The Minimum Dilation Triangulation Problem

Introduction and Theoretical Aspects

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Triangulations

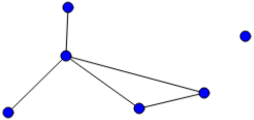
Let $S \subset \mathbb{R}^2$ finite.

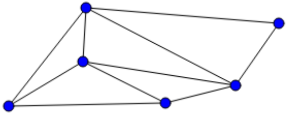
Definition

A *planar network* on S is a set of line segments with endpoints in S, where no two segments intersect nontrivially (they may only share an endpoint).

Definition

A *triangulation* of S is a planar network which is maximal for inclusion.





How good is a triangulation?

Let T be a triangulation of S. For $p, q \in S$, write $d_T(p,q)$ for the minimal possible sum of euclidean lengths of edges forming a path from p to q (i.e. the distance in the weighted graph (S, T)).

- **Diameter:**

$$\mathrm{diam}(T) := \max_{p,q\in S} d_T(p,q) \quad \left(\text{or } \frac{\max_{p,q\in S} d_T(p,q)}{\max_{p,q\in S} \|p-q\|}, \cdots \right)$$

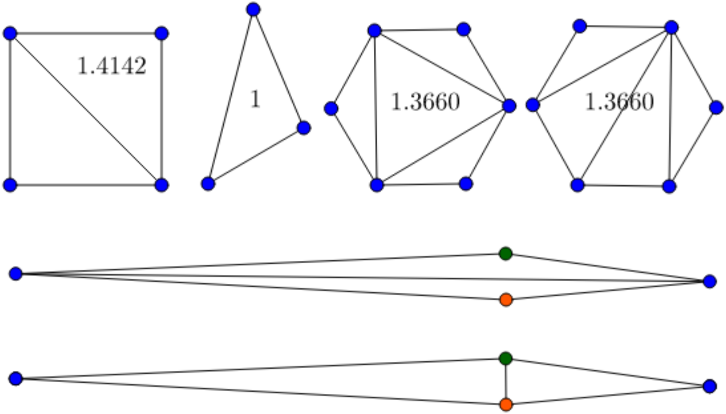
- **Dilation:**

$$\boxed{\mathrm{dil}(T) := \max_{p\neq q\in S} \frac{d_T(p,q)}{\|p-q\|}} \in [1, +\infty)$$

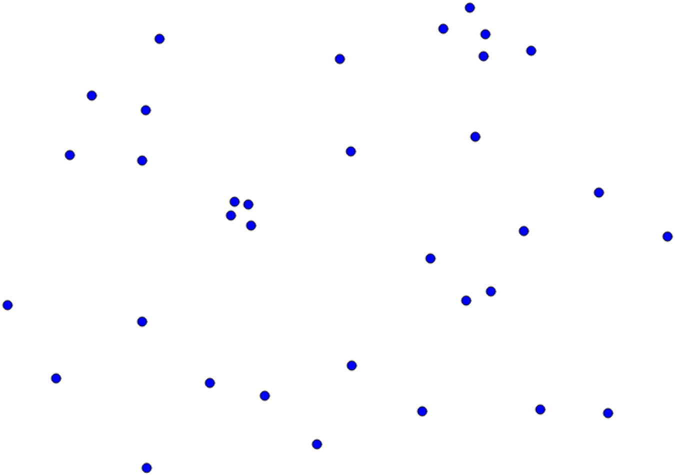
Goal

Find T such that $\mathrm{dil}(T)$ is minimal.

Examples



Can you always find a triangulation with bounded dilation?



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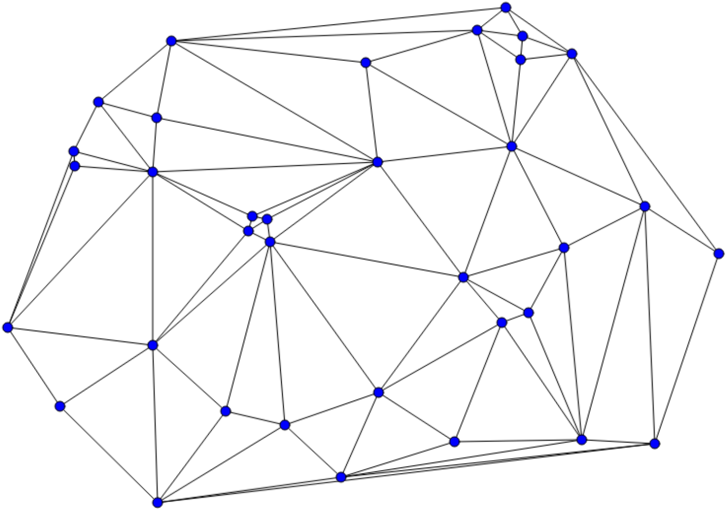

General upper bounds

Definition

The dilation of a finite set $S \subset \mathbb{R}^2$ is defined as

$$\mathrm{dil}(S) := \min_{T \in \mathcal{T}} \mathrm{dil}(T)$$

where $\mathcal{T}$ is the set of all triangulations of S.

Theorem (1989; Chew)

For every finite $S \subset \mathbb{R}^2$,

$$\mathrm{dil}(S) \leq 2.$$

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Theorem (2011; Xia)

For every finite $S \subset \mathbb{R}^2$,

$$\mathrm{dil}(S) \leq 1.998.$$

What is the true bound?

So, we know that every S has a triangulation with dilation ≤ 1.998.

Proposition (Trivial)

If S is the set of vertices of a square, then every triangulation of S has dilation at least $\sqrt{2} \approx 1.4142$.

Proposition (2004; Multzer)

If S is the set of vertices of a regular 21-gon, then every triangulation of S has dilation at least 1.4161.

Proposition (2016; Dumitrescu)

If S is the set of vertices of a regular 23-gon, then every triangulation of S has dilation at least 1.4308.

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Proposition (2019; Galant, P.)

If S is the set of vertices of a regular 53-gon, then every triangulation of S has dilation at least 1.4336.

Convex case

Convex polygons seem to be special for the study of dilation:

- The known examples S with largest dilation are convex (even regular) polygons
- General algorithms often have much lower performance with convex polygons than with other (e.g. random) point sets

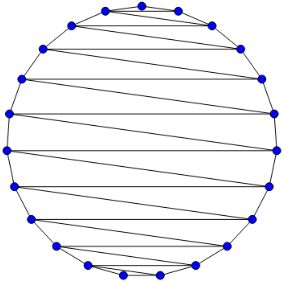

Figure: Output of the approximation algorithm based on Delaunay triangulations. This algorithm is proven to always output a triangulation with dilation less than 1.998. But in this case, the output is actually the worst triangulation (dilation $\approx \pi/2$, the actual answer is ≈ 1.43)

Upper Bounds in the convex case

Theorem (2011; Xia)

For every finite $S \subset \mathbb{R}^2$,

$$\mathrm{dil}(S) \leq 1.998.$$

Theorem (2016; Bose et al.)

For every finite $S \subset \mathbb{R}^2$ **in convex position**,

$$\mathrm{dil}(S) \leq 1.88.$$

Conversely, there is a set S in convex position with $\mathrm{dil}(S) \geq 1.4336$.

Dilation of regular polygons

Let c_n be the dilation of the regular n-gon.

n	c_n	n	c_n	n	c_n	n	c_n
4	1.4142	12	1.3836	20	1.4142	**28**	**1.4147**
5	1.2360	13	1.3912	21	1.4161	**29**	**1.4199**
6	1.3660	14	1.4053	22	1.4047	**30**	**1.4236**
7	1.3351	15	1.4089	23	1.4308	**31**	**1.4120**
8	1.4142	16	1.4092	24	1.4013	**32**	**1.4161**
9	1.3472	17	1.4084	**25**	**1.4050**	**33**	**1.4185**
10	1.3968	18	1.3816	**26**	**1.4169**	**34**	**1.4167**
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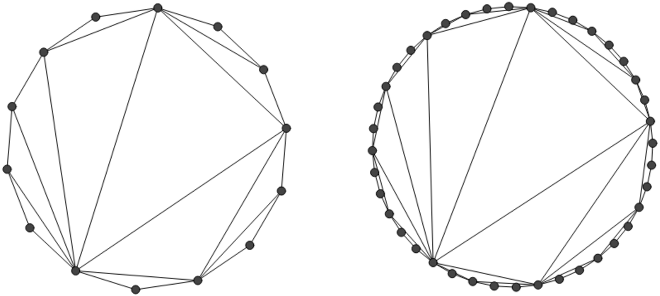
Theorem (2019; P.)

The sequence c_n converges.

Proof sketch.

Let $\Delta := \liminf c_n$. Let $\varepsilon > 0$. It is sufficient to show that $c_n \leq \Delta + \varepsilon$ for all n sufficiently large.
By definition of Δ, there exists a m for which $c_m \leq \Delta + \varepsilon/2$. Let T_m be an optimal triangulation of the regular m-gon (i.e. $\mathrm{dil}(T_m) = c_m$).
Let n_0 be much larger than m. Let $n \geq n_0$.
We construct a triangulation T_n of the regular n-gon which is as similar to T_m as possible. It is possible to prove that $\mathrm{dil}(T_n) \leq \mathrm{dil}(T_m) + \varepsilon/2$, which concludes the proof. □



Generalisation

Theorem (2019; P.)

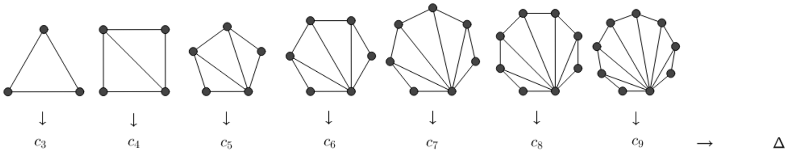
Let $\mathcal{C}$ be a strictly convex differentiable closed curve. There is a real number Δ, with the following property.
Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of finite subsets of $\mathcal{C}$ such that $\forall\varepsilon > 0$, S_n is an ε-covering of $\mathcal{C}$ for all but finitely many n.
Then, $\mathrm{dil}(S_n)$ converges to Δ.

Thus, the limit is independent of the sequence of polygons, and depends only on the curve $\mathcal{C}$.

Summary

To simplify, let S_n be the regular n-gon. For each n, there is at least one triangulation T_n of S_n with smallest possible dilation.

1. We know that the sequence of dilations $(c_n) := (\mathrm{dil}(T_n))$ converges to some value Δ.
2. What can be said about the triangulations T_n themselves?



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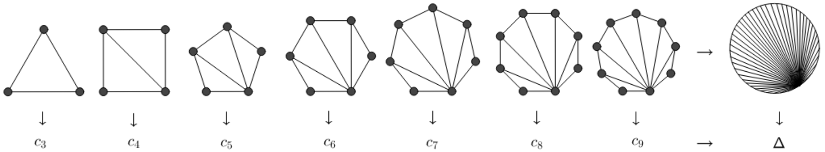
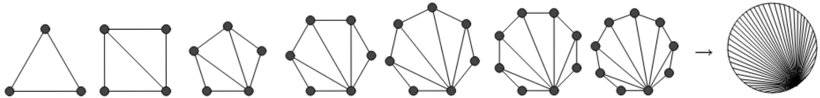


Infinite triangulations



Definition (Infinite triangulations)

- Let $\mathcal{C}$ be a strictly convex closed curve. A *geometric network* on $\mathcal{C}$ is a set of segments with both endpoints in $\mathcal{C}$, which are pairwise non-intersecting (except possibly at the endpoints).
- A *triangulation* of $\mathcal{C}$ is a geometric network which is maximal for inclusion.

Definition (Distance in an infinite triangulation)

Let T be a triangulation of a strictly convex closed curve $\mathcal{C}$. The *distance* $d_T(p, q)$ between two points $p, q \in \mathcal{C}$ is the length of the shortest path[a] from p to q which uses only edges of T *and arcs of the curve* $\mathcal{C}$.

[a] Infimum over all finite paths.

Infinite triangulations

Definition (Dilation of an infinite triangulation)

Let T be a triangulation of a strictly convex closed curve $\mathcal{C}$.

- The *dilation* of T is[a]

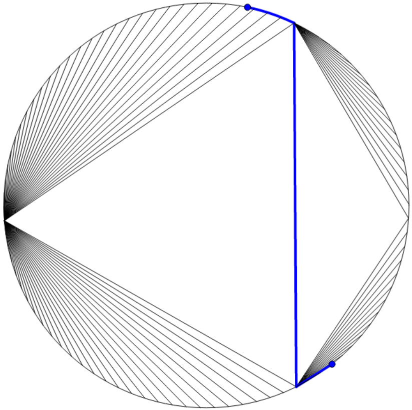
$$\operatorname{dil}(T) := \sup_{p,q \in \mathcal{C}} \frac{d_T(p,q)}{\|p-q\|}.$$

- The dilation of $\mathcal{C}$ is

$$\operatorname{dil}(\mathcal{C}) := \inf_{T \in \mathcal{T}} \operatorname{dil}(T),$$

where $\mathcal{T}$ is the set of triangulations of $\mathcal{C}$.

[a] As $\mathcal{C}$ is compact, $\cancel{\sup} \to \max$.

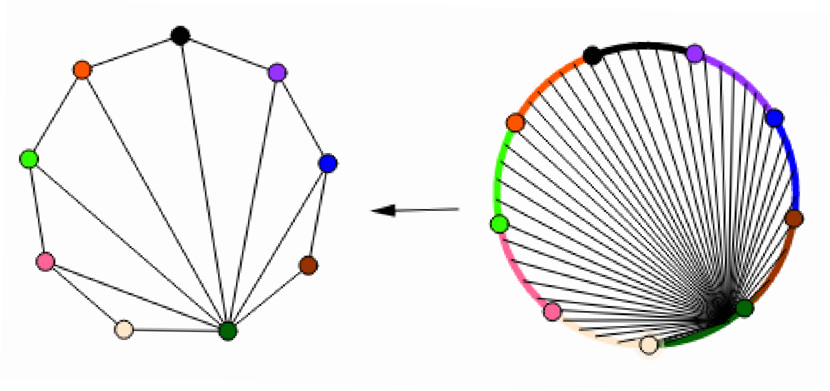


Question

Is there a triangulation T of $\mathcal{C}$ that achieves $\operatorname{dil}(T) = \operatorname{dil}(\mathcal{C})$?

The quotient operation

We need a way to transfer triangulations of “larger sets” (e.g. the circle or a regular polygon with many sides) to triangulations of “smaller sets” (e.g. a regular polygon with less sides).



The quotient operation

Lemma

Let T be a triangulation (of the circle or a regular N-gon with $N \geq n$). Let T' be the triangulation obtained after a quotient of T into n evenly spaced equivalence classes. Then

$$|\mathrm{dil}(T') - \mathrm{dil}(T)| = O\left(\frac{\log n}{n}\right).$$

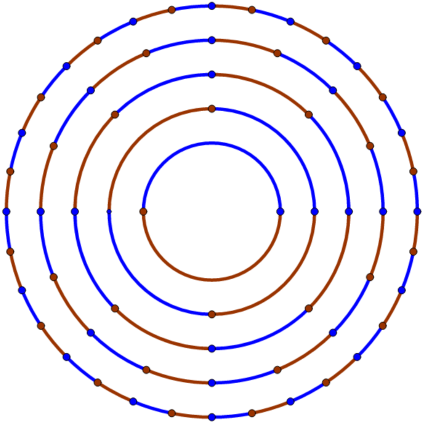
Corollary

Let T be a triangulation of the circle. Let $T_{\phi(n)}$ be a sequence of quotients of T into $\phi(n)$ evenly spaced equivalence classes (where $\phi : \mathbb{N} \to \mathbb{N}$ strictly increasing). Then $\lim \mathrm{dil}(T_{\phi(n)}) = \mathrm{dil}(T)$.

Profinite triangulations

Let T be a triangulation of the circle. We consider a special sequence of quotients of T. Let T_{2^n} be the quotient of T into 2^n classes as shown on the picture.

We have the following property: for all $n \geq k$, the quotient of T_{2^n} into the appropriate 2^k equivalence classes is exactly T_{2^k}.



Profinite triangulations

Definition

[a] For $n \geq 1$, let $\mathcal{T}_{2^n}$ be the set of triangulations of a regular 2^n-gon. We define

$$\varprojlim \mathcal{T}_{2^n}$$

to be the set of sequences of triangulations

$$(T_2, T_4, T_8, \ldots)$$

where

- $T_{2^n} \in \mathcal{T}_{2^n}$,
- for all $n \geq k$, the quotient of T_{2^n} into the appropriate 2^k equivalence classes is exactly T_{2^k}.

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[a] This definition can be generalised to any strictly convex closed curve.

Theorem (2019; P.)

Let $\mathcal{C}$ be a differentiable strictly convex closed curve and let $\mathcal{T}$ be the set of triangulations of $\mathcal{C}$.

1. *We have a natural bijection*

$$\mathcal{T} \cong \varprojlim \mathcal{T}_{2^n}.$$

2. *The maps $\mathrm{dil}_1 : \mathcal{T} \to \mathbb{R}$, $T \mapsto \mathrm{dil}(T)$ and $\mathrm{dil}_2 : \varprojlim \mathcal{T}_{2^n} \to \mathbb{R}$, $(T_{2^n})_n \mapsto \lim_n \mathrm{dil}(T_{2^n})$ agree under this bijection.*
3. *The space $\varprojlim \mathcal{T}_{2^n}$ is compact and dil_2 is continuous, so there is a triangulation $T \in \mathcal{T}$ such that $\mathrm{dil}(T) = \mathrm{dil}(\mathcal{C})$.*

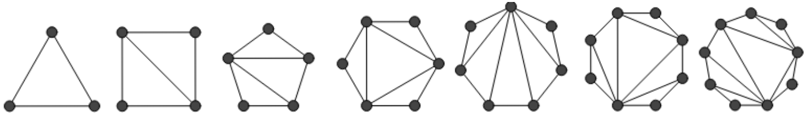
Corollary

Let c_n be the dilation of the regular n-gon, and $\Delta = \lim c_n$. Then $\Delta = \mathrm{dil}(\mathit{circle})$ and

$$|c_n - \Delta| = O\left(\frac{\log n}{n}\right).$$

Summary

For each n, let T_n be a triangulation of the regular n-gon with minimal dilation.



- **Dilations:**

$$\lim_{n\to\infty} \mathrm{dil}(T_n)$$

exists.

- **Triangulations:** A *subsequence* $(T_{\phi(n)})$ of (T_n) converges to a triangulation T of the circle.[1]

Furthermore:

$$\mathrm{dil}(T) = \mathrm{dil}(\text{circle}) = \mathrm{dil}(T_n) + O\left(\frac{\log n}{n}\right).$$

[1] The convergence is in the space $\mathcal{T}$ (choose an arbitrary lift for every $T_{\phi(n)}$).

Open problems

Question 1

How does the dilation of a curve change as the curve changes?

Question 2

Can we improve the bound 1.88 for the dilation of convex curves?

Question 3

For the regular polygons, is minimizing the dilation equivalent to minimizing the diameter of the graph? If not, how close are those two problems?

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Thank you for your attention!