n=1}^\infty\|_\infty$ . □

Eg.  $(C[0, 2], \|\cdot\|_p)$ ,  $\|f\|_p = (\int_0^2 |f|^p)^{\frac{1}{p}}$ ,  $1 \leq p < \infty$ .  
 NOT a Banach space!

Let

$$f_n(t) = \begin{cases} 1 & 0 \leq t \leq \frac{1}{2} \\ n(\frac{1}{2} + \frac{1}{n} - t) & \frac{1}{2} < t \leq \frac{1}{2} + \frac{1}{n} \\ 0 & \frac{1}{2} + \frac{1}{n} < t \end{cases}.$$

Then for  $m < n \in \mathbb{N}$ ,

$$\begin{aligned} \|f_n - f_m\|_p &= \left( \int_0^2 |f_n - f_m|^p \right)^{\frac{1}{p}} \\ &= \left( \underbrace{\int_0^{\frac{1}{2}} |f_n - f_m|^p}_0 + \underbrace{\int_{\frac{1}{2}}^{\frac{1}{2} + \frac{1}{m}} |f_n - f_m|^p}_{\leq \frac{1}{m}} + \underbrace{\int_{\frac{1}{2} + \frac{1}{m}}^2 |f_n - f_m|^p}_0 \right)^{\frac{1}{p}} \\ &\leq \frac{1}{m^{\frac{1}{p}}}. \end{aligned}$$

Hence  $(f_n)_{n=1}^\infty$  is Cauchy in  $(C[0, 2], \|\cdot\|_p)$ .

Consider

$$\chi_{[0, \frac{1}{2}]}(t) = \begin{cases} 1 & 0 \leq t \leq \frac{1}{2} \\ 0 & \frac{1}{2} < t \end{cases}.$$

$\chi_{[0, \frac{1}{2}]}$  is bounded, piecewise continuous, so Riemann integrable.

$$\begin{aligned} \|f_n - \chi_{[0, \frac{1}{2}]} \|_p &= \left( \int_0^2 |f_n - \chi_{[0, \frac{1}{2}]}|^p \right)^{\frac{1}{p}} \leq \frac{1}{n^{\frac{1}{p}}} \\ \implies \lim_{n \rightarrow \infty} \|f_n - \chi_{[0, \frac{1}{2}]} \|_p &= 0. \end{aligned}$$

If  $g \in C[0, 1]$  s.t.  $\lim_{n \rightarrow \infty} \|f_n - g\|_p = 0$ , then  $\|g - \chi_{[0, \frac{1}{2}]} \|_p = 0$ .

Using Riemann integration theory,

$$g(t) = \begin{cases} 1 & 0 \leq t \leq \frac{1}{2} \\ 0 & \frac{1}{2} < t \end{cases}.$$

Then  $\lim_{t \rightarrow \frac{1}{2}} g$  does not exist!

## 11.1 COMPLETENESS OF METRIC SPACES

$(X, d)$  metric space.

Remark:  $|d(x, z) - d(y, z)| \leq d(x, y)$ .

If  $x = \lim_{n \rightarrow \infty} x_n$ ,  $y = \lim_{n \rightarrow \infty} y_n$  in  $(X, d)$ , then  $\lim_{n \rightarrow \infty} d(x_n, y_n) = d(x, y)$ . (See solution to A3Q2).

Def:  $(X, d_X), (Y, d_Y)$  metric spaces.  $i : X \rightarrow Y$  is an isometry if  $d_Y(i(x), i(y)) = d_X(x, y) \forall x, y \in X$ .

Notes: An isometry is injective. Consider  $i : X \rightarrow i(X) \subseteq Y \implies i^{-1} : i(X) \rightarrow X$  isometry.

**Theorem 11.1.**  $(X, d)$  metric space.

i) Existence of completion: there exists a metric space  $(\overline{X}, \overline{d})$  s.t.

- a)  $(\overline{X}, \overline{d})$  is complete
- b)  $\exists$  isometry  $\bar{i} : X \rightarrow \overline{X}$
- c)  $\overline{i(X)} = \overline{X}$ ; i.e.  $i(X)$  is dense in  $\overline{X}$

- ii) Uniqueness up to isometry: if  $(\tilde{X}, \tilde{d})$  is a metric space with map  $\tilde{i} : X \rightarrow \tilde{X}$  s.t.  $(\tilde{X}, \tilde{d}), \tilde{i}$  satisfy (a),(b),(c), then  $\exists$  a surjective isometry  $\varphi : \tilde{X} \rightarrow \overline{X}$  s.t.  $\varphi \circ \tilde{i} = \bar{i}$ .

*Proof.* 1. Fix  $x_0 \in X$ . For  $u \in X$ , let  $f_u : X \rightarrow \mathbb{R}$ ,  $f_u(x) = d(x, u) - d(x, x_0)$

$\implies f_u$  is continuous and  $|f_u(x)| \leq d(u, x_0)$

$\implies \|f_u\|_\infty = \sup_{x \in X} |f_u(x)| \leq d(u, x_0) < \infty \implies f_u$  is bounded

$\implies f_u \in C_b(X)$ .

For  $u, v \in X, x \in X$ ,

$$|f_u(x) - f_v(x)| = |d(x, u) - d(x, v)| \leq d(u, v).$$

Thus  $\|f_u - f_v\|_\infty \leq d(u, v)$ . Finally,

$$\begin{aligned} |f_u(u) - f_v(u)| &= |d(u, u) - d(u, x_0) - d(u, v) + d(u, x_0)| \\ &= d(u, v). \end{aligned}$$

Thus  $\|f_u - f_v\|_\infty \geq d(u, v) \implies \|f_u - f_v\|_\infty = d(u, v)$ .

Define  $\tau : X \rightarrow C_b(X)$ ,  $\tau(u) = f_u$ ,  $\tau$  isometry.

Let  $\overline{X} = \overline{\tau(X)} = \{\overline{f_u} : u \in X\} \subseteq C_b(X)$ .

By A3Q2(a),  $(\overline{X}, \bar{d})$  is complete, where  $\bar{d}$  is relativized from the metric on  $C_b(X)$ .

2. Let  $\varphi_0 = \tau \circ \tau^{-1} : \tau(X) \rightarrow \tau(X)$ .  $\varphi_0$  an isometry  $\implies$  uniformly continuous. Hence it admits an extension  $\varphi = \overline{\varphi_0} : \tilde{X} = \overline{\tau(X)} \rightarrow \overline{X} = \tau(X)$ .

Verify  $\varphi$  is an isometry:

If  $\tilde{x}, \tilde{y} \in \tilde{X}$ , let  $\tilde{x} = \lim_{n \rightarrow \infty} \tau(x_n), \tilde{y} = \lim_{n \rightarrow \infty} \tau(y_n), x_n, y_n \in X$ . Then

$$\varphi(\tilde{x}) = \lim_{n \rightarrow \infty} \varphi_0(\tau(x_n)) = \lim_{n \rightarrow \infty} \tau(x_n).$$

Hence

$$\begin{aligned} \bar{d}(\varphi(\tilde{x}), \varphi(\tilde{y})) &= \lim_{n \rightarrow \infty} \bar{d}(\tau(x_n), \tau(y_n)) \\ &= \lim_{n \rightarrow \infty} d(x_n, y_n) \\ &= \lim_{n \rightarrow \infty} \tilde{d}(\tau(x_n), \tau(y_n)) = \tilde{d}(\tilde{x}, \tilde{y}). \end{aligned}$$

$\implies \varphi$  is an isometry.  $\varphi \circ \tau = \tau$  comes for free.

□

## 12 2017-10-23

Assignment discussion – the completion vs A4,Q1:

Suppose  $(V, \|\cdot\|)$  is a non-complete normed vector space, eg.  $(C[0, 2], \|\cdot\|_p)$  ( $1 \leq p < \infty$ ). Consider the map

$$\tau : V \rightarrow C_b(V)$$

$$\tau(v) \in C_b(V), \tau(v)(x) = \|x - v\| - \|x\|$$

We saw that  $\tau$  is an isometry, hence we let

$$\overline{V} = \overline{\tau(V)}_{\text{complete}} \subseteq C_b(V)$$

Problem:  $\tau$  is not linear,  $\overline{\tau(V)}$  not evidently a subspace of  $C_b(V)$ .

A4, Q1 shows that an addition and a scalar multiplication may be imposed on  $\overline{V} = \overline{\tau(V)}$  which makes it a Banach (complete normed vector) space. These two operations are not the same as addition and scalar multiplication in  $C_b(V)$ . (The only linear property that  $\tau$  enjoys seems to be that it takes 0 to 0.)

## 12.1 COMPACTNESS

Let  $(X, d)$  be a metric space, and  $K \subseteq X$ . We say that  $K$  is compact if given a family of open sets  $\{U_i\}_{i \in I}$  for which

$$K \subseteq \bigcup_{i \in I} U_i \text{ -- we say } \{U_i\}_{i \in I} \text{ is an "open cover"}$$

there is a finite subfamily  $\{U_{i_1}, \dots, U_{i_n}\}$  such that

$$K \subseteq \bigcup_{k=1}^n U_{i_k} \text{ -- we say } \{U_i\}_{i \in I} \text{ admits a "finite subcover" .}$$

If  $X = K$  itself is compact, we will call  $(X, d)$  a compact metric space.

**Remark:** If  $K \subseteq X$  is compact, the relativized metric space  $(K, d_K)$  is a compact metric space.

**Proposition 12.1.** Let  $(X, d)$  be a metric space and  $K \subseteq X$ . If  $K$  is compact, then it must be closed.

*Proof.* Let us suppose, for sake of contradiction that there is  $x \in \overline{K} \setminus K$ . Then for  $n$  in  $\mathbb{N}$ ,

$$B(x, \frac{1}{n}) \cap K \neq \emptyset \implies B[x, \frac{1}{n}] \cap K \neq \emptyset. \quad (\star)$$

Further,  $\bigcap_{n=1}^{\infty} B[x, \frac{1}{n}] = \{x\}$ . Let  $U_n = X \setminus B[x, \frac{1}{n}]$ , which is open.

We have that

$$\bigcup_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} (X \setminus B[x, \frac{1}{n}]) = X \setminus \bigcap_{n=1}^{\infty} B[x, \frac{1}{n}] = X \setminus \{x\} \supseteq K.$$

But, for any finite  $m$  we have

$$\bigcup_{n=1}^m U_n = X \setminus \bigcap_{n=1}^m B[x, \frac{1}{n}] = X \setminus B[x, \frac{1}{m}] \not\supseteq K$$

by  $(\star)$ . Hence if  $\overline{K} \setminus K \neq \emptyset$ ,  $K$  cannot be compact. So we are done.  $\square$

**Proposition 12.2.** Let  $(X, d)$  be a compact metric space and  $C \subseteq X$  is closed. Then  $C$  is compact.

*Proof.* Suppose  $\{U_i\}_{i \in I}$  is an open cover of  $C$ . Then  $\{U_i\}_{i \in I} \cup \{X \setminus C\}$  is an open cover of  $X$ . Hence  $X$  admits finite subcover  $\{U_{i_1}, \dots, U_{i_n}\} \cup \{X \setminus C\}$ , hence,  $\{U_{i_1}, \dots, U_{i_n}\}$  is a finite subcover of  $C$ .  $\square$

**Theorem 12.1** (continuous image of compact is compact). Let  $(X, d_X)$  be a compact metric space,  $(Y, d_Y)$  be a metric space, and  $f : X \rightarrow Y$  be continuous. Then  $f(X) = \{f(x) : x \in X\}$  is compact.

*Proof.* Let  $\{V_i\}_{i \in I}$  be an open cover of  $f(X)$ . Then  $U_i = f^{-1}(V_i)$  is open, and  $\{U_i\}_{i \in I}$  is an open cover of  $X$ . Hence there is a finite subcover,  $X \subseteq \bigcup_{k=1}^n U_{i_k}$  so  $f(X) \subseteq \bigcup_{k=1}^n f(U_{i_k}) = \bigcup_{k=1}^n V_{i_k}$ , so  $\{V_{i_1}, \dots, V_{i_n}\}$  is a finite subcover of  $f(X)$ .  $\square$

**Corollary 12.1** (Extreme Value Theorem). If  $(X, d)$  is a compact metric space,  $f : X \rightarrow \mathbb{R}$  is continuous, then there are  $x_{\min}, x_{\max} \in X$  for which

$$f(x_{\min}) \leq f(x) \leq f(x_{\max}) \quad \forall x \in X.$$

*Proof.* We have  $f(X) \subseteq \mathbb{R}$  is compact. Hence  $f(X)$  is closed. Also  $\{(-n, n)\}_{n=1}^{\infty}$  (open intervals), then  $f(X) \subseteq \mathbb{R} = \bigcup_{n=1}^{\infty} (-n, n)$  admits a finite subcover,  $\{(-1, 1), \dots, (-n, n)\}$  and hence  $f(X) \subseteq (-n, n)$ . Thus we have  $\inf(f(X)), \sup(f(X))$  exist.

Since  $f(X)$  is closed we have

$$\inf(f(X)), \sup(f(X)) \in f(X)$$

(use meet-set of closure). Let  $x_{\min}, x_{\max}$  be so  $f(x_{\min}) = \inf(f(X)), f(x_{\max}) = \sup(f(X))$ .  $\square$

– Assignment line –

**Theorem 12.2** (finite intersection property). Let  $(X, d)$  be a metric space. Then  $(X, d)$  is compact  $\iff$  for any family  $\{F_i\}_{i \in I}$  of closed subsets of  $X$  for which  $\bigcap_{k=1}^n F_{i_k} \neq \emptyset$ ,  $\{i_1, \dots, i_n\}$  finite in  $I$ , we must have  $\bigcap_{i \in I} F_i \neq \emptyset$ .

*Proof.* ( $\implies$ ) (contrapositive) Let us suppose that  $\{F_i\}_{i \in I}$  is a family of closed subsets with  $\bigcap_{i \in I} F_i = \emptyset$ . Then if  $U_i = X \setminus F_i$ , we have that  $\{U_i\}_{i \in I}$  is an open cover (De Morgan's law) and hence admits finite subcover  $\{U_{i_1}, \dots, U_{i_n}\}$ . Again, by DeMorgan's law,  $\bigcap_{k=1}^n F_{i_k} = \emptyset$ . Hence we are done.

( $\impliedby$ ) Very similar, interchange roles of  $U_i$ s and  $F_i = X \setminus U_i$ .  $\square$

Example: Let  $X = B[0, 1]$  in  $\ell_p$  ( $1 \leq p \leq \infty$ ).

Let  $e_n = (0, \dots, 0, \underbrace{1}_{n\text{-th place}}, 0, \dots)$  and let  $F_n = \{e_k\}_{k \geq n}$  (seen before on Oct 18).

Each  $F_n$  is closed. Also

$$\bigcap_{n=1}^{\infty} F_n = \emptyset$$

$$\bigcap_{n=1}^m F_n = F_m \neq \emptyset$$

Conclusion:  $(B[0, 1], d_p)$  ( $d_p(x, y) = \|x - y\|_p$ ) is not compact.

## 13 2017-10-25

Def: Let  $(X, d)$  be a metric space. Then we say it is

- bounded if there are  $x_0$  in  $X$ , and  $R > 0$  such that  $X \subseteq B[x_0, R]$  (of course “=” holds) (equivalently, for any  $x \in X$ , there is  $R_x > 0$  such that  $X \subseteq B[x, R_x]$ ; or, equivalently,  $\text{diam}(X) < \infty$ )
- totally bounded if, for any  $\varepsilon > 0$ , there are  $x_1, \dots, x_n \in X$  such that  $X \subseteq \bigcup_{k=1}^n B[x_k, \varepsilon]$

Totally bounded  $\implies$  bounded. [with  $\varepsilon > 0$ ,  $x_1, \dots, x_n$  in defn, check that  $\bigcup_{k=1}^n B[x_k, \varepsilon] \subseteq B[x_1, \varepsilon + \max_{k=2, \dots, n} d(x_1, x_k)]$ ]

Example: (bounded  $\not\implies$  totally bounded)

In  $\ell_p$  ( $1 \leq p \leq \infty$ ),  $e_n = (0, \dots, 0, \underbrace{1}_{n\text{-th place}}, 0, \dots)$ ,  $F_n = \{e_k\}_{k \geq n} \subseteq \ell_p$ ,

$F_n$  int,  $F_n \subseteq B[0, 1] \subseteq B[e, 2]$  so  $F_n$  is bounded. But  $n \neq m$ ,  $d(e_n, e_m) = \begin{cases} 2^{\frac{1}{p}} & 1 \leq p < \infty \\ 1 & \text{otherwise} \end{cases} =: R$ .

If  $0 < \varepsilon < \frac{1}{2}R$ , we see that  $F_n \not\subseteq \bigcup_{k=1}^n B[e_k, \varepsilon]$  for any  $n$ .

**Theorem 13.1** (Characterizations of compact metric spaces). Let  $(X, d)$  be a metric space. TFAE:

- $(X, d)$  is compact,
- any sequence  $(x_n)_{n=1}^{\infty} \subseteq X$  admits a subsequence which converges in  $X$
- $(X, d)$  is complete and totally bounded

*Proof.* (i)  $\implies$  (ii): Let  $F_n = \overline{\{x_k\}_{k=n}^{\infty}}$ . Then each  $F_n$  is closed, and  $F_1 \supseteq F_2 \supseteq \dots$ , so if  $n_1 < n_2 < \dots < n_m$ , then  $\bigcap_{j=1}^m F_{n_j} = F_{n_m} \neq \emptyset$ . Thus, by finite intersection property, we have that  $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$ . Let  $x \in \bigcap_{n=1}^{\infty} F_n$ . Now let

$$n_1 = \min\{k : x_k \in B(x, 1)\} \text{ (exists by meet-set closure definition)}$$

and, inductively,

$$n_{m+1} = \min\{k : k > n_m \text{ and } x_k \in B(x, \frac{1}{m+1})\}.$$

Then, as is easy to check,  $\lim_{m \rightarrow \infty} x_{n_m} = x$ .

(ii)  $\implies$  (iii): If  $(x_n)_{n=1}^{\infty} \subseteq X$  is Cauchy, it admits a converging subsequence (by assumption), and hence itself converges



(earlier proposition). Thus  $(X, d)$  is complete.

Let us suppose that  $(X, d)$  is not totally bounded.

Thus, there exists  $\varepsilon > 0$  so no finite collection of closed  $\varepsilon$ -balls covers  $X$ . Let

$$x_1 \in X \setminus B[x_1, \varepsilon], \dots, x_{n+1} \in X \setminus \bigcup_{k=1}^n B[x_k, \varepsilon] \text{ (always possible by assumption).}$$

Thus  $d(x_n, x_m) > \varepsilon$  for  $n \neq m$ . Thus, this sequence  $(x_n)_{n=1}^\infty$  admits no Cauchy subsequences, hence no subsequences which converge, violating assumption (ii). Thus (ii)  $\implies (X, d)$  is totally bounded.

(iii)  $\implies$  (ii): We first use total boundedness. Given  $n$  in  $\mathbb{N}$ , there exist  $y_{n1}, \dots, y_{nm_n} \in X$  such that the closed balls

$$B_{n1} = B[y_{n1}, \frac{1}{n}], \dots, B_{nm_n} = B[y_{nm_n}, \frac{1}{n}]$$

satisfy that  $X \subseteq \bigcup_{k=1}^{m_n} B_{nk}$ . Let

- $B_1$  be a ball from  $B_{11}, \dots, B_{1m_1}$  such that

$$|\{n \in \mathbb{N} : x_n \in B_1\}| = \aleph_0 \text{ (pigeonhole principle)}$$

•  $\vdots$

- $B_k$  be a ball from  $B_{11}, \dots, B_{1m_1}$  such that

$$|\{n \in \mathbb{N} : x_n \in \bigcap_{j=1}^k B_j\}| = \aleph_0$$

(we've covered  $X$  by 1-balls,  $B_1$  by  $\frac{1}{2}$ -balls, then  $B_2 \cap B_1$  covered by  $\frac{1}{3}$ -balls, ...)

Now we use completeness. Let  $F_n = \bigcap_{k=1}^n B_k$  so each  $F_n$  is closed.

- $F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$
- $\text{diam}(F_n) \leq \text{diam}(B_n) = \frac{2}{n} \xrightarrow{n \rightarrow \infty} 0$

Thus, by nested sets theorem,  $\bigcap_{n=1}^\infty F_n \neq \emptyset$ .

Let  $n_1 = \min\{k \in \mathbb{N} : x_k \in F_1\}$ , inductively,  $n_{m+1} = \min\{k \in \mathbb{N} : k > n_m \text{ and } x_k \in F_k\}$ .

Then, if  $x \in \bigcap_{n=1}^\infty F_n$ ,  $d(x, x_m) \leq \text{diam}(F_m) \leq \text{diam}(B_m) = \frac{2}{m} \xrightarrow{m \rightarrow \infty} 0$  so  $x = \lim_{n \rightarrow \infty} x_{n_m}$ . □

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Office hours:

Mon 2:30 – 4:30

Tue 2 – 3:30

*Proof.* Continuing theorem from last time:

So far we did (i)  $\xRightarrow{\text{F.I.P.}}$  (ii)  $\xRightarrow{\text{routine}}$  (iii)  $\xRightarrow{\text{harder, nested sets thm}}$  (ii)

(ii)  $\implies$  (i): Let  $\{U_i\}_{i \in I}$  be an open cover of  $X$ .

(LN) There exists  $r > 0$  s.t. for any  $x$  in  $X$  there exists  $i$  in  $I$  so  $B(x, r) \subseteq U_i$ .

(This number  $r$  is sometimes called the “Lebesgue number” of the covering; its existence is based on (ii).)

Suppose (LN) fails. Then for choice of  $r = \frac{1}{n}$ , there exists  $x_n$  in  $X$  s.t.  $B(x, \frac{1}{n}) \not\subseteq U_i$  for all  $i$  in  $I$ .

Our assumption is that  $(x_n)_{n=1}^\infty \subseteq X$  admits a subsequence  $(x_{n_k})_{k=1}^\infty$  such that  $x_0 = \lim_{k \rightarrow \infty} x_{n_k}$  exists.

Then  $x_0 \in U_{i_0}$  for some  $i_0$ , so there is  $\varepsilon > 0$  such that  $B(x, \varepsilon) \subseteq U_{i_0}$ . Now, there is  $k_\varepsilon$  in  $\mathbb{N}$  so  $k \geq k_\varepsilon \implies x_{n_k} \in B(x_0, \frac{\varepsilon}{2})$ . Hence, let us choose  $k \geq k_\varepsilon$  and  $\frac{1}{n_k} < \frac{\varepsilon}{2}$ . Thus, if  $x \in B(x_{n_k}, \frac{1}{n_k})$ , we have

$$d(x, x_0) \leq d(x, x_{n_k}) + d(x_{n_k}, x_0) < \frac{1}{n_k} + \frac{\varepsilon}{2} < \varepsilon$$

and hence  $B(x_{n_k}, \frac{1}{n_k}) \subseteq B(x_0, \varepsilon) \subseteq U_{i_0}$ , contradiction the choice of the elements  $x_n$ .

Hence, we must conclude that (LN) holds.

We saw in (ii)  $\implies$  (iii) above, that our assumption gives total boundedness of  $(X, d)$ . Hence there are  $y_1, \dots, y_m$  such that  $X \subseteq \bigcup_{j=1}^m B[y_j, \frac{r}{2}] \subseteq \bigcup_{j=1}^m B(y_j, r)$ . Now, for each  $j = 1, \dots, m$ , (LN) tells us that there is  $i_j \in I$  so  $B(y_j, r) \subseteq U_{i_j}$ .

Thus  $X \subseteq \bigcup_{i=1}^n B(y_j, r) \subseteq U_{i_j}$ , so  $\{U_{i_1}, \dots, U_{i_m}\}$  is a finite subcover.

Remark: On  $\mathbb{R}^n$ , norms  $\|\cdot\|_p$  ( $1 \leq p \leq \infty$ ) are equivalent, and from A2, each gives the same open sets, and hence the same compact sets.

### Corollary 14.1.

(i) (Bolzano-Weierstrauss Theorem for  $\mathbb{R}^n$ )

If  $(x^{(n)})_{n=1}^\infty \subseteq [-R, R]^n = B_\infty[0, R]$ , then it admits a converging subsequence.

(ii) (Heine-Borel Theorem)

A subset  $K \subseteq \mathbb{R}^n$  is compact  $\iff K$  is closed &  $K$  is bounded (with respect to any  $\|\cdot\|_\infty$ ).

*Proof.* (i) We consider, first  $(x_1^{(n)})_{n=1}^\infty \subseteq [-R, R] \subseteq \mathbb{R}$ . By Bolzano-Weierstrauss for  $\mathbb{R}$ , this admits converging subsequence  $(x_1^{(n_k)})_{n=1}^\infty$ . Then  $(x_2^{(n)})_{n=1}^\infty \subseteq [-R, R] \subseteq \mathbb{R}$  admits a converging subsequence  $(x_2^{(n_k)})_{n=1}^\infty$ . Etc. Hence, after finitely many  $(n)$  iterations, we get a subsequence of  $(x^{(n)})_{n=1}^\infty$  which converges ( $\mathbb{R}^n, \|\cdot\|_\infty$ ).

(ii) If  $K$  is compact, then  $K$  is closed by a result at the beginning of the section, and totally bounded by last theorem, hence bounded. Conversely, if  $K$  is closed and bounded,  $K \subseteq [-R, R]^n$  for some  $R > 0$ . Let us consider a sequence  $(x^{(n)})_{n=1}^\infty \subseteq K$ . First,  $(x^{(n)})_{n=1}^\infty$  admits a converging subsequence, by (i). Since  $K$  is closed, the limit of the subsequence is in  $K$ . □

Example:  $P = \prod_{k=1}^\infty \{0, \frac{1}{2^k}\} \subseteq \ell_1$  is compact in  $(\ell_1, \|\cdot\|_1)$ .

First soln: The Cantor set  $C$  is closed and bounded in  $\mathbb{R}$ , so thus compact. And there is a continuous function  $f : C \rightarrow \ell_1$  with  $f(C) = P$  (A4, Q3), so  $P$  is compact. [In fact  $f$  is a bijection from  $C$  to  $P$  so  $f^{-1} : P \rightarrow C$  is also continuous.]

Second soln:  $P$  is closed (A3). Hence the relativised metric space  $(P, d_P)$  is complete. Let us show total boundedness.

Let  $\varepsilon > 0$ , and  $n$  be so  $\frac{1}{2^n} < \varepsilon$ . For  $(b_1, \dots, b_m) \in \{0, 1\}^n$ , let  $x_{b_1 \dots b_m} = \sum_{k=1}^\infty \frac{b_k}{2^k} e_k \in P$ . If  $b = (b_1, b_2, \dots) \in \{0, 1\}^\mathbb{N}$ , then  $x_b = \sum_{k=1}^\infty \frac{b_k}{2^k} e_k \in P$  (generic element of  $P$ ).

Then for  $b = (b_1, b_2, \dots)$  as above,

$$\|x_b - x_{b_1 \dots b_n}\|_1 = \sum_{k=n+1}^\infty \frac{1}{2^k} b_k \leq \sum_{k=n+1}^\infty \frac{1}{2^k} = \frac{1}{2^n} \leq \varepsilon.$$

Thus,  $P \subseteq \bigcup_{(b_1, \dots, b_n) \in \{0, 1\}^n} B[x_{b_1 \dots b_n}, \varepsilon]$ . □

– MIDTERM CUTOFF –

15 2017-10-30

Midterm: Wed evening

See info sheet on website

Office hours:

– 2:30 - 4:30

– 1:30 - 3:30

A5 - will be posted Friday

**Theorem 15.1** (sequential characterization of uniform continuity). Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces,  $f : X \rightarrow Y$ . Then

$$f \text{ is uniformly continuous} \iff \text{whenever } d_X(x_n, y_n) \xrightarrow{n \rightarrow \infty} 0, \ x_n, y_n \in X, \\ \text{we must have } d_Y(f(x_n), f(y_n)) \xrightarrow{n \rightarrow \infty} 0.$$

*Proof.* ( $\implies$ ) Given  $\varepsilon > 0$ , there is  $\delta > 0$  such that  $d_X(x, y) < \delta \ (x, y \text{ in } X) \implies d_Y(f(x), f(y)) < \varepsilon$ . Now suppose  $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty \subseteq X$  such that  $\lim_{n \rightarrow \infty} d_X(x_n, y_n) = 0$ . Then there is  $n_\varepsilon$  in  $\mathbb{N}$  such that

$$n \geq n_\varepsilon \implies d_X(x_n, y_n) < \delta \\ \implies d_Y(f(x_n), f(y_n)) < \varepsilon.$$

I.e.  $\lim_{n \rightarrow \infty} d_Y(f(x_n), f(y_n)) = 0$ .

( $\impliedby$ ) (contrapositive) Suppose  $f$  is not uniformly continuous, so there exists  $\varepsilon > 0$  such that for all  $\delta > 0$  there are  $x, y$  in  $X$  with  $d_X(x, y) < \delta$  but  $d_Y(f(x), f(y)) \geq \varepsilon$ . For each choice  $\delta = \frac{1}{n}$ , let  $x_n, y_n$  in  $X$  so  $d_X(x_n, y_n) < \frac{1}{n}$  for which  $d_Y(f(x_n), f(y_n)) \geq \varepsilon$ .

Plainly,  $\lim_{n \rightarrow \infty} d_X(x_n, y_n) = 0$  while  $\lim_{n \rightarrow \infty} d_Y(f(x_n), f(y_n)) \neq 0$  (if the limit exists).

Ex: Let  $f(x) = x^2$  on  $\mathbb{R}$ . Let  $x_n = n, y_n = n + \frac{1}{n}$ . Then  $|x_n - y_n| = \frac{1}{n} \xrightarrow{n \rightarrow \infty} 0$ , while  $|f(x_n) - f(y_n)| = 2 + \frac{1}{n^2} \not\xrightarrow{n \rightarrow \infty} 0$ . Hence  $f$  is not uniformly continuous.  $\square$

**Theorem 15.2** (continuous on compact is uniformly continuous). Let  $(X, d_X), (Y, d_Y)$  be metric spaces, with  $(X, d_X)$  compact, and  $f : X \rightarrow Y$  continuous. Then  $f$  is uniformly continuous.

*Proof.* Let us suppose not. Then there is  $\varepsilon > 0$  and  $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty \subseteq X$  such that  $d_X(x_n, y_n) \xrightarrow{n \rightarrow \infty} 0$  while  $d_Y(f(x_n), f(y_n)) \geq \varepsilon$ . Let  $(x_{n_k})_{k=1}^\infty$  be a converging subsequence. Then let  $(y_{n_k})_{k=1}^\infty$  be a sequence in  $X$ , hence admits converging subsequence  $(y_{n_{k_\ell}})_{\ell=1}^\infty$ . Then if  $x = \lim_{k \rightarrow \infty} x_{n_k} = \lim_{\ell \rightarrow \infty} x_{n_{k_\ell}}$  then

$$d_X(x, y_{n_{k_\ell}}) \leq d_X(x, x_{n_{k_\ell}}) + d_X(x_{n_{k_\ell}}, y_{n_{k_\ell}}) \\ \xrightarrow{\ell \rightarrow \infty} 0$$

so  $x = \lim_{\ell \rightarrow \infty} y_{n_{k_\ell}}$ . Then we have  $f(x) = \lim_{\ell \rightarrow \infty} f(y_{n_{k_\ell}})$ , by continuity, so

$$0 = d_Y(f(x), f(x)) = \lim_{\ell \rightarrow \infty} d_Y(f(x_{n_{k_\ell}}), f(y_{n_{k_\ell}}))$$

contradicts  $(\star)$ . Thus, we conclude that  $f$  is uniformly continuous.  $\square$

Definition: A map  $f : X \rightarrow Y$  ( $(X, d_X), (Y, d_Y)$ ) is called Lipschitz if there is  $L \geq 0$  such that

$$d_Y(f(x), f(y)) \leq L d_X(x, y) \text{ for all } x, y \in X.$$

Notice that

$$\sup_{x, y \in X, x \neq y} \frac{d_Y(f(x), f(y))}{d_X(x, y)} = \inf \{ L \geq 0 : (\text{Lip}) \text{ is satisfied} \}$$

so there exists a minimum  $L$  satisfying (Lip). We call this the “Lipschitz constant”.

Remark: Lipschitz  $\xrightarrow{\text{exercise}}$  uniform continuity  $\implies$  continuity  
 Lipschitz  $\not\xrightarrow{\text{assignment}}$  uniform continuity  $\not\implies$  continuity

**Theorem 15.3.** Any two norms on  $\mathbb{R}^n$  are equivalent, i.e. if  $\|\cdot\|, \|\cdot\|$  on  $\mathbb{R}^n$  satisfy  $\|\cdot\| \approx \|\cdot\|$ , i.e., there are  $m, M > 0$  for which  $m\|x\| \leq \|x\| \leq M\|x\|$ .

*Proof.* Let  $\|\cdot\|$  be a norm on  $\mathbb{R}^n$ . We will see that  $\|\cdot\| \approx \|\cdot\|_1$  ( $\|x\|_1 = \sum_{j=1}^n |x_j|$ ). Since  $\approx$  is an equivalence relation, we get  $\|\cdot\| \approx \|\cdot\|_1$  so  $\|\cdot\| \approx \|\cdot\|$ .

Let  $\{e_1, \dots, e_n\}$  be the standard basis, so if  $x \in \mathbb{R}^n$ ,  $x = \sum_{j=1}^n x_j e_j$ . Then

$$\|x\| = \left\| \sum_{j=1}^n x_j e_j \right\| \underbrace{\leq}_{\text{properties of norm}} \sum_{j=1}^n |x_j| \|e_j\| \leq M \|x\|_1 \text{ where } M = \max_{j=1, \dots, n} \|e_j\|.$$

Notice, then, for  $x, y$  in  $\mathbb{R}^n$  we have

$$\| \|x\| - \|y\| \| \underbrace{\leq}_{\text{standard } \leq \text{ (shown before completeness of } C_b(X))} \|x - y\| \leq M \|x - y\|_1$$

so  $\|\cdot\| : \mathbb{R}^n \rightarrow \mathbb{R}$  is Lipschitz with respect to  $d_1(x, y) = \|x - y\|_1$  and thus continuous.

Let  $S_1 = \{x \in \mathbb{R}^n : \|x\|_1 = 1\} = B_1[0, 1] \setminus \underbrace{B_1(0, 1)}_{\subseteq B_1[0, 1]}$  so  $S_1$  is closed in  $B_1[0, 1]$ . Hence by Heine-Borel Theorem, it is compact.

Hence, by Extreme Value Theorem, there is  $x_{\min}$  in  $S_1$  such that

$$\|x_{\min}\| = \inf\{\|x\| : x \in S_1\}.$$

Let  $m = \|x_{\min}\| > 0$  (as  $x_{\min} \neq 0$ , since  $\|x_{\min}\|_1 = 1 \neq 0$ ).

Now, if  $x \in \mathbb{R}^n \setminus \{0\}$ , then

$$m \leq \left\| \frac{1}{\underbrace{\|x\|_1}_{\in S_1}} x \right\| \implies m \|x\|_1 \leq \|x\| \quad (\ddagger)$$

Then  $(\dagger)$  and  $(\ddagger)$  show that  $\|\cdot\| \approx \|\cdot\|_1$ . □

**Corollary 15.1.** If  $\|\cdot\|$  is a norm on  $\mathbb{R}^n$ ,  $\|\cdot\|$  on  $\mathbb{R}^m$  and  $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is linear. Then  $A$  is Lipschitz from  $(\mathbb{R}^n, \|\cdot\|)$  to  $(\mathbb{R}^m, \|\cdot\|)$ , and hence continuous.

*Proof.* Let  $\{e_1, \dots, e_n\}$  be the standard basis of  $\mathbb{R}^n$ ,  $\{e_1, \dots, e_m\}$  be the standard basis of  $\mathbb{R}^m$ . Then there is a matrix  $[a_{ij}]$  such that  $Ae_j = \sum_{i=1}^m a_{ij} e_i$ .

Then for  $x = \sum_{j=1}^n x_j e_j$  in  $\mathbb{R}^n$  we have

$$\begin{aligned} Ax &= \sum_{j=1}^n x_j Ae_j \\ &= \sum_{j=1}^n x_j \sum_{i=1}^m a_{ij} e_i \\ &= \sum_{i=1}^m \left( \sum_{j=1}^n a_{ij} x_j \right) e_i \in \mathbb{R}^m \end{aligned}$$

so

$$\begin{aligned}
\|Ax\| &\leq \sum_{j=1}^n \left| \sum_{i=1}^n a_{ij}x_j \right| \|e_i\|, & M &= \max_{j=1,\dots,n} \|e_j\| \\
&\leq M \sum_{j=1}^n \sum_{i=1}^m |a_{ij}| |x_j|, & \|A\|_\infty &= \max_{i=1,\dots,m, j=1,\dots,n} |a_{ij}| \\
&= M \sum_{i=1}^m \sum_{j=1}^n |a_{ij}| |x_j| \\
&\leq M \sum_{i=1}^m |A|_\infty |x|_1 \\
&= M \|x\|_1 \leq M
\end{aligned}$$

$$\|x\|_1 \leq M \|x\|$$

□

16 2017-11-01

**Proposition 16.1.** Let  $(V, \|\cdot\|_V)$ ,  $(W, \|\cdot\|_W)$  be normed linear spaces,  $A : V \rightarrow W$  be linear. Then TFAE:

1.  $A$  is continuous
2.  $\|A\| := \sup\{\|Ax\|_W : x \in \underbrace{B_V[0,1]}_{\text{closed ball, center 0 in } V}\} < \infty$
3.  $A$  is Lipschitz map with Lipschitz constant  $\|A\|$

Moreover, in the case of (ii) (hence (iii)), above,  $\|Ax\|_W \leq \|A\| \|x\|_V$  for any  $x$  in  $V$ .

*Proof.* (i)  $\implies$  (ii)  $A$  is continuous at 0 in  $V$ . Thus, letting  $\varepsilon = 1$ , there is  $\delta > 0$  s.t.  $A(B_V(0, \delta)) \subseteq B_W(0, 1)$ . Now, if  $x \in B_V[0, 1]$ , then  $\frac{\delta}{2}x \in B_V(0, \delta)$ , so

$$\|Ax\|_W = \frac{2}{\delta} \left\| \underbrace{A\left(\frac{\delta}{2}x\right)}_{\in B_W(0,1)} \right\|_W < \frac{2}{\delta} 1 = \frac{2}{\delta} < \infty$$

so  $\|A\| = \sup_{x \in B_V[0,1]} \|Ax\|_W \leq \frac{2}{\delta} < \infty$ .

(ii)  $\implies$  (iii) If  $x \in V \setminus \{0\}$ , so  $\frac{1}{\|x\|_V}x \in B_V[0, 1]$  and

$$(\star) \quad \|Ax\|_W = \|x\|_V \underbrace{\left\| A\left(\frac{1}{\|x\|_V}x\right) \right\|_W}_{\leq \|A\|} \leq \|A\| \|x\|_V.$$

Clearly,  $(\star)$  holds for  $x = 0$  in  $V$ . Hence if  $x, y \in V$ ,

$$\|Ax - Ay\|_W = \|A(x - y)\|_W \leq \|A\| \|x - y\|_V.$$

Thus  $A$  is Lipschitz and “Moreover...” holds. Furthermore, by  $(\star)$ ,

$$\|A\| = \sup_{x \in V \setminus \{0\}} \frac{\|Ax\|_W}{\|x\|_V} = \sup_{x \neq y \text{ in } V} \frac{\|Ax - Ay\|_W}{\|x - y\|_V}$$

which is the definition of the Lipschitz constant.

(iii)  $\implies$  (i) Obvious.

□

Remark: Let  $B(V, W) = \{A : V \rightarrow W \mid A \text{ is linear and continuous}\}$ . Notice that (ii) above shows that  $A$  must be bounded on  $B_V[0, 1]$  and we call  $A$  a “bounded linear operator”.

$B(V, W)$  is a  $\mathbb{R}$ -vector space (pointwise addition and scalar multiplication) and  $\|\cdot\|$  is a norm on  $B(V, W)$ , called “bounded operator norm”. (Exercise.)

Question: Is continuity automatic for linear operators?

Example: Consider the vector space  $C[0, 1]$  of continuous  $\mathbb{R}$ -valued functions on  $[0, 1]$ . Let

$$\varphi : C[0, 1] \rightarrow \mathbb{R}, \quad \varphi(f) = f\left(\frac{1}{2}\right) \text{ (evaluation at } \frac{1}{2}\text{)}.$$

Then  $\varphi$  is linear: let  $f, g \in C[0, 1]$ ,  $\alpha \in \mathbb{R}$ , then

$$\begin{aligned} \varphi(f + \alpha g) &= f\left(\frac{1}{2}\right) + \alpha g\left(\frac{1}{2}\right) \\ &= \varphi(f) + \alpha \varphi(g) \end{aligned}$$

(i) Consider  $(C[0, 1], \|\cdot\|_\infty)$ . Then

$$|\varphi(f)| = |f\left(\frac{1}{2}\right)| \leq \max_{t \in [0, 1]} |f(t)| = \|f\|_\infty.$$

Thus  $\|\varphi\| \leq 1$  (easy to show that  $\|\varphi\| = 1$ ), i.e.,  $\varphi \in B((C[0, 1], \|\cdot\|_\infty), \mathbb{R})$ .

(ii) Now consider  $(C[0, 1], \|\cdot\|_p)$  ( $1 \leq p < \infty$ ). Let

$$f_n(t) = \begin{cases} 0 & \text{if } t \leq \frac{1}{2} - \frac{1}{n^{2p}} \\ n^{2p+1}(t - \frac{1}{2} + \frac{1}{n^{2p}}) & \text{if } \frac{1}{2} - \frac{1}{n^{2p}} < t \leq \frac{1}{2} \\ n^{2p+1}(\frac{1}{2} + \frac{1}{n^{2p}} - t) & \text{if } \frac{1}{2} < t \leq \frac{1}{2} + \frac{1}{n^{2p}} \\ 0 & \text{if } t > \frac{1}{2} + \frac{1}{n^{2p}} \end{cases}$$

[triangular spike at  $[\frac{1}{2} - \frac{1}{n^{2p}}, \frac{1}{2} + \frac{1}{n^{2p}}$  with peak at  $\frac{1}{2}$  having value  $n$ .] Notice

$$\varphi(f_n) = f_n\left(\frac{1}{2}\right) = n$$

while

$$\begin{aligned} \|f_n\|_p &= \left( \int_0^1 f_n^p \right)^{\frac{1}{p}} \\ &= \left( \int_{\frac{1}{2} - \frac{1}{n^{2p}}}^{\frac{1}{2} + \frac{1}{n^{2p}}} \underbrace{f_n^p}_{0 \leq f_n^p \leq n^p} \right)^{\frac{1}{p}} \\ &\leq \left( \int_{\frac{1}{2} - \frac{1}{n^{2p}}}^{\frac{1}{2} + \frac{1}{n^{2p}}} \underbrace{n^p}_{\text{constant}} \right)^{\frac{1}{p}} \\ &= \left( n^p \frac{2}{n^{2p}} \right)^{\frac{1}{p}} = \frac{2^{\frac{1}{p}}}{n}. \end{aligned}$$

Thus

$$\frac{|\varphi(f_n)|}{\|f_n\|_p} = \frac{n}{\frac{2^{\frac{1}{p}}}{n}} = \frac{n^2}{2^{\frac{1}{p}}} \xrightarrow{n \rightarrow \infty} \infty.$$

Hence

$$\varphi \notin B((C[0, 1], \|\cdot\|_p), \mathbb{R}).$$

Example: (Axiom of choice) If  $(V, \|\cdot\|)$  is an infinite dimensional normed vector space, then it admits an infinite linearly independent family  $\{v_n\}_{n=1}^\infty$ . There exists a basis  $\{w_i\}_{i \in I}$  s.t.  $\{v_n\}_{n=1}^\infty \subseteq \{w_i\}_{i \in I}$ .

Define  $f : V \rightarrow \mathbb{R}$

$$f(w_i) = \begin{cases} \frac{n}{\|v_n\|} & \text{if } w_i = v_n \\ 0 & \text{otherwise} \end{cases}$$

and extend uniquely to a linear operator on  $V$ .

Check that  $f \notin B(V, \mathbb{R})$ .

Why isn't  $B[0, 1]$  in  $(C[0, 1], \|\cdot\|_\infty)$  compact?

Reason: existence of subsequence with no converging subsequence [similar holds on  $(\ell_p, \|\cdot\|_p)$ ].

Picture: [triangle spike to height  $f_n(t) = 1$  on  $[\frac{1}{n+1}, \frac{1}{n}]$ , 0 elsewhere.]

Calculate that if  $m \neq n$ ,  $\|f_n - f_m\|_\infty = 1$ . Conclude that  $(f_n)_{n=1}^\infty \subset B[0, 1]$  admits no converging subsequence.

## 17 2017-11-03

**Theorem 17.1** (Banach's Contraction Mapping Theorem). Let  $(X, d)$  be a complete metric space and let  $\Gamma : X \rightarrow X$  be a strict contraction, i.e., there is  $0 < c < 1$  s.t.  $d(\Gamma(x), \Gamma(y)) < cd(x, y)$  for  $x, y$  in  $X$  ( $\Gamma$  is  $c$ -Lipschitz). Then

- (i) there is a unique fixed point  $x_{\text{fix}}$  for  $\Gamma$ , i.e.  $\Gamma(x_{\text{fix}}) = x_{\text{fix}}$ ,
- (ii) given any  $x_0$  in  $X$ , if we define a sequence by  $x_n = \Gamma(x_{n-1})$ ,  $n \in \mathbb{N}$ , then it satisfies

$$d(x_n, x_{\text{fix}}) \leq \frac{c^n}{1-c} d(x_0, \Gamma(x_0))$$

and hence  $\lim_{n \rightarrow \infty} x_n = x_{\text{fix}}$ .

*Proof.* Let  $x_0 \in X$ . We define  $(x_n)_{n=1}^\infty \subseteq X$  as in (ii), above. We note that  $d(x_1, x_2) = d(\Gamma(x_0), \Gamma(x_1)) \leq cd(x_0, x_1) = cd(x_0, \Gamma(x_0))$ .

Now, if

$$(\star) \quad d(x_n, x_{n+1}) \leq c^n d(x_0, \Gamma(x_0)),$$

then

$$d(x_{n+1}, x_{n+2}) = d(\Gamma(x_n), \Gamma(x_{n+1})) \leq cd(x_n, x_{n+1}) \leq c^{n+1} d(x_0, \Gamma(x_0))$$

so  $(\star)$  holds generally. Thus, if  $m < n$  in  $\mathbb{N}$  we have

$$\begin{aligned} d(x_m, x_n) &\leq \sum_{j=m}^{n-1} d(x_j, x_{j+1}) \\ &\leq \sum_{j=m}^{n-1} c^j d(x_0, \Gamma(x_0)), \text{ by } (\star) \\ &\leq \sum_{j=m}^{\infty} c^j d(x_0, \Gamma(x_0)), \text{ by } (\star) = \frac{c^m}{1-c} d(x_0, \Gamma(x_0)). \end{aligned}$$

It follows that  $(x_n)_{n=1}^\infty$  is Cauchy, and hence  $x_{\text{fix}} = \lim_{n \rightarrow \infty} x_n$  exists. Then

$$x_{\text{fix}} = \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \Gamma(x_n) \underset{\substack{\Gamma \text{ Lipschitz} \\ \implies \text{continuous}}}{=} \Gamma(\lim_{n \rightarrow \infty} x_n) = \Gamma(x_{\text{fix}}).$$

Hence  $x_{\text{fix}}$  is a fixed point. If  $y_{\text{fix}}$  is any other fixed point then

$$\begin{aligned} d(x_{\text{fix}}, y_{\text{fix}}) &= d(\Gamma(x_{\text{fix}}), \Gamma(y_{\text{fix}})) \\ &\leq cd(x_{\text{fix}}, y_{\text{fix}}) \\ &< d(x_{\text{fix}}, y_{\text{fix}}), \text{ if } d(x_{\text{fix}}, y_{\text{fix}}) > 0 \end{aligned}$$

so we must have  $d(x_{\text{fix}}, y_{\text{fix}}) = 0$ , i.e.  $x_{\text{fix}} = y_{\text{fix}}$ . Thus (i) holds.  
Also we have for  $m, n$ , as above,

$$d(x_m, x_n) \leq \frac{c^m}{1-c} d(x_0, \Gamma(x_0)) \implies d(x_n, x_{\text{fix}}) = \lim_{n \rightarrow \infty} d(x_m, x_n) \leq \frac{c^m}{1-c} d(x_0, \Gamma(x_0))$$

so (ii) holds. □

### Application: Some differentiable equations

Let  $F : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$  be continuous, and  $y_0 \in \mathbb{R}$ . We consider the following initial value problem:  
Want  $f \in C[a, b]$ , with  $\underbrace{f(a) = y_0}_{\text{initial value}}$  and  $\underbrace{f'(t) = F(t, f(t))}_{\text{differential equation}}$  (IVP).

We use the Fundamental Theorem of Calculus to convert this to an integral equation:

Want  $f \in C[a, b]$ ,  $f(t) = y_0 + \int_a^t F(s, f(s)) ds$  (IE).

**Theorem 17.2** (Picard-Lindelof Theorem). Let  $F, y_0$  be as above and suppose that  $F$  is Lipschitz in the second variable: for all  $t \in [a, b]$ ,  $y, z \in \mathbb{R}$ ,

$$|F(t, y) - F(t, z)| \leq L|y - z|, \text{ for some } L > 0.$$

Then (IVP) admits a unique solution,  $f_{\text{sol}}$  in  $C[a, b]$ .

*Proof.* (I) Let us assume that  $(b-a)L < 1$ . Define  $\Gamma : C[a, b] \rightarrow C[a, b]$  by, for  $t \in [a, b]$ ,

$$\Gamma(F)(t) = y_0 + \int_a^t F(s, f(s)) ds.$$

Then for  $f, g \in C[a, b]$ , and  $t \in [a, b]$ , then

$$\begin{aligned} |\Gamma(f)(t) - \Gamma(g)(t)| &= \left| \int_a^t [F(s, f(s)) - F(s, g(s))] ds \right| \\ &\leq \int_a^t \underbrace{|F(s, f(s)) - F(s, g(s))|}_{\leq L|f(s) - g(s)|} ds \\ &\leq L \int_a^t \underbrace{|f(s) - g(s)|}_{\leq \|f - g\|_\infty} ds \\ &\leq L \|f - g\|_\infty \int_a^t 1 ds \\ &= L \|f - g\|_\infty (t - a) \leq (b - a)L \|f - g\|_\infty. \end{aligned}$$

In summary,

$$\begin{aligned} \|\Gamma(f) - \Gamma(g)\|_\infty &= \sup_{t \in [a, b]} \|\Gamma(f)(t) - \Gamma(g)(t)\| \\ &\leq \underbrace{(b-a)L}_{< 1} \|f - g\|_\infty. \end{aligned}$$

Hence, by the Contraction Mapping Theorem, applied to  $\Gamma$  on  $(C[a, b], \|\cdot\|_\infty)$ , there is a unique  $f_{\text{sol}}$  such that  $\Gamma(f_{\text{sol}}) = f_{\text{sol}}$ .

(II) Let

$$a = a_1 < a_2 < b_1 < b_3 < b_2 < \cdots < a_n < b_{n-1} < b_n = b$$

so that  $(b_j - a_j)L < 1$  for  $j = 1, \dots, n$ .

Notice that  $[a_j, b_j] \cap [a_{j+1}, b_{j+1}] = [a_j, b_{j+1}]$  has non-empty interior.

Let  $f_1 \in C[a_1, b_1]$  be the unique solution to (IVP) with  $f_1(a) = y_0$ , by (I).



Then, let  $f_2$  in  $C[a_2, b_2]$  satisfy (IVP) with  $f_2(a_2) = f_1(a_2)$ . Then, let  $f_3$  in  $C[a_3, b_3]$  satisfy (IVP) with  $f_3(a_3) = f_2(a_3)$ . Etc. Let  $f : [a, b] \rightarrow \mathbb{R}$  be given by

$$f(t) = f_j(t) \text{ for } t \in [a_j, b_j], j = 1, \dots, n.$$

Check that this is well-defined. Its value is uniquely determined on each  $[a_{j+1}, b_j]$ , thanks to uniqueness in (I).  $\square$

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Example: (IVP) Want  $f \in C[0, 1]$  s.t.

$$f(0) = 1, \quad f'(t) = tf(t).$$

We convert to

$$(IE) \quad f(t) = 1 + \int_0^t sf(s)ds.$$

This fits into Picard-Lindelof Theorem. Let  $F(t, y) = ty$ , so  $f(t) = 1 + \int_0^t F(s, f(s))ds$  with  $|F(t, y) - F(t, z)| = \underbrace{|t|}_{\leq 1} |y - z| \leq$

$|y - z|$ . (Case (II) of Picard-Lindelof.)

However, let  $\Gamma : C[0, 1] \rightarrow C[0, 1]$  by, for  $t \in [0, 1]$ ,

$$\Gamma(f)(t) = 1 + \int_0^t sf(s)ds.$$

Let us see that  $\Gamma$ , itself, is a strict contraction. Let  $f, g \in C[0, 1], t \in [0, 1]$ ,

$$\begin{aligned} |\Gamma(f)(t) - \Gamma(g)(t)| &\leq \int_0^t s \underbrace{|f(s) - g(s)|}_{\leq \|f - g\|_\infty} ds \\ &\leq \int_0^t s ds \|f - g\|_\infty \\ &= \underbrace{\frac{t^2}{2}}_{\leq \frac{1}{2}} \|f - g\|_\infty \\ &\leq \frac{1}{2} \|f - g\|_\infty. \end{aligned}$$

$$(\|\Gamma(f) - \Gamma(g)\|_\infty \leq \frac{1}{2} \|f - g\|_\infty)$$

Hence, contraction mapping theorem tells us that  $\Gamma$  has a unique fixed point, ie (IE) and (IVP) have a unique solution,  $f_{\text{sol}}$ . Furthermore, if we choose  $f_0 \in C[0, 1]$  and let  $f_n = \Gamma(f_{n-1})$  ( $n \in \mathbb{N}$ ) then

$$\|f_{\text{sol}} - f_n\|_\infty \leq \underbrace{\frac{(\frac{1}{2})^n}{1 - \frac{1}{2}}}_{=\frac{1}{2^{n-1}}} \|f_0 - \Gamma(f_0)\|_\infty.$$

We can compute  $f_{\text{sol}}$ .

Let  $f_0(t) = 0$  (constant zero).

$$\begin{aligned} f_1(t) &= \Gamma(f_0)(t) = 1 + \int_0^t s \cdot 0 \, ds = 1 \\ f_2(t) &= \Gamma(f_1)(t) = 1 + \int_0^t s \cdot 1 \, ds = 1 + \frac{t^2}{2} \\ f_3(t) &= \Gamma(f_2)(t) = 1 + \int_0^t s \left(1 + \frac{t^2}{2}\right) ds = 1 + \frac{t^2}{2} + \frac{t^4}{4 \cdot 2} \end{aligned}$$

(Use induction to check)

$$f_n(t) = 1 + \frac{t^2}{2} + \frac{t^4}{4 \cdot 2} + \cdots + \frac{t^{2(n-1)}}{[2(n-1)][2(n-2)] \cdots 2} = \sum_{k=1}^n \frac{t^{2(k-1)}}{2^{k-1}(k-1)!}.$$

Thus, at any  $t$  in  $[0, 1]$ ,

$$f_{\text{sol}} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{t^{2(k-1)}}{2^{k-1}(k-1)!} = \sum_{k=1}^{\infty} \frac{t^{2(k-1)}}{2^{k-1}(k-1)!}.$$

Furthermore, for each  $n$ ,

$$\begin{aligned} \|f_{\text{sol}} - f_n\|_{\infty} &= \max_{t \in [0,1]} |f_{\text{sol}}(t) - f_n(t)| \\ &\leq \frac{1}{2^{n-1}} \left\| 0 - \underbrace{\Gamma(0)}_{=1} \right\|_{\infty} = \frac{1}{2^{n-1}}. \end{aligned}$$

Question: Suppose we only knew that

$$d(\Gamma(x), \Gamma(y)) < d(x, y) \text{ for } x \neq y \text{ in } X.$$

(“proper contraction” instead of “strict contraction”)

Does  $\Gamma$  necessarily admit a fixed point?

Answer #1: No.

Example: On  $X = [1, \infty) \subset \mathbb{R}$ , let  $\Gamma(x) = x + \frac{1}{x}$ . If  $x < y$ , we have there is  $x < c_{x,y} < y$  such that

$$|\Gamma(x) - \Gamma(y)| = |\Gamma'(c_{x,y})| |x - y| = \left| 1 - \frac{1}{c_{x,y}^2} \right| |x - y| < |x - y|.$$

Notice: if  $\Gamma(x) = x$  we'd have  $x = x + \frac{1}{x} \implies 0 = \frac{1}{x}$ . Hence  $\Gamma$  admits no fixed point in  $[1, \infty)$ .

Answer #2: Yes, provided we limit  $(X, d)$ .

**Theorem 18.1** (Edelstein). Let  $(X, d)$  be compact, and  $\Gamma : X \rightarrow X$  satisfy  $d(\Gamma(x), \Gamma(y)) < d(x, y)$  for  $x \neq y$  in  $X$ . Then

- (i)  $\Gamma$  admits a unique fixed point  $x_{\text{fix}}$ , and
- (ii) if  $x_0 \in X$ , and  $x_n = \Gamma(x_{n-1})$  ( $n \in \mathbb{N}$ ), then  $x_{\text{fix}} = \lim_{n \rightarrow \infty} x_n$ .

*Proof.* (i) Let  $f : X \rightarrow \mathbb{R}, f(x) = d(x, \Gamma(x))$ . Since  $\Gamma$  is continuous,  $f$  is continuous. [Check that  $f$  is 2-Lipschitz.] Hence, by EVT, there is  $x_{\text{min}}$  in  $X$  so  $f(x_{\text{min}}) = \min f(X)$ . Suppose  $x_{\text{min}} \neq \Gamma(x_{\text{min}})$ , then

$$\begin{aligned} f(\Gamma(x_{\text{min}})) &= d(\Gamma(x_{\text{min}}), \Gamma \circ \Gamma(x_{\text{min}})) \\ &< d(x_{\text{min}}, \Gamma(x_{\text{min}})) = f(x_{\text{min}}) \end{aligned}$$

violating choice of  $x_{\min}$ . Hence  $x_{\min} = \Gamma(x_{\min})$ , so write  $x_{\min} = x_{\text{fix}}$ .

If, also,  $y = \Gamma(y)$  in  $X$ , with  $y \neq x_{\text{fix}}$ , then

$$d(y, x_{\text{fix}}) = d(\Gamma(y), \Gamma(x_{\text{fix}})) < d(y, x_{\text{fix}})$$

which is absurd.

(ii) Let  $x_0 \in X, (x_n)_{n=1}^\infty$  be as above. Notice that

$$0 \leq d(x_{\text{fix}}, x_{n+1}) = d(\Gamma(x_{\text{fix}}), \Gamma(x_0)) < d(x_{\text{fix}}, x_0)$$

so  $L = \lim_{n \rightarrow \infty} d(x_{\text{fix}}, x_n)$  exists (decreasing, bounded sequence in  $\mathbb{R}$ ).

Consider any converging subsequence  $(x_{n_k})_{k=1}^\infty$  of  $(x_n)_{n=1}^\infty$ , with  $x = \lim_{k \rightarrow \infty} x_{n_k}$ . Then  $d(x_{\text{fix}}, x) = \lim_{k \rightarrow \infty} d(x_{\text{fix}}, x_{n_k}) = L$ .

If  $x \neq x_{\text{fix}}$ , then

$$\begin{aligned} L &= \lim_{k \rightarrow \infty} d(x_{\text{fix}}, x_{n_k} + 1) = \lim_{k \rightarrow \infty} d(x_{\text{fix}}, \Gamma(x_{n_k})) \\ &= d(x_{\text{fix}}, \Gamma(x)) < d(x_{\text{fix}}, x) = L \end{aligned}$$

which is absurd. Hence the sequence  $(x_n)_{n=1}^\infty$  has that  $x_{\text{fix}}$  is the only possible limit of a subsequence. Thus  $\lim_{n \rightarrow \infty} x_n = x_{\text{fix}}$  (check!).  $\square$

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Office hours:

Today	2:30-3:30
Tomorrow	2:30-4
Friday	2:30-3:30

## 19.1 BAIRE CATEGORY THEOREM

Definition: Let  $(X, d)$  be a metric space.

- (i) A subset  $N \subset X$  is called nowhere dense if  $(\overline{N})^\circ = \emptyset$  (ie. the interior of the closure of  $N$  is the empty set). [Equivalently, for any  $x \in N, \varepsilon > 0, B(x, \varepsilon) \setminus \overline{N} \neq \emptyset$ ].
- (ii) A set  $S \subseteq X$  will be called meager (or is 1st category) if  $S$  is a countable union of nowhere dense sets: i.e.

$$S = \bigcup_{n=1}^{\infty} N_n, \text{ each } (\overline{N}_n)^\circ = \emptyset.$$

(ii')  $S \subseteq X$  is non-meager (or is 2nd category) provided that it is not meager.

- (iii) A set  $R \subseteq X$  is residual if  $X \setminus R$  is meager.

Remarks:

nowhere dense  $\implies$  meager

residual  $\implies$  non-meager (provided  $(X, d)$  is complete;

consequence of B.C.T, Baire Category Theorem)

If  $(X, d)$  is complete, we think of meager = “small”, non-meager = “not small”  $\iff$  residual.

Examples:

- (i) If  $x_0 \in X, \{x_0\}$  is nowhere dense  $\iff x_0$  is an accumulation point.

(ii) In  $(\mathbb{R}^2, \|\cdot\|_2)$ ,  $\mathbb{R} \times \{0\}$  is meager (exercise).

(iii) In  $(\mathbb{R}, |\cdot|)$ , the Cantor set  $C$  is nowhere dense.

Indeed,  $C$  is closed. If  $t = 0.t_1t_2\cdots \in C$  (ternary representation), then given  $\varepsilon > 0$ , find  $k$  so  $\frac{1}{3^k} < \varepsilon$  and then

$$t' = 0.t_1t_2\cdots t_{k-1}1t_{k+1}\cdots \in B(t, \varepsilon) \setminus C.$$

(iv)  $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$  is meager in  $(\mathbb{R}, |\cdot|)$  (using (i)).

(v)  $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$  is meager in  $(\mathbb{Q}, |\cdot|)$  (using (i)).

Note: if  $(X, d)$  is not complete, it may be meager in itself. [meager sets are interesting in complete settings.]

**Remark:** If  $(X, d)$  is a metric space,  $U \subseteq X$  is open and  $x_0 \in U$ , then there is  $\varepsilon > 0$ , s.t.  $B[x, \varepsilon] \subseteq U$  (Indeed, let  $\varepsilon' > 0$  be so  $B(x, \varepsilon') \subseteq U$ , and  $\varepsilon \in (0, \varepsilon')$ ).

**Lemma 19.1.** Let  $(X, d)$  be a metric space,  $N \subset X$ . Then  $N$  is nowhere dense  $\iff \overline{X \setminus \overline{N}} = X$ .

*Proof.*

$$\begin{aligned} N \text{ is nowhere dense} &\iff \text{for any } x \in \overline{N}, \varepsilon > 0, B(x, \varepsilon) \setminus \overline{N} \neq \emptyset \\ &\iff x \in \overline{X \setminus \overline{N}} \text{ for any } x \in \overline{N} \cup (X \setminus \overline{N}). \end{aligned}$$

□

**Theorem 19.1** (Baire Category Theorem). Let  $(X, d)$  be a complete metric space.

(i) Suppose  $\{U_n\}_{n=1}^\infty$  is a sequence of open sets, each dense in  $X$ . Then  $\bigcap_{n=1}^\infty U_n$  is dense in  $X$ .

(ii) If  $M \subset X$  is meager, then  $M^\circ = \emptyset$ .

*Proof.* (i) Let  $x_0 \in X$  and  $\varepsilon_0 > 0$ . We wish to show that  $B(x_0, \varepsilon_0) \cap \bigcap_{n=1}^\infty U_n \neq \emptyset$ .

Since  $\overline{U_1} = X$ , there is  $x_1 \in B(x_0, \varepsilon_0) \cap U_1$  (using meet set characterization of closure). Let  $\varepsilon_1 > 0$  be chosen so  $B[x_1, \varepsilon_1] \subseteq B(x_0, \varepsilon_0) \cap U_1$ .

Since  $\overline{U_2} = X$ , there is  $x_2 \in B(x_1, \varepsilon_1) \cap U_2$ .

Let  $\varepsilon_2 \in (0, \frac{\varepsilon_1}{2}]$  be so  $B[x_2, \varepsilon_2] \subseteq B(x_1, \varepsilon_1) \cap U_2$ .

Inductively, having chosen  $x_n, \varepsilon_n$ , we appeal to the fact that  $\overline{U_{n+1}} = X$  to find  $x_{n+1} \in B(x_n, \varepsilon_n) \cap U_{n+1}$ , then choose  $\varepsilon_{n+1} \in (0, \frac{\varepsilon_n}{2}]$  and  $B[x_{n+1}, \varepsilon_{n+1}] \subseteq B(x_n, \varepsilon_n) \cap U_{n+1}$ .

Thus, we have  $(x_n)_{n=1}^\infty \subseteq X$ ,  $(\varepsilon_n)_{n=1}^\infty \subset (0, \infty)$  s.t.

$$(a) \ B[x_{n+1}, \varepsilon_{n+1}] \subseteq B(x_n, \varepsilon_n) \subseteq B[x_n, \varepsilon_n]$$

$$(b) \ \text{diam } B[x_n, \varepsilon_n] = 2\varepsilon_n \leq \varepsilon_{n-1} \leq \frac{\varepsilon_{n-2}}{2} \leq \cdots \leq \frac{\varepsilon_1}{2^{n-1}}.$$

$$(c) \ B[x_n, \varepsilon_n] \subseteq U_n \cap B(x_0, \varepsilon_0).$$

Then (a) & (b), with the Nested Sets Theorem, show that  $\bigcap_{n=1}^\infty B[x_n, \varepsilon_n] \neq \emptyset$ .

Further, (c) shows that  $\emptyset \neq \bigcap_{n=1}^\infty B[x_n, \varepsilon_n] \subseteq \bigcap_{n=1}^\infty U_n \cap B(x_0, \varepsilon_0)$ .

Hence, for any  $x_0 \in X$ ,  $\varepsilon_0 > 0$ ,  $B(x_0, \varepsilon_0) \cap \bigcap_{n=1}^\infty U_n \neq \emptyset$ , so  $\bigcap_{n=1}^\infty \overline{U_n} = X$ .

(ii) Write  $M = \bigcup_{n=1}^\infty N_n$ , each  $(\overline{N_n})^\circ = \emptyset$ . Then  $U_n = X \setminus \overline{N_n}$  is open, and dense in  $X$ , by Lemma. We have

$$\begin{aligned} X \setminus M &= X \setminus \bigcup_{n=1}^\infty N_n \supseteq X \setminus \bigcup_{n=1}^\infty \overline{N_n} \text{ (as each } N_n \subseteq \overline{N_n}) \\ &= \bigcap_{n=1}^\infty (X \setminus \overline{N_n}) = \bigcap_{n=1}^\infty U_n \end{aligned}$$

so  $\overline{X \setminus M} = X$ . Thus if  $x \in M, \varepsilon > 0$ , we have  $B(x, \varepsilon) \setminus M = B(x, \varepsilon) \cap (X \setminus M) \neq \emptyset$ . Thus  $x \notin M^\circ$ , i.e.  $M^\circ = \emptyset$ .  $\square$

Question: Let  $\{q_k\}_{k=1}^\infty = \mathbb{Q}$ . Let for  $n$  in  $\mathbb{N}$

$$U_n = \underbrace{\bigcup_{k=1}^\infty \underbrace{\left(q_k - \frac{1}{2^{kn+1}}, q_k + \frac{1}{2^{kn+1}}\right)}_{\text{length is } \frac{1}{2^{nk}}}}_{U_n \text{ is a union of intervals, sum of lengths is } \sum_{k=1}^\infty \frac{1}{(2^n)^k} = \frac{1}{1 - \frac{1}{2^n}}}$$

Is  $\mathbb{Q} = \bigcap_{n=1}^\infty U_n$ ?

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Remark: In particular, a nonempty open subset in a complete metric space is nonmeager. The whole of  $X$  is a nonempty open set.

**Corollary 20.1.** A residual set in a complete metric space is nonmeager.

*Proof.* Let  $R \subset X$  be residual, so  $M = X \setminus R$  is meager, so  $X \setminus R = \bigcup_{n=1}^\infty N_n$ , each  $(\overline{N_n})^\circ = \emptyset$ . If we had that  $R$  was meager, i.e.  $R = \bigcup_{n=1}^\infty N'_n$ ,  $(\overline{N'_n})^\circ = \emptyset$ , then

$$X = R \cup (X \setminus R) = \underbrace{\bigcup_{n=1}^\infty N'_n}_{\text{countable union of nowhere dense sets}} \cup \bigcup_{n=1}^\infty N_n.$$

But  $X^\circ = X$ , so this contradicts B.C.T.  $\square$

meager = “small”, residual = “bigness”, “typical elements”

Definition: Let  $(X, d)$  be a metric space.

1.  $G \subseteq X$  is a  $G_\delta$ -set if  $G = \bigcap_{n=1}^\infty U_n$ , each  $U_n$  open
2.  $F \subseteq X$  is an  $F_\sigma$ -set if  $F = \bigcup_{n=1}^\infty F_n$ , each  $F_n$  closed

Examples:

1. In A4, Q2, we saw that any closed set is  $G_\delta$   
(i') Any open set  $U \subseteq X$  is  $F_\sigma$  (De Morgan's law)

2.  $\mathbb{R} \setminus \mathbb{Q}$  is not  $F_\sigma$ .

First,  $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$  is  $F_\sigma$ . Second, if  $F \subset \mathbb{R} \setminus \mathbb{Q}$  is closed, then  $F$  is nowhere dense (this just follows density of  $\mathbb{Q}$ ). Thus if we had an  $F_\sigma$  realization  $\mathbb{R} \setminus \mathbb{Q} = \bigcup_{n=1}^\infty F_n$ ,  $F_n \subset \mathbb{R} \setminus \mathbb{Q}$  closed, then  $\mathbb{R} \setminus \mathbb{Q}$  is meager. Thus,

$$\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q}) = \bigcup_{q \in \mathbb{Q}} \{q\} \cup \bigcup_{n=1}^\infty F_n$$

would be meager which violates B.C.T. (Corollary just stated).

(ii')  $\mathbb{Q}$  is not  $G_\delta$  (De Morgan, from (ii)).

In particular

$$\mathbb{Q} \not\subseteq \bigcap_{n=1}^\infty \bigcup_{k=1}^\infty \underbrace{\left(q_k - \frac{1}{2^{kn+1}}, q_k + \frac{1}{2^{kn+1}}\right)}_{U_n}.$$

$$\{q_k\}_{k=1}^\infty = \mathbb{Q}.$$

**Corollary 20.2.** In a complete metric space, a dense  $G_\delta$ -subset is residual.

*Proof.* In complete  $(X, d)$ , if  $G = \bigcap_{n=1}^\infty U_n$ , each  $U_n$  open, and  $\overline{G} = X$ , then each  $\overline{U_n} = X$ . Thus, by lemma before B.C.T., each  $X \setminus U_n$  is nowhere dense hence  $X \setminus G = X \setminus \bigcap_{n=1}^\infty U_n = \bigcup_{n=1}^\infty (X \setminus U_n)$  is meager.  $\square$

**Theorem 20.1** (Uniform Boundedness Principle). Let  $(X, d)$  be a complete metric space and  $\{f_i\}_{i \in I} \subset C(X)$  (continuous  $\mathbb{R}$ -valued functions) which satisfies for each  $x$

$$\sup_{i \in I} |f_i(x)| < \infty \text{ (pointwise boundedness).}$$

Then there exists an open  $\emptyset \neq U \subseteq X$  s.t.

$$\sup_{i \in I} \sup_{x \in U} |f_i(x)| < \infty \text{ (uniform boundedness on } U\text{).}$$

*Proof.* For  $n$  in  $\mathbb{N}$ , let

$$F_n = \{x \in X : |f_i(x)| \leq n \text{ for all } i \in I\}.$$

By our pointwise boundedness assumption,

$$X = \bigcup_{n=1}^\infty F_n \quad (\star).$$

Each  $F_n$  is closed:

$$F_n = \bigcap_{i \in I} |f_i|^{-1}((-\infty, n]) = \bigcap_{i \in I} (X \setminus \underbrace{|f_i|^{-1}(n, \infty)}_{\substack{\text{open, as } |f_i(\cdot)| \text{ is continuous} \\ \text{closed}}})$$

But B.C.T. tells us that our complete  $X$  is non-meager, so for some  $n_0$ ,  $F_{n_0}^\circ \neq \emptyset$ . Let  $U = F_{n_0}^\circ$ , and for all  $x \in U \subseteq F_n$

$$\begin{aligned} &|f_i(x)| \leq n_0 \text{ for all } i \in I \\ \implies &\sup_{x \in U} |f_i(x)| \leq n_0 \text{ for all } i \in I \\ \implies &\sup_{i \in I} \sup_{x \in U} |f_i(x)| \leq n_0 < \infty. \end{aligned}$$

$\square$

**Corollary 20.3** (Banach-Stenhaus Theorem). Let  $(V, \|\cdot\|_V)$  be a Banach space,  $(W, \|\cdot\|_W)$  a normed vector space, and  $\{T_i\}_{i \in I} \subset B(V, W)$  satisfies

$$\sup_{i \in I} \|T_i x\|_W < \infty \text{ for each } x \in V.$$

Then

$$\sup_{i \in I} \|T_i\| < \infty. \text{ [Recall } \|T_i\| = \sup_{x \in B_V[0,1]} \|T_i x\|_W.]$$

*Proof.* Let  $f_i(x) = \|T_i x\|_W$ , for  $i \in I, x \in V$ , so  $\{f_i\}_{i \in I} \subset C(V)$ . Our assumption on  $\{T_i\}_{i \in I}$ , gives pointwise boundedness of  $\{f_i\}_{i \in I}$ , so U.B.P provides  $\emptyset \neq U \subset V$  for which

$$M = \sup_{i \in I} \sup_{x \in U} \|T_i x\| < \infty.$$

As  $U$  is open, if  $x_0 \in U$ , there is  $\varepsilon > 0$ ,  $B[x_0, \varepsilon] \subset U$ .

Now if  $z \in B_V[0, 1]$ , then we may write

$$z = \frac{1}{2\varepsilon}(-x_0 + \varepsilon z) + \frac{1}{2\varepsilon}(x_0 + \varepsilon z)$$

and, for  $i$  in  $I$ , we have

$$\begin{aligned}\|T_i z\|_W &\leq \frac{1}{2\varepsilon} \left\| T_i \left( \underbrace{x_0 - \varepsilon z}_{\in B[x, \varepsilon] \subset U} \right) \right\|_W + \frac{1}{2\varepsilon} \left\| T_i \left( \underbrace{x_0 + \varepsilon z}_{\in B[x, \varepsilon] \subset U} \right) \right\|_W \\ &\leq \frac{1}{2\varepsilon} M + \frac{1}{2\varepsilon} M = \frac{M}{\varepsilon}. \\ \Rightarrow \|T_i\| &= \sup_{z \in B_V[0,1]} \|T_i z\|_W \leq \frac{M}{\varepsilon} < \infty.\end{aligned}$$

□

## 21 2017-11-13

### 21.1 BAIRE-1 FUNCTIONS

Def: Let  $\emptyset \neq X \subseteq \mathbb{R}$ , so  $(X, d)$  is a metric space with relativized metric from  $\mathbb{R}$ .

A function  $f : X \rightarrow \mathbb{R}$  is called Baire-1 if there is a sequence  $(f_n)_{n=1}^\infty \subset C(X)$  such that for  $t \in X$ ,

$$f(t) = \lim_{n \rightarrow \infty} f_n(t) \text{ (pointwise limit).}$$

Remark: Unlike uniform limits, pointwise limits of continuous functions need not be continuous.

Example: Let  $X = [0, 1]$ ,  $f_n(t) = t^n$ . Then

$$\lim_{n \rightarrow \infty} f_n(t) = \begin{cases} 0 & t \in [0, 1) \\ 1 & t = 1. \end{cases}$$

Question: Let for  $t$  in  $[0, 1]$ ,

$$f_n(t) = \cos(n! \pi t)^{n!}.$$

If  $t = \frac{k}{\ell} \in \mathbb{Q}$ ,  $\ell \in \mathbb{N}$ , then  $f_n(t) = 1$ , if  $t \geq \ell + 1$ .

Does  $\lim_{n \rightarrow \infty} f_n(t) = \chi_{\mathbb{Q} \cap [0, 1]}(t)$  for  $t$  in  $[0, 1]$ ?

Answer: No. (Probably the limit does not exist.)

The answer will follow from (corollary to) the next theorem and B.C.T.

**Theorem 21.1** (Baire). Let  $a < b$ , and  $f : (a, b) \rightarrow \mathbb{R}$  be a Baire-1 function, then there is  $t_0$  in  $(a, b)$  such that  $f$  is continuous at  $t_0$ .

$$\chi_{\mathbb{Q}}(t) = \lim_{n \rightarrow \infty} \underbrace{\lim_{m \rightarrow \infty} |\cos(n! \pi t)^m|}_{\chi_{\{\frac{k}{n!}, k \in \mathbb{Z}\}}(t)}$$

Baire-2 = pointwise limit of Baire-1 functions.

At no  $t_0$  is  $\chi_{\mathbb{Q}}$  continuous, thus not Baire-1.

*Proof.* Let  $f(t) = \lim_{n \rightarrow \infty} f_n(t)$ ,  $t \in (a, b)$ ,  $(f_n)_{n=1}^\infty \subset C(a, b)$ .

(I) Given  $\varepsilon > 0$ , we will show that there are  $\alpha < \beta$  in  $(a, b)$ , and  $N_\varepsilon$  in  $\mathbb{N}$  such that for all  $n, m \geq N_\varepsilon$ ,

$$|f_n(t) - f_m(t)| \leq \varepsilon \text{ for } t \in [\alpha, \beta].$$

Let us proceed by contradiction. Hence, there exists  $t_1$  in  $(a, b)$ , and  $n_1, m_1 \in \mathbb{N}$  such that

$$|f_{n_1}(t_1) - f_{m_1}(t_1)| > \varepsilon.$$

Since each  $f_{n_1}, f_{m_1}$  is continuous, there is an open interval  $I_1 \subset \overline{I_1} \subset (a, b)$  such that

$$|f_{n_1}(t) - f_{m_1}(t)| > \varepsilon \text{ for } t \in I_1.$$

$[t \mapsto |f_{n_1}(t) - f_{m_1}(t)| \text{ is continuous.}]$

Next, by assumption, there is  $t_2 \in I_1$  such that there exist  $n_2, m_2 > \max\{n_1, m_1\}$  such that

$$|f_{n_2}(t_2) - f_{m_2}(t_2)| > \varepsilon.$$

Again, as  $f_{n_2}, f_{m_2}$  are continuous, there is an open interval  $I_2 \subset \overline{I_2} \subset I_1$  such that

$$|f_{n_2}(t) - f_{m_2}(t)| > \varepsilon \text{ for } t \in I_2.$$

Inductively, we obtain

- a sequence of intervals

$$\overline{I_1} \supset I_1 \supset \overline{I_2} \supset I_2 \supset \cdots \supset \overline{I_n} \supset I_n \supset \cdots, \text{ and}$$

- two increasing sequences  $(n_k)_{k=1}^\infty, (m_k)_{k=1}^\infty \subseteq \mathbb{N}$  such that

$$|f_{n_k}(t) - f_{m_k}(t)| > \varepsilon \text{ for } t \in I_k.$$

Thus, by Nested Intervals Theorem, there exists

$$t_0 \in \bigcap_{k=1}^\infty \overline{I_k} = \bigcap_{k=2}^\infty \overline{I_k} \subseteq \bigcap_{k=1}^\infty I_k$$

so  $t_0 \in I_k$  for each  $k$ , so

$$|f_{n_k}(t) - f_{m_k}(t)| > \varepsilon. \quad (\dagger)$$

But, by pointwise convergence,  $f(t_0) = \lim_{k \rightarrow \infty} f_k(t_0)$  so  $(f_n(t_0))_{n=1}^\infty \subset \mathbb{R}$  is Cauchy. This violates  $(\dagger)$ . Hence (I) holds.

(II) We use (I), with  $\varepsilon = 1$ , to find  $\alpha_1 < \beta_1$  in  $(a, b)$  and  $N_1$  in  $\mathbb{N}$  so

$$|f_n(t) - f_m(t)| \leq 1 \text{ for } t \in [\alpha_1, \beta_1], \text{ if } n, m \geq N_1.$$

We again use (I), with  $\varepsilon = \frac{1}{2}$ , to find  $\alpha_2 < \beta_2$  in  $(a, b)$  and  $N_2$  in  $\mathbb{N}$  so

$$|f_n(t) - f_m(t)| \leq \frac{1}{2} \text{ for } t \in [\alpha_2, \beta_2], \text{ if } n, m \geq N_2.$$

Inductively, we obtain

- intervals

$$(a, b) \supset [\alpha_1, \beta_1] \supset (\alpha_1, \beta_1) \supset [\alpha_2, \beta_2] \supset (\alpha_2, \beta_2) \supset \cdots \supset [\alpha_n, \beta_n] \supset (\alpha_n, \beta_n) \supset \cdots, \text{ and}$$

- an increasing sequence  $(N_k)_{k=1}^\infty \subset \mathbb{N}$  such that

$$|f_n(t) - f_m(t)| \leq \frac{1}{k} \text{ for } t \in [\alpha_k, \beta_k], \text{ if } n, m \geq N_k. \quad (\ddagger)$$

By N.I.T. (Nested Intervals Theorem), there exists

$$t_0 \in \bigcap_{k=1}^\infty [\alpha_k, \beta_k] \subseteq \bigcap_{k=1}^\infty (\alpha_k, \beta_k).$$

Now, given  $\varepsilon > 0$ , let  $k$  in  $\mathbb{N}$  so  $\frac{1}{k} < \varepsilon$ , and then let  $\delta = \min\{t_0 - \alpha_k, \beta_k - t_0\} > 0$  so  $(t_0 - \delta, t_0 + \delta) \subset (\alpha_k, \beta_k) \subset [\alpha_k, \beta_k]$ . Hence by  $(\ddagger)$ , we have that

$$|f_n(t) - f_m(t)| \leq \frac{1}{k} < \varepsilon \text{ whenever } t \in (t_0 - \delta, t_0 + \delta), n, m \geq N_k.$$



Hence  $(f_n)_{n=1}^\infty$  converges “uniformly at  $t_0$ ” (see Assignment 6), so  $f$  is continuous at  $t_0$  (Assignment 6).  $\square$

**Corollary 21.1.** Let  $a < b$  in  $\mathbb{R}$ ,  $f : (a, b) \rightarrow \mathbb{R}$  be a Baire-1 function. The set  $G = \{t \in (a, b) : f \text{ is continuous at } t\}$  is a dense  $G_\delta$ -subset of  $(a, b)$ . [By B.C.T.,  $G \subset [a, b]$  is residual.]

*Proof.* If  $t_0 \in (a, b)$  and  $\varepsilon > 0$ , then there exists  $t \in (t_0 - \varepsilon, t_0 + \varepsilon) \cap (a, b) \cap G$ . I.e.  $G \cap (t_0 - \varepsilon, t_0 + \varepsilon) \neq \emptyset$ , so  $\overline{G} = (a, b)$  (relativized topology). Furthermore, the set  $G$  is always  $G_\delta$  (Assignment 6).  $\square$

Example:  $\underbrace{\chi_{\mathbb{Q}}}_{\text{nowhere continuous}}$  is not Baire-1 on any interval.

## 22 2017-11-15

**Corollary 22.1.** Let  $f \in C(a, b)$  ( $a < b$  in  $\mathbb{R}$ ) be right differentiable on  $(a, b)$ . Then  $f'_+$  (right derivative) is continuous on a dense  $G_\delta$ -subset of  $(a, b)$ . [In particular, if  $f$  is differentiable,  $f'$  is continuous on a dense  $G_\delta$ -subset.]

*Proof.* Let  $h_n(t) = \min\{b - t, \frac{1}{n}\}$  for  $n$  in  $\mathbb{N}$ ,  $t$  in  $(a, b)$ . Then

$$f_n(t) = \frac{f(t + h_n(t)) - f(t)}{h_n(t)} \quad \left( = \frac{f(t + \frac{1}{n}) - f(t)}{\frac{1}{n}}, n \text{ large} \right)$$

satisfies that each  $f_n \in C(a, b)$  and

$$f'_+(t) = \lim_{n \rightarrow \infty} f_n(t) \text{ for each } t \in (a, b).$$

$\square$

### 22.1 ON THE BANACH SPACES $C(X)$ , $X$ COMPACT

First case  $X = [a, b]$ , compact interval in  $\mathbb{R}$ .

**Lemma 22.1.** For  $n$  in  $\mathbb{N}$  let  $q_n(t) = c_n(1 - t^2)^n$  where  $c_n$  satisfies

$$1 = c_n \int_{-1}^1 (1 - t^2)^n dt.$$

Then

(q1)  $q_n(t) \geq 0$  for  $t \in [-1, 1]$ ,  $n$  in  $\mathbb{N}$  (non-negative)

(q2)  $\int_{-1}^1 q_n(t) dt = 1$ ,  $n$  in  $\mathbb{N}$  (total mass 1)

(q3) if  $\delta \in (0, 1)$ , then  $\left( \int_{-1}^{-\delta} + \int_{\delta}^1 \right) q_n(t) dt \xrightarrow{n \rightarrow \infty} 0$  (concentration of mass near 0)

*Proof.* (q1) and (q2) are obvious. Now for  $t \in [0, 1]$ ,

$$\begin{aligned} t^2 \leq t &\implies 1 - t \leq 1 - t^2 \\ &\implies (1 - t)^n \leq (1 - t^2)^n \end{aligned}$$

and hence

$$\begin{aligned} \frac{1}{c_n} &= \int_{-1}^1 (1 - t^2)^n dt = 2 \int_0^1 (1 - t^2)^n dt \\ &\leq 2 \int_0^1 (1 - t)^n dt = \frac{-2}{n+1} (1 - t)^{n+1} \Big|_0^1 = \frac{2}{n+1} \end{aligned}$$

so  $c_n \leq \frac{n+1}{2}$ . Hence, for  $|t| \in (\delta, 1)$ , we have

$$\begin{aligned} q_n(t) &= c_n(1-t^2)^n \leq c_n(1-t^2)^n \\ &\leq \frac{n+1}{2} \underbrace{(1-t^2)^n}_{<1} \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Thus

$$\begin{aligned} \left( \int_{-1}^{-\delta} + \int_{\delta}^1 \right) q_n(t) dt &\leq \left( \int_{-1}^{-\delta} + \int_{\delta}^1 \right) \frac{n+1}{2} (1-t^2)^n dt \\ &= (1-\delta)(n+1)(1-\delta^2)^n \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

□

**Theorem 22.1** (Weierstrauss approximation theorem). Given  $a < b$  in  $\mathbb{R}$ ,  $f \in C[a, b]$ , there exists a sequence  $(p_n)_{n=1}^{\infty}$  of polynomial functions such that

$$(WA) \quad \|p_n - f\|_{\infty} = \max_{t \in [a, b]} |p_n(t) - f(t)| \xrightarrow{n \rightarrow \infty} 0.$$

*Proof.* (I) We condition  $f$ . Let  $\tilde{f} \in C[0, 1]$  be given by

$$\tilde{f}(t) = f(a + t(b-a)) - [f(b) - f(a)]t - f(a).$$

So

- $\tilde{f}(0) = f(b) - f(a) = 0$
- $\tilde{f}(1) = f(b) - [f(b) - f(a)]1 - f(a) = 0.$

If we can find a sequence  $(\tilde{p}_n)_{n=1}^{\infty}$  of polynomials,

$$\left\| \tilde{p}_n - \tilde{f} \right\|_{\infty} = \sup_{t \in [0, 1]} |\tilde{p}_n(t) - \tilde{f}(t)| \xrightarrow{n \rightarrow \infty} 0$$

we are done. Indeed, if  $s \in [a, b]$ , then define each  $p_n(s) = \tilde{p}_n\left(\frac{1}{b-a}(s-a)\right) + \frac{f(b)-f(a)}{b-a}(s-a) + f(a)$ ; may be easily shown to satisfy (WA).

(II) Let us assume that

$$f \in C[0, 1], f(0) = 0 = f(1).$$

We can extend  $f$  to  $\mathbb{R}$  by letting  $f(t) = 0$  for  $t \in (-\infty, 0) \cup (1, \infty)$ , so  $f \in C_b(\mathbb{R})$ , but  $f(t) \neq 0$  only possibly for  $t \in [0, 1]$ , and  $f$  is uniformly continuous [any function in  $C[0, 1]$  is uniformly continuous].

Let  $(q_n)_{n=1}^{\infty}$  be as in the last lemma, and let for each  $n$  in  $\mathbb{N}$  and each  $t$  in  $[0, 1]$ ,

$$p_n(t) = \int_0^1 q_n(s-t)f(s)ds.$$

Let us compute, for each  $n, t$ ,

$$\begin{aligned} \frac{d^{2n+1}}{dt^{2n+1}} p_n(t) &= \int_0^1 \frac{\partial^{2n+1}}{\partial t^{2n+1}} \underbrace{q_n(s-t)}_{\text{function is } 2n+2\text{-times continuously differentiable}} f(s) ds \\ &= 0, \text{ since } \deg q_n(t) = \deg(1-t^2)^n = 2n. \end{aligned}$$

$\implies p_n$  is a polynomial,  $\deg p_n(t) \leq 2n$ .

By change of variable  $u = s - t$ ,

$$\begin{aligned} p_n(t) &= \int_0^1 q_n(s-t)f(s)ds \\ &= \int_{-t}^{1-t} q_n(u)f(u+t)du \\ &= \int_{-1}^1 q_n(u)f(u+t)du, \text{ since } f(u+t) \geq 0 \text{ possibly only on } [-t, 1-t]. \end{aligned}$$

Hence for  $t$  in  $[0, 1]$ ,

$$\begin{aligned} |p_n(t) - f(t)| &= \left| \int_{-1}^1 q_n(u)f(u+t)du - \underbrace{\int_{-1}^1 q_n(u)f(t)du}_{\text{property (q2)}} \right| \\ &\leq \int_{-1}^1 q_n(u)|f(u+t) - f(t)|du. \end{aligned}$$

Given  $\varepsilon > 0$ , let  $\delta > 0$  be so  $|x - y| < \delta(x, y \in \mathbb{R}) \implies |f(x) - f(y)| < \frac{\varepsilon}{2}$  and then

$$\begin{aligned} |p_n(t) - f(t)| &\leq \int_{-\delta}^{\delta} q_n(u) \underbrace{|f(u+t) - f(t)|}_{< \frac{\varepsilon}{2}, \text{ by choice of } \delta} du + \left( \int_{-1}^{-\delta} + \int_{\delta}^1 \right) q_n(u) \underbrace{|f(u+t) - f(t)|}_{\leq 2\|f\|_{\infty}} du \\ &\leq \frac{\varepsilon}{2} \int_{-1}^1 q_n(u)du + \left( \int_{-1}^{-\delta} + \int_{\delta}^1 \right) q_n(u)2\|f\|_{\infty} du \text{ by (q1)} \xrightarrow{n \rightarrow \infty} \frac{\varepsilon}{2} + 0. \end{aligned}$$

(Continued next lecture.)

## 23 2017-11-17

We saw  $p_n$  is polynomial, i.e.  $d^{2n+1}/dt^{2n+1}p_n(t) = 0$ . Need approx.

Using (q2) we saw for  $t \in [0, 1]$

$$|p_n(t) - f(t)| \leq \int_{-1}^1 \underbrace{q_n(u)}_{(q1)} |f(u+t) - f(t)| du$$

Given  $\varepsilon > 0$ , use uniform continuity of  $f$  to find  $\delta > 0$  s.t.  $|x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{2}$ .

$$\begin{aligned}
|p_n(t) - f(t)| &\leq \int_{-1}^1 q_n(u) |f(u+t) - f(t)| du \\
&= \int_{-\delta}^{\delta} q_n(u) |f(u+t) - f(t)| du + \left( \int_{-1}^{-\delta} + \int_{\delta}^1 \right) q_n(u) \underbrace{|f(u+t) - f(t)|}_{\leq 2\|f\|_{\infty}} du \\
&\leq \int_{-\delta}^{\delta} q_n(u) \frac{\varepsilon}{2} du + \left( \int_{-1}^{-\delta} + \int_{\delta}^1 \right) q_n(u) 2\|f\|_{\infty} du \\
&\leq \underbrace{\frac{\varepsilon}{2} \int_{-\delta}^{\delta} q_n(u) du}_{=1(q2)} + 2\|f\|_{\infty} \left( \int_{-1}^{-\delta} + \int_{\delta}^1 \right) q_n(u) du.
\end{aligned}$$

Hence, if  $n_{\varepsilon}$  is so  $n \geq n_{\varepsilon} \implies \left( \int_{-1}^{-\delta} + \int_{\delta}^1 \right) q_n(u) du \leq \frac{\varepsilon}{2(2\|f\|_{\infty}+1)}$   
we have for  $n \geq n_{\varepsilon}$ ,

$$|p_n(t) - f(t)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and we thus have

$$\|p_n - f\|_{\infty} = \max_{t \in [0,1]} |p_n(t) - f(t)| < \varepsilon$$

and we thus see that  $\lim_{n \rightarrow \infty} p_n = f$  in  $(C[0,1], \|\cdot\|_{\infty})$ . □

**Corollary 23.1.** If  $f \in C^1[a, b]$  (differentiable on  $[a, b]$ , with continuous derivative). Then, given  $\varepsilon > 0$ , there is a polynomial  $p$  s.t.

$$\begin{aligned}
\|p' - f\|_{\infty} &< \varepsilon \\
\|p - f\|_{\infty} &< (b-a)\varepsilon.
\end{aligned}$$

*Proof.* By Weierstrauss approximation theorem, find a polynomial  $q$  s.t.  $\|f' - q\|_{\infty} < \varepsilon$ . Let  $p(t) = f(a) + \int_a^t q(s) ds$ . Check that this works. (Remember Fundamental Theorem of Calculus.) □

**Corollary 23.2.**  $(C[a, b], \|\cdot\|_{\infty})$  is separable.

*Proof.* Let  $f \in C[a, b], \varepsilon > 0$ .

By Weierstrauss approximation theorem, find polynomial  $p$  s.t.

$$\|f - p\|_{\infty} < \frac{\varepsilon}{2}.$$

Write  $p(t) = a_0 + a_1 t + \dots + a_n t^n$ . For  $j = 1, \dots, n$  let  $q_j \in \mathbb{Q}$  be such that

$$|a_j - q_j| < \frac{\varepsilon}{2(n+1) \max\{|a|^j, |b|^j\}}$$

then let  $r(t) = q_0 + q_1 t + \dots + q_n t^n$ .

Check that for each  $t$  in  $[a, b]$ ,

$$|p(t) - r(t)| < \frac{\varepsilon}{2}$$

so  $\|p - r\|_{\infty} = \max_{t \in [a,b]} |p(t) - r(t)| < \frac{\varepsilon}{2}$ ,  
and thus

$$\|f - r\|_{\infty} \leq \|f - p\|_{\infty} + \|p - r\|_{\infty} < \varepsilon.$$

□

**Theorem 23.1** (nowhere differentiable functions are generic). Let  $ND[0, 1]$  denote the set of  $f$  in  $C[0, 1]$  which are nowhere differentiable. Then  $ND[0, 1]$  is residual in  $C[a, b]$ .

*Proof.* Recall for  $M, \delta > 0$ ,

$$F_{M,\delta} = \{f \in C[0,1] : \text{there is } x \text{ in } [0,1] \text{ so } \frac{|f(x) - f(t)|}{|x - t|} \leq M \\ \text{for all } t \in [0,1] \cap [(x - \delta, x) \cup (x, x + \delta)] \}$$

(A5,Q1).

(I) Let us see that each  $F_{M,\delta}$  is nowhere dense in  $(C[0,1], \|\cdot\|_\infty)$ .

To this end, let  $f \in F_{M,\delta}, \varepsilon > 0$ .

First, use Weierstrauss approximation to get a polynomial  $p$  so  $\|f - p\|_\infty < \frac{\varepsilon}{2}$ . In particular,  $p'$  exists everywhere, let  $M' = \sup_{t \in [0,1]} \|p'(t)\|$ .

Let

$$\varphi : [0, \infty) \rightarrow [0, 1], \varphi(t) = \begin{cases} t - n & t \in [n, n+1], n \in \{0\} \cup \mathbb{N} \text{ is even} \\ n+1 - t & t \in [n, n+1], n \in \mathbb{N} \text{ is odd} \end{cases}$$

For each  $k$  in  $\mathbb{N}$  let  $\varphi_k(t) = \frac{1}{k} \varphi(k^2 t)$ .

For  $s, t \in [\frac{n-1}{k^2}, \frac{n}{k^2}], n \in \mathbb{N}$ ,

$$\frac{|\varphi_k(s) - \varphi_k(t)|}{|s - t|} = k \quad (\dagger).$$

Now let  $k$  be so  $\frac{1}{k} < \frac{\varepsilon}{2}$  and  $k - M' > M, \frac{1}{k^2} < \delta$ .

Let  $\psi_k = p + \varphi_k$  and we have for  $s, t$  satisfying  $(\dagger)$ ,

$$\begin{aligned} \frac{|\psi_k(s) - \psi_k(t)|}{|s - t|} &= \left| \frac{p(s) - p(t)}{s - t} - \frac{\varphi_k(s) - \varphi_k(t)}{s - t} \right| \\ &\geq \left| \underbrace{\frac{|\psi_k(s) - \psi_k(t)|}{|s - t|}}_k - \underbrace{\frac{|p(s) - p(t)|}{|s - t|}}_{\leq M', \text{ by Mean Value Theorem}} \right| \\ &\geq |k - M'| = k - M' > M. \end{aligned}$$

Hence  $\psi_k \notin F_{M,\delta}$ . And  $\|f - \psi_k\|_\infty \leq \|f - p\|_\infty + \left\| \underbrace{p - \psi_k}_{-\varphi_k} \right\|_\infty < \frac{\varepsilon}{2} + \frac{1}{k} < \varepsilon$ . □

24 2017-11-20

**Theorem 24.1.**  $ND[0,1] = \{f \in C[0,1] : f \text{ is nowhere differentiable}\}$  is a residual set in  $(C[0,1], \|\cdot\|_\infty)$ .

*Proof.* We saw:

Each

$$F_{M,\delta} = \{f \in C[0,1] : \exists x \text{ in } [0,1], \frac{|f(x) - f(t)|}{|x - t|} \leq M \text{ for } t \in [0,1] \cap [(x - \delta, x) \cup (x, x + \delta)]\}$$

is closed (A5), nowhere dense (I).

(II) Let  $SD[0,1] = C[0,1] \setminus ND[0,1]$  ("somewhere differentiable"). If  $f \in SD[0,1]$ , in A5, it was shown that  $f \in F_{M,\delta}$  for some  $M > 0, \delta > 0$ . If  $n \in \mathbb{N}$ , with  $n > \max\{M, \frac{1}{\delta}\}$ , then  $F_{M,\delta} \subseteq F_{n, \frac{1}{n}}$ . Then

$$SD[0,1] = \bigcup_{n=1}^{\infty} F_{n, \frac{1}{n}}, \text{ each } F_{n, \frac{1}{n}} \text{ closed, } F_{n, \frac{1}{n}}^\circ = \emptyset.$$

Thus  $SD[0,1]$  is meager, so  $ND[0,1] = C[0,1] \setminus SD[0,1]$  is residual. □

Remark: Baire Category Theorem tells us that in the complete metric space  $(C[0,1], \|\cdot\|_\infty)$ .  
residual = "large" = "generic"

## 24.1 TOWARDS STONE-WEIERSTRAUSS THEOREM

Notation: (lattice structure)

Let  $X$  be non-empty,  $f, g : X \rightarrow \mathbb{R}$ . Define

$$\begin{aligned} \text{("join")} \quad & f \vee g : X \rightarrow \mathbb{R}, f \vee g(x) = \max\{f(x), g(x)\} \\ \text{("meet", min)} \quad & f \wedge g : X \rightarrow \mathbb{R}, f \wedge g(x) = \min\{f(x), g(x)\}. \end{aligned}$$

**Proposition 24.1.** Let  $(X, d)$  be a (compact) metric space,  $f, g \in C(X)$ . Then  $f \vee g, f \wedge g \in C(X)$ .

*Proof.* If  $a, b \in \mathbb{R}$ , then  $\max\{a, b\} = \frac{1}{2}(a + b) + \frac{1}{2}|a - b|$ .

Hence

$$f \vee g = \frac{1}{2}(f + g) + \frac{1}{2} \underbrace{|f - g|}_{f-g \text{ compact with } |\cdot|} \in C(X).$$

Also  $\min\{a, b\} = -\max\{-a, -b\}$ , so

$$f \wedge g = -(-f) \vee (-g) \in C(X).$$

□

Notation: A family  $\mathcal{L} \subseteq C(X)$  is called a lattice if for each  $f, g \in \mathcal{L}$ ,  $f \vee g, f \wedge g \in \mathcal{L}$ . Notice if  $f_1, \dots, f_n \in \mathcal{L}$ ,

$$\begin{aligned} f_1 \vee f_2 &\in \mathcal{L} \\ \implies f_1 \vee f_2 \vee f_3 &\in \mathcal{L} \\ &\vdots \text{ (obvious induction)} \\ \implies f_1 \vee \dots \vee f_n &\in \mathcal{L}. \end{aligned}$$

Likewise  $f_1 \wedge \dots \wedge f_n \in \mathcal{L}$ .

**Theorem 24.2** (Stone). Let  $(X, d)$  be a compact metric space and let the lattice  $\mathcal{L} \subseteq C(X)$  satisfy

- $\mathcal{L}$  is a  $\mathbb{R}$ -space
- $1 \in \mathcal{L}$  (contains constant function)
- $\mathcal{L}$  separates points: if  $x \neq y$  in  $X$ , there exists  $\varphi \in \mathcal{L}$ , so  $\varphi(x) \neq \varphi(y)$ .

Then  $\overline{\mathcal{L}} = C(X)$  ( $\mathcal{L}$  is uniformly dense in  $C(X)$ ).

*Proof.* Suppose  $x \neq y$  in  $X$  and  $\alpha, \beta \in \mathbb{R}$ . Since  $\mathcal{L}$  separates points, there is  $\varphi \in \mathcal{L}$  with  $\varphi(x) \neq \varphi(y)$ . Then

$$g = \alpha 1 + \frac{\beta - \alpha}{\varphi(y) - \varphi(x)} [\varphi - \varphi(x)1] \in \mathcal{L} \text{ as } 1 \in \mathcal{L}, \mathcal{L} \text{ is a } \mathbb{R}\text{-subspace}$$

with  $g(x) = \alpha, g(y) = \beta$ .

Fix  $f \in C(X), \varepsilon > 0$ .

(I) Fix  $x$  in  $X$ . For each  $y$  in  $X$ , letting  $\alpha = f(x), \beta = f(y)$ , if  $y \neq x$ , we have that there is

$$g_{x,y} \in \mathcal{L} \text{ s.t. } g_{x,y}(x) = f(x), g_{x,y}(y) = f(y).$$

Since each  $f, g_{x,y}$  are continuous (near  $y$ ), there are  $\delta_y > 0$  so that

$$d(z, y) < \delta_y \implies g_{x,y}(z) < f(z) + \varepsilon \text{ i.e. } g_{x,y} < f + \varepsilon \text{ on } B(x, \delta_y)$$

$$\text{(i.e. } g_{x,y} - f \text{ is 0 at } y \text{ so } < \varepsilon \text{ in a neighbourhood of } y)$$

Since  $X = \bigcup_{y \in X} B(y, \delta_y)$ , by compactness, there are  $y_1, \dots, y_m$  s.t.  $X = \bigcup_{j=1}^m B(y_j, \delta_{y_j})$ . Let

$$g_x = g_{x, y_1} \wedge \dots \wedge g_{x, y_m} \in \mathcal{L}$$

and we have  $g_x \leq g_{x, y} < f + \varepsilon 1$ .

Notice that  $g_x(x) = \min\{f_{x, y_1}(x), \dots, f_{x, y_m}(x)\} = f(x)$ . □

## 25 2017-11-22

Small goof up:

Then we let  $g_x = g_{x, y_1} \wedge \dots \wedge g_{x, y_m} \in \mathcal{L}$ .

Now, if  $z \in X$ , then  $z \in B(y_j, \delta_{y_j})$  for some  $j = 1, \dots, m$  and then

$$g_x(z) = g_{x, y_1} \wedge \dots \wedge g_{x, y_m} \leq g_{x, y_j}(z) < f(z) + \varepsilon, \text{ property of } \delta_{y_j} \text{ w.r.t. } y_j$$

so we have

$$g_x < f + \varepsilon 1, \text{ and } g_x(x) = f(x).$$

(II) For each  $x$  in  $X$ , we found  $g_x \in \mathcal{L}$  s.t.  $g_x < f + \varepsilon 1, g_x(x) = f(x)$ .

Hence  $g_x(x) = f(x) < f(x) + \varepsilon$  at each  $x$ , so there is  $\delta_x > 0$ , s.t.

$$g_x(z) > f(z) - \varepsilon \text{ for } z \in B(x, \delta_x).$$

We have  $X = \bigcup_{x \in X} B(x, \delta_x)$  so there are  $x_1, \dots, x_n \in X$  so  $X = \bigcup_{j=1}^n B(x_j, \delta_{x_j})$ . We then let

$$g = g_{x_1} \vee \dots \vee g_{x_n} \in \mathcal{L}.$$

For  $z \in X, z \in B(x_j, \delta_{x_j})$  for some  $j = 1, \dots, n$ , so

$$g(z) \geq g_{x_j}(z) > \dots > f(z) - \varepsilon$$

and thus

$$g > f - \varepsilon 1.$$

Furthermore, each  $g_{x_j} < f + \varepsilon 1$ , so if  $z \in X$ , then  $g(z) = g_{x_j}(z)$  for some  $j$ , so

$$g(z) = g_{x_j}(z) < f(z) + \varepsilon \implies g < f + \varepsilon 1$$

i.e.  $f - \varepsilon 1 < g < f + \varepsilon 1$ , so  $g \in B(f, \varepsilon)$  in  $(C(X), \|\cdot\|_\infty)$ .

In summary, given  $f \in C(X), \varepsilon > 0, B(f, \varepsilon) \cap \mathcal{L} \neq \emptyset$ . Hence,  $\overline{\mathcal{L}} = C(X)$ . □

**Corollary 25.1.** (i) Let  $\mathcal{L} = \{f \in C[a, b] : f \text{ is piecewise affine (A5)}\}$ . Then  $\overline{\mathcal{L}} = C[a, b]$ .

(ii) Let  $C$  be the Cantor set and  $\mathcal{L} = \{f \in C(C) : |f(C)| < \aleph_0\}$ . Then  $\overline{\mathcal{L}} = C(C)$ .

Definition: Let  $(X, d)$  be a (compact) metric space. A subset  $A \subseteq C(X)$  is called an algebra if for  $f, g \in A, \alpha \in \mathbb{R}$ , we have

$$\begin{aligned} f + \alpha g &\in A && (A \text{ is a } \mathbb{R}\text{-subspace}) \\ fg &\in A && (A \text{ is closed under pointwise multiplication}) \end{aligned}$$

(If  $f, g \in C(X)$ , then  $fg \in C(X)$ , too.) If  $f_1, \dots, f_n \in A, f_1 \cdots f_n \in A$  too.

If  $1 \in A$ , and  $p(t) = \sum_{i=1}^n a_i t^i$ , then for  $f \in A$ ,

$$p \circ f = a_0 1 + a_1 f + a_2 f^2 + \dots + a_n f^n \in A.$$

$(f^k(x) = f(x)^k \text{ for } x \in X.)$

**Theorem 25.1** (Stone-Weierstrauss Theorem). If  $(X, d)$  is a compact metric space,  $A \subseteq C(X)$  satisfies

- $A$  is an algebra
- $1 \in A$
- $A$  separates points: for  $x \neq y$  in  $X$ , there is  $g \in A$  so  $g(x) \neq g(y)$

Then  $\overline{A} = C(X)$  (uniform closure).

*Proof.* (I) If  $f \in A$ , then  $|f| \in \overline{A}$ . First, since  $(X, d)$  is compact,  $f$  continuous,  $f(X) \subset \mathbb{R}$  is compact, hence bounded, so there is  $a > 0$  s.t.  $f(X) \subseteq [-a, a]$ . Now, the Weierstrauss approximation theorem provides  $(p_n)_{n=1}^\infty$  of polynomials s.t.  $\|p_n - |\cdot|\|_\infty = \max_{t \in [-a, a]} |p_n(t) - |t|| \rightarrow 0$ . Hence  $\|p_n \circ f - |f|\|_\infty = \max_{x \in X} |p_n(f(x)) - |f(x)|| \rightarrow 0$ . Each  $p_n \circ f \in A$ .

(II) Since  $A$  is a  $\mathbb{R}$ -subspace, so is  $\overline{A}$  (A4 Q1). If  $f, g \in \overline{A}$ , let  $f = \lim_{n \rightarrow \infty} f_n, g = \lim_{n \rightarrow \infty} g_n$  under uniform limits, each  $f_n, g_n \in A$ . Then

$$\begin{aligned} f \vee g &= \frac{1}{2}(f + g) + \frac{1}{2}|f - g| \\ &= \lim_{n \rightarrow \infty} \underbrace{\frac{1}{2}(f_n + g_n)}_{\in A \subseteq \overline{A}} + \underbrace{\frac{1}{2}|f_n - g_n|}_{\in A \text{ by (I)}} \in \overline{A} \end{aligned}$$

since  $\overline{A}$  is closed.

Also,  $f \wedge g = -(-f) \vee (-g) \in \overline{A}$  as well.

$\implies \overline{A}$  is a  $\mathbb{R}$ -subspace and a lattice. Also,  $1 \in A \subseteq \overline{A}$ , and  $A$  separates points, hence  $\overline{A}$  separates points.

Thus  $\overline{A}$  is dense in  $C(X)$ , but is closed, so  $\overline{A} = C(X)$ . □

## 26 2017-11-24

Example: Let  $I = [a_1, b_1] \times \cdots \times [a_n, b_n]$  be a non-empty compact interval in  $\mathbb{R}^n$ . A polynomial on  $I$  is any function

$$p(t_1, \dots, t_n) = \sum_{j_1, \dots, j_n=1}^N a_{j_1, \dots, j_n} t_1^{j_1} \cdots t_n^{j_n}$$

where each  $a_{j_1, \dots, j_n} \in \mathbb{R}, N \in \mathbb{N}$ . By Stone-Weierstrauss Theorem, the family  $P(I)$  of polynomial functions is dense in  $C(I)$ .

Example: Let  $(X, d_X), (Y, d_Y)$  be compact metric spaces. Let  $\|\cdot\|$  be a norm on  $\mathbb{R}^2$ . Define

$$\rho(X \times Y) \times (X \times Y) \rightarrow [0, \infty) \text{ by}$$

$$\rho((x_1, y_1), (x_2, y_2)) = \|(d_X(x_1, x_2), d_Y(y_1, y_2))\|.$$

It is “obvious” that  $\rho$  is a metric on  $X \times Y$ .

(Usually,  $\|\cdot\| = \|\cdot\|_\infty, \|\cdot\|_1, \|\cdot\|_2$  on  $\mathbb{R}^2$ .)

Furthermore,  $(X \times Y, \rho)$  is compact. Indeed, let  $((x_n, y_n))_{n=1}^\infty \subseteq X \times Y$  be a sequence. Then  $(x_n)_{n=1}^\infty \subseteq X$  admits a converging subsequence: let  $x = \lim_{k \rightarrow \infty} x_{n_k} \in X$ . Then  $(y_{n_k})_{k=1}^\infty \subseteq Y$  admits a converging subsequence: let  $y = \lim_{\ell \rightarrow \infty} y_{n_{k_\ell}} \in Y$ .

Notice that

$$\begin{aligned} &\rho((x, y), (x_{n_{k_\ell}}, y_{n_{k_\ell}})) \\ &= \left\| (d_X(x, x_{n_{k_\ell}}), d_Y(y, y_{n_{k_\ell}})) \right\| \\ &\leq d_X(x, x_{n_{k_\ell}}) \|(1, 0)\| + d_Y(y, y_{n_{k_\ell}}) \|(0, 1)\| \\ &\xrightarrow{\ell \rightarrow \infty} 0. \end{aligned}$$



Hence  $((x_{n_{k_\ell}}, y_{n_{k_\ell}}))_{\ell=1}^\infty$  is a converging subsequence of  $((x_n, y_n))_{n=1}^\infty$ . Suppose that each  $A_X \subseteq C(X)$  and  $A_Y \subseteq C(Y)$ , each satisfy assumptions of Stone-Weierstrauss Theorem. If  $f \in A_X, g \in A_Y$ ,

$$f \otimes g : X \times Y \rightarrow \mathbb{R}, f \otimes g(x, y) = f(x)g(y).$$

Let  $A_X \otimes A_Y = \text{span}_{\mathbb{R}}\{f \otimes g : f \in A_X, g \in A_Y\}$ . Convince yourself that  $A_X \otimes A_Y \subseteq C(X \times Y)$  and satisfies assumptions of Stone-Weierstrauss Theorem.

Hence  $\overline{A_X \otimes A_Y} = C(X \times Y)$  (uniform closure).

**Corollary 26.1** (Stone-Weierstrauss without constant functions). Let  $(X, d)$  be a compact metric space, and  $A \subseteq C(X)$  satisfy

- $A$  is an algebra
- $A$  separates points
- there is  $x_0$  in  $X$  s.t.  $f(x_0) = 0$  for  $f$  in  $A$ .

Then  $\overline{A} = C_{x_0}(X) := \{f \in C(X) : f(x_0) = 0\}$ .

*Proof.* First,  $C_{x_0}(X)$  is closed in  $C(X)$ . (Let  $\varphi : C(X) \rightarrow \mathbb{R}, \varphi(f) = f(x_0)$ , which is linear and continuous:  $\|\varphi\| \leq 1$  (seen before). Then  $C_{x_0}(X) = \varphi^{-1}(\{0\}) = C(X) \setminus \underbrace{\varphi^{-1}(\mathbb{R} \setminus \{0\})}_{\substack{\text{open} \\ \text{open}}} \underbrace{\phantom{C(X) \setminus \varphi^{-1}(\mathbb{R} \setminus \{0\})}}_{\text{closed}}$ . Since  $A \subseteq C_{x_0}(X) \implies \overline{A} \subseteq C_{x_0}(X)$ .)

Second, note that  $\mathbb{R}1 + A = \{\alpha 1 + f : \alpha \in \mathbb{R}, f \in A\}$  satisfies  $\overline{\mathbb{R}1 + A} = C(X)$ . If  $g \in \mathbb{R}1 + A$ , write  $g = \alpha 1 + h, \alpha \in \mathbb{R}, h \in A$ , and  $g(x_0) = \alpha + h(x_0) = \alpha$  so  $g = g(x_0)1 + h$ .

Now, if  $f \in C_{x_0}(X)$ , there exists  $(g_n)_{n=1}^\infty \subseteq \mathbb{R}1 + A$  s.t.  $\|f - g\|_\infty \xrightarrow{n \rightarrow \infty} 0$  (Stone-Weierstrauss Theorem). Write each  $g_n = g_n(x_0)1 + h_n$  where  $h_n \in A$ . Notice that  $0 = f(x_0) = \lim_{n \rightarrow \infty} g_n(x_0)$ . Hence

$$\begin{aligned} \|f - h_n\|_\infty &\leq \|f - (g_n(x_0)1 + h_n)\|_\infty + \|g_n(x_0)1\|_\infty \\ &= \|f - g_n\|_\infty + |g_n(x_0)| \quad (\|1\|_\infty = 1) \\ &\xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Thus  $C_{x_0}(X) \subseteq \overline{A}$ . □

Def: Let  $C_\infty(\mathbb{R}) = \{\bar{f} \in C(\mathbb{R}) : \lim_{|t| \rightarrow \infty} f(t) = 0\}$ . Then  $C_\infty(\mathbb{R}) \subseteq C_b(\mathbb{R})$  and is a closed subspace. ( $L_\pm : C_b(\mathbb{R}) \rightarrow \mathbb{R}, L_\pm(f) = \lim_{t \rightarrow \pm\infty} f(t)$ , then  $L_+, L_-$  are linear and with  $\|L_\pm\| \leq 1$ . Then  $C_\infty(\mathbb{R}) = L_+^{-1}(\{0\}) \cap L_-^{-1}(\{0\})$  is closed.)

**Corollary 26.2.** Let  $A \subseteq C_\infty(\mathbb{R})$  satisfy that

- $A$  is an algebra
- $A$  separates points
- for each  $t$  of  $\mathbb{R}$ , there is  $f \in A$  s.t.  $f(t) \neq 0$ .

Then  $\overline{A} = C_\infty(\mathbb{R})$  (uniform closure).

*Proof.* (Sketch of proof)  $\psi : \mathbb{R} \rightarrow (-1, 1), \psi(t) = \frac{t}{|t|+1}$ , then  $\psi$  is continuous and bijective with  $\psi^{-1}(-1, 1) \rightarrow \mathbb{R}$  continuous. Let  $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ .

$$\begin{aligned} \varphi(-1, 1) &\rightarrow S \setminus \{(-1, 0)\} \\ \varphi(s) &= (\cos(\pi s), \sin(\pi s)) \end{aligned}$$

so  $\varphi$  is a continuous bijection with continuous inverse. Hence,  $\varphi \circ \psi : \mathbb{R} \rightarrow S \setminus \{(-1, 0)\}$  is a homeomorphism, i.e. continuous bijection with continuous inverse.

Define

$$\begin{aligned}\Psi : C_\infty(\mathbb{R}) &\rightarrow C_{(-1,0)}(S) \\ \Psi(f)(x, y) &= f(\psi^{-1} \circ \varphi^{-1}(x, y)).\end{aligned}$$

Check that  $\Psi$  is a surjective isometry, between  $(C_\infty(\mathbb{R}), \|\cdot\|_\infty)$  and  $(C_{(-1,0)}(S), \|\cdot\|_\infty)$ , and hence has isometric inverse.

We have  $\Psi(A) \subseteq C_{(-1,0)}(S)$  satisfies assumptions of last corollary, so  $\overline{\Psi(A)} = C_{(-1,0)}(S)$  but it follows that  $\overline{A} = \Psi^{-1}(\overline{\Psi(A)}) = C_\infty(\mathbb{R})$ .  $\square$

27 2017-11-27

Today's subject: towards Arzela-Ascoli Theorem (by guest lecturer)

Def: Let  $(X, d)$  be a complete metric space. Let  $F \subseteq X$  be a subset. We say  $F$  is relatively compact if  $\overline{F}$  is compact. (Here  $\overline{F}$  means the closure of  $F$ .)

**Proposition 27.1** (Properties of relatively compact subsets). Let  $(X, d)$  be a metric space,  $F \subseteq X$ . TFAE:

1.  $F$  is relatively compact
2. Every sequence  $(x_n)$  admits a Cauchy subsequence  $(x_{n_k})$
3.  $F$  is totally bounded

*Proof.* (i)  $\implies$  (ii) Let  $(x_n)$  be a sequence in  $F$ . Since  $(x_n)$  is in  $\overline{F}$  and  $\overline{F}$  is compact,  $(x_n)$  has a Cauchy subsequence  $(x_{n_k})$  (that may converge to a point in  $\overline{F} \setminus F$ ).

(ii)  $\implies$  (i) Let  $(x_n)$  be a sequence in  $\overline{F}$ . We want to show there is a subsequence  $(x_{n_k})$  converging to a point in  $\overline{F}$  (note this is nonempty by characterization of the closure).

Now, by (ii), there is a Cauchy subsequence  $(y_{n_k})$ .

Claim:  $(x_{n_k})$  is Cauchy.

For  $k, \ell \geq 1$ ,

$$\begin{aligned}d(x_{n_k}, x_{n_\ell}) &\leq d(x_{n_k}, y_{n_k}) + d(x_{n_k}, y_{n_\ell}) + d(x_{n_\ell}, y_{n_\ell}) \\ &\leq \frac{1}{n_k} + d(y_{n_k}, y_{n_\ell}) + \frac{1}{n_\ell} \xrightarrow{k, \ell \rightarrow \infty} 0.\end{aligned}$$

(i)  $\implies$  (iii)  $\overline{F}$  is totally bounded since it is compact. So for  $\frac{\varepsilon}{2} > 0$ , there are  $x_1, \dots, x_n \in \overline{F}$  s.t.  $B(x_i, \frac{\varepsilon}{2})$  covers  $\overline{F}$  (i.e.  $\bigcup_{i=1}^n B(x_i, \frac{\varepsilon}{2}) \supseteq \overline{F}$ ).

For each  $i$ , choose  $y_i \in B(x_i, \frac{\varepsilon}{2}) \cap F$ . Then  $B(y_i, \varepsilon) \supseteq B(x_i, \frac{\varepsilon}{2})$  so  $y_1, \dots, y_n$  is an  $\varepsilon$ -net for  $F$ .

(iii)  $\implies$  (i) Since  $F$  is totally bounded, there is an  $\varepsilon$ -net  $y_1, \dots, y_n \in F$ . So

$$\begin{aligned}F &\subseteq \bigcup_{i=1}^n B(y_i, \varepsilon) \\ \implies \overline{F} &\subseteq \bigcup_{i=1}^n \overline{B(y_i, \varepsilon)} \\ \implies \overline{F} &\subseteq \bigcup_{i=1}^n B(y_i, 2\varepsilon).\end{aligned}$$

So  $\overline{F}$  is totally bounded.  $\square$

Def: [Equicontinuity] Let  $(X, d)$  be a (compact) metric space. A subset  $F \subseteq C(X)$  is equicontinuous if for  $\varepsilon > 0$  and  $x \in X$  there is  $\delta > 0$  s.t. if  $d(x, y) < \delta$  then  $|f(y) - f(x)| < \varepsilon \forall f \in F$  (holds for all  $f$  simultaneously).

**Lemma 27.1.** If  $(X, d)$  is compact and  $F \subseteq C(X)$  then  $F$  is equicontinuous  $\iff F$  is uniformly equicontinuous meaning for  $\varepsilon > 0$  there is  $\delta > 0$  s.t. if  $x, y \in X$  and  $d(x, y) < \delta$  then  $|f(x) - f(y)| < \varepsilon \forall f \in F$ .

*Proof.* If  $F$  is uniformly equicontinuous it is clearly equicontinuous.

For the other direction, fix  $\varepsilon > 0$ . For each  $x$  there is  $\delta_x$  s.t. if  $d(x, y) < \delta_x$  then  $|f(y) - f(x)| < \varepsilon/2 \forall f \in F$ . Then  $(B(x, \delta_x))_{x \in X}$  is an open cover. Let  $\delta > 0$  be the corresponding Lebesgue covering number. So for any  $y \in X$ ,  $B(y, \delta) \subseteq B(x, \delta_x)$  for some  $x \in X$ . So if  $y, z \in X$  with  $d(y, z) < \delta$ , choose  $x \in X$  s.t.  $B(y, \delta) \subseteq B(x, \delta_x)$ , then

$$\begin{aligned} |f(y) - f(z)| &\leq |f(y) - f(x)| + |f(x) - f(z)| \quad (z \in B(x, \delta_x)) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

□

Ex: Let  $F$  be a set of differentiable functions from  $[0, 1]$  to  $\mathbb{R}$  s.t.  $|f'(x)| \leq M \forall f \in F, x \in [0, 1]$  for some  $M$ . By the MVT, for  $x, y \in [0, 1]$  there is  $z \in [0, 1]$  s.t.  $M \geq |f'(z)| = \frac{|f(y) - f(x)|}{|y - x|}$ .

$$|f(y) - f(x)| \leq M|y - x| \forall y, x \in [0, 1], \forall f \in F.$$

Now take  $\delta = \frac{\varepsilon}{M}$ . Then if  $|x - y| < \delta$  then

$$\begin{aligned} |f(x) - f(y)| &\leq M|x - y| \\ &< M \frac{\delta}{M} = \varepsilon. \end{aligned}$$

28 2017-11-29

Office Hours:

Today: 2:30-4:30

Tomorrow: 2-4 pm

Last time:

In complete  $(X, d)$ , TFAE:

- (i) relative compactness
- (ii) every sequence admits a Cauchy subsequence
- (iii) total boundedness

Discussed for  $F \subset C(X)$ :

- equicontinuity  $\implies$  uniform equicontinuity if  $(X, d)$  compact
- pointwise boundedness

**Theorem 28.1** (Arzela-Ascoli Theorem). Let  $(X, d)$  be a compact metric space,  $F \subset C(X)$ . Then

$F$  is relatively compact in  $(C(X), \|\cdot\|_\infty) \iff F$  is both equicontinuous and pointwise bounded.

*Proof.*  $(\implies)$   $F$  is totally bounded. In particular,  $F$  is bounded:  $\sup_{f \in F} \|f\|_\infty < \infty$  (totally bounded  $\implies$  bounded). Hence for  $x$  in  $X$ ,  $\sup_{f \in F} |f(x)| < \sup_{f \in F} \sup_{x \in X} |f(x)| = \sup_{f \in F} \|f\|_\infty < \infty$ .

Given  $\varepsilon > 0$ , let  $f_1, \dots, f_n \in F$  s.t.  $F \subseteq \bigcup_{j=1}^n B[f_j, \frac{\varepsilon}{3}]$ . Let for  $j = 1, \dots, n$ ,  $\delta_j > 0$  be so for  $x, y$  in  $X$ ,  $d(x, y) < \delta_j \implies |f_j(x) - f_j(y)| < \frac{\varepsilon}{3}$  (uniform continuity of  $f_j$ ). Then let  $\delta = \min\{\delta_1, \dots, \delta_n\}$  and then for  $x, y$  in  $X$ ,  $d(x, y) < \delta$ , we have for  $f$  in  $F$ , then  $f \in B[f_j, \frac{\varepsilon}{3}]$  for some  $j$ . Then

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f(y)| \\ &< \|f - f_j\|_\infty + \frac{\varepsilon}{3} + \|f - f_j\|_\infty \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Hence,  $F$  is (uniformly) equicontinuous, thus equicontinuous.

( $\Leftarrow$ ) Let  $(x_n)_{n=1}^\infty \subset X$  satisfy that there are  $n_1 < n_2 < n_3 < \dots$  for which

$$X = \bigcup_{k=1}^\infty \bigcup_{j=1}^{n_k} B[x_j, \frac{1}{k}] \quad (\dagger)$$

(assignment 5,  $(X, d)$  compact  $\implies (X, d)$  separable).

Now, let  $(f_n)_{n=1}^\infty \subseteq F$ . We wish to extract a uniformly Cauchy subsequence, hence showing  $F$  is relatively compact.

(I) Let us extract a candidate Cauchy subsequence. This technique is a variant of “Cantor’s diagonalization argument”. First,  $(f_n(x_1))_{n=1}^\infty \subset \mathbb{R}$  is bounded (pointwise bounded assumption) so by Bolzano-Weierstrauss admits a Cauchy subsequence  $(f_{n_k}(x_1))_{k=1}^\infty \subset \mathbb{R}$ . Let  $f_{1,k} = f_{n_k}$  for each  $k$ . Second,  $(f_{1,n}(x_2))_{n=1}^\infty \subset \mathbb{R}$  is bounded, and again admits a Cauchy subsequence  $(f_{1,n_k}(x_2))_{k=1}^\infty \subset \mathbb{R}$ . Let  $f_{2,k} = f_{1,n_k}$ .

Inductively, we continue. We build sequences  $(f_{1,k})_{k=1}^\infty, (f_{2,k})_{k=1}^\infty, \dots, (f_{n,k})_{k=1}^\infty, \dots \subseteq F$  which satisfy

- $m < n$ ,  $(f_{n,k})_{k=1}^\infty$  is a subsequence of  $(f_{m,k})_{k=1}^\infty$
- $(f_{n,k}(x_n))_{k=1}^\infty \subset \mathbb{R}$  is Cauchy.

We now let

$$g_n = f_{n,n}.$$

Then  $(g_n)_{n=m}^\infty$  is a subsequence of  $(f_{m,n})_{n=1}^\infty$  so  $(g_n(x_m))_{n=1}^\infty$  is Cauchy in  $\mathbb{R}$ , (being a subsequence of  $(f_{m,n}(x_m))_{n=1}^\infty$ ). Thus  $(g_n(x_m))_{m=1}^\infty$  is Cauchy for each  $m$  in  $\mathbb{N}$ , and  $(g_k)_{k=1}^\infty$  is a subsequence of  $(f_n)_{n=1}^\infty$ .

(II) Let us show that  $(g_n)_{n=1}^\infty$  is Cauchy in  $(C(X), \|\cdot\|_\infty)$ , i.e., Cauchy in  $\|\cdot\|_\infty$ .

Given  $\varepsilon > 0$ , our set  $F$ , being equicontinuous on compact  $(X, d)$ , is uniformly equicontinuous (lemma Monday), so there is  $\delta > 0$  s.t.  $|f(x) - f(y)| < \frac{\varepsilon}{3}$  whenever  $x, y \in X$ ,  $d(x, y) < \delta$  and  $f \in F$ .

Now, let  $k$  in  $\mathbb{N}$  satisfy  $\frac{1}{k} < \delta$ , and we have from  $(\dagger)$  that  $X = \bigcup_{j=1}^{n_k} B[x_j, \frac{\varepsilon}{3}]$ .

Now, for  $j = 1, \dots, n_k$ , let  $N_j$  in  $\mathbb{N}$  be s.t.  $m, n \geq N_j \implies |g_m(x_j) - g_n(x_j)| < \frac{\varepsilon}{3}$  (i.e.  $(g_n(x_j))_{n=1}^\infty$  is Cauchy). Let  $N = \max\{N_1, \dots, N_{n_k}\}$ . If  $x \in X$ , so  $x \in B[x_j, \frac{\varepsilon}{3}]$  for some  $j = 1, \dots, n_k$ , and we have for  $m, n \geq N$  that

$$\begin{aligned} |g_m(x) - g_n(x)| &\leq |g_m(x) - g_m(x_j)| + |g_m(x_j) - g_n(x_j)| + |g_n(x_j) - g_n(x)| \\ &< \underbrace{\frac{\varepsilon}{3}}_{\text{thanks to uniform equicontinuity of } F; g_n \in F} + \underbrace{\frac{\varepsilon}{3}}_{n, m \geq N \geq N_j \text{ Cauchy at } x_j} \\ &\quad + \underbrace{\frac{\varepsilon}{3}}_{\text{thanks to uniform equicontinuity of } F; g_n \in F} = \varepsilon. \end{aligned}$$

Hence  $\|g_m - g_n\|_\infty = \max_{x \in X} |g_m(x) - g_n(x)| < \varepsilon$ .

– END OF FINAL LINE (except Assignment 7) –

□

## 29 2017-12-01

**Theorem 29.1** (Peano’s Theorem). Let  $D \subset \mathbb{R}^2$  be open and  $F : D \rightarrow \mathbb{R}$  be continuous, and  $(t_0, y_0) \in D$ . Then there are  $a < b$  in  $\mathbb{R}$  so  $t_0 \in (a, b)$  for which

$$(IVP) \quad f'(t) = F(t, f(t)), f(t_0) = y_0, t \in (a, b)$$

admits a solution.

(This is stronger than Picard-Lindelof, which required a Lipschitz condition on the second variable of a two variable function.) The solution here may not be unique.

*Proof.* (Most of proof):

(I) (Get  $a < b$ .) Let  $R = [a_1, b_1] \times [a_2, b_2] \subset D$  (compact interval) so  $(t_0, y_0) \in R^\circ$  (interior), and let  $M = \max_{(t,y) \in R} |F(t, y)|$ .

We let

$$W = \{(t, y) \in D : |y - y_0| \leq M|t - t_0|\}$$

and  $a < b$  in  $\mathbb{R}$  so

$$([a, b] \times \mathbb{R}) \cap W \subset R.$$

(II) (Work on  $[t_0, b]$ , find a particular family of piecewise affine functions.) Given  $\varepsilon > 0$ , the uniform continuity of  $F$  on  $R$  provides  $\delta > 0$  such that

$$\begin{aligned} (s, x), (t, y) \in R \text{ with } \max\{|s - t|, |x - y|\} = \|(s, x) - (t, y)\|_\infty < \delta \\ \implies |F(s, x) - F(t, y)| < \varepsilon. \end{aligned}$$

We partition  $[t_0, b]$ ,  $t_0 < t_1 < \dots < t_n = b$ , so  $\max_{j=1, \dots, n} (t_j - t_{j-1}) < \frac{\delta}{M+1}$  (let  $M = 0$ ).

We define  $f_\varepsilon : [t_0, b] \rightarrow \mathbb{R}$  inductively by

$$f_\varepsilon(t) = \begin{cases} y_0 + F(t_0, y_0)(t - t_0) & t \in [t_0, t_1] \\ f_\varepsilon(t_1) + F(t_1, f_\varepsilon(t_1))(t - t_1) & t \in (t_1, t_2] \\ \vdots & \\ f_\varepsilon(t_{n-1}) + F(t_{n-1}, f_\varepsilon(t_{n-1}))(t - t_{n-1}) & t \in (t_{n-1}, t_n] \end{cases}.$$

Two nice properties (exercise):

- graph of  $f_\varepsilon$  on  $[t_0, b]$  is in  $R$ , so  $\max_{t \in [t_0, b]} |f_\varepsilon(t)| \leq \max\{|a_2|, |b_2|\}$
- if  $s < t$  in  $[t_0, b]$ , then  $|f_\varepsilon(t) - f_\varepsilon(s)| \leq M|t - s|$  ( $\dagger$ ).

These estimates are independent of  $\varepsilon$ . I.e. if we form  $K = \{f_\varepsilon\}_{\varepsilon \in (0, \infty)}$  it is

- pointwise bounded & equi-Lipschitz  $\implies$  (uniformly) equicontinuous.

Hence  $K$  is relatively compact.

(III) (Relate  $K = \{f_\varepsilon\}_{\varepsilon \in (0, \infty)}$  to the (IVP).) Fix  $f_\varepsilon, \varepsilon$  and  $\delta$  as in  $(\varepsilon - \delta)$  above. If  $t \in (t_j, t_{j+1})$ ,  $j = 0, \dots, n-1$  then

$$f'_\varepsilon(t) = F(t_j, f_\varepsilon(t_j)). \quad (\star)$$

Also, for such  $t$  as above, then  $|t - t_j| < \frac{\delta}{M+1}$  so by ( $\dagger$ )

$$|f_\varepsilon(t) - f_\varepsilon(t_j)| \leq M|t - t_j| \leq \delta \frac{M}{M+1} < \delta$$

so, by choice of  $\delta$ ,

$$\begin{aligned} |F(t, f_\varepsilon(t)) - F(t_j, f_\varepsilon(t_j))| < \varepsilon \\ (\text{using } (\star)) \implies |F(t, f_\varepsilon(t)) - f'_\varepsilon(t)| < \varepsilon \quad (\star\star). \end{aligned}$$

Thus for  $t \in [t_0, b]$  we have

$$\begin{aligned} f_\varepsilon(t) &= y_0 + \int_{t_0}^t f'_\varepsilon(s) ds \text{ (piecing together F.T. of C., as } f'_\varepsilon(t) \text{ exists except at } t_1, \dots, t_{n-1}) \\ &= y_0 + \int_{t_0}^t F(s, f_\varepsilon(s)) ds + \int_{t_0}^t [f'_\varepsilon(s) - F(s, f_\varepsilon(s))] ds \end{aligned}$$

Let  $\tilde{f}_\varepsilon(t) = y_0 + \int_{t_0}^t F(s, f_\varepsilon(s))ds$ , and we have for  $t \in [t_0, b]$

$$|f_\varepsilon(t) - \tilde{f}_\varepsilon(t)| \leq \int_{t_0}^t \underbrace{|f'_\varepsilon(s) - F(s, f_\varepsilon(s))|}_{< \varepsilon} ds$$

$$(\star \star \star) \quad \leq (t - t_0)\varepsilon \leq (b - t_0)\varepsilon.$$

We now consider a sequence  $(f_{\frac{1}{n}})_{n=1}^\infty \subseteq K$ . By relative compactness, we get a uniformly Cauchy, hence uniformly converging subsequence  $(f_{\frac{1}{n_k}})_{k=1}^\infty, f = \lim_{k \rightarrow \infty} f_{\frac{1}{n_k}}$  (uniform limit). Let  $\tilde{f}(t) = y_0 + \int_{t_0}^t F(s, f(s))ds$ .

We have

$$\|f - \tilde{f}\|_\infty \leq \|f - f_{\frac{1}{n_k}}\|_\infty + \|f_{\frac{1}{n_k}} - \tilde{f}_{\frac{1}{n_k}}\|_\infty + \|\tilde{f}_{\frac{1}{n_k}} - \tilde{f}\|_\infty$$

We have  $\lim_{k \rightarrow \infty} f_{\frac{1}{n_k}}(s) = f(s)$  uniformly for  $s \in [t_0, b]$ , so, by uniform continuity  $\lim_{k \rightarrow \infty} |F(s, f_{\frac{1}{n_k}}(s)) - F(s, f(s))| = 0$  uniformly for  $s$  in  $[t_0, b]$ , and thus  $(\ddagger) \xrightarrow{k \rightarrow \infty} 0$ . In conclusion

$$\|f - \tilde{f}\|_\infty \leq \|\tilde{f}_{\frac{1}{n_k}}\|_\infty + (b - t_0)\frac{1}{n_k} + (\ddagger)$$

$\implies f(t) = \tilde{f}(t) = y_0 + \int_{t_0}^t F(s, f(s))ds$ , i.e.  $f$  satisfies (IE)  $\implies$  (IVP).

□