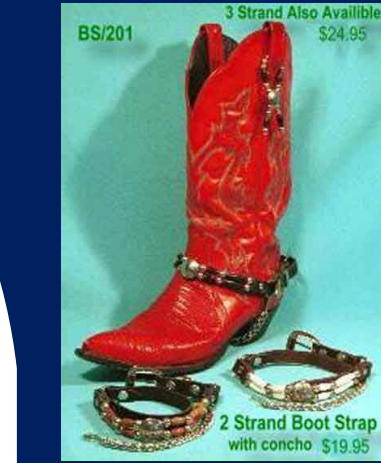
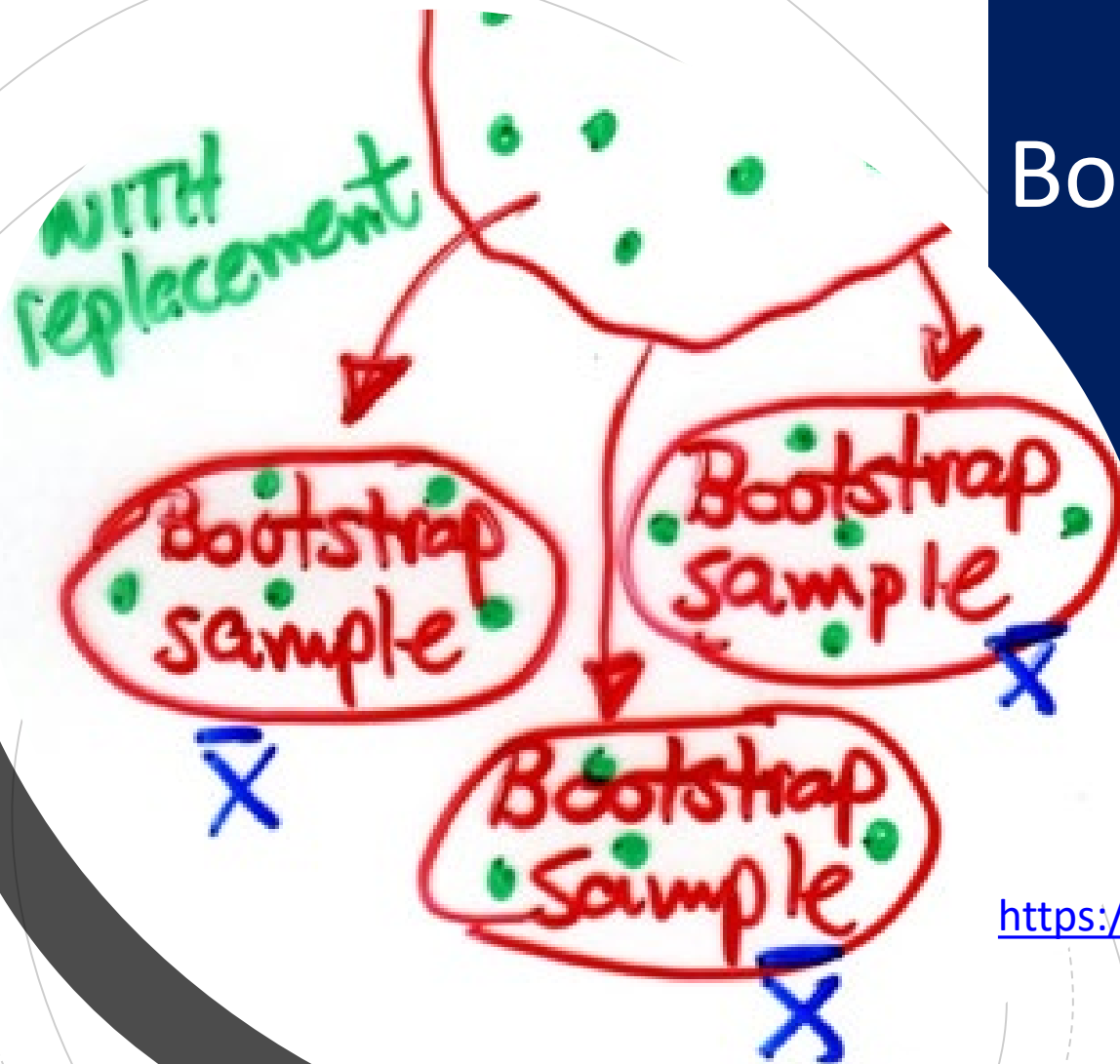


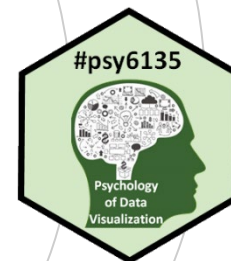
Bootstrapping



Michael Friendly

Psych 6135

<https://friendly.github.io/6135/>



Bootstrapping

- Classical statistical inference relies on
 - Distributional assumptions, e.g., $\varepsilon \sim N(0, \sigma^2)$
 - Asymptotic results, e.g., in SEM: $F_{ML} \sim \chi^2$ as $n \rightarrow \infty$
- Bootstrapping is a non-parametric approach to inference that substitutes **computation** for **assumptions**



← **Functional** bootstraps: help to pull you up from where you are (data), to where you want to be (reasonable conclusions)

bootstrap (v): help oneself, often through improvised means

← **Decorative** bootstraps: we don't need these

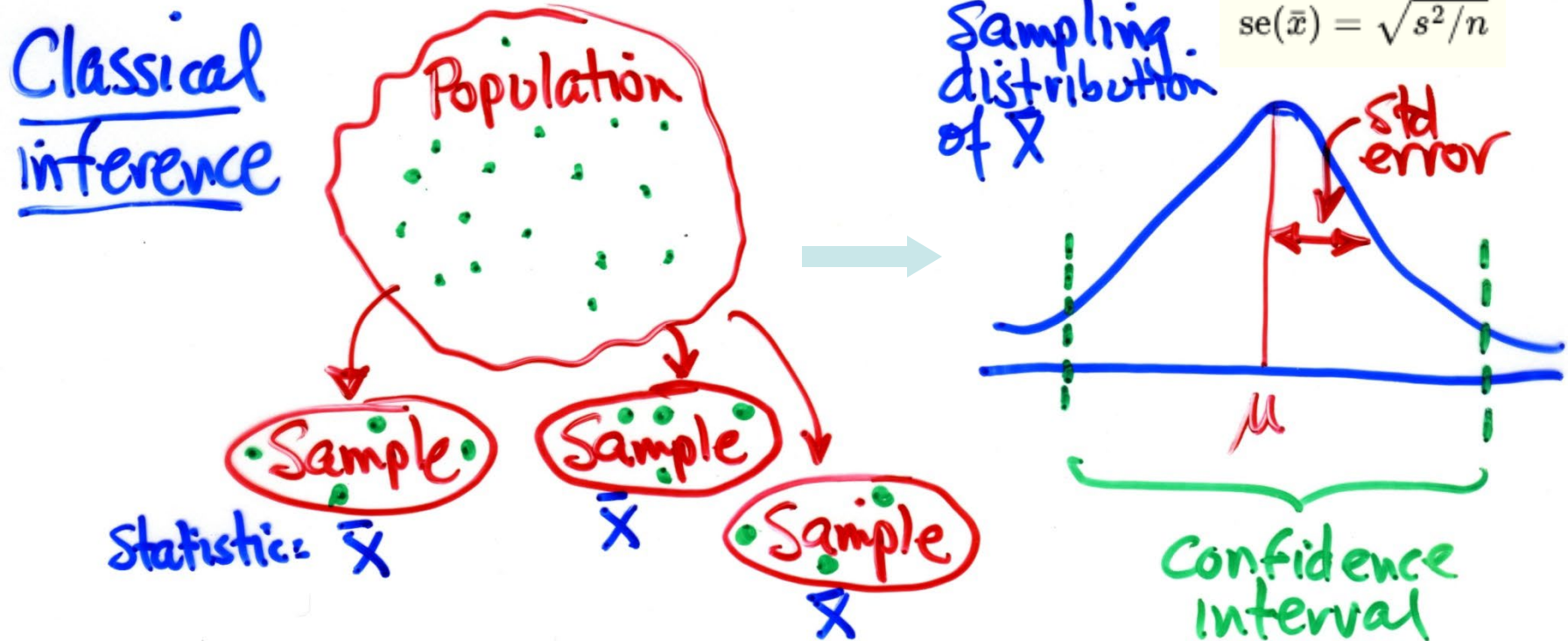
Bootstrapping

- Can provide more accurate inferences when data is badly behaved or *n* is small
 - linear models, SEM, ...
- Can be applied when *no sampling theory* is available
 - Tests of equality of ratios: $(y_1/x_1) =_? (y_2/x_2)$
 - fMRI studies: differences among patterns of brain activation
 - Shoeless Joe Jackson: how did he hit in clutch situations?
- Can be applied to complex data-collection plans (stratified/clustered samples)

More general ideas: Resampling

- The bootstrap is an example of the general idea of *resampling* from an original data set for statistical inference
- Other examples:
 - Jackknife: leave-one-out analysis
 - Cross-validation: choosing optimal model fitting parameters
 - Permutation tests: totally non-parametric
- Uses:
 - Std errors, CIs with small samples
 - Subset selection in linear models
 - Dealing with missing data
 - Complex algorithms: ML neural networks

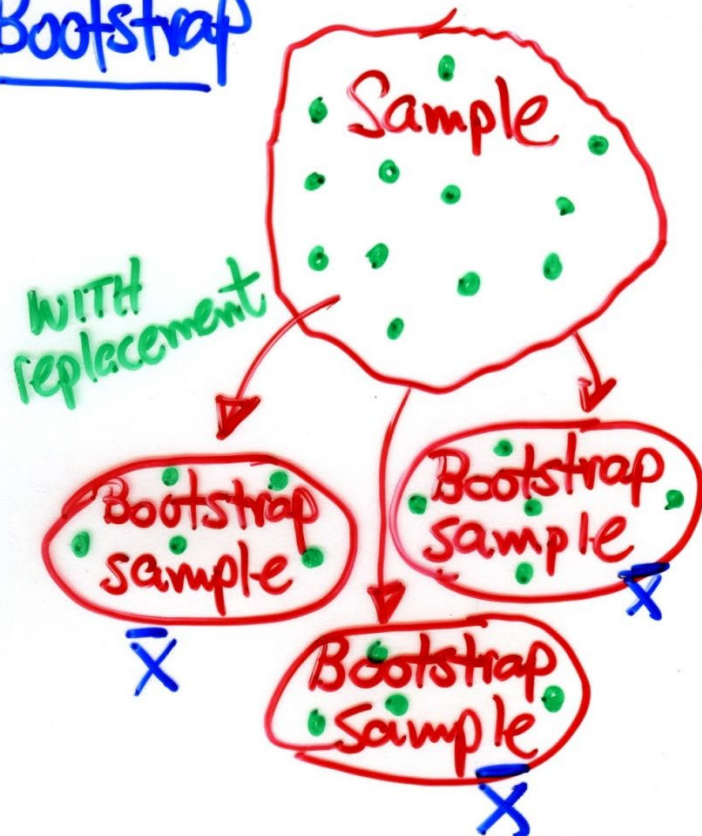
Classical statistical inference



Here, we rely on statistical theory (CLT) & assumptions (independence, normality, constant variance) to take us to the sampling distribution of the statistic of interest.

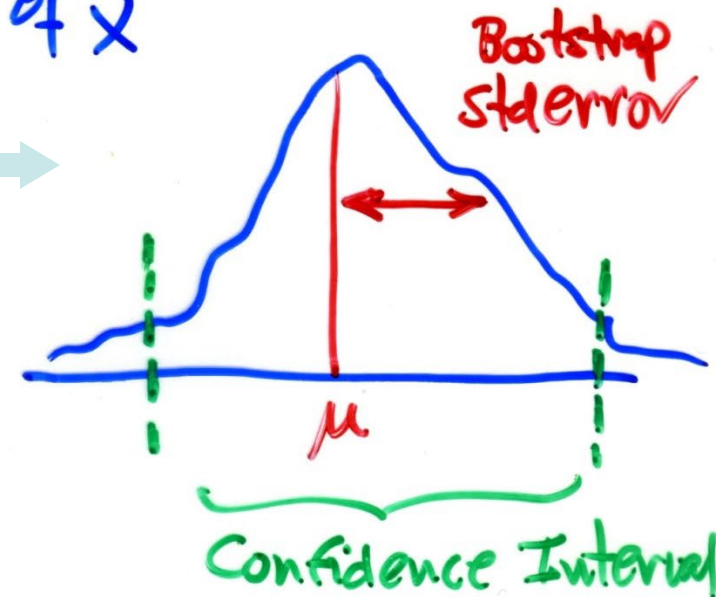
Bootstrap

Bootstrap



Bootstrap distribution
of \bar{x}

$$se(\bar{x}) = \sqrt{\text{Var}(\bar{x})}$$



Key idea:

Population is to the
sample

AS

Sample is to the
bootstrap sample

R Packages

ungevis Tools for visualizing uncertainty with ggplot2

- `bootstrapper()`
- `stat_smooth_draws()`
- Animations over bootstrap samples

rsample Tidy resampling methods

- Random, stratified, grouped resampling
- Cross validation (train, test)
- Bootstrapping
 - `bootstraps()`
 - `purrr::map()` over samples
 - CI methods
 - plots

Bootstrap resampling demo

```
# devtools::install_github("wilkelab/ungeviz")
library(ungeviz)
bs <- bootstrapper(3)      # create 3 draws
(draws <- bs(data.frame(letter = LETTERS[1:4])))
```

The bootstrapper function
creates a **function** to create
bootstrap samples
-- here 3 draws of 4 letters

```
# A tibble: 12 x 6
# Groups:   .draw [3]
  .draw .id .original_id letter .copies .row
  <int> <int>         <int> <chr>    <dbl> <int>
1     1     1             1 A         1     1
2     1     2             4 D         2     2
3     1     3             4 D         2     3
4     1     4             3 C         1     4
5     2     1             4 D         1     5
6     2     2             1 A         2     6
7     2     3             1 A         2     7
8     2     4             2 B         1     8
9     3     1             1 A         1     9
10    3     2             2 B         1    10
11    3     3             3 C         2    11
12    3     4             3 C         2    12
```

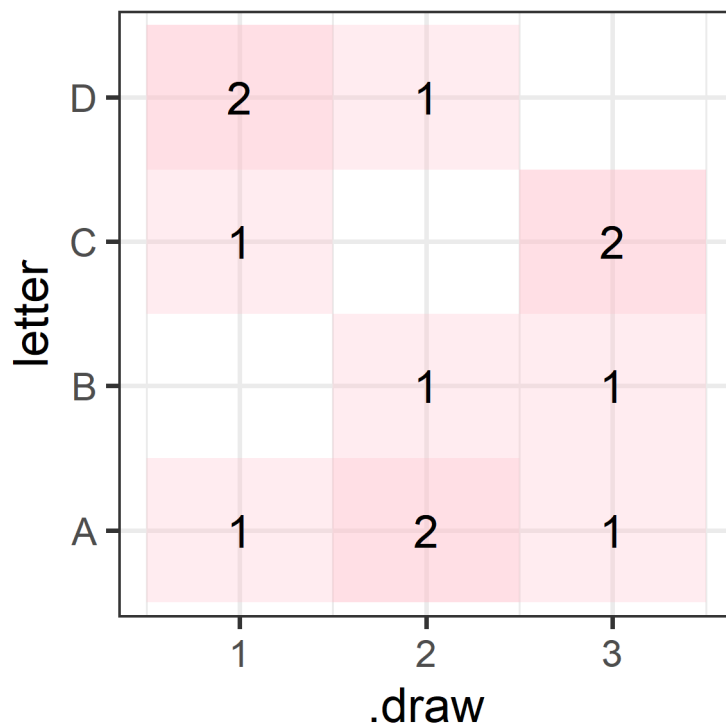
letter is the data
value.

Other variables
identify all aspects of
the bootstrap

Bootstrap resampling demo

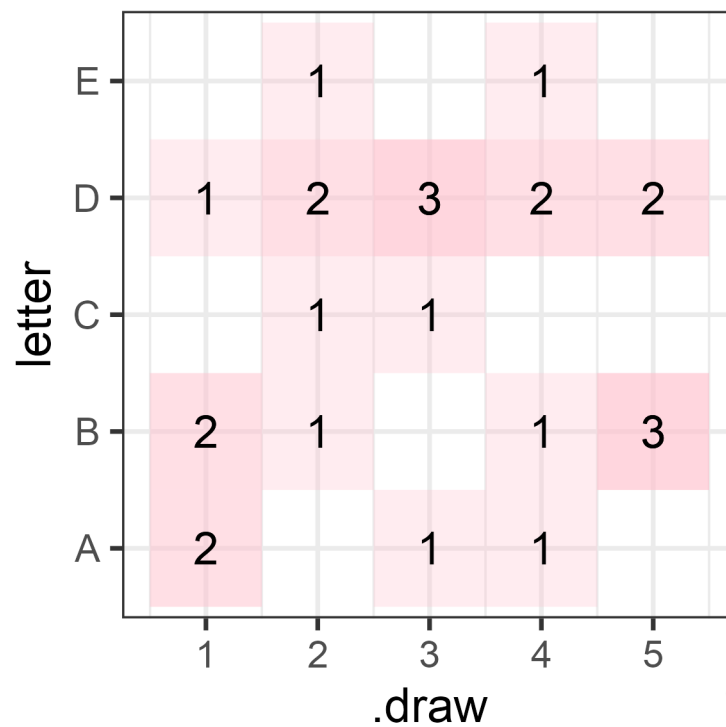
```
ggplot(draws, aes(x=.draw, y=letter)) +  
  geom_tile(fill="pink", alpha=0.3) +  
  geom_text(aes(label=.copies), size=6)
```

Each tile shows the number of times that letter was picked in a given .draw



The same for 5 draws of LETTERS[1:5]

```
bs2 <- bootstrapper(5)  
draws2 <- bs2(data.frame(letter = LETTERS[1:5]))
```

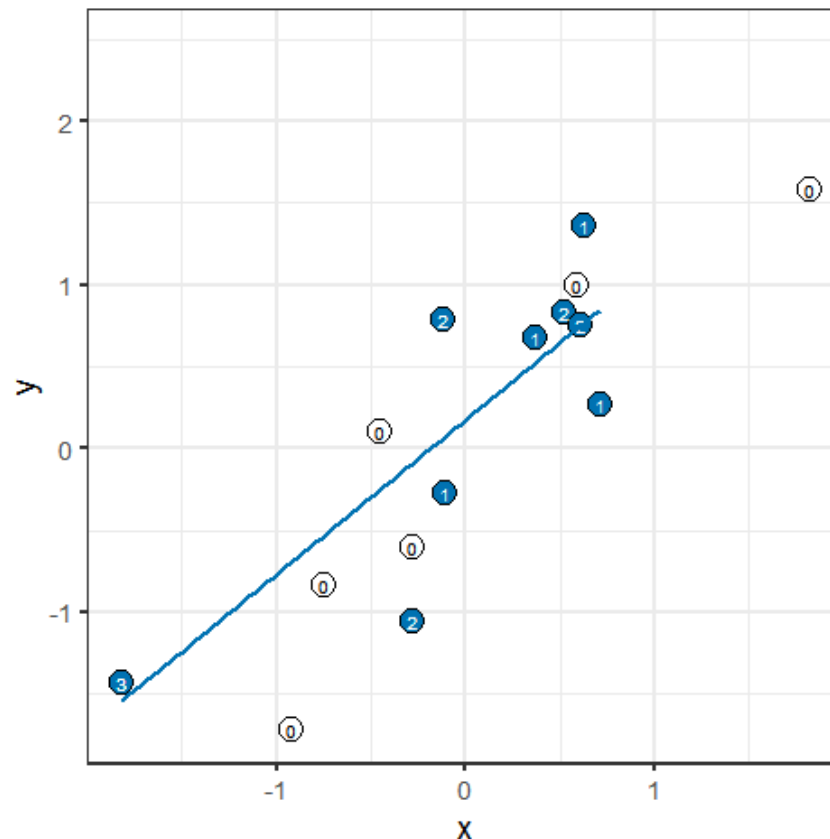


Regression illustration

```
# randomly generate dataset
set.seed(12345)
x <- rnorm(15)
df <- data.frame(x,
                 y = x + 0.5*rnorm(15))
# bootstrapper object
bsr <- bootstrapper(10)
```

Animated plot, by **.draw**:

```
ggplot(df, aes(x,y) +
  geom_point(data=bsr, aes(group= .row) +
  geom_text(data = bsr, aes(label = .copies) +
  geom_smooth(data = bsr, aes(group = .draw),
              method = "lm") +
  # animation
  transition_states(.draw, 1, 2) +
  enter_fade() + exit_fade())
```

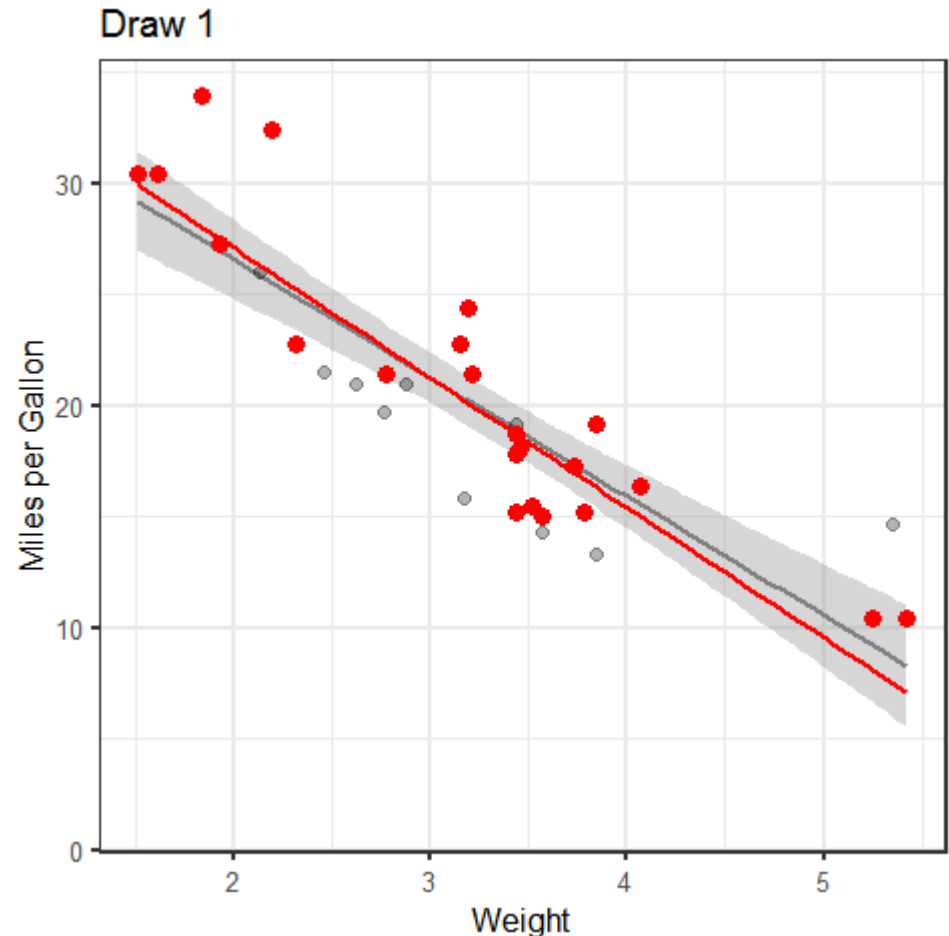


Bootstrapped confidence bands

The same method can be used to illustrate the uncertainty around the regression line, as reflected in the confidence band

However, the std conf. band is calculated using classical normal theory

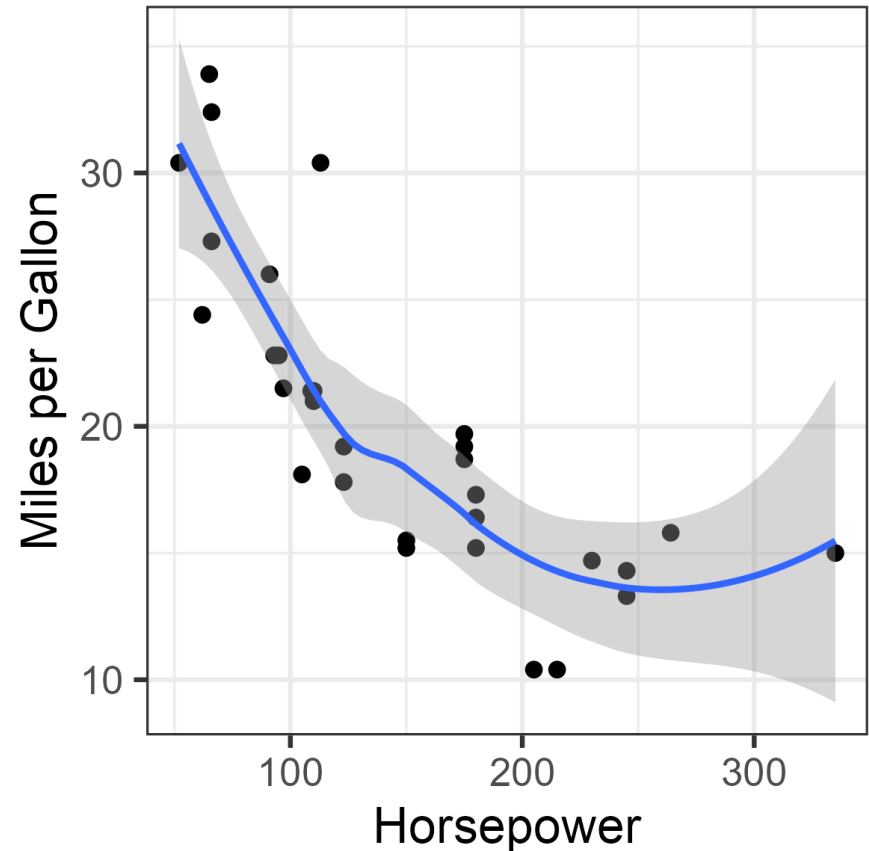
The bootstrapped fits trace out an empirical confidence band.



Non-linear relations: smoothing

We know how to use loess to estimate a non-parametric smoothed curve
There is also theory that allows calculation of a (approx.) confidence envelope

```
ggplot(mtcars, aes(hp, mpg)) +  
  geom_point(size = 2) +  
  geom_smooth(method = "loess") +  
  labs(x = "Horsepower",  
       y = "Miles per Gallon")
```

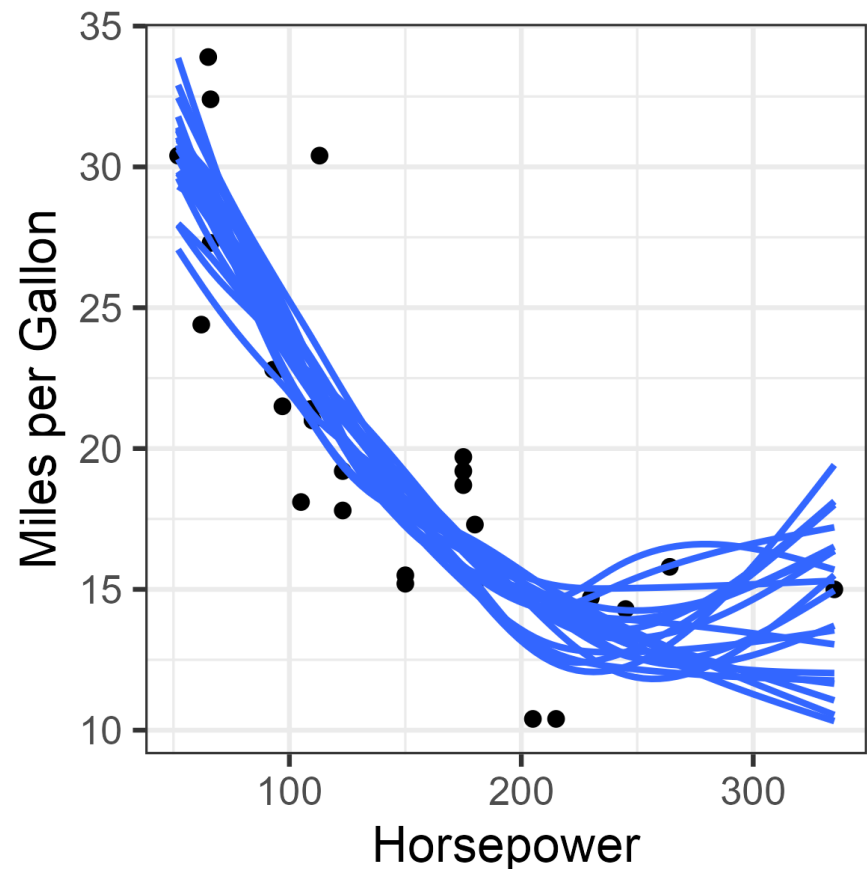


Resampling: smooth draws

Instead, resampling methods generate outcome draws from a smooth fit using `mgcv::gam()`. The collection of draws provide an empirical confidence envelope

```
plt <-  
ggplot(mtcars, aes(hp, mpg)) +  
  geom_point(size = 2) +  
  stat_smooth_draws(times = 20,  
    aes(group = stat(.draw)))
```

plt



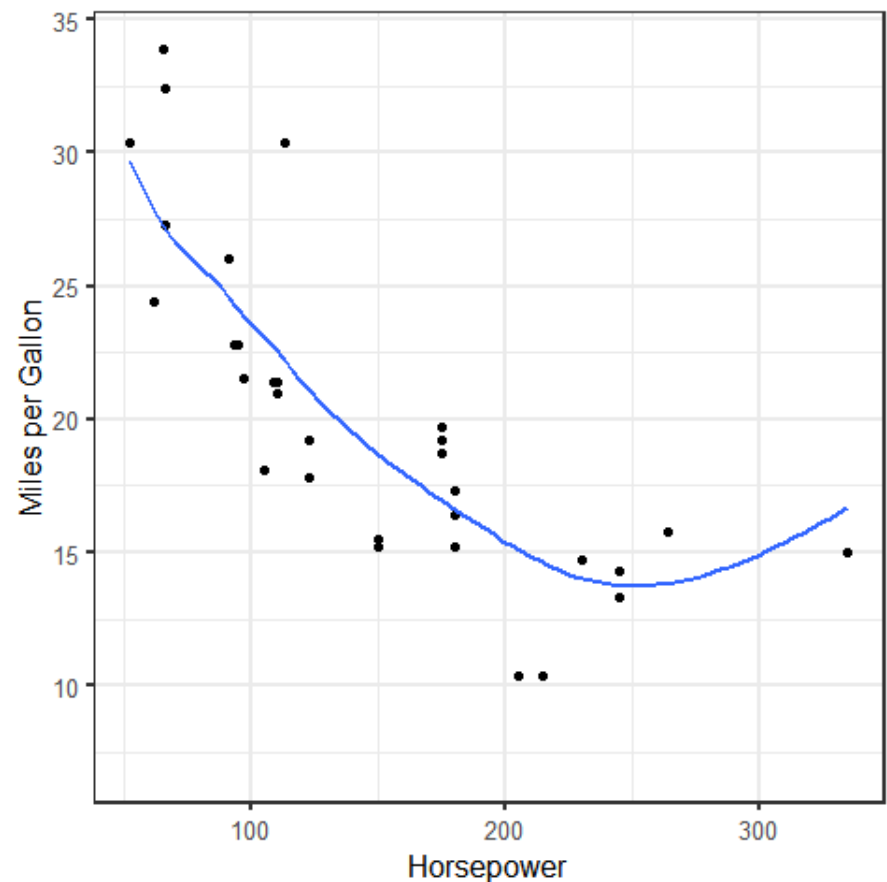
Resampling: smooth draws

Animation shows how the collection of sampled smooths develop over time

The animation transitions over draws (`.draw`)

`shadow_trail()` keeps the previous curves

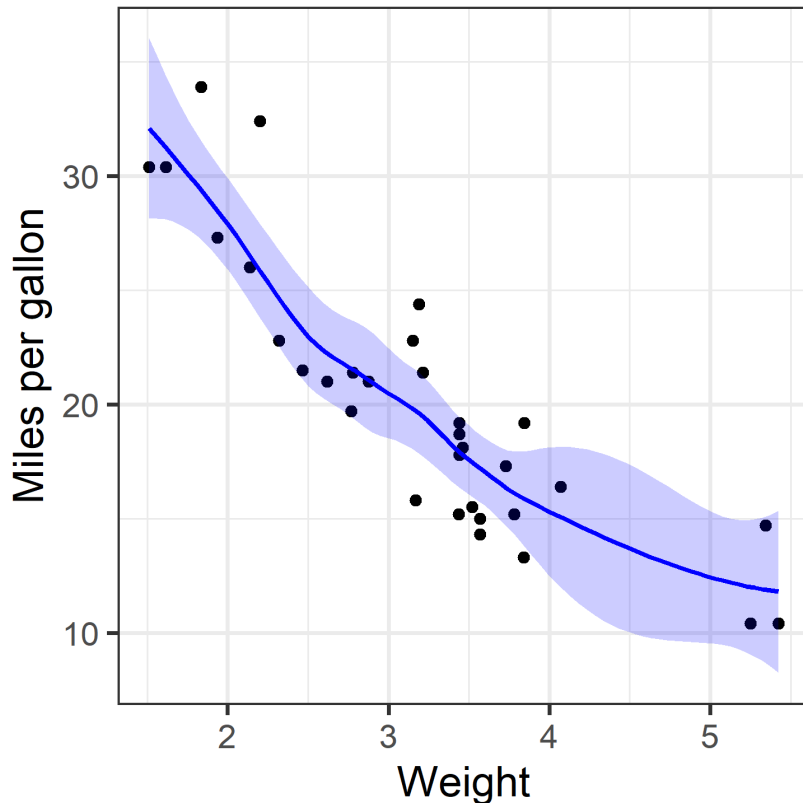
```
plt +  
  transition_states(stat(.draw)) +  
  enter_fade() +  
  exit_fade(alpha=0.8) +  
  shadow_trail()
```



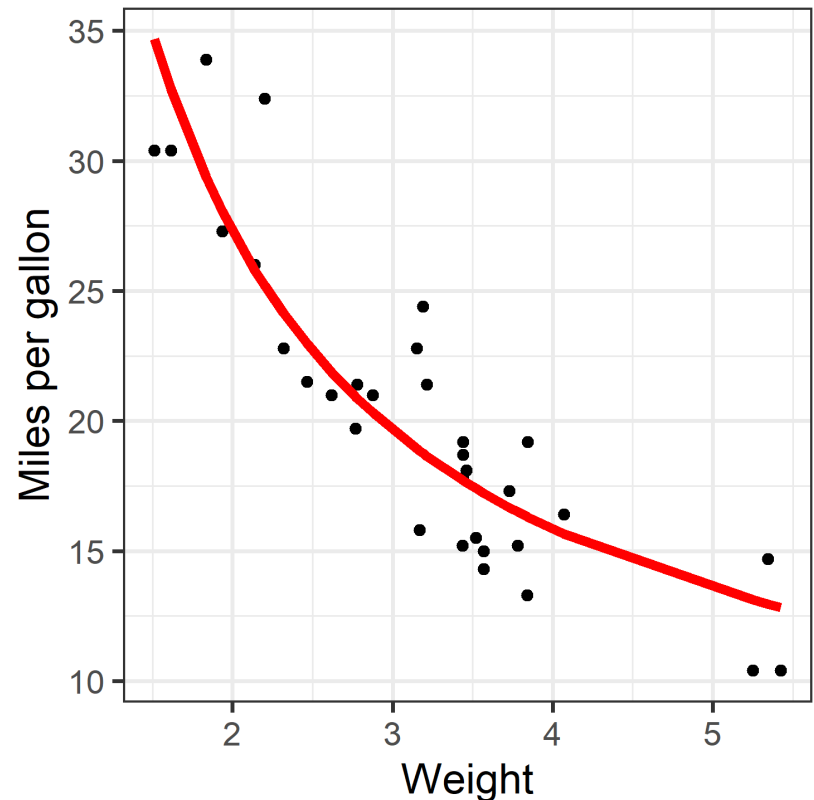
Bootstrapping models

Rather than fitting a nonparametric smoothed curve, we might want to fit a **parametric** but **nonlinear** model, perhaps for substantive interpretation

loess: nonparametric



An inverse relation: $y = \frac{k}{x} + b$



Nonlinear model: nls()

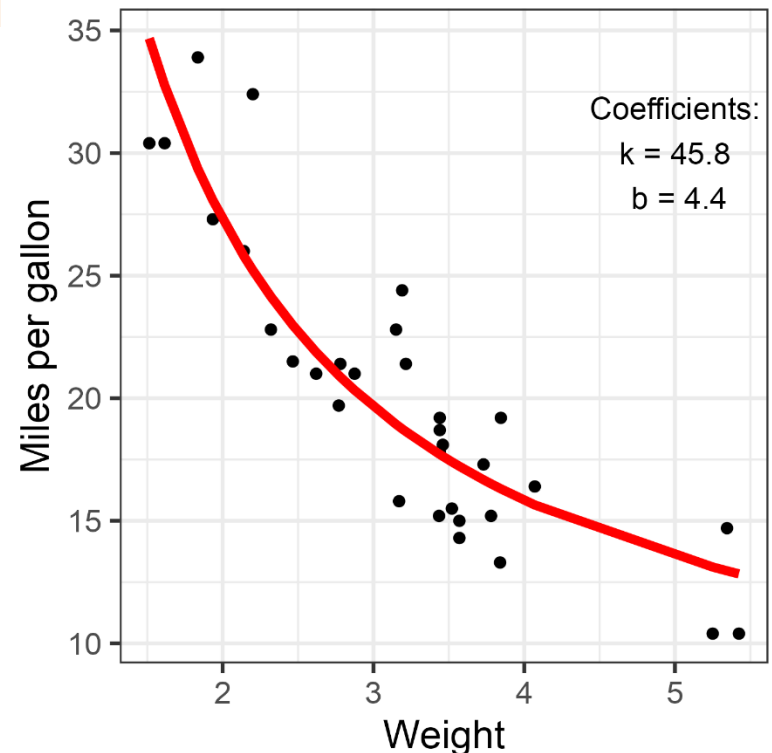
```
nlsfit <- nls(mpg ~ k / wt + b, mtcars,  
             start = list(k = 1, b = 0))  
summary(nlsfit)
```

Formula: $\text{mpg} \sim k/\text{wt} + b$

Parameters:

	Estimate	Std. Error	t value	Pr(> t)	
k	45.829	4.249	10.786	7.64e-12	***
b	4.386	1.536	2.855	0.00774	**

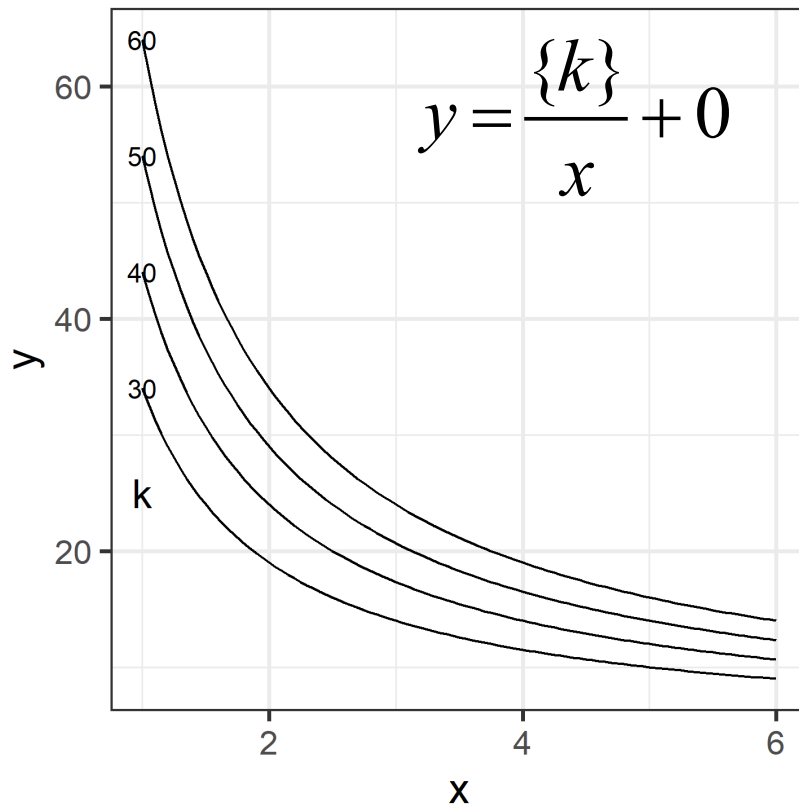
This uses `stats::nls()` to fit nonlinear models
There is also a `{nlstools}` package (that does bootstrapping)



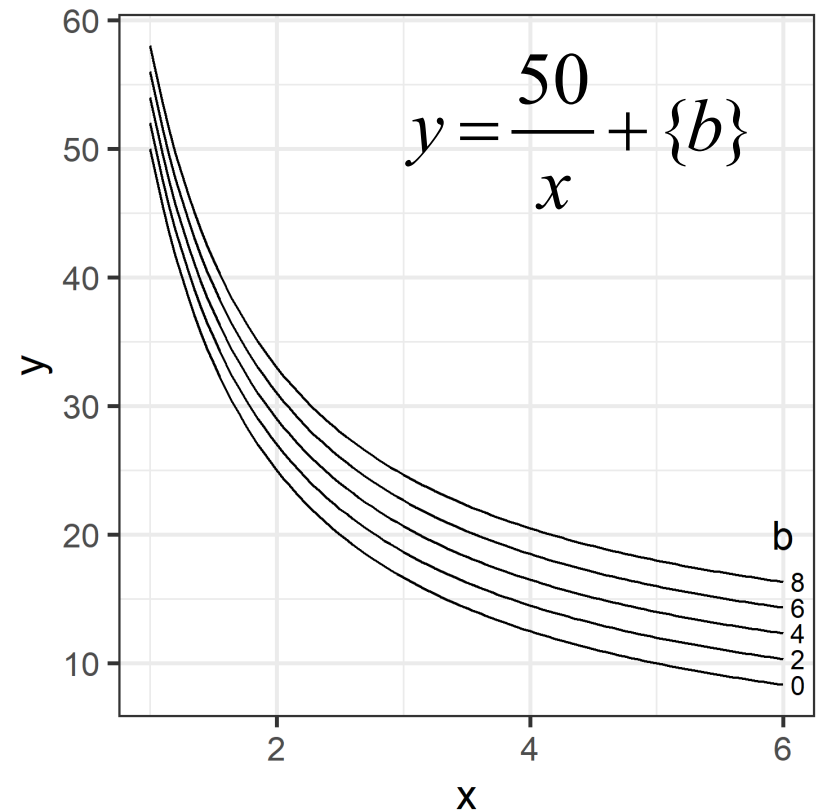
Inverse model

What are the parameters in this model?

k: starting value



b: asymptote



rsample package



```
set.seed(27)
boots <- bootstraps(mtcars, times = 500)
```

```
boots
# Bootstrap sampling
# A tibble: 500 x 2
  splits          id
  <list>         <chr>
1 <split [32/10]> Bootstrap001
2 <split [32/12]> Bootstrap002
3 <split [32/10]> Bootstrap003
4 <split [32/10]> Bootstrap004
5 <split [32/11]> Bootstrap005
6 <split [32/14]> Bootstrap006
7 <split [32/11]> Bootstrap007
8 <split [32/8]>  Bootstrap008
9 <split [32/11]> Bootstrap009
10 <split [32/13]> Bootstrap010
# ... with 490 more rows
```

Generate 'times' bootstrapped samples

{rsample} provides a more general approach, allowing cross-validation

For bootstrapping, each split[n/m] contains:

[n] sample with replacement
[m] items **not** selected in that sample

rsample package



Schema for bootstrapping using the rsample package

```
set.seed(42)
boots <- bootstraps(data, times = 2000)

# specify the model
mod <- as.formula(y ~ x1 + x2 + x3)

# function to fit for one <split>
fit <- function(split, ...) {
  glm(mod, data = analysis(split))
}

# run the bootstrap
boot_models <- boots |>
mutate(model = map(splits, fit),
       coefs = map(model, tidy))

# Confidence intervals
int_pctl(boot_models, coefs)
int_bca(boot_models, coefs)
int_t(boot_models, coefs)
```

Generate 'times' bootstrapped samples, indexed by splits

Scheme to fit the model for one bootstrap sample. `analysis()` extracts the data.

Generates a **nested** data structure containing <model>, <coefs> for each split

Functions to calculate confidence intervals from the bootstrapped models

Running the bootstrap

```
fit_nls<- function(split) {  
  nls(mpg ~ k / wt + b,  
      data= analysis(split),  
      start = list(k = 1, b = 0))  
}
```

Create a helper function to fit an nls() model on each bootstrap sample. `rsample::analysis()` extracts that sample.

```
boot_models <- boots |>  
  mutate(model = map(splits, fit_nls),  
         coef_info = map(model, tidy))
```

Use `purrr::map()` to apply this function to all the bootstrap samples at once. Similarly, create a column of tidy coefficients

```
boot_coefs <-  
  boot_models |>  
  unnest(coef_info)
```

Extract the coefficients for all models

Bootstrapped coefficients

The result is a nested data frame of coefficient statistics for each bootstrap sample

```
> boot_coefs
# A tibble: 1,000 x 8
  splits      id      model term estimate std.error statistic p.value
<list>    <chr>    <list> <chr>    <dbl>    <dbl>    <dbl>    <dbl>
1 <split [32/10]> Bootstrap001 <nls> k      47.1     3.49     13.5 2.74e-14
2 <split [32/10]> Bootstrap001 <nls> b       3.60    1.23     2.92 6.62e- 3
3 <split [32/12]> Bootstrap002 <nls> k      50.0     5.64     8.87 6.95e-10
4 <split [32/12]> Bootstrap002 <nls> b       3.29    2.09     1.57 1.26e- 1
5 <split [32/10]> Bootstrap003 <nls> k      42.0     4.38     9.59 1.20e-10
6 <split [32/10]> Bootstrap003 <nls> b       5.89    1.51     3.89 5.20e- 4
7 <split [32/10]> Bootstrap004 <nls> k      56.7     5.01    11.3 2.36e-12
8 <split [32/10]> Bootstrap004 <nls> b       1.49    1.75     0.852 4.01e- 1
9 <split [32/11]> Bootstrap005 <nls> k      48.6     3.22    15.1 1.48e-15
10 <split [32/11]> Bootstrap005 <nls> b       3.01    1.22     2.46 1.98e- 2
# ... with 990 more rows
```

From this we can find confidence intervals (& test hypotheses)

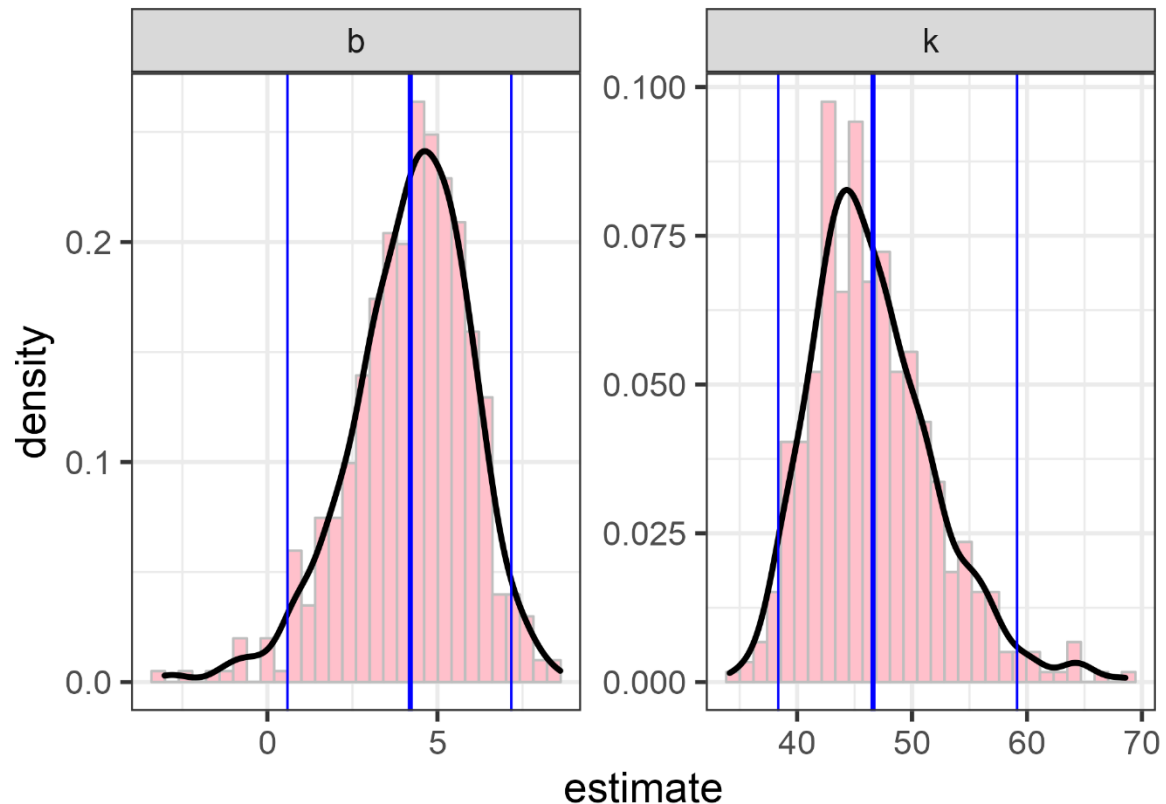
```
> int_pctl(boot_models, coef_info)
# A tibble: 2 x 6
  term .lower .estimate .upper .alpha .method
<chr> <dbl>    <dbl> <dbl> <dbl> <chr>
1 b     0.312     4.20   7.04   0.05 percentile
2 k    38.0     46.4  59.0   0.05 percentile
```

Percentile intervals use the (.025, .975) quantiles, but require >1000 samples

Bootstrapped distributions

```
ggplot(boot_coefs, aes(estimate)) +  
  geom_histogram(aes(y = ..density..),  
    bins = 30, fill="pink", color="gray") +  
  geom_density(size = 1.2) +  
  facet_wrap( ~ term, scales = "free") + ...
```

Plots of bootstrapped coefficients
show their shape
-- not quite normal as assumed by
std theory



Scatterplot of coefficients

Finally, a fancy scatterplot of the joint distribution of the (b, k) estimates

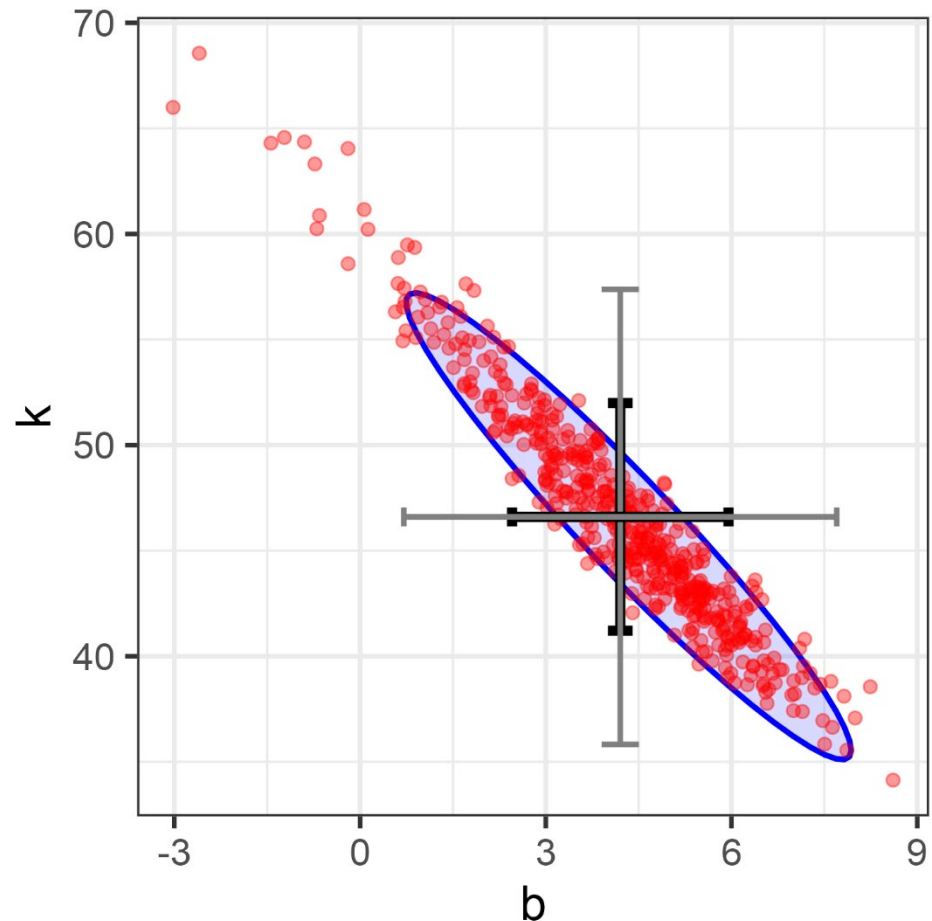
How did I do this?

Processing:

1. spread coefs -> wide to plot $k \sim b$
2. find means, se of b & k

Plotting:

1. ellipse: `stat_ellipse()`
2. `geom_point()` – after ellipse!
3. `geom_errorbar()`: `se * (1, 2)`



1. pivot wider

```
boot_coefs_wide <- boot_coefs %>%  
  select(id, term, estimate) %>%  
  tidyr::pivot_wider(  
    names_from = term,  
    values_from = estimate)
```

```
> boot_coefs_wide  
# A tibble: 500 x 3  
   id          k      b  
   <chr>    <dbl> <dbl>  
1 Bootstrap001  47.1  3.60  
2 Bootstrap002  50.0  3.29  
3 Bootstrap003  42.0  5.89  
4 Bootstrap004  56.7  1.49  
5 Bootstrap005  48.6  3.01  
6 Bootstrap006  42.7  4.46  
7 Bootstrap007  49.1  3.56  
8 Bootstrap008  49.6  3.19  
9 Bootstrap009  51.8  2.66  
10 Bootstrap010  54.0  1.94  
# ... with 490 more rows
```

2. find means , se of b & k

```
mean_se <- boot_coefs_wide %>%  
  summarise(  
    sk = sd(k),  sb = sd(b),  
    k = mean(k), b = mean(b))
```

```
> mean_se, digits=4  
   sk    sb    k    b  
1 5.511 1.737 46.37 4.204
```



```
boot_coefs_wide %>%
  ggplot(aes(b, k)) +
    stat_ellipse(level = .95,
                 geom = "polygon", alpha = 0.15,
                 fill = "blue", color = "blue") +
    geom_point(color = "red", size=2, alpha=0.4) +
```

①

②

Error bars

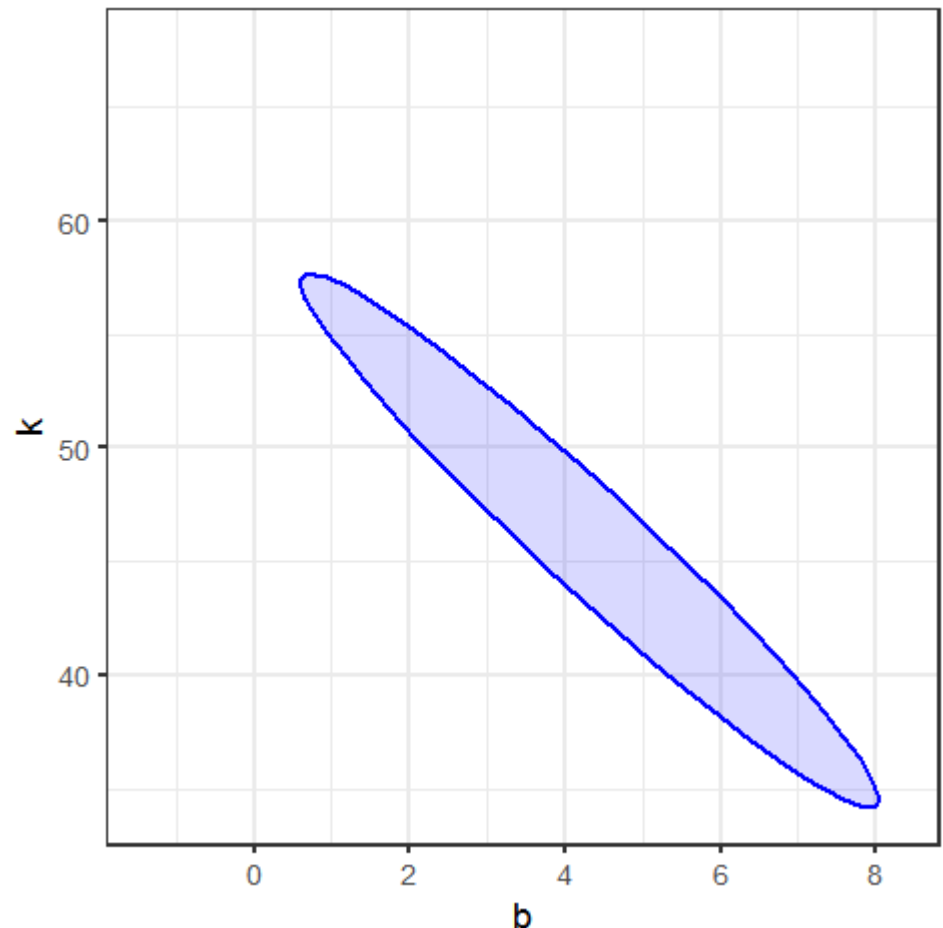
```
geom_errorbar(data = mean_se,
              aes(ymin = k - sk,
                  ymax = k + sk, x = b), size=2) +
```

③

```
geom_errorbarh(data = mean_se,
               aes(xmin = b - sb,
                   xmax = b + sb, y = k), size=2) +
```

Redraw error bars at $m \pm 2$ sd, but thinner

④



Visualize the fitted curves

```
boot_aug <-  
  boot_models %>%  
  sample_n(200) %>%  
  mutate(augmented =  
    map(model, augment)) %>%  
  unnest(augmented)
```

```
ggplot(boot_aug, aes(wt, mpg)) +  
  geom_line(aes(y = .fitted, group = id),  
    alpha = 0.1) +  
  geom_line(data=mtcars,  
    aes(x = wt, y = predict(nlsfit)), color="red") +  
  geom_point() +  
  labs(x = "Weight", y = "Miles per gallon")
```

Use `augment()` to visualize the uncertainty in the fitted curve

Use `sample_n()` to plot only 200

