

Visualizing Uncertainty



Michael Friendly

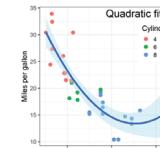
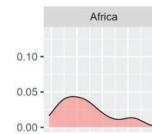
Psych 6135

<https://friendly.github.io/6135/>



Topics

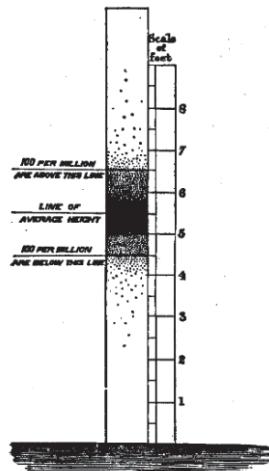
- Uncertainty in statistics & visualization
- Visualizing distributions
- “Error bars”
- Bayesian uncertainty
- Uncertainty in fitted curves
- Hypothetical outcome plots
- Cartographic uncertainty



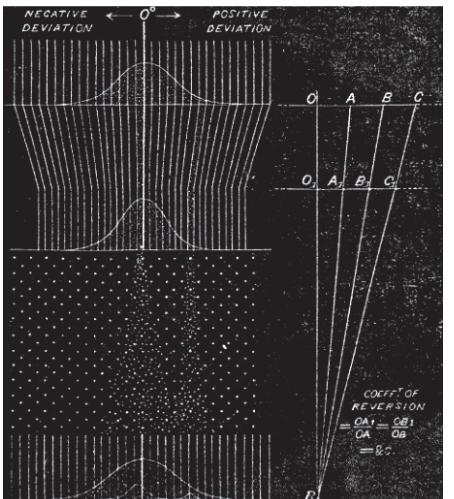
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Galton: Variation & Uncertainty

Distribution of human height
(*Hereditary Genius*, 1869)



Quincunx: How many small effects → Normal



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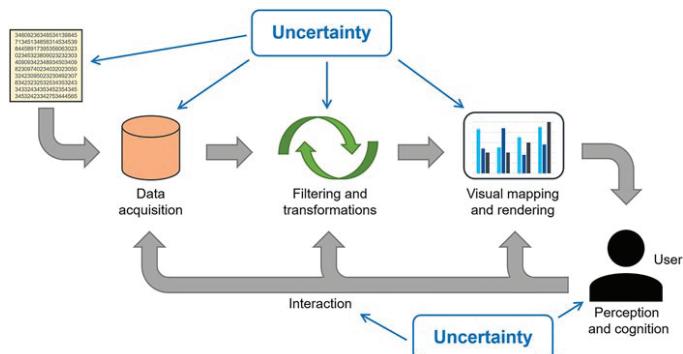
Sources of uncertainty

- Where does the uncertainty in statistics come from? There are four main sources:
 - **Data:** data can contain random error or have missing entries.
 - **Assumptions:** model assumptions assess plausible values from distributions.
 - **Models:** there is choice over the techniques and models we use.
 - Different analysts may choose different methods, yielding different estimates.
 - **Replications:** Estimates of effects can vary from study to study. How to synthesize these?

See: [Uncertainty Toolkit, Ch 3](#) for other terms to understand uncertainty

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Where does uncertainty come from?



From: D. Weiskopf, <https://www.frontiersin.org/articles/10.3389/fbinf.2022.793819/full>

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Problems: data, models, graphics

- Uncertainty is fundamental to data analysis & models
 - **data:** IQR, std dev., std error, ... (variation)
 - **assumptions:** we assume some distribution for errors, e.g., $\varepsilon \sim \mathcal{N}(0, \sigma^2)$, independent with constant variance
 - **models:**
 - classical: confidence intervals, p-values;
 - Bayesian: credible intervals, posterior distributions
- In data graphics,
 - Easy to show “fit” – means, regression estimates, ...
 - Harder to show the uncertainty in these numbers

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P-values, significance & uncertainty

ASA President’s Task Force on Statistical Significance

- “Much of the controversy surrounding **statistical significance** can be dispelled by better understanding of uncertainty, variability, multiplicity & replicability”
- “Different measures of uncertainty can **complement** each other; no single measure serves all purposes”
- “Controlling and accounting for uncertainty begins with the **design** of the study”
- “The **theoretical basis** of statistical science offers general strategies for dealing with uncertainty”
 - **Frequentist** approach: p-values, confidence intervals & prediction intervals
 - **Bayesian** approach: Bayes factors, posterior probability distributions, credible intervals

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Model fits as uncertainties

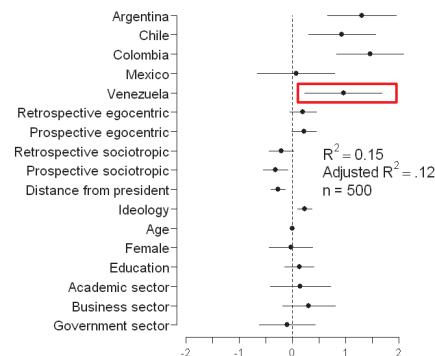
Table 2 from Stevens (2006): Determinants of Authoritarian Aggression

Variable	Coefficient (Standard Error)
Constant	.41 (.93)
Countries	
Argentina	.131 (.33)*** B.M
Chile	.93 (.32)*** B.M
Colombia	1.46 (.37) *** B.M
Mexico	.87 (.32) *** C.O.V
Venezuela	-.96 (.35) *** B.M
Threat	
Retrospective egocentric economic perceptions	-.20 (.13)
Prospective egocentric economic perceptions	-.22 (.12)*
Retrospective sociotropic economic perceptions	-.21 (.12)*
Prospective sociotropic economic perceptions	-.32 (.12)*#
Ideological Distance from president	
Ideology	
Ideology	.23 (.07) ***
Individual Differences	
Age	.00 (.01)
Female	-.03 (.21)
Education	.13 (.14)
Academic Sector	.15 (.29)
Business Sector	.31 (.25)
Government Sector	-.10 (.27)
R ²	.15
Adjusted R ²	.12
n	500

***p < .01, **p < .05, *p < .10 (two-tailed)
 A Coefficient is significantly different from Argentina's at p < .05;
 B Coefficient is significantly different from Brazil's at p < .05;
 C# Coefficient is significantly different from Chile's at p < .05;
 CO Coefficient is significantly different from Colombia's at p < .05;
 M Coefficient is significantly different from Mexico's at p < .05;
 V Coefficient is significantly different from Venezuela's at p < .05;

Coefficients** & std errors express uncertainty

Can we do better?

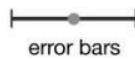


Source: tables2graphs.com

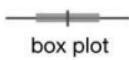
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Graphical annotations for uncertainty

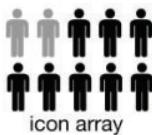
Intervals and Ratios



error bars



box plot



icon array

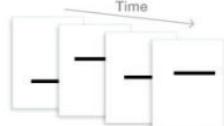
Distributions



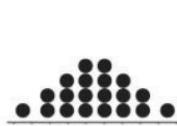
violin plot



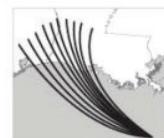
gradient plot



hypothetical outcome plot



quantile dot plot



ensemble plot

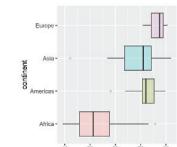
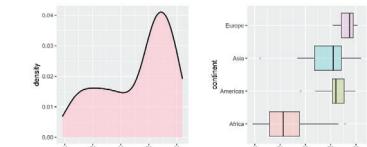
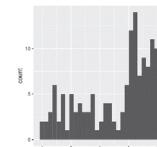
From: Padilla, Kay & Hullman (2021), *Uncertainty Visualization*, DOI: 10.1002/9781118445112.stat08296

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Visualizing distributions

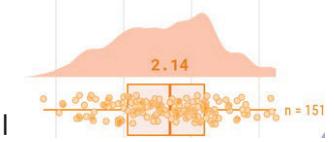
The basics:

- Histograms
- Density plots
- Boxplots



Doing better:

- violin plots
- rainclouds
- {ggdist}: data, distribution, interval



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Histograms

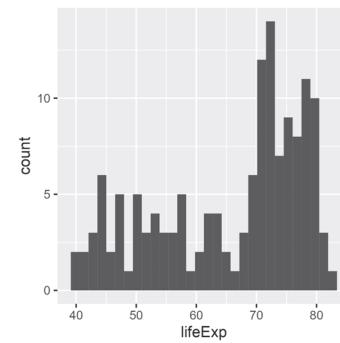
Perhaps the simplest display

- divide the data into **bins**: [40-42), [42-44), ...
- bar plot of the frequencies: length \sim frequency in bin

```
library(gapminder)
gapminder_2002 <- gapminder %>%
  filter(year == 2002)
```

```
ggplot(gapminder_2002,
       aes(x = lifeExp)) +
  geom_histogram()
```

`stat_bin()` using `bins = 30`. Pick better value with `binwidth`.



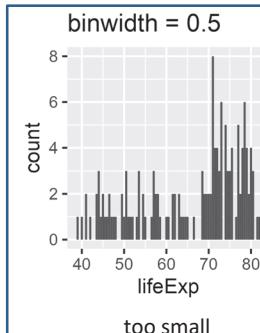
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Histograms: bin width

Explicitly selecting the binwidth shows:

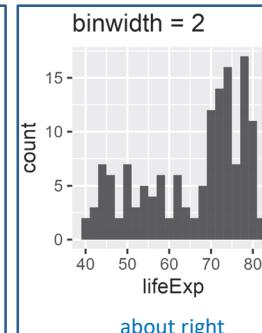
- the Goldilocks principle
- the default is often OK, but optimal “best” is harder to define

```
ggplot(gapminder_2002, aes(x = lifeExp)) + geom_histogram(binwidth = )
```



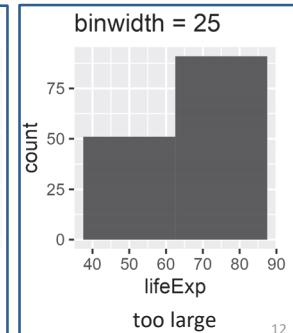
binwidth = 0.5

too small



binwidth = 2

about right



binwidth = 25

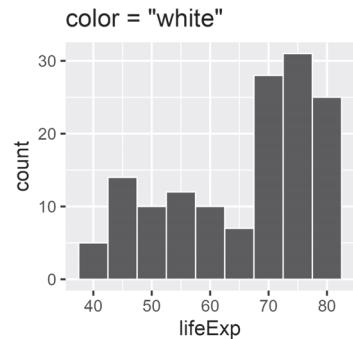
too large

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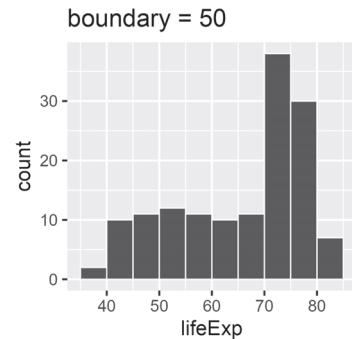
Histograms: other properties

- Pay attention to graphic details
 - border color to make bars distinct
 - set bar boundaries: to edges? – it can make a difference

`geom_histogram(..., color = "white")`



`geom_histogram(..., boundary=50)`

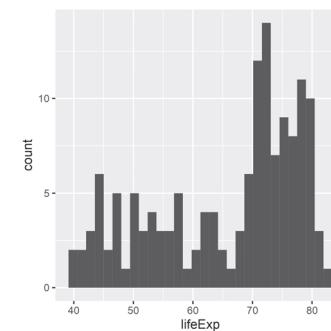


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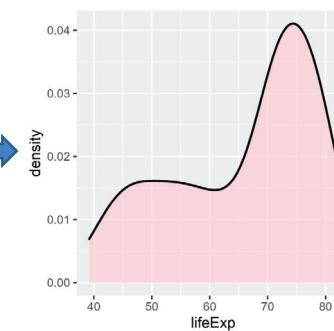
Density plots

- Basic idea: Smooth the distribution to avoid artifacts of discrete bins and bin centers
 - Uses a “kernel”, e.g, gaussian, averaged over a moving window

`geom_histogram()`



`geom_density()`



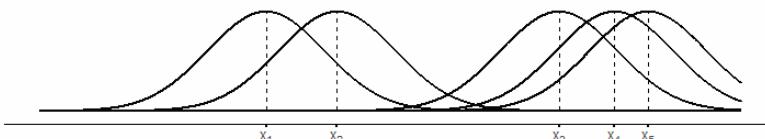
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Kernel density estimation

Imagine a distribution of potential density centered at each X_i , w/ $sd = h$ (**bandwidth**)
 $x \sim \mathcal{N}(\mu=X_i, \sigma=h)$

Five observations, each with a distribution

A **moving window** sweeps across, averaging the density for all observations

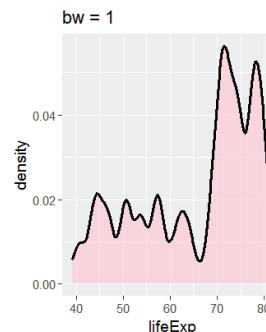


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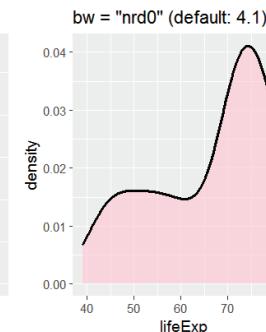
Density plots: bandwidth

- The result depends on the width of the moving window – **bandwidth**
 - The default calculation is usually reasonable, but beware of weird data

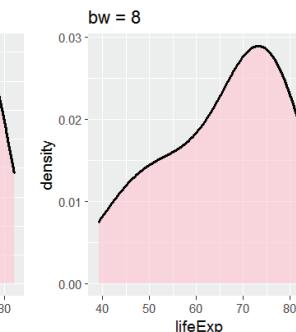
`geom_density(bw=1)`



`geom_density()`



`geom_density(bw=8)`



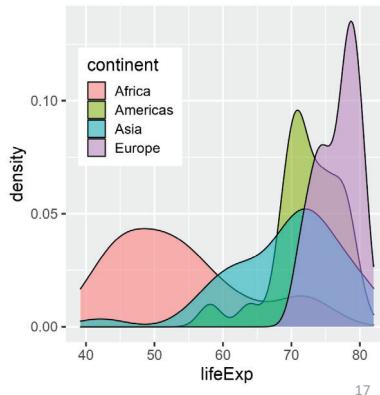
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Comparing groups

For multiple groups, using the fill aesthetic → overlaid curves -- is a decent start
But even with transparency it may be hard to see the separate curves

```
gap_2002c <-  
  gapminder_2002 %>%  
  filter(continent != "Oceania")  
  
ggplot(gap_2002c,  
       aes(x = lifeExp,  
            fill = continent)) +  
  geom_density(alpha = 0.5) +  
  theme(legend.position = c(.2, .7))
```

NB: ggplot picks a **joint** bandwidth, here: 2.52

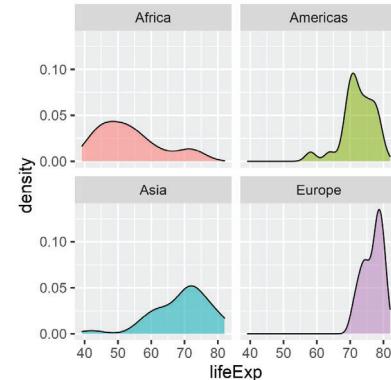


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Comparing groups: Facets

Faceting solves the overlap problem, but the eye has to move from panel to panel to make comparisons.

```
ggplot(gap_2002c,  
       aes(x = lifeExp,  
            fill = continent)) +  
  geom_density(alpha = 0.5) +  
  facet_wrap(~ continent) +  
  theme(legend.position = "none")
```



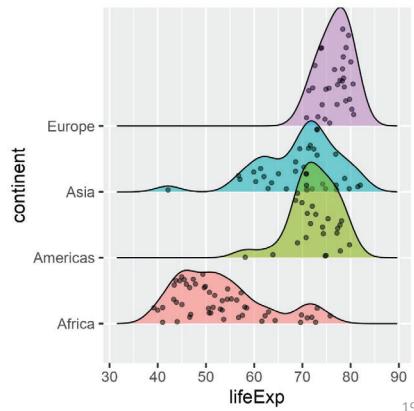
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{ggridges}: Ridgeline plots

Ridgeline plots are partially overlapping density plots, suggesting a mountain range.

- Useful for comparing distributions over time or circumstances
- Adding jittered points helps to show where the data are

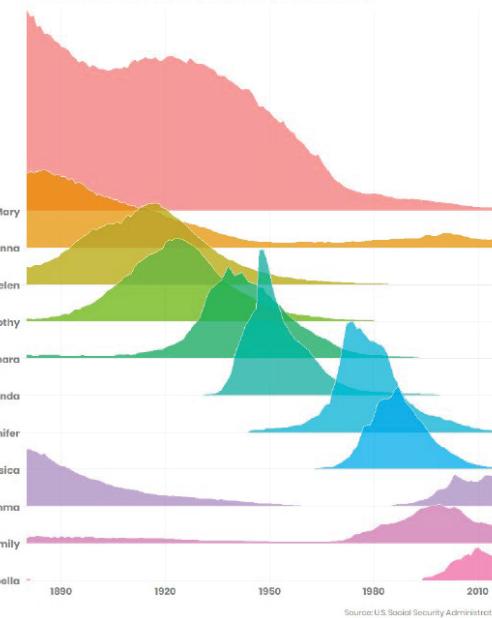
```
library(ggridges)  
ggplot(gap_2002c,  
       aes(x=lifeExp,  
            y=continent,  
            fill=continent)) +  
  geom_density_ridges(  
    alpha = 0.5,  
    jittered_points=TRUE) +  
  theme(legend.position = "none")
```



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Most popular girl names in the U.S.

Top 2 names with the highest mean and/or maximum per quarter are shown.



Baby names

Ridgeline plots are particularly effective with more than a few categories, and when the distributions differ in shape as well as central location

Which names stand out from the rest?

What is the role of color here?

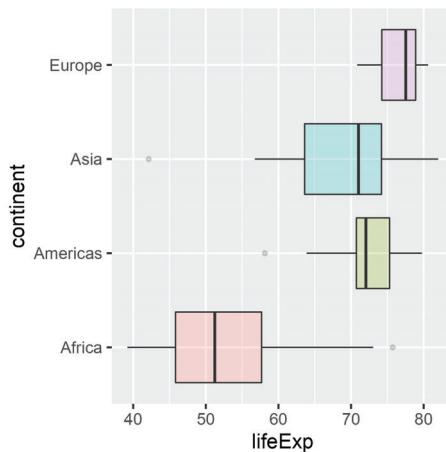
Note the subtle use of white to outline each distribution

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Boxplots

Boxplots give a more schematic summary of a dataset— median, quartiles, whiskers & outliers

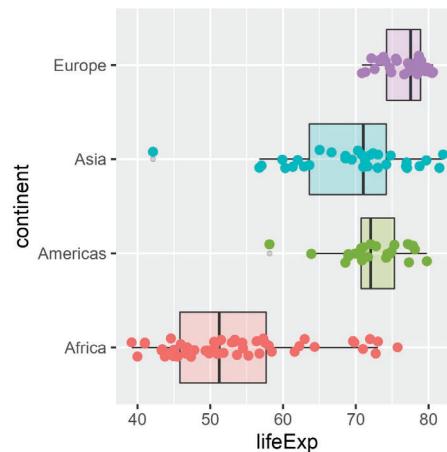
```
ggplot(gap_2002c,  
       aes(x=lifeExp,  
            y=continent,  
            fill=continent)) +  
  geom_boxplot(alpha = 0.2) +  
  theme(legend.position = "none")
```



Boxplots

But perhaps too schematic— it sometimes helps to see the data as jittered points

```
ggplot(gap_2002c,  
       aes(x=lifeExp,  
            y=continent,  
            fill=continent)) +  
  geom_boxplot(alpha = 0.2) +  
  geom_point(aes(color = continent),  
             position =  
               position_jitter(height=0.1)) +  
  theme(legend.position = "none")
```

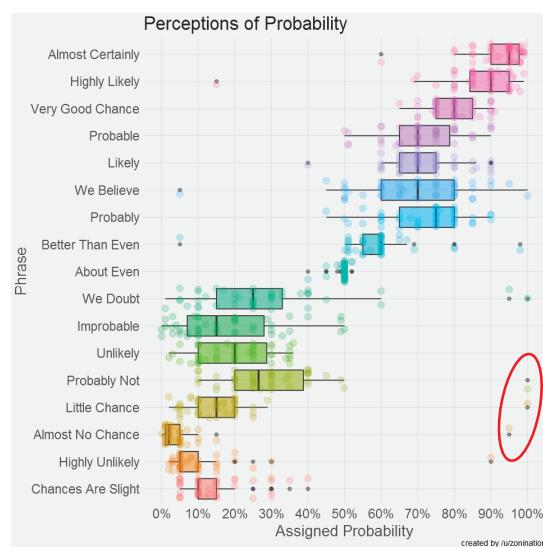


How people view “probability”

What makes this graph successful?

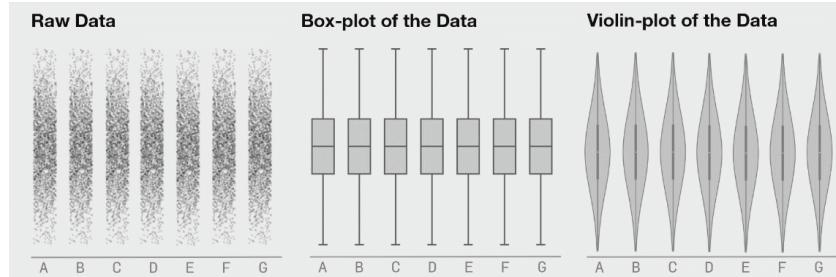
Note the wide range of variability (uncertainty) in the estimates:
“about even” vs. “we believe”

Outliers: individuals who misunderstood instructions?



Problems with boxplots revealed

Boxplots are fine for unimodal distributions – well summarized by Q1, Median, Q3
They are insensitive to **multi-modal** data

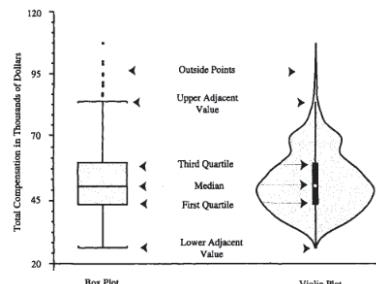


Violin plots

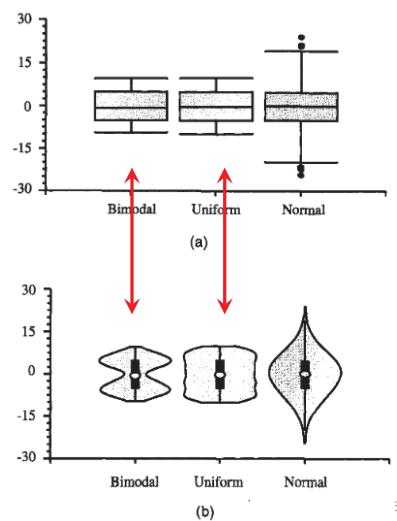
Boxplots are great for ~ normal data

- Shows center, spread, outliers

Violin plots add a (reflected) density curve to show the **shape** of the distribution



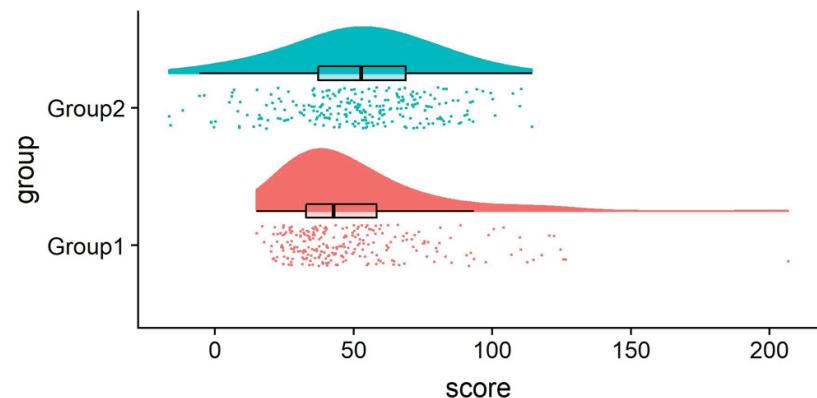
Hintze & Nelson (1998), *American Statistician*, 52:2, 181-184



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Raincloud plots

Raincloud plots combine **density curve** & **boxplot**, but also show the observations as jittered **points**



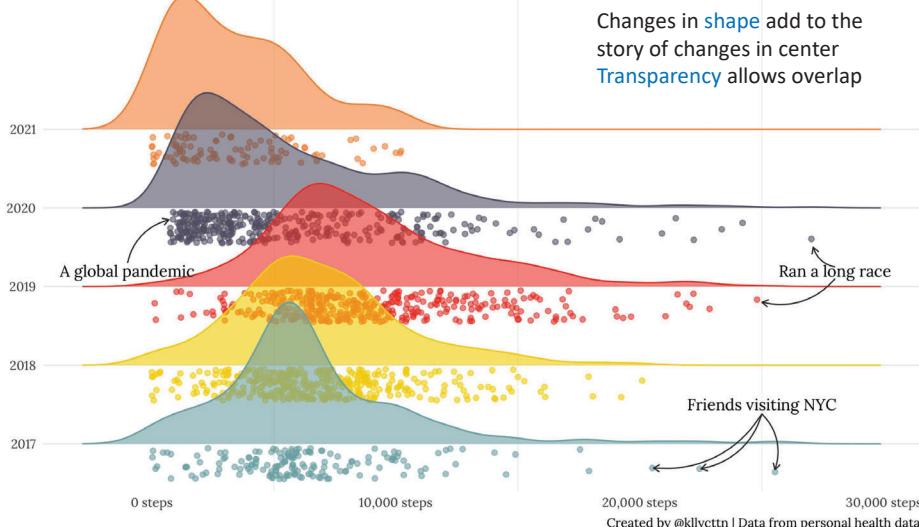
Allen M, Poggiali D, Whitaker K et al. Raincloud plots: a multi-platform tool for robust data visualization [version 2]. Wellcome Open Res 2021, 4:63 (doi: 10.12688/wellcomeopenres.15191.2)

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How many steps have I taken since 2017?

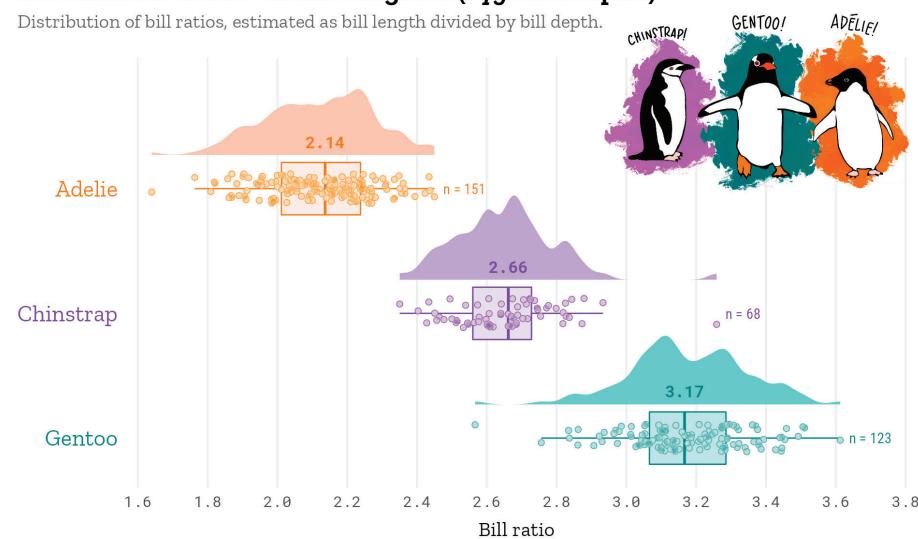
Since July 2017, I have tracked the number of steps I've taken (almost) every day. In a little over 4 years, I have taken 9,232,798 steps. This includes days spent walking around New York with visiting friends, running a half-marathon, and a pandemic that dropped my step count to nearly 0.

Changes in **shape** add to the story of changes in center
Transparency allows overlap



Bill Ratios of Brush-Tailed Penguins (*Pygoscelis* spec.)

Distribution of bill ratios, estimated as bill length divided by bill depth.



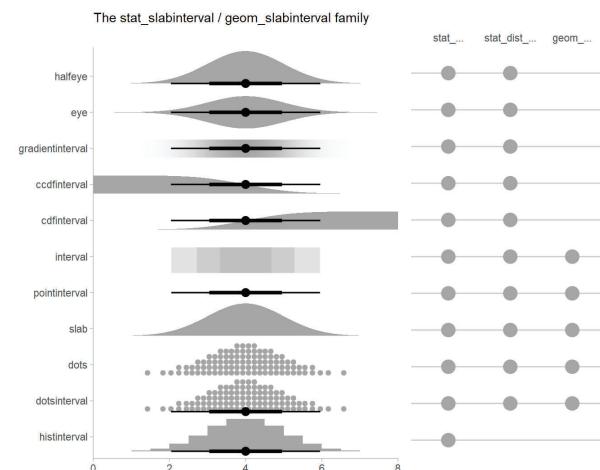
Graphical excellence!

Gorman, Williams & Fraser (2014) PLoS ONE DOI: 10.1371/journal.pone.0090081
Visualization: Cédric Scherer • Illustration: Alison Horst

{ggdist} package

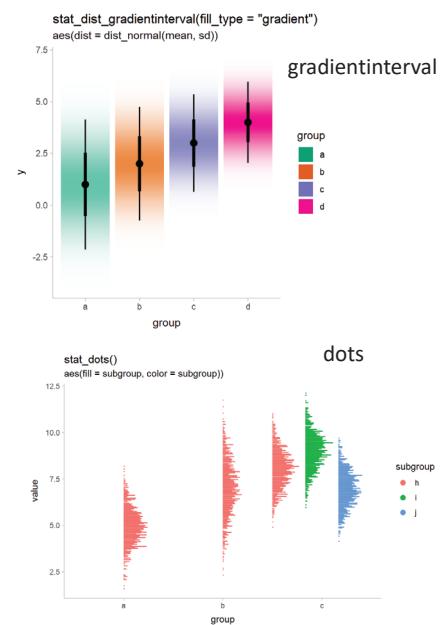
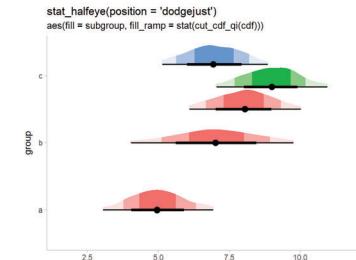


The {ggdist} package provides a wide variety of ggplot stats to display distributions & intervals



Design of {ggdist} makes it easy to combine two or more graphical representations –
data + distribution + interval

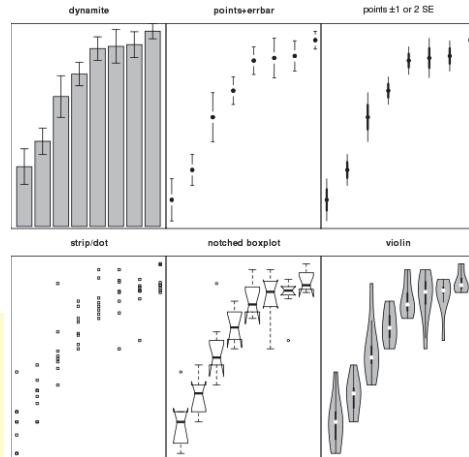
halfeye, varying fill



Comparing groups: Summary + Uncertainty

Six different graphs for comparing groups in a one-way design

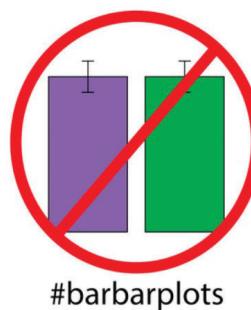
- which group means differ?
- equal variability?
- distribution shape?
- what do error bars mean?
- unusual observations?



Never use dynamite plots

Always explain what error bars mean

Consider tradeoff between summarization & exposure

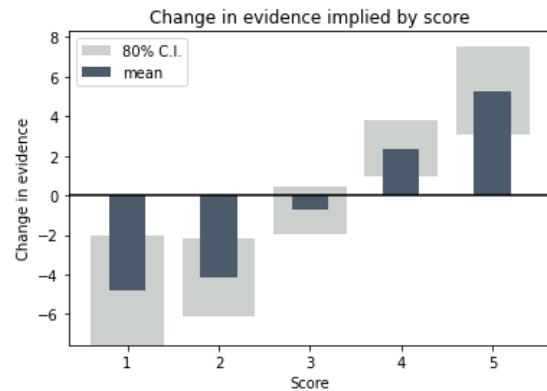


"Friends don't let friends make barplots" (video)

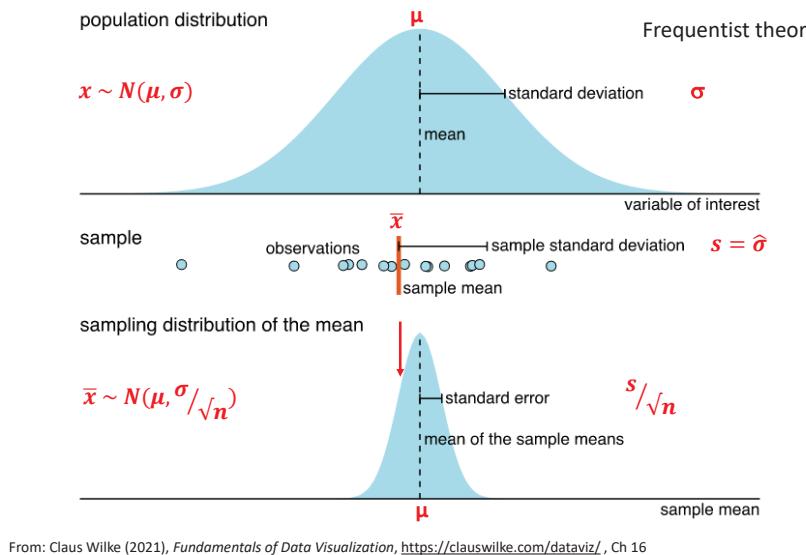
<https://barbarplots.github.io/>

De-fusing the barplot

If you insist on bars, use a better visual representation of uncertainty or CI



Key ideas of statistical sampling



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What's a “confidence interval”?



Steve Haroz @steveharoz.com · 2h
A 95% confidence interval could be

- a) 1.96 * the standard error
- b) 1.96 * the standard error of means
- c) 95% inner quantile of bootstrapped means
- d) 95% inner quantile of bootstrapped subject means
- e) 1.96 * the standard deviation
- f) 95% inner quantile of values
- g) 95% of damn near anything

#stats



3



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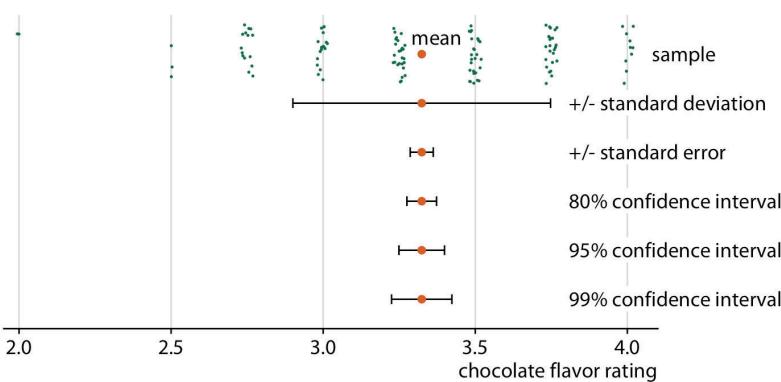
...

<https://bsky.app/profile/steveharoz.com/post/3ko5xd7waa42m>

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Visualizing distributions: Error bars

There are many ways to show variability in a single sample

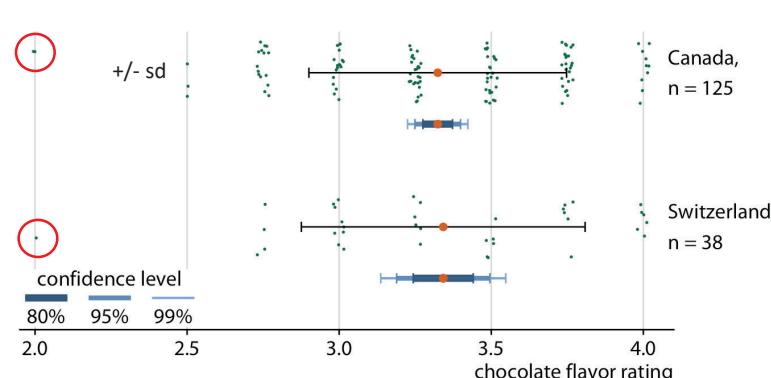


From: Claus Wilke (2021), *Fundamentals of Data Visualization*, <https://clauswilke.com/dataviz/>, Ch 16

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Comparing distributions: Sample size

- means and standard deviations are similar for Canada & Switzerland
- confidence interval widths $\sim 1/\sqrt{n}$
- can show different sized confidence bands together
- Jittered dots show the data: sample size & are there any outliers?

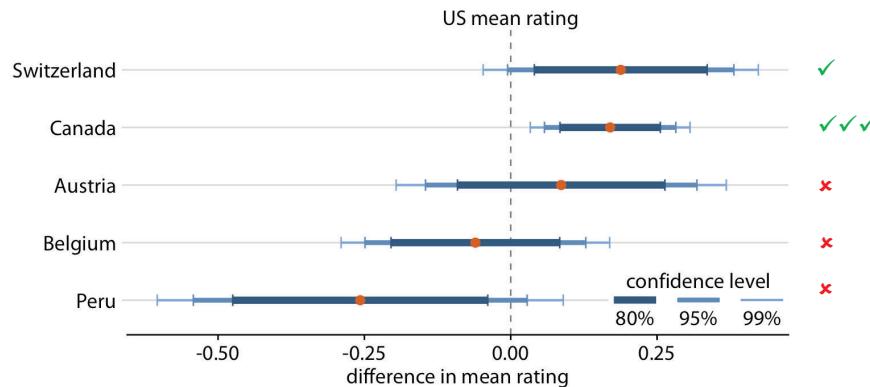


From: Claus Wilke (2021), *Fundamentals of Data Visualization*, <https://clauswilke.com/dataviz/>, Ch 16

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Comparing distributions: Contrasts

- For comparison of one group to all others, plot the **difference** directly
- Easy to see which differences exclude 0, at what confidence level



From: Claus Wilke (2021), *Fundamentals of Data Visualization*, <https://clauswilke.com/dataviz/>, Ch 16

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Intervals: Direct vs. Differences

The standard error for the **difference** between two means is always **larger than the standard error of either mean**

$$SE(\bar{x}) = \sqrt{s^2 / n}$$

$$SE(\bar{x}_1 - \bar{x}_2) = \sqrt{s_1^2 / n_1 + s_2^2 / n_2}$$

When separate intervals are shown, the visual inference is that groups **differ** significantly if intervals do not **overlap**.

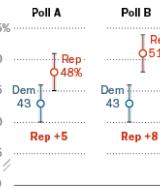
For election polls, different measures of the race have different margins of error

The margin of error reported for most polls applies to support for individual candidates ...

... while the margin of error for a candidate's lead is nearly twice as large.

Margin of error for single candidate support (MOE +/- 3 pct. points)

Margin of error for difference between two candidates' level of support (%Rep - %Dem) (MOE +/- 6 pct. points)



Source: Hypothetical polling results from a fictitious election.
PEW RESEARCH CENTER

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What kind of intervals?



Frequentist

- Confidence interval
- Scope: repeated (hypothetical) samples
- Center: parameter estimate
 - $\mu \rightarrow \bar{x}; \beta \rightarrow \hat{\beta}$
- Width: \sim std. error = $\hat{\sigma} / \sqrt{n}$
- Interpretation: true parameter w/in this interval $1-\alpha$ % (in repeated samples)



Bayesian

- Credibility interval
- Scope: repeated draws from the posterior distribution
- Center: median of posterior distribution
- Width: MAD sd of posterior
- Interpretation: Given prior, expect parameter w/in this interval $1-\alpha$ % of draws

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Bayesian intervals

Distribution of repeated draws from posterior distribution

Switzerland

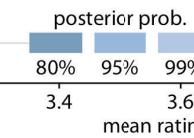
Canada

Austria

US

Belgium

Peru



From: Claus Wilke (2021), *Fundamentals of Data Visualization*, <https://clauswilke.com/dataviz/>, Ch 16

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tidybayes: Bayesian analysis + tidy data + geoms

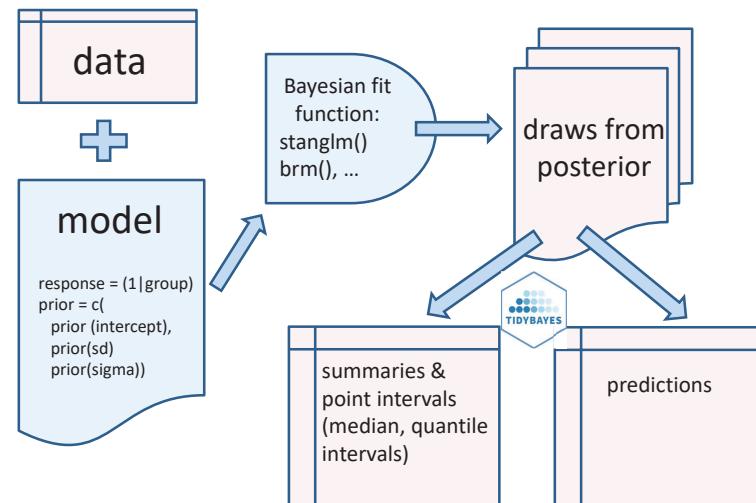
- The {tidybayes} package makes it easier to combine Bayesian analysis with insightful ggplot visualization
 - Bayesian packages: JAGS, Stan (rstanarm), brms
 - Inputs: data, model specifications aren't tidy
 - Need to translate data into forms these packages expect
 - Outputs: Posterior draws, distributions aren't tidy
 - Need to translate these into form suitable for summaries & plotting
 - Extract tidy fits and predictions from models
 - Summarize posterior distributions
 - Visualize priors and posteriors



Docs: <http://mjskay.github.io/tidybayes/>

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The Bayesian process



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Posterior = Prior × Likelihood

We have: Data, some model, some parameter(s) of interest, θ

Can calculate likelihood, $p(Data|\theta)$

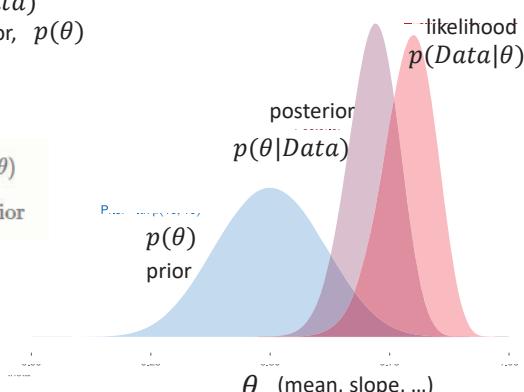
Want: posterior: $p(\theta|Data)$

Previous research: some prior, $p(\theta)$

Bayes theorem:

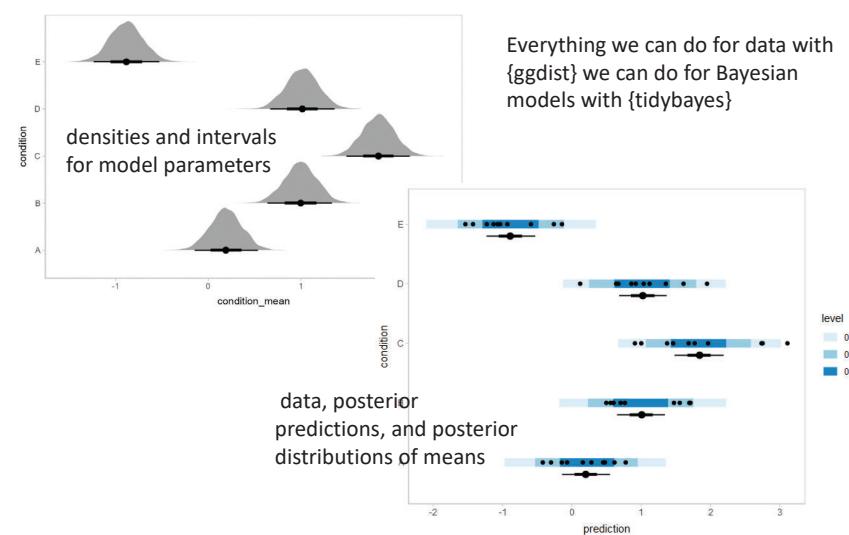
$$p(\theta|Data) \propto p(Data|\theta) \cdot p(\theta)$$

$$\text{posterior} \propto \text{likelihood} \cdot \text{prior}$$

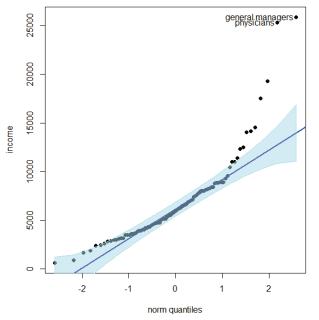


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tidybayes plots



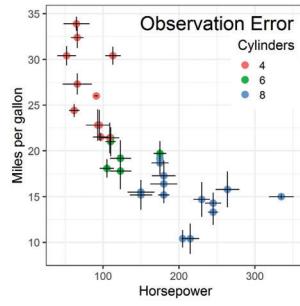
45



- QQplots
- Model fit plots

Uncertainty in fits & curves

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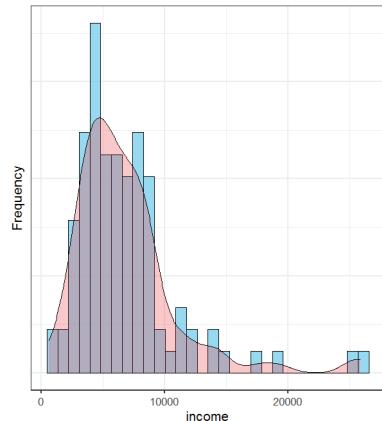
QQ plots

- How close is my data to a {Normal | exponential | χ^2 } distribution?
- There are lots of statistical tests, but these don't tell **why** or **where** a distribution is rejected.
- These tests are also overly sensitive to small departures
- Plot observed Quantiles vs. theoretical Quantiles
 - If observed \sim theoretical with **slope = 1**, OK
 - **Confidence bands** help to identify deviation from model & outliers
- Use cases:
 - Is a single variable reasonably normally distributed?
 - Are the residuals from my linear model Normal?
 - Outliers in multivariate data? $D^2 \sim \chi^2 \rightarrow \text{chisq QQ plot}$

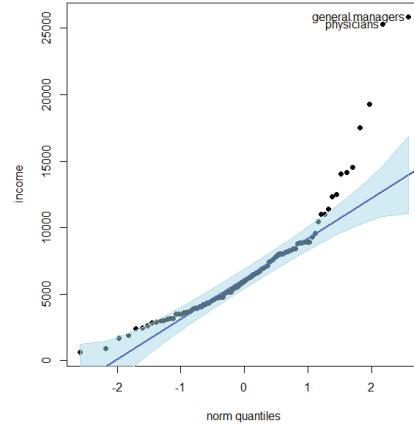
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Prestige data: income

Income is clearly **positively skewed**.
(But normality is not required for predictors.)



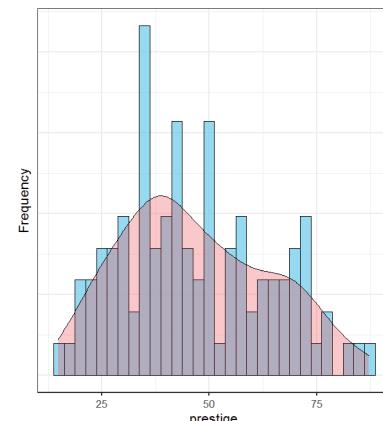
This shows up as a **U-shaped** pattern
The 95 % confidence band shows greatest departure in the upper tail



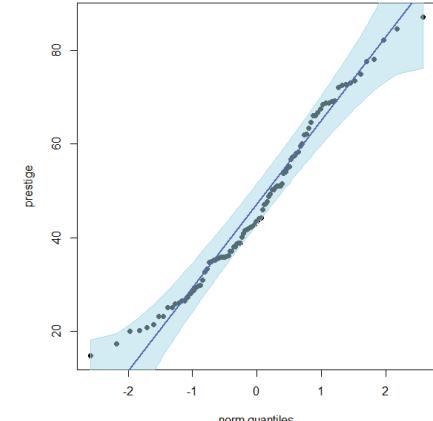
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Prestige data: prestige

Occupational prestige doesn't look precisely normal, but not that bad.



The 95% confidence band includes all the observations



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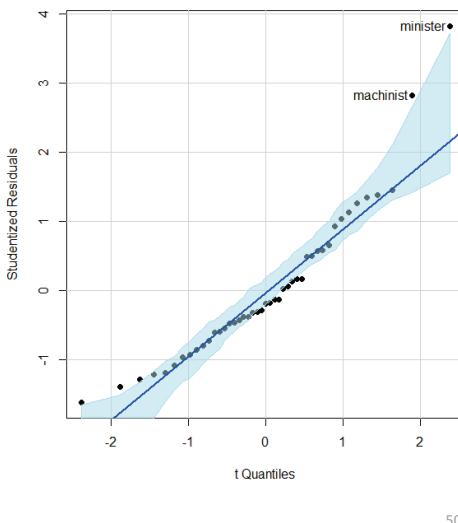
Prestige data: residuals

Normality of **residuals** is more important for linear models

Some small evidence of + skew

Confidence bands help to identify potential outliers – badly fitted pts

```
qqPlot(lm(prestige ~ income + education + type, data=Duncan))
```



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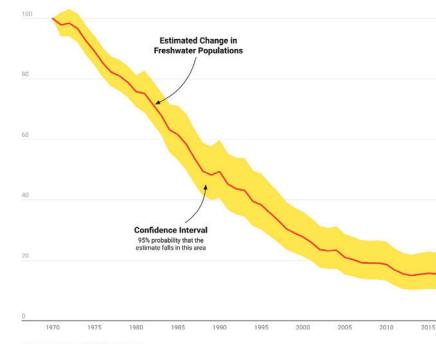
Curves + Uncertainty

Cedric Scherer used this graphic to argue about the decline of animal & freshwater populations.

Details aside, the confidence band gives visual evidence that the decline is systematic.

Humanity has wiped out 60% of animal populations since 1970 – and freshwater habitats are the worst hit with populations having collapsed by more than 80%

The Living Planet Index, produced for WWF by the Zoological Society of London, uses data on 16,704 populations of mammals, birds, fish, reptiles and amphibians to track the decline of wildlife. It underscores how the vast and growing consumption of food and resources by the global population is destroying the web of life upon which human society ultimately depends on.



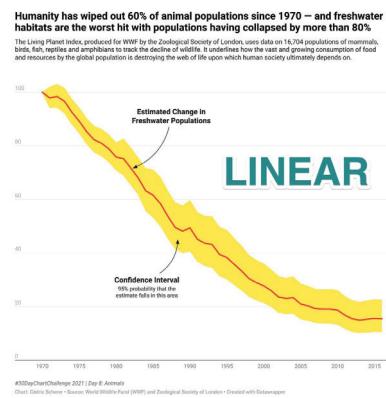
From: <https://twitter.com/CedScherer/status/1380211291466399744>

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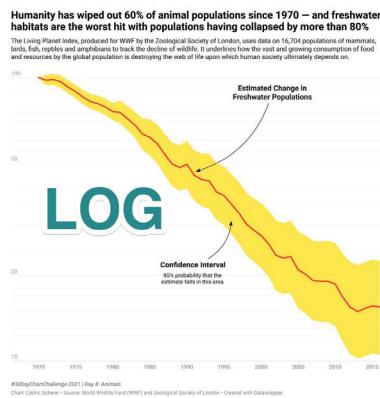
Curves + Uncertainty: Scale

Arguably, **percent** reduction in animal population should be viewed on a **log** scale.

Transformed uncertainty intervals are here the logs of the Upper/Lower levels



LINEAR

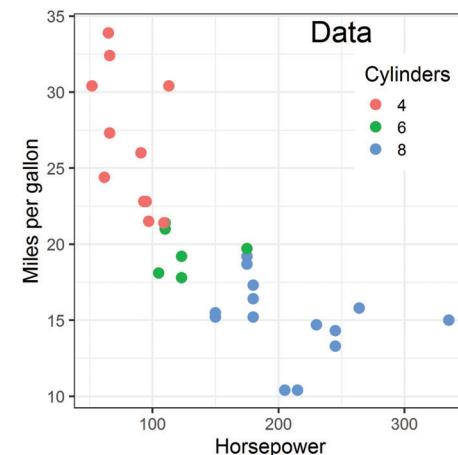


LOG

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Fitted curves

Data on gas mileage of *Motor Trend* 1974 cars

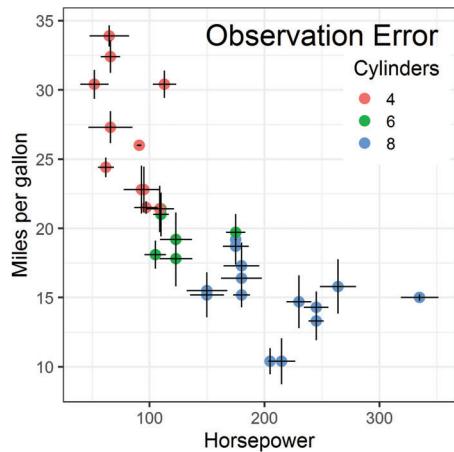


Sources of uncertainty:

- Observations: **measurement error** in MPG and/or HP?
- **Model form**: Linear? Quadratic? Interaction with cylinders
- **Model fit uncertainty**: normal theory CIs? Bootstrap? Bayesian?

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Measurement uncertainty



Sometimes, we can quantify the uncertainty ("error") in values of x and or y .
e.g., each point is the average of $n > 1$ cars.

Fitted models allow for errors in y :
 $y = f(x) + \text{error}$
and find estimates to minimize error

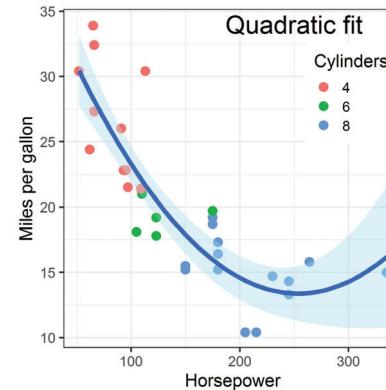
Most fitted models assume x is measured w/o error.

Big problem if error $\sim f(x, \text{other } xs)$

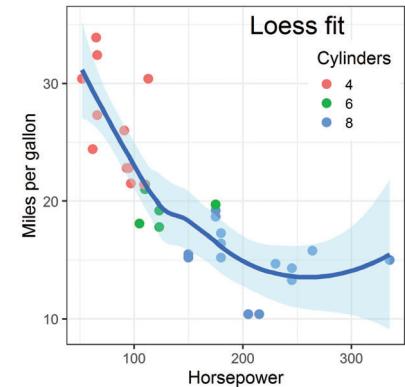
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Model forms: nonlinear fits

When a relation is clearly non-linear, we can fit alternative models.
The CI bands tell us where the data is too thin to rely on the predicted value.



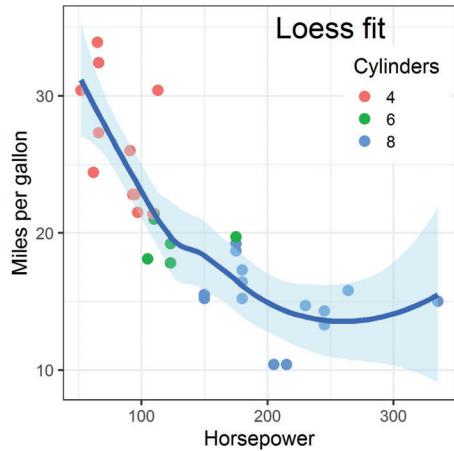
`p1 + geom_smooth(method = lm, formula = y~poly(x,2), ...)`



`p1 + geom_smooth(method = loess, formula = y~x, ...)`

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Fitted curves: smoothers



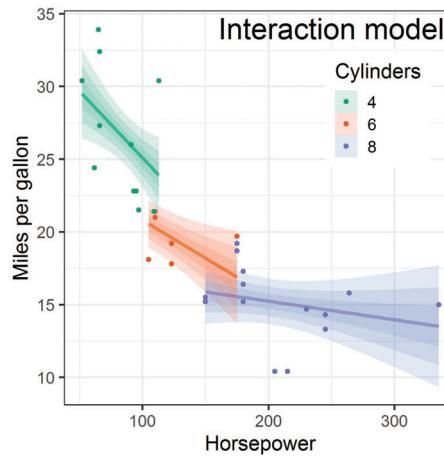
In each case, the confidence band gives visual evidence for uncertainty of the predicted values.

But, uncertainty may be expressed differently.

- a formula for std. error based on normal/large sample theory
- envelope of (normal) simulations
- Bayesian predictive distribution

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Interaction models



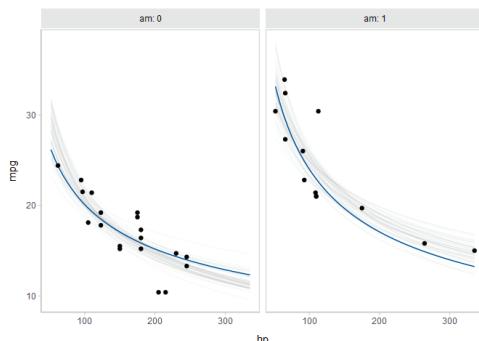
The non-linear relation between hp & mpg can (arguably) be better explained by a model that allows different slopes for 4, 6, 8 cylinders.

The graph shows normal theory CIs at 95%, 90%, and 80% for each cylinder level

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Hypothetical Outcome Plots

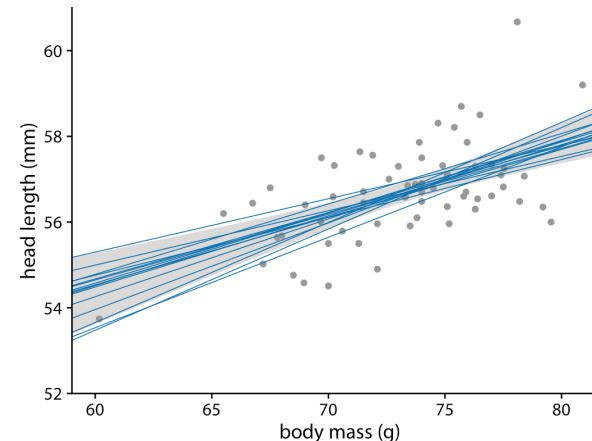
- Rather than showing a complete distribution or point estimate and error bars, Hypothetical Outcome Plots (HOPs) visualize a set of **draws** from a distribution
 - each draw is shown as a new plot in either a small multiples or animated form.



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Simulations to convey uncertainty

Simulating fits from the data (e.g., bootstrap, Bayesian estimation) shows the variability. Doesn't rely on classical, normal theory.

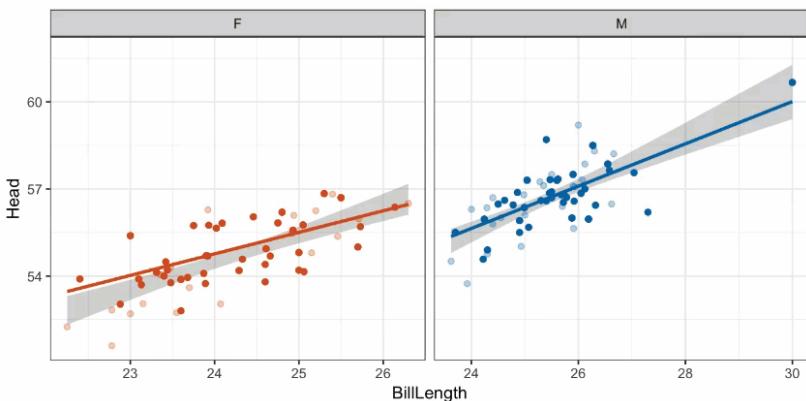


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Animation to understand uncertainty

- All assessments of uncertainty rely on a comparison: data vs. what could have been
- Sampling distributions, simulations, Bayesian posterior distributions, ...

Sometimes useful to appreciate the variability with animated graphics



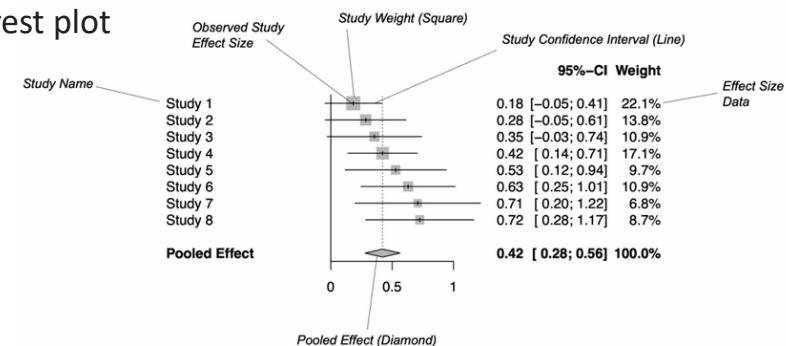
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Replication uncertainty: Meta analysis

In meta analysis, we have multiple studies reporting the same effect.

- How to visualize/compare effect sizes?
- How to calculate a **pooled**, overall effect?
- How to assess **heterogeneity** of effects?

Forrest plot



From: Harrer, [Doing Meta Analysis in R](#)

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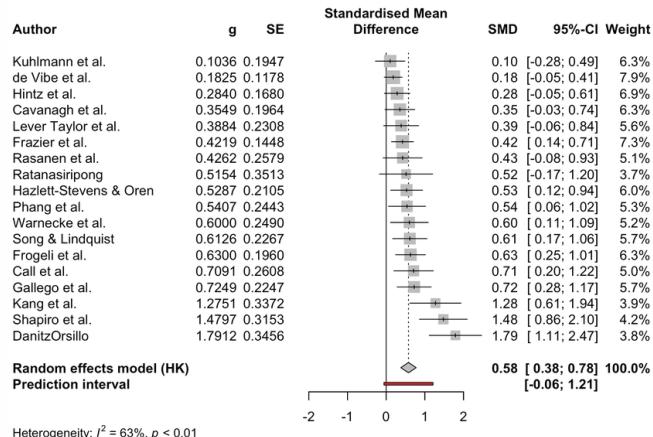
Forrest plots

Effect size: std. measure of size of effect

Study weight: $\sim 1 / \text{SE}^2$

Pooled effect: weighted average of effect sizes

`meta::forrest()`



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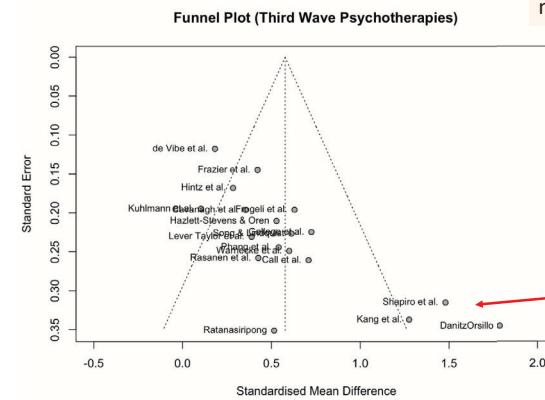
Publication bias: Funnel plots

Publication bias: NS studies less likely to be published → effect overestimated

Funnel plot: Plot std. error vs. observed effect size

Should be **symmetric** when no publication bias

`meta::funnel()`

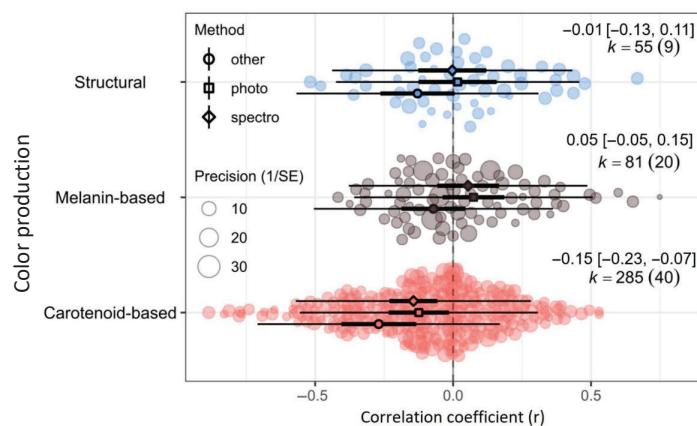


Lucky studies?
HARKing?

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Orchard plots

Plot for interaction (moderator) effects between mechanism of colour production and pollutant type



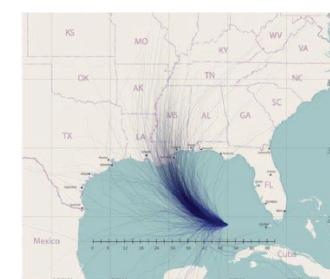
Janas et al (2024) Avian colouration in a polluted world: a meta-analysis/ <https://doi.org/10.1111/brv.13067>

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Geographic uncertainty

Predicting the path of hurricanes:

- Given what we can measure today (location, wind speed, direction, ...) where is **this** hurricane likely to be in 1 day, 3 days, 5 days?
- Most forecasts are based on an **ensemble of predictions**, representing the uncertainty in initial conditions, model physics, ...
- Often this is represented as a “cone of uncertainty”



(a) Storm path ensemble



(b) Uncertainty cone

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What is the Cone of Uncertainty?

As seen on TV:

- The center is meant to track the average prediction, either over models or history
- The cone size generally represents some “2/3 confidence interval”
- Does this mean I am safe if I lived in Tallahassee FL  in 2005? 2020?



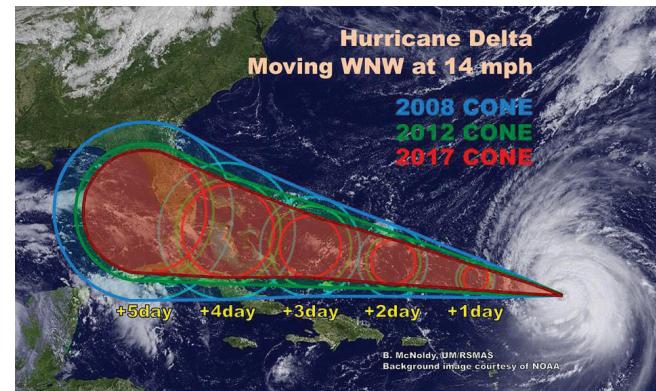
From: <https://www.youtube.com/watch?v=nE6XaHtpm04>

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The Incredible Shrinking Cone

Changes in presumed accuracy are often shown as below

- The cone represents the probable track of the center of a tropical cyclone, formed by enclosing the area swept out by a set of circles along the forecast track (at 12, 24, 36 hours, etc).
- The size of each circle is set so that two-thirds of historical official forecast errors over a 5-year sample fall within the circle.



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Sharpiegate

In Sept. 2019, Donald Trump went live with “extrapolated” predictions of the path of Hurricane Dorian.

- He had earlier predicted it would hit Alabama & Georgia.
- Let it be said, let it be written (with a sharpie)

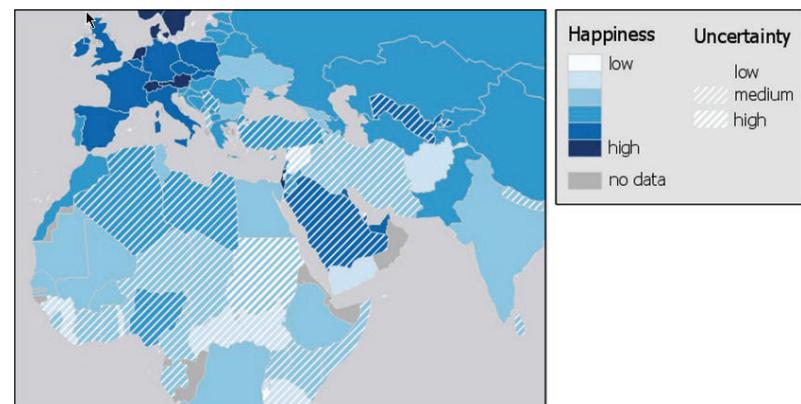


From: <https://www.theguardian.com/us-news/2019/sep/05/trump-hurricane-dorian-alabama-map-sharpiegate>

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Coding maps for uncertainty

In choropleth maps we can show uncertainty with another visual attribute



From: <https://www.e-education.psu.edu/geog486/node/693>

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Summary

- Uncertainty is fundamental to data analysis & models
- Showing variation in distributions a basic problem
 - histograms, density plots, boxplots
 - Better: violin, raincloud, ...
 - `{ggdist}` offers many alternatives
- Error bars: many flavors; can show [multiple intervals](#)
- Bayesian methods, bootstrap, simulation
 - Different methods, but similar ways to show uncertainty
- Geographic data: need to be careful about interpretations