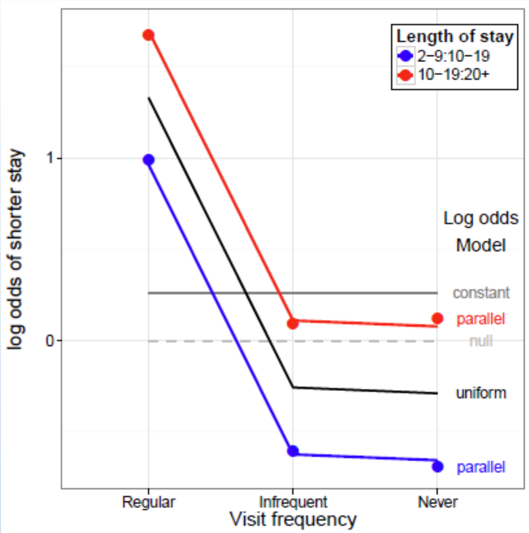

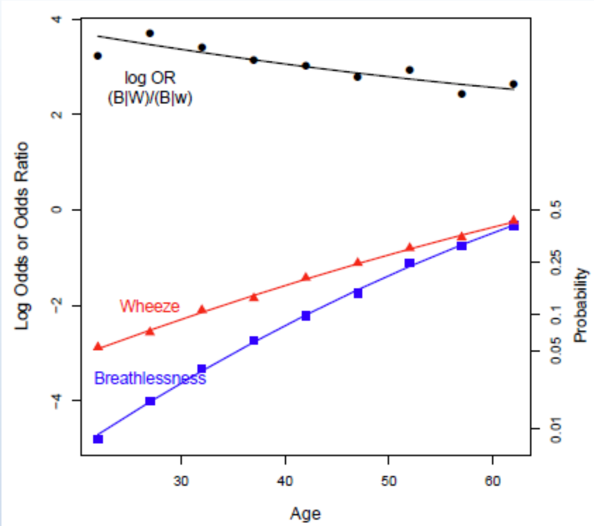

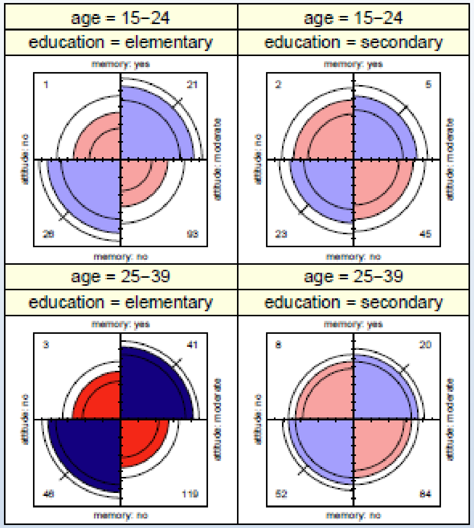

Models & graphs for log odds and log odds ratios

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Psych 6136

http://friendly.github.io/6136

Today's topics

- Logit models $\rightarrow$ log odds models
  - Two-way tables
  - Three-way + tables
  - Log odds plots
- Models for generalized odds ratios
  - Log odds ratios
  - Bivariate response models

Main ideas

- Familiar case— Binary responses:
  - Every loglinear model for a binary response has an equivalent form in terms of log odds ["logit" models]
  - Log odds models have simple interpretations
  - Data + model plots give simple descriptions of data and models
- Extend to two-way ($I \times J$) and three-way + ($I \times J \times K_1 \ldots$) tables:
  - Log odds as contrasts in log($\boldsymbol{n}$)
  - Variety of simple models for log odds (ANOVA-like)
  - Easily incorporate ordinal variables
  - Data + model plots give simple descriptions of data and models
- Generalized log odds ratios capture associations between two focal variables
  - Simple linear models for LOR
  - Direct visualization (Data + model plots) $\Longrightarrow$ more sensitive comparisons

Based on my CARME (2015) presentation, https://www.datavis.ca/papers/CARME2015-2x2.pdf

Logit models → Log odds models

- In an $I \times 2$ table for variables[A B], where B is a binary response, the logit model expresses the log odds that B=1 vs. B=2

$$\psi_i^A = \log\left(\frac{m_{i1}}{m_{i2}}\right)$$

  - Models pertain to the one-way log odds

- This generalizes to $I \times J$ tables, where we consider (J-1) log odds for each level of A, e.g.,
  - Adjacent categories

$$\psi_{ij}^{A\bar{B}} = \log\left(\frac{m_{ij}}{m_{i(j+1)}}\right) \quad j = 1, 2, \ldots, J-1$$

  - In general, $I \times J \rightarrow$ (J-1) log odds contrasts of the B categories for each level of A
  - Similar to how polytomous responses treated in logistic regression
  - Can also use comparisons with a baseline category

2-way example: Hospital visits

How does the length of stay in hospital differ among schizophrenic patients, classified by the frequency of visiting by friends and relatives?

```
data(HospVisits, package="vcdExtra")
HospVisits

##              stay
## visit        2-9 10-19 20+
##   Regular     43    16   3
##   Infrequent   6    11  10
##   Never        9    18  16
```

- Length of stay is the response, and it is ordered
- Can model the adjacent odds or log odds that stay is category j vs (j+1)
  - E.g., stay= 2-9 vs. 10-19; stay= 10-19 vs. 20+

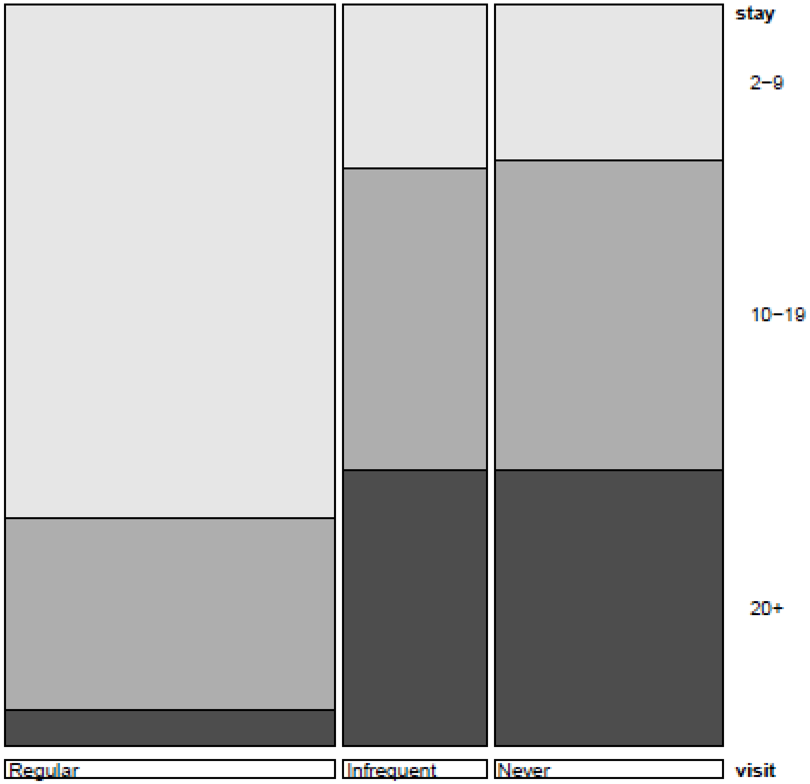
- In general, $I \times J \rightarrow I \times (J\text{-}1)$ adjacent comparisons
- visit is also ordered. Can consider simpler (e.g., linear) models for the log odds

Exploratory plots: Doubledecker

Doubledecker plot

```
doubledecker(HospVisits)
```

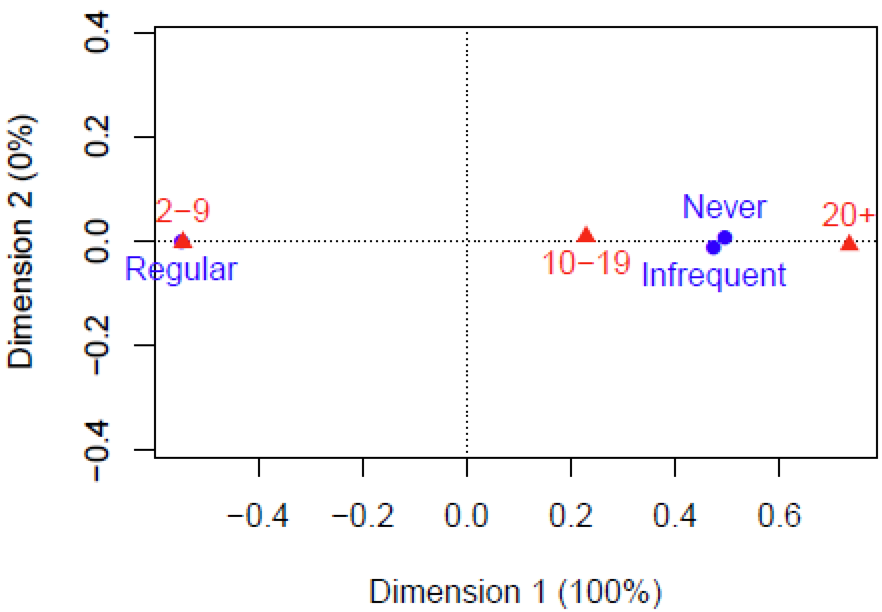
- Shows directly the conditional distributions of `stay` given `visit`
- Length of stay is shorter with frequent visits
- Infrequent and Never don't differ very much



Exploratory plots: ca

What does CA tell us?

```
plot(ca(HospVisits))
```



- Association is entirely 1D!
- Infrequent and Never category points don't differ much
- Greater visit frequency associated with shorter stay

But, how can we test and and visu-alize these ideas with models?

Models for log odds

- Start with the saturated loglinear model for the two-way table

$$\log m_{ij} = \mu + \lambda_i^A + \lambda_j^B + \lambda_{ij}^{AB}$$

- For adjacent categories of the response variable B, the odds, $\omega_{ij}^{A\overline{B}}$ and log odds, $\psi_{ij}^{A\overline{B}}$, that the response is in category j rather than $j+1$ are:

$$\text{odds: } \omega_{ij}^{A\overline{B}} = \frac{m_{ij}}{m_{i,j+1}} \qquad \text{log odds: } \psi_{ij}^{A\overline{B}} = \log\left(\frac{m_{ij}}{m_{i,j+1}}\right) \quad , j = 1, \ldots, J-1$$

- For the hospital visits data, this gives:

```
> t(lodds(HospVisits, response = "stay"))
log odds for stay by visit

            stay
visit         2-9:10-19 10-19:20+
  Regular         0.989    1.6740
  Infrequent     -0.606    0.0953
  Never          -0.693    0.1178
```

Models for log odds

A variety of simple models can be specified in terms of log odds:

Table: Models for adjacent log odds in an $I \times J$ table with B as the response

Model	log odds parameters	degrees of freedom
null log odds	$\psi_{ij}^{A\bar{B}} = 0$	$I(J-1)$
constant log odds	$\psi_{ij}^{A\bar{B}} = \psi$	$I(J-1)-1$
uniform B log odds	$\psi_{ij}^{A\bar{B}} = \psi_i^A$	$I(J-2)$
parallel log odds	$\psi_{ij}^{A\bar{B}} = \psi_i^A + \psi_j^B$	$(I-1)(J-2)$
saturated	$\psi_{ij}^{A\bar{B}}$ unspecified	

- The log odds, $\psi_{ij}^{A\bar{B}}$ can be viewed as entries in an $I \times (J-1)$ table
- These models are analogous to ANOVA tests of the A, B and $A * B$ effects in this table.

Fit some models

I'm simply using lm() here. Should use WLS: weights = $1/ASE^2$

```
mod.null  <- lm(logodds ~ -1,     data=hosp.lodds)     # null
mod.const <- lm(logodds ~ 1,      data=hosp.lodds)     # constant
mod.unif  <- lm(logodds ~ visit, data=hosp.lodds)      # uniform
mod.par   <- lm(logodds ~ visit + stay, data=hosp.lodds)  # parallel
```

Compare models:

```
anova(mod.null, mod.const, mod.unif, mod.par)

## Analysis of Variance Table
##
## Model 1: logodds ~ -1
## Model 2: logodds ~ 1
## Model 3: logodds ~ visit
## Model 4: logodds ~ visit + stay
##   Res.Df  RSS Df Sum of Sq   F Pr(>F)
## 1      6 4.65
## 2      5 4.24  1      0.41 177 0.0056 **
## 3      4 3.43  1      0.81 345 0.0029 **
## 4      2 0.00  2      3.43 734 0.0014 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Ordinal variables

When the levels of A are ordinal, we can also test for linear effects.

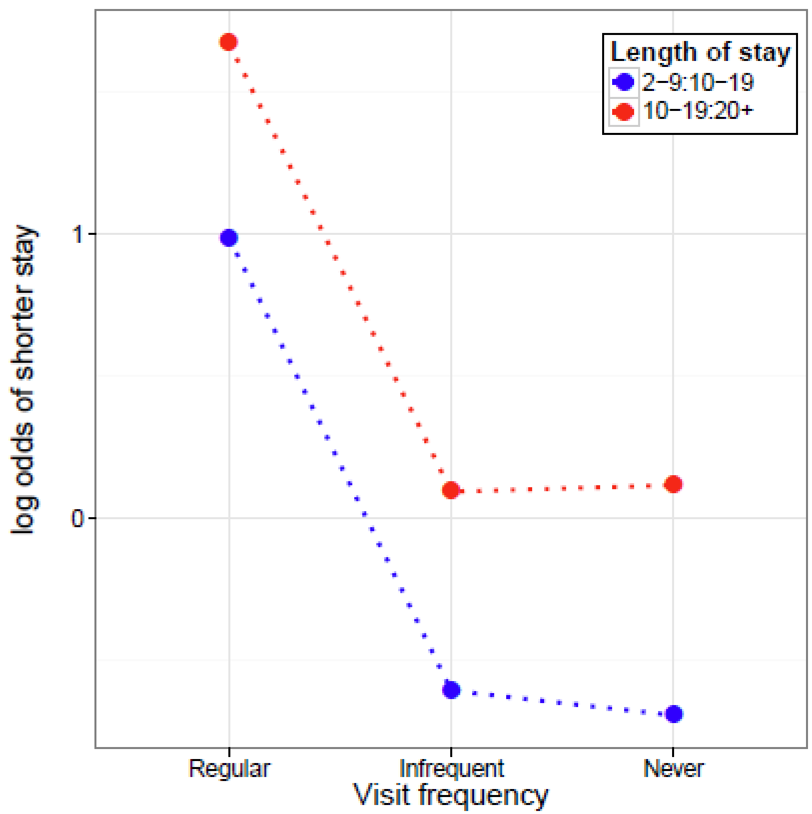
```
mod1a <- lm(logodds ~ as.numeric(visit), data=hosp.lodds)
mod2a <- lm(logodds ~ as.numeric(visit) + stay, data=hosp.lodds)
# compare parallel log odds models
anova(mod.const, mod2a, mod.par)

## Analysis of Variance Table
##
## Model 1: logodds ~ 1
## Model 2: logodds ~ as.numeric(visit) + stay
## Model 3: logodds ~ visit + stay
##   Res.Df  RSS Df Sum of Sq   F Pr(>F)
## 1      5 4.24
## 2      2 0.00  3      4.23 604 0.0017 **
## 3      2 0.00  0      0.00
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

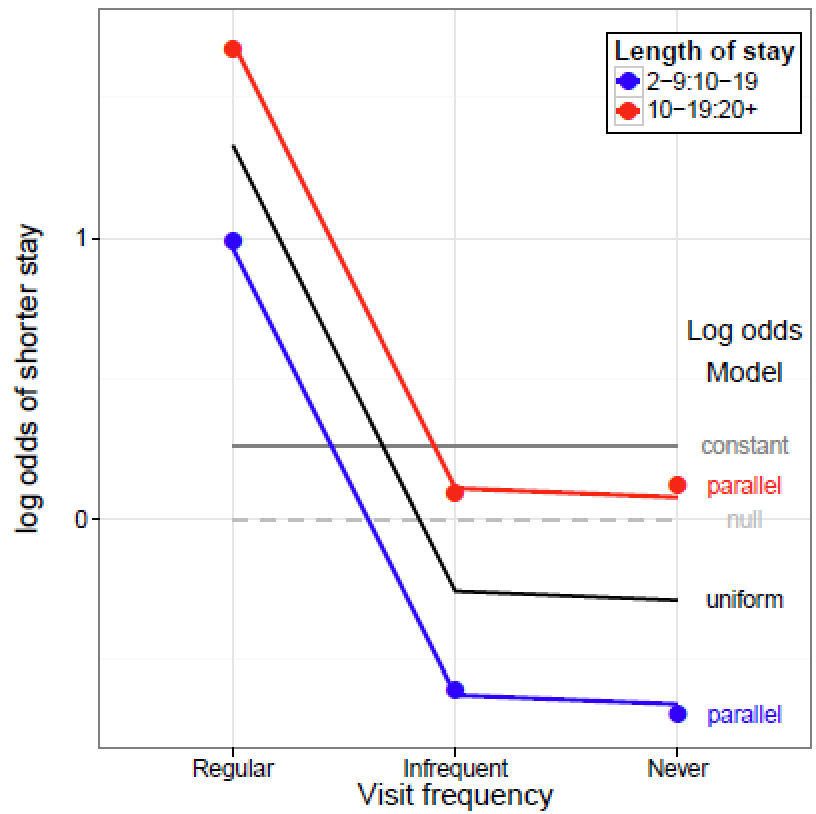
Effects of `visit` are certainly not linear.

Visualizing log odds and models

Plots of observed and fitted log odds: easy interpretation of data and models



Data plot: Observed log odds



Data + Model plot (fitted log odds)

Visualizing log odds and models

Basic plot:

```
gg <- ggplot(hosp.lodds, aes(x=visit, y=logodds,
                              group=stay, color=stay)) +
        geom_point(size=5) +
        geom_line(size=1.2, linetype="dotted")
         ylab("log odds of shorter stay\n") +
         xlab("Visit frequency") + theme_bw() + …
```

Add lines for predicted values from the models

```
grid <- hosp.lodds[,1:2]
gg_lines <- function(grid, mod, size=1.2, color=NULL, ...) {
         grid$logodds <- stats::predict(mod, grid)
         if(is.null(color)) geom_line(data=grid, size=size, ...)
         else geom_line(data=grid, size=size, color=color, ...)
         }

gg + gg_lines(grid, mod.null, color="gray", size=1, linetype="dashed") +
     gg_lines(grid, mod.const, color=gray(.5), size=1) +
     gg_lines(grid, mod.unif, color="black", size=1) +
     gg_lines(grid, mod.par)
```

Three-way+ tables: Log odds

These methods naturally extend to three- and higher-way tables:

- Consider a three-way $I \times J \times K$ table of variables A, B and C, where C is the response (or focal variable)
- The standard loglinear model is:

$$\log m_{ijk} = \mu + \lambda_i^A + \lambda_j^B + \lambda_k^C + \lambda_{ij}^{AB} + \lambda_{ik}^{AC} + \lambda_{jk}^{BC} + \lambda_{ijk}^{ABC}$$

- For categories k and $k+1$ the adjacent log odds for C are

$$\text{log odds: } \psi_{ijk}^{AB\overline{C}} = \log\left(\frac{m_{ijk}}{m_{i,j+1}}\right) \quad , \quad k = 1, \ldots, K-1$$

- These log odds can be viewed as entries in a two-way, $IJ \times (K-1)$ table.

Three-way+ tables: Log odds

- The parallel log odds model is

$$\begin{aligned}\psi_{ijk}^{AB\overline{C}} &= \Psi_{ij}^{AB} + \psi_k^{C} \\ &= \psi + \psi_i^{A} + \psi_j^{B} + \psi_{ij}^{AB} + \psi_k^{C}\end{aligned}$$

  where the Ψ_{ij}^{AB} are unspecified and the ψ parameters obey standard (sum-to-zero) constraints.

- Simpler models:

$$\begin{aligned}\text{uniform log odds:} \quad & \psi_k^{C} = 0 \\ \text{joint independence:} \quad & \Psi_{ij}^{AB} = \psi\end{aligned}$$

- Even simpler models: null effects of A ($\psi_i^A = 0$) or B ($\psi_j^B = 0$)
- Linear effects models: An ordinal A can use $\psi_i^A = i \times \beta_A$ to test for linearity

3-way example: Mice depletion data

- Kastenbaum and Lamphiear (1959) gave a $3 \times 5 \times 2$ table of the number of deaths (0, 1, 2+) in 657 litters of mice, classified by litter size (7–11) and treatment (“A”, “B”)
- How does number of deaths depend on litter size and treatment?

```
data(Mice, package="vcdExtra")
mice.tab <- xtabs(Freq ~ litter + treatment + deaths, data=Mice)
ftable(litter + treatment ~ deaths, data=mice.tab)

##          litter     7      8      9     10     11
##          treatment  A  B   A  B   A  B   A  B   A  B
## deaths
## 0                  58 75  49 58  33 45  15 39   4  5
## 1                  11 19  14 17  18 22  13 22  12 15
## 2+                  5  7  10  8  15 10  15 18  17  8
```

→ Adjacent categories:

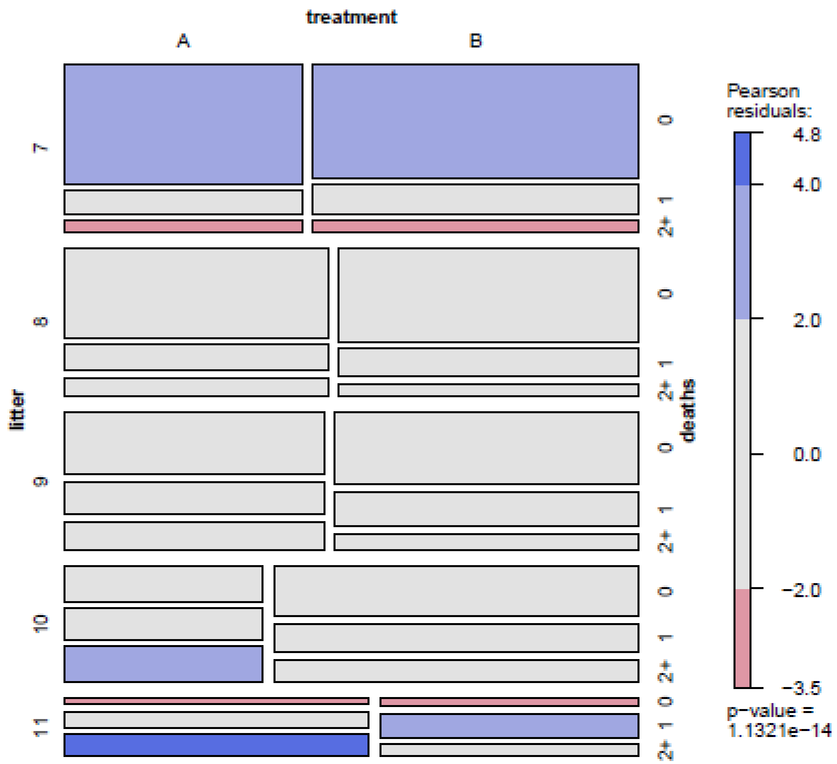
- Odds or log odds of 0 vs. 1 deaths
- Odds or log odds of 1 vs. 2+ deaths

How do these differ with litter size & treatment?

Mice data: mosaic plot

Fit and display the model of joint independence, [litter, treatment] [deaths]

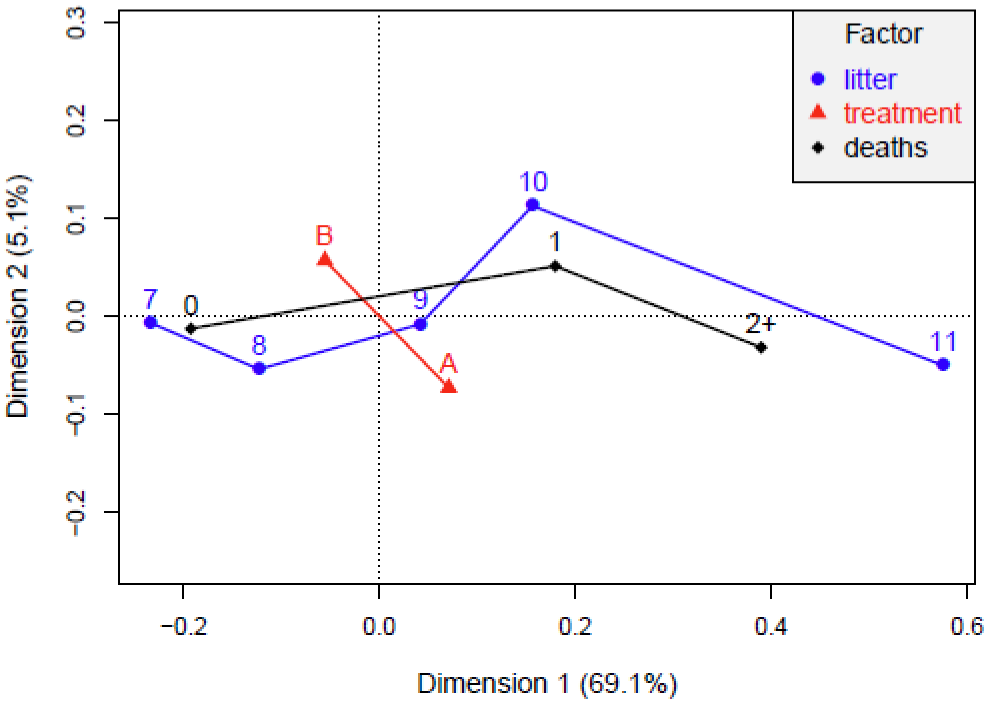
```
mosaic(mice.tab, expected= ~ litter * treatment + deaths)
```



- What can we see?
- Small litters more likely to have 0 deaths
- Large litters more likely to have 2+ deaths
- More deaths with treatment A than B

Mice data: MCA

```
mice.mca <- mjca(mice.tab)
plot(mice.mca)
```



What can we see?

- Larger litter size associated with more deaths
- More deaths with treatment A than B
- What model? How to simplify?

Calculating log odds

For a three-way table, a simple way to calculate all (log) odds is to reshape the data as a two-way matrix, $\boldsymbol{T}$, with $I \times J$ rows and K columns.

```
##         0  1 2+
## 7:A    58 11  5
## 8:A    49 14 10
## 9:A    33 18 15
## 10:A   15 13 15
## 11:A    4 12 17
...
```

The $IJ \times (K-1)$ table of adjacent log odds can then be calculated as $\log(\boldsymbol{T})\boldsymbol{C}$, where $\boldsymbol{C}$ is the $K \times K-1$ matrix of contrasts,

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$$

Adjacent categories

$$\mathbf{C} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Reference level = 0

In general, any set of K-1 {1, 0, -1} contrasts can be used

Calculating log odds

```
mice.tab <- xtabs(Freq ~ litter + treatment + deaths, data=Mice)

# reshape table to matrix
T <- matrix(mice.tab,
            nrow=prod(dim(mice.tab)[1:2]),
            ncol=dim(mice.tab)[3])
colnames(T) <- dimnames(mice.tab)[[3]]
rn <- expand.grid(dimnames(mice.tab)[1:2])
rownames(T) <- apply(rn, 1, paste, collapse=":")
```

```
C <- matrix(c(1, -1, 0,
              0,  1, -1), nrow=3)
lodds <- log(T) %*% C
colnames(lodds) <- c("0:1", "1:2+")
```

```
> lodds
         0:1   1:2+
7:A    1.663  0.788
8:A    1.253  0.336
9:A    0.606  0.182
10:A   0.143 -0.143
11:A  -1.099 -0.348
7:B    1.373  0.999
8:B    1.227  0.754
9:B    0.716  0.788
10:B   0.573  0.201
11:B  -1.099  0.629
```

Calculating log odds

More generally,

- Consider an $R \times K_1 \times K_2 \times \ldots$ frequency table $n_{ij\ldots}$, with factors $K_1, K_2 \ldots$ considered as strata.
- Let $\boldsymbol{n} = \mathrm{vec}(n_{ij\ldots})$ be the $N \times 1$ vectorization of the table.
- Then, all log odds and their asymptotic covariance matrix $\boldsymbol{S}$ can be calculated as:
  - $\widehat{\psi} = \boldsymbol{C} \log(\boldsymbol{n})$
  - $\boldsymbol{S} = \mathrm{Var}[\psi] = \boldsymbol{C}\, \mathrm{diag}\, \boldsymbol{n}^{-1}\, \boldsymbol{C}^{\mathsf{T}}$

  where $\boldsymbol{C}$ is an N-column matrix containing all zeros, except for one $+1$ elements and one -1 elements in each row.
- With strata, $\boldsymbol{C}$ can be calculated as the Kronecker product $\boldsymbol{C} = \boldsymbol{C}_R \otimes \boldsymbol{I}_{K_1} \otimes \boldsymbol{I}_{K_2} \otimes \cdots$
- Linear models for log odds: $\psi = \boldsymbol{X}\beta$

Mice data: Log odds

The vcd package contains a general implementation of these ideas:

- `odds()` and `lodds()`: calculate odds or log odds for 1 variable in an n-way table
- Provides methods (`coef()`, `vcov()`, `confint()`, …) for “lodds” objects

```
> (mice.lodds <- as.data.frame(lodds(mice.tab, response="deaths")))
   deaths litter treatment logodds   ASE
1     0:1      7         A   1.663 0.329
2    1:2+      7         A   0.788 0.539
3     0:1      8         A   1.253 0.303
4    1:2+      8         A   0.336 0.414
5     0:1      9         A   0.606 0.293
6    1:2+      9         A   0.182 0.350
7     0:1     10         A   0.143 0.379
8    1:2+     10         A  -0.143 0.379
9     0:1     11         A  -1.099 0.577
10   1:2+     11         A  -0.348 0.377
```

Mice data: Fit models

Use WLS, with weights $\sim ASE^{-2}$

```
mod0 <- lm(logodds ~ 1, weights=1/ASE^2, data=mice.lodds)
mod1 <- lm(logodds ~ litter + treatment, weights=1/ASE^2, data=mice.lodds)
mod2 <- lm(logodds ~ litter * treatment, weights=1/ASE^2, data=mice.lodds)
mod3 <- lm(logodds ~ litter * treatment + deaths, weights=1/ASE^2, data=mi
```

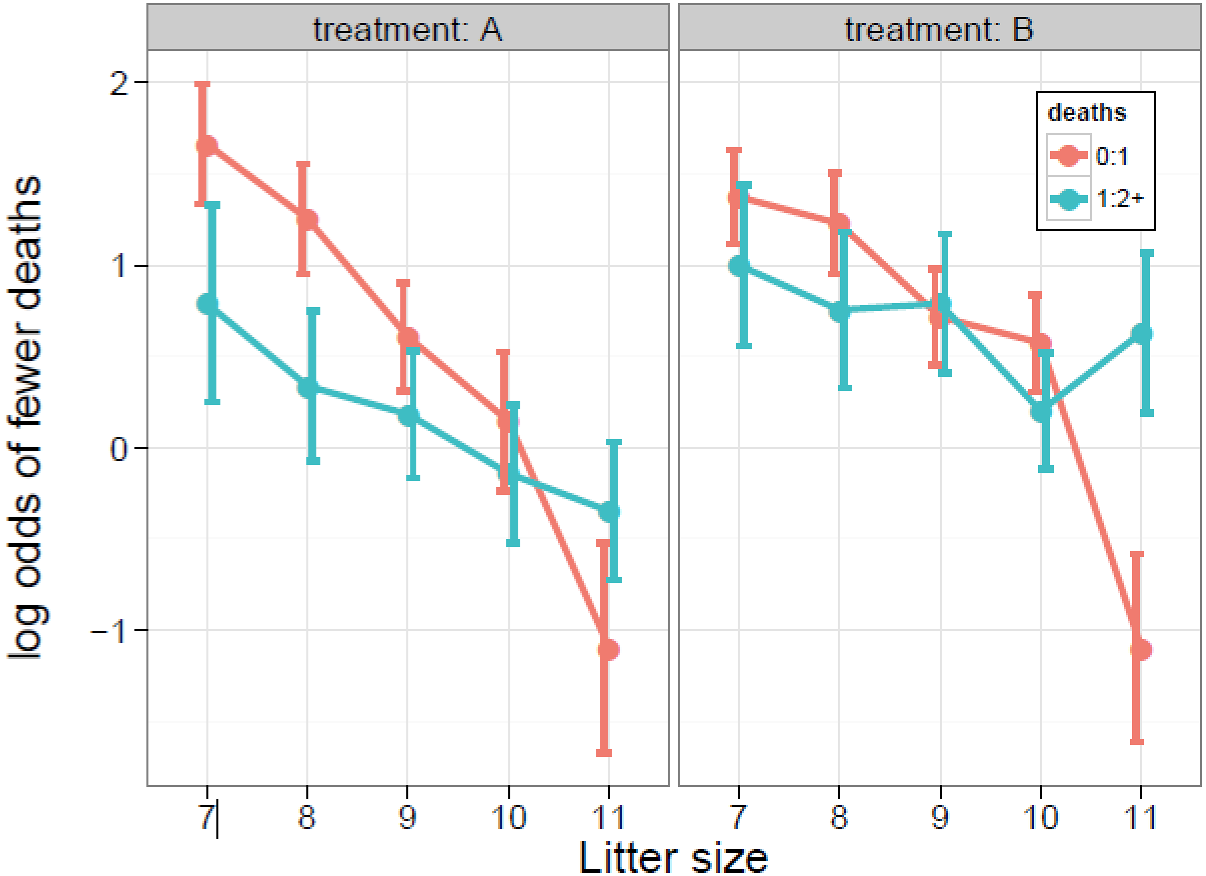
Compare models:

```
anova(mod0, mod1, mod2, mod3)

## Analysis of Variance Table
##
## Model 1: logodds ~ 1
## Model 2: logodds ~ litter + treatment
## Model 3: logodds ~ litter * treatment
## Model 4: logodds ~ litter * treatment + deaths
##   Res.Df  RSS Df Sum of Sq     F  Pr(>F)
## 1     19 65.0
## 2     14 17.8  5      47.2 18.22 0.00018 ***
## 3     10  6.7  4      11.1  5.36 0.01737 *
## 4      9  4.7  1       2.1  3.98 0.07723 .
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Visualize log odds & models: Data plot

- Data plot: log odds with error bars: $\psi_{ijk}^{AB\overline{C}} \pm 1 ASE_{\psi}$
- This is equivalent to the saturated model for log odds

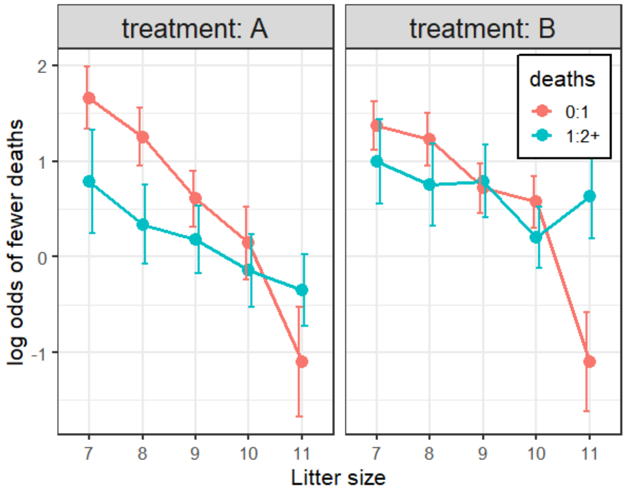


Basic plot:

```
gg <- ggplot(mice.lodds, aes(x=litter, y=logodds,
                              color=deaths, group=deaths)) +
  geom_point(size=4) +
  ylab("log odds of fewer deaths") +
  xlab("Litter size") +
  theme_bw(base_size = 16) +
  theme(legend.position = c(.9, .85),
        legend.background = element_rect(colour = "black")) +
  facet_grid(. ~ treatment, labeller=label_both) +
  theme(strip.text = element_text(size = rel(1.2)))
```

Add error bars, dodged

```
bars <- aes(ymin=logodds-ASE,
            ymax=logodds+ASE)
gg + geom_line(size=1.2) +
  geom_errorbar(bars,
        width=0.25, size=1,
        position=position_dodge(width=.2))
```

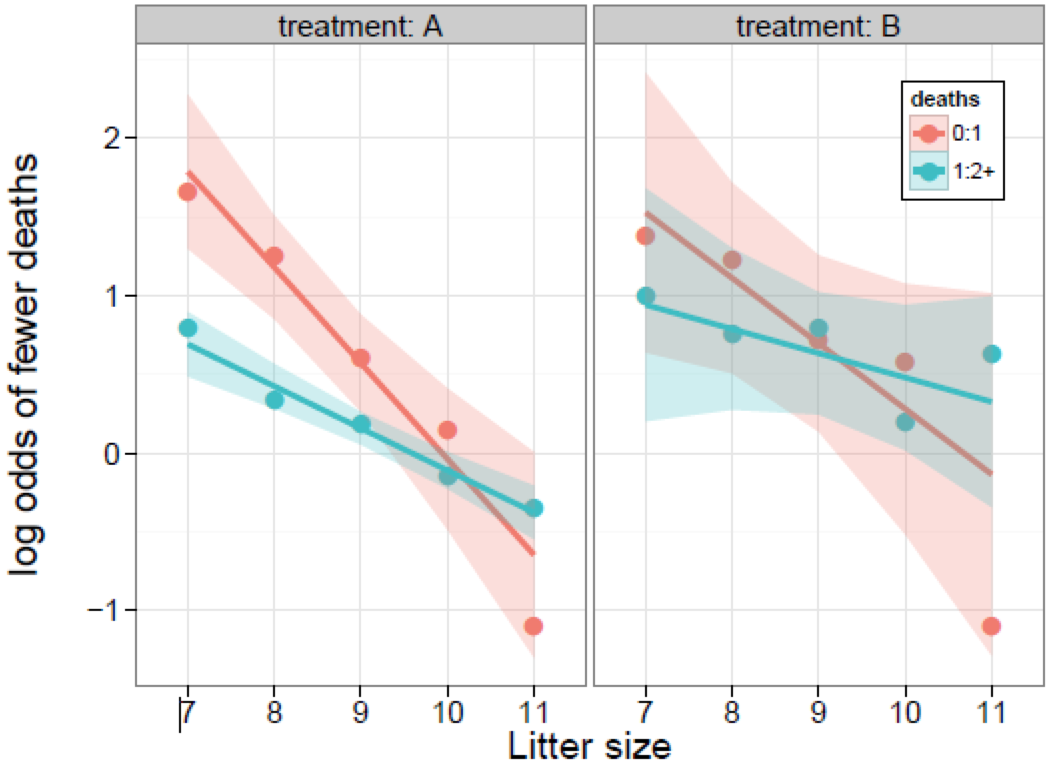


ggplot thinking:

- gg is my basic plot of points
- I can add other layers to it

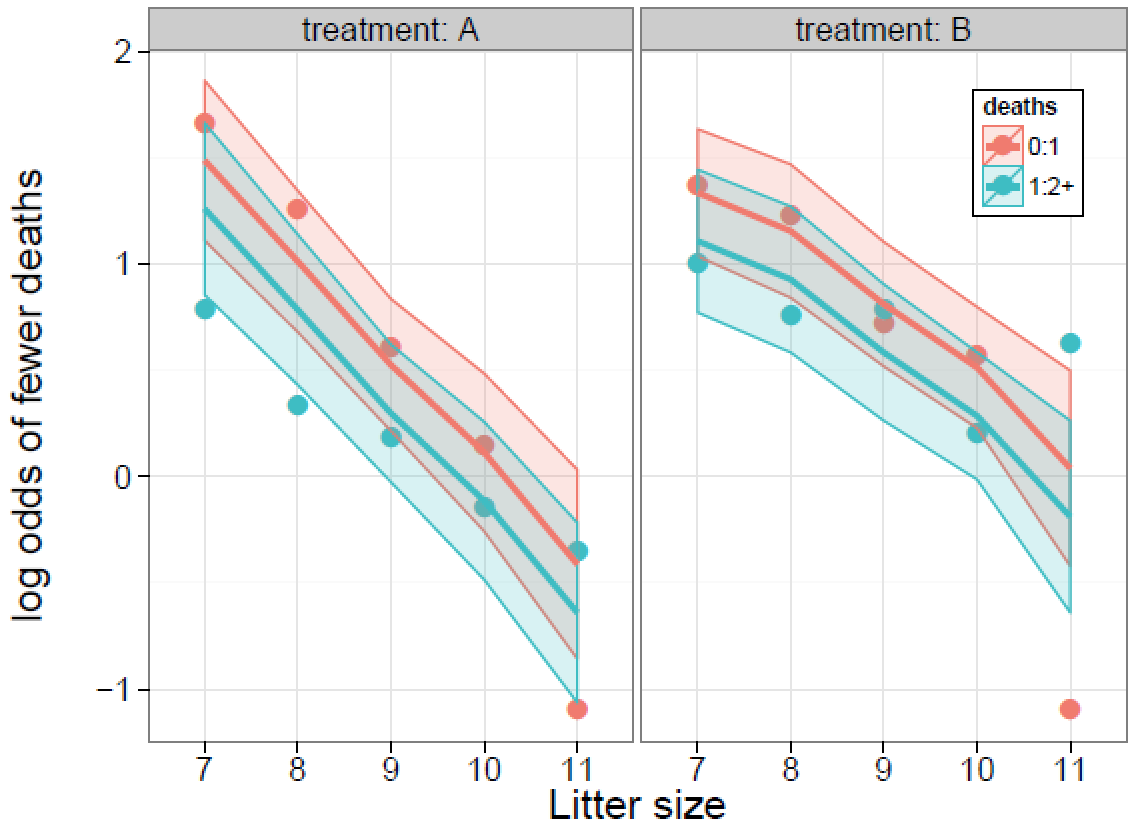
Visualize log odds & models: Smoothing

- Apply a linear smoother (weighed linear regression) to each
- This is equalvalent to a model with a three-way term,
  `as.numeric(litter)*treatment*deaths`
- Error bands show model uncertainty



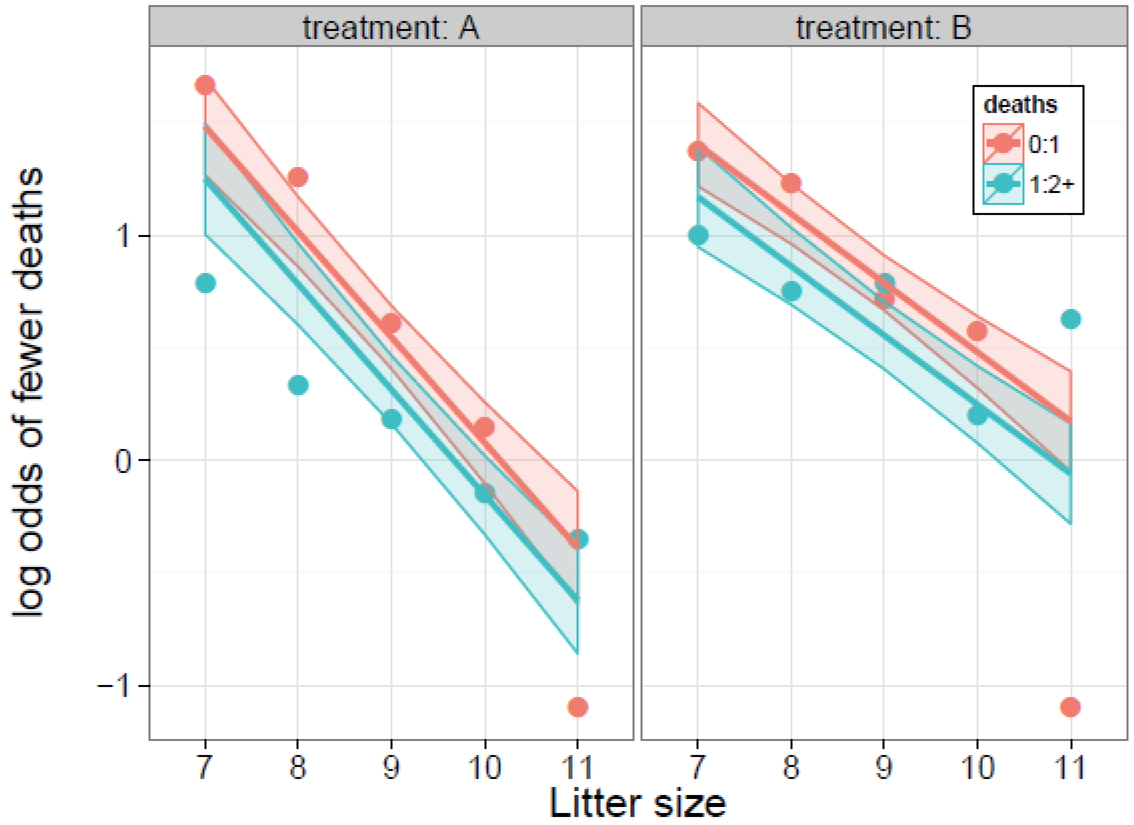
Visualize log odds & models: Data + Model

- Display the fit of the parallel log odds model, $\psi_{ijk}^{AB\overline{C}} = \Psi_{ij}^{AB} + \psi_k^{C}$



Visualize log odds & models: Data + Model

- Simplify the model: fit only linear effects of `litter`
- `lm(logodds   as.numeric(litter)*treatment + deaths)`
- Error bands show smaller model uncertainty



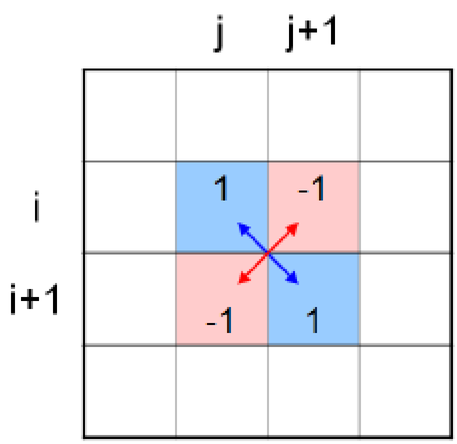
Generalized log odds ratios

- In any two-way, $R \times C$ table, *all* associations can be represented by a set of $(R-1) \times (C-1)$ odds ratios,

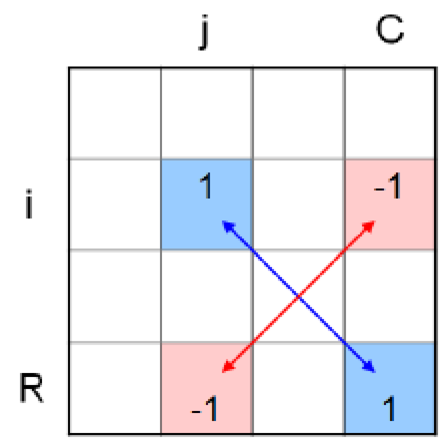
$$\theta_{ij} = \frac{n_{ij} / n_{i+1,j}}{n_{i,j+1} / n_{i+1,j+1}} = \frac{n_{ij} \times n_{i+1,j+1}}{n_{i+1,j} \times n_{i,j+1}}$$

Simpler in terms of log odds ratios:

$$\log(\theta_{ij}) = \begin{pmatrix} 1 & -1 & -1 & 1 \end{pmatrix} \log \begin{pmatrix} n_{ij} & n_{i+1,j} & n_{i,j+1} & n_{i+1,j+1} \end{pmatrix}^{\mathsf{T}}$$



local odds ratios



ref='last' odds ratios

Generalized log odds ratios

- $\log \theta_{ij} \sim \mathcal{N}(0, \sigma^2)$, with estimated asymptotic standard error:

$$\hat{\sigma}(\log \theta_{ij}) = (n_{ij}^{-1} + n_{i+1,j}^{-1} + n_{i,j+1}^{-1} + n_{i+1,j+1}^{-1})^{1/2}$$

- This extends naturally to $\theta_{ij\,|\,k}$ in higher-way tables, stratified by one or more "control" variables.

- Many models have a simpler form expressed in terms of $\log(\theta_{ij})$.
  - e.g., Uniform association model

$$\log(m_{ij}) = \mu + \lambda_i^A + \lambda_j^B + \gamma \boldsymbol{a}_i \boldsymbol{b}_j \equiv \log(\theta_{ij}) = \gamma$$

- Direct visualization of log odds ratios permits more sensitive comparisons than area-based displays.

Models for log odds ratios: Computation

- Consider an $R \times C \times K_1 \times K_2 \times \ldots$ frequency table $n_{ij\ldots}$, with factors $K_1, K_2 \ldots$ considered as strata.
- Let $\boldsymbol{n} = \text{vec}(n_{ij\ldots})$ be the $N \times 1$ vectorization of the table.
- Then, all log odds ratios and their asymptotic covariance matrix $\boldsymbol{S}$ can be calculated as:
  - $\log(\widehat{\theta}) = \boldsymbol{C} \log(\boldsymbol{n})$
  - $\boldsymbol{S} = \text{Var}[\log(\theta)] = \boldsymbol{C} \, \text{diag} \, \boldsymbol{n}^{-1} \, \boldsymbol{C}^{\mathsf{T}}$

  where $\boldsymbol{C}$ is an N-column matrix containing all zeros, except for two $+1$ elements and two -1 elements in each row.
- With strata, $\boldsymbol{C}$ can be calculated as $\boldsymbol{C} = \boldsymbol{C}_{RC} \otimes \boldsymbol{I}_{K_1} \otimes \boldsymbol{I}_{K_2} \otimes \cdots$
- `loddsratio()` in vcd provides generic methods (`coef()`, `vcov()`, `confint()`, ...)
- `plot()` method gives reasonable data and model plots.

Models for log odds ratios: Computation

For example, for a 2×3 table, there are two adjacent odds ratios

```
##     Age
## Sex Yng Mid Old
##   M  30  20  10
##   F   5  15  25
## log odds ratios for Sex and Age
##
## Yng:Mid Mid:Old
##   1.504   1.204
```

These are calculated as:

$$\log(\theta) = \boldsymbol{C} \log(\boldsymbol{n}) = \begin{bmatrix} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix} \log \begin{pmatrix} n_{11} \\ n_{21} \\ n_{12} \\ n_{21} \\ n_{13} \\ n_{23} \end{pmatrix}$$

Models for log odds ratios: Estimation

- A log odds ratio linear model for the $\log(\theta)$ is

$$\log(\theta) = \boldsymbol{X}\beta$$

  where $\boldsymbol{X}$ is the design matrix of covariates
- The (asymptotic) ML estimates $\widehat{\beta}$ are obtained by GLS via

$$\widehat{\beta} = \left(\boldsymbol{X}^{\mathsf{T}}\boldsymbol{S}^{-1}\boldsymbol{X}\right)^{-1}\boldsymbol{X}^{\mathsf{T}}\boldsymbol{S}^{-1}\log\left(\widehat{\theta}\right)$$

  where $\boldsymbol{S} = \mathrm{Var}[\log(\theta)]$ is the estimated covariance matrix
- $\Longrightarrow$ Standard graphical and diagnostic methods can be adapted to this case.
  - visualization: full-model plots, effect plots, . . .
  - diagnostics: influence plots, added-variable plots, . . .

Technical note: for simplicity, I use `lm()` for WLS, with S^{-1} = diag(1/ASE^2)
Should probably use `nlme::gls()` instead

Example: Breathlessness & wheeze in coal miners

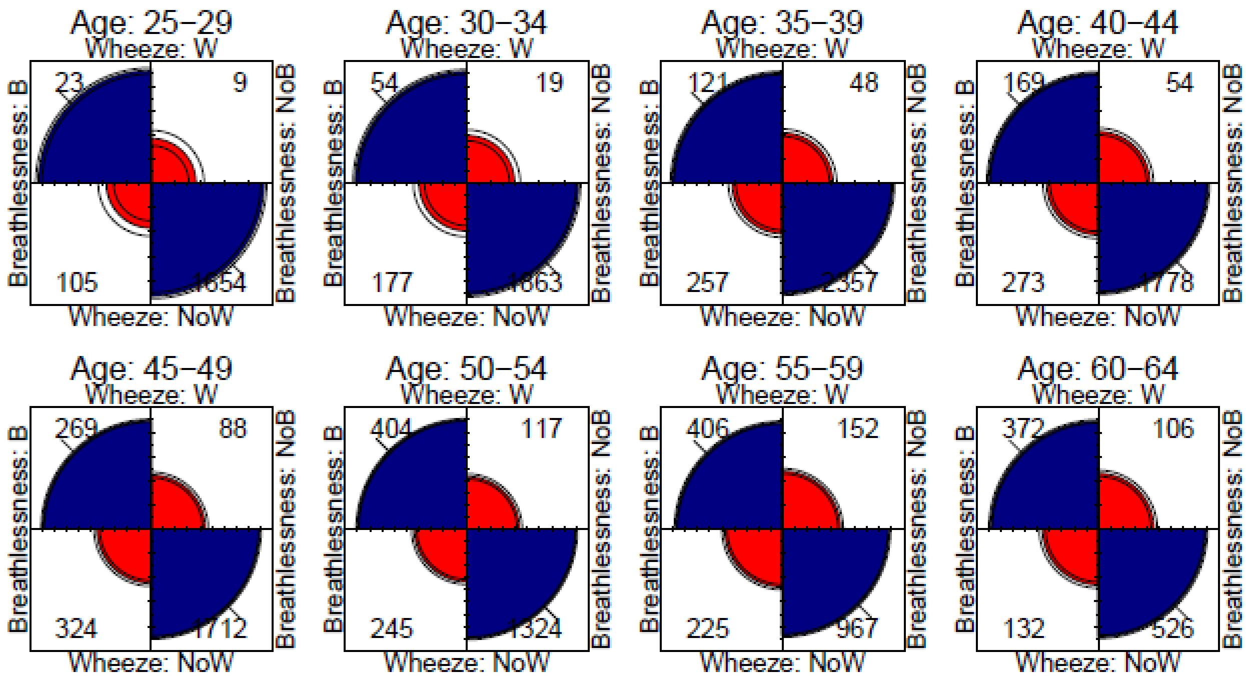
- Ashford & Sowden (1970) gave data on the association between two pulmonary conditions: breathlessness and wheeze, in a large sample of coal miners
- Age is the primary covariate
- How does the association between breathlessness and wheeze vary with age?

```
ftable(CoalMiners)

##                            Age 25-29 30-34 35-39 40-44 45-49 50-54 55-59 60-
## Breathlessness Wheeze
## B              W                  23    54   121   169   269   404   406   3
##                NoW                 9    19    48    54    88   117   152   1
## NoB            W                 105   177   257   273   324   245   225   1
##                NoW              1654  1863  2357  1778  1712  1324   967   5
```

Example: Breathlessness & wheeze in coal miners

```
fourfold(CoalMiners, mfcol=c(2,4), fontsize=18)
```



- There is a strong + association at all ages
- But can you see the trend?

Coal miners: Log odds & models

```
(lor.CM <- loddsratio(CoalMiners))

## log odds ratios for Breathlessness and Wheeze by Age
##
## 25-29 30-34 35-39 40-44 45-49 50-54 55-59 60-64
## 3.695 3.398 3.141 3.015 2.782 2.926 2.441 2.638
```

How does LOR vary with Age?

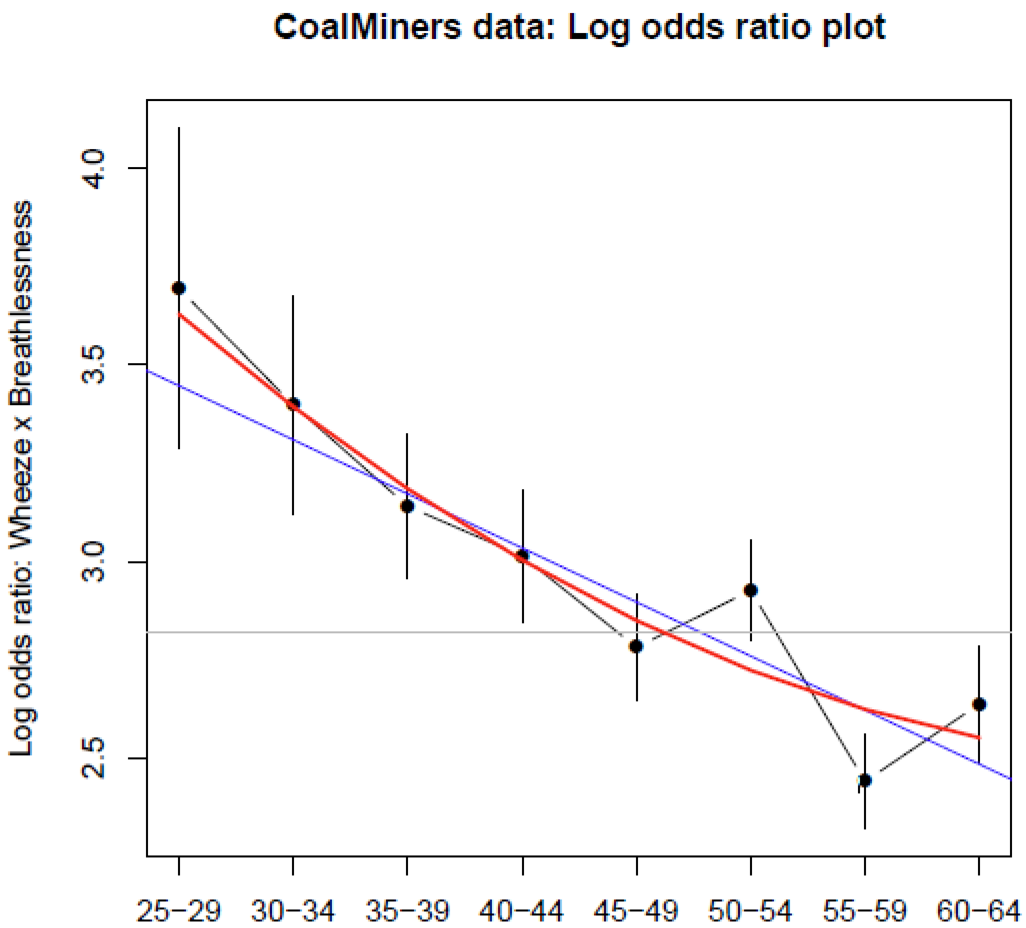
- Uniform association: $\ln(\theta) = \beta_0$
- Linear association: $\ln(\theta) = \beta_0 + \beta_1 \text{ Age}$
- Quadratic association: $\ln(\theta) = \beta_0 + \beta_1 \text{ Age} + \beta_2 \text{ Age}^2$

Fit models using WLS:

```
lor.CM.df <- as.data.frame(lor.CM)
age <- seq(25, 60, by = 5)
CM.mod0 <- lm(LOR ~ 1,            weights=1/ASE^2, data=lor.CM.df)
CM.mod1 <- lm(LOR ~ age,          weights=1/ASE^2, data=lor.CM.df)
CM.mod2 <- lm(LOR ~ poly(age,2), weights=1/ASE^2, data=lor.CM.df)
```

Coal miners: LOR plot

Plot log odds ratios and fitted regressions: The trend is now clear!



Coal miners: Model comparisons

Standard ANOVA procedures allow tests of nested competing models:

```
anova(CM.mod0, CM.mod1, CM.mod2)

## Analysis of Variance Table
##
## Model 1: LOR ~ 1
## Model 2: LOR ~ age
## Model 3: LOR ~ poly(age, 2)
##   Res.Df   RSS Df Sum of Sq     F Pr(>F)
## 1      7 25.61
## 2      6  6.34  1     19.28 17.23 0.0089 **
## 3      5  5.60  1      0.74  0.66 0.4525
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

(vcdExtra::`LRstats()` gives direct tests of each model, and AIC, BIC)
The linear model, $\ln(\theta) = \beta_0 + \beta_1$ Age, gives the best fit.

Going further: Bivariate response models

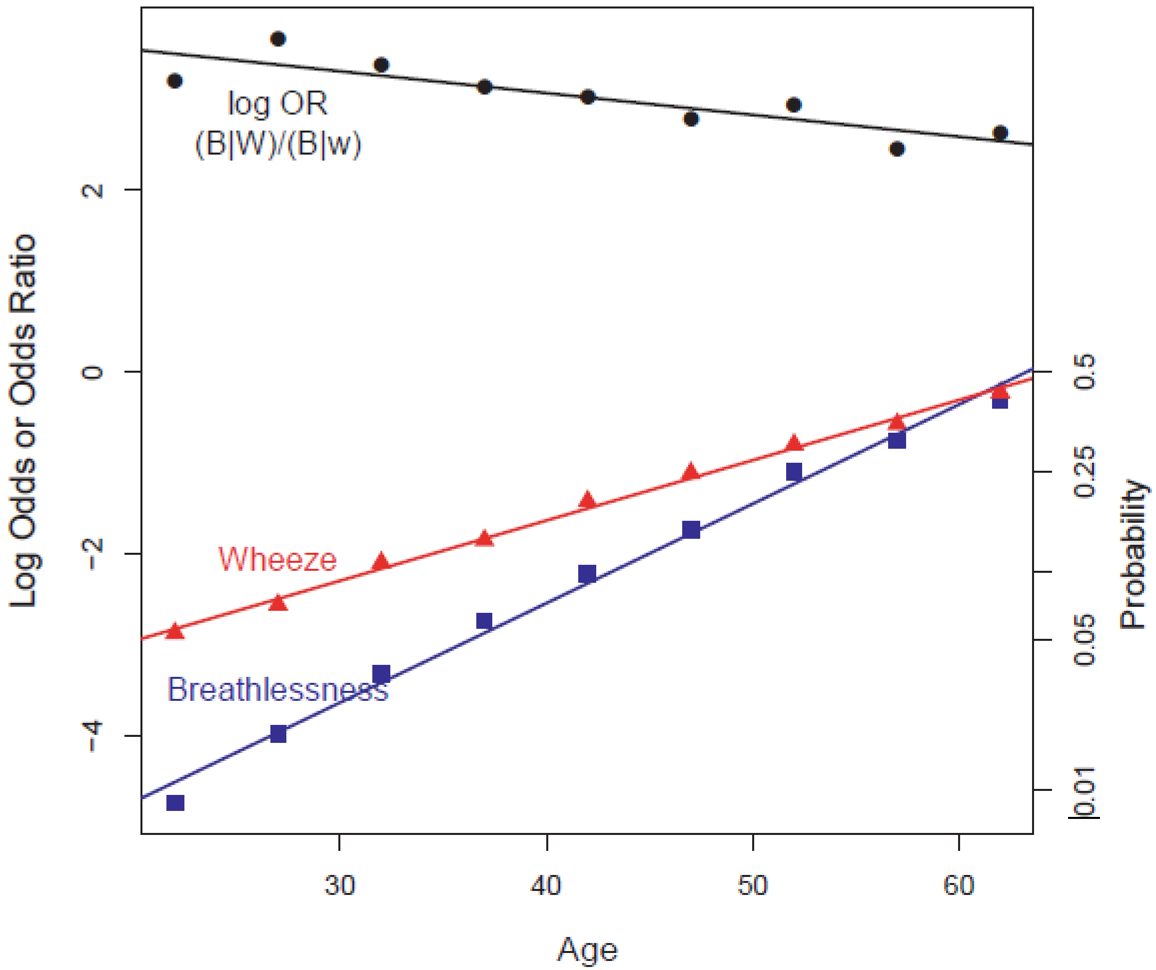
- In this example, breathlessness and wheeze are two binary responses
- A bivariate logistic response model fits simultaneously
  - the marginal log odds of each response, ψ_1, ψ_2 vs. predictors ($\boldsymbol{x}$)
  - the joint log odds ratio, ϕ_{12}, vs. $\boldsymbol{x}$
- This model has the form

$$\eta(\boldsymbol{x}) = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_{12} \end{pmatrix} \equiv \begin{pmatrix} \text{log odds}_1(\boldsymbol{x}) \\ \text{log odds}_2(\boldsymbol{x}) \\ \log \text{OR}_{12}(\boldsymbol{x}) \end{pmatrix} \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \\ \log \theta_{12} \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}_1^\mathsf{T} \beta_1 \\ \boldsymbol{x}_2^\mathsf{T} \beta_2 \\ \boldsymbol{x}_{12}^\mathsf{T} \beta_{12} \end{pmatrix}$$

  where $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_{12} \subset \boldsymbol{x}$
- For example, with one x, the following model allows linear effects on log odds, with a constant log odds ratio

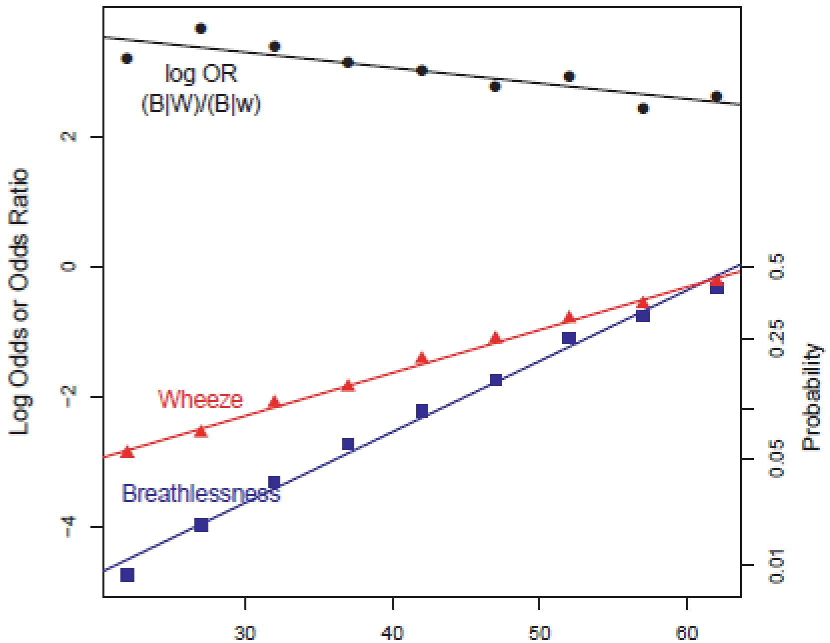
$$\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_{12} \end{pmatrix} = \begin{pmatrix} \alpha_1 + \beta_1 x \\ \alpha_2 + \beta_2 x \\ \log(\theta) \end{pmatrix} \quad (1)$$

Linear model for log odds and log odds ratios



Log odds & LORs have similar scales, so it is not terrible to plot them together

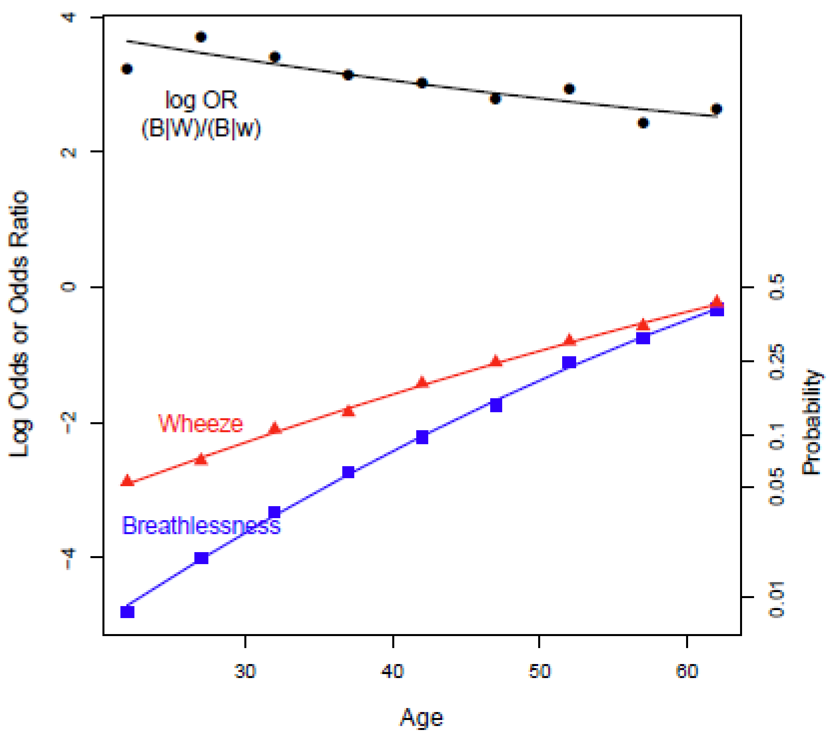
Linear model for log odds and log odds ratios



This data + model plot has a simple interpretation:

- Prevalence of breathlessness and wheeze both increase with age
- Breathlessness is less prevalent at young age, but increases faster
- Their association decreases approx. linearly, but is still strong

Quadratic model for log odds and log odds ratios



- Allowing quadratic fits in age serves as a sensitivity check
- The story is pretty much the same

Example: Attitudes toward corporal punishment

A four-way table, classifying 1,456 persons in Denmark (`Punishment` data in vcd).

- Attitude: approves moderate punishment of children ("moderate"), or refuses any punishment ("no")
- Memory: Person recalls having been punished as a child?
- Education: highest level (elementary, secondary, high)
- Age group: (15–24, 25–39, 40+)

		Age	15–24		25–39		40+	
Education	**Attitude**	**Memory**	Yes	No	Yes	No	Yes	No
Elementary	No		1	26	3	46	20	109
	Moderate		21	93	41	119	143	324
Secondary	No		2	23	8	52	4	44
	Moderate		5	45	20	84	20	56
High	No		2	26	6	24	1	13
	Moderate		1	19	4	26	8	17

Attitudes: Questions

Interest focuses on several questions:

- How does Attitude toward punishment depend on Memory, Education and Age?
  - Model log odds approve of moderate corporal punishment
  - Standard logit model:

```
glm(attitude ~ memory + education + age, data=Punishment,
weight=Freq, family=binomial)
```

- How does association between Attitude and Memory vary with Education and Age?
  - Model log odds ratio (Attitude, Memory)
  - Visualize: LOR plots

Log odds model for attitude

Fit the main-effects model for Attitude on other predictors:

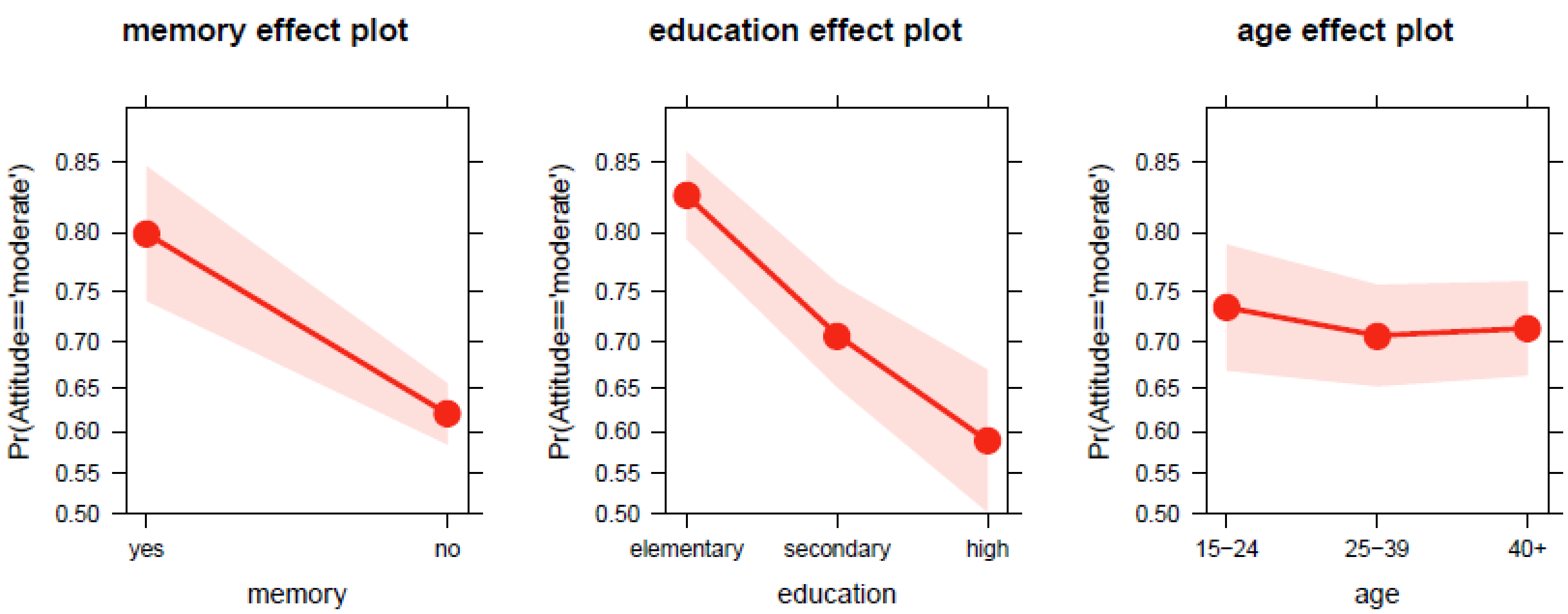
```
pun.logit <- glm(attitude ~ memory + education + age,
                 data=Punishment, weight=Freq, family=binomial)
Anova(pun.logit)

## Analysis of Deviance Table (Type II tests)
##
## Response: attitude
##           LR Chisq Df Pr(>Chisq)
## memory        29.5  1    5.6e-08 ***
## education     50.3  2    1.2e-11 ***
## age            0.6  2       0.73
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

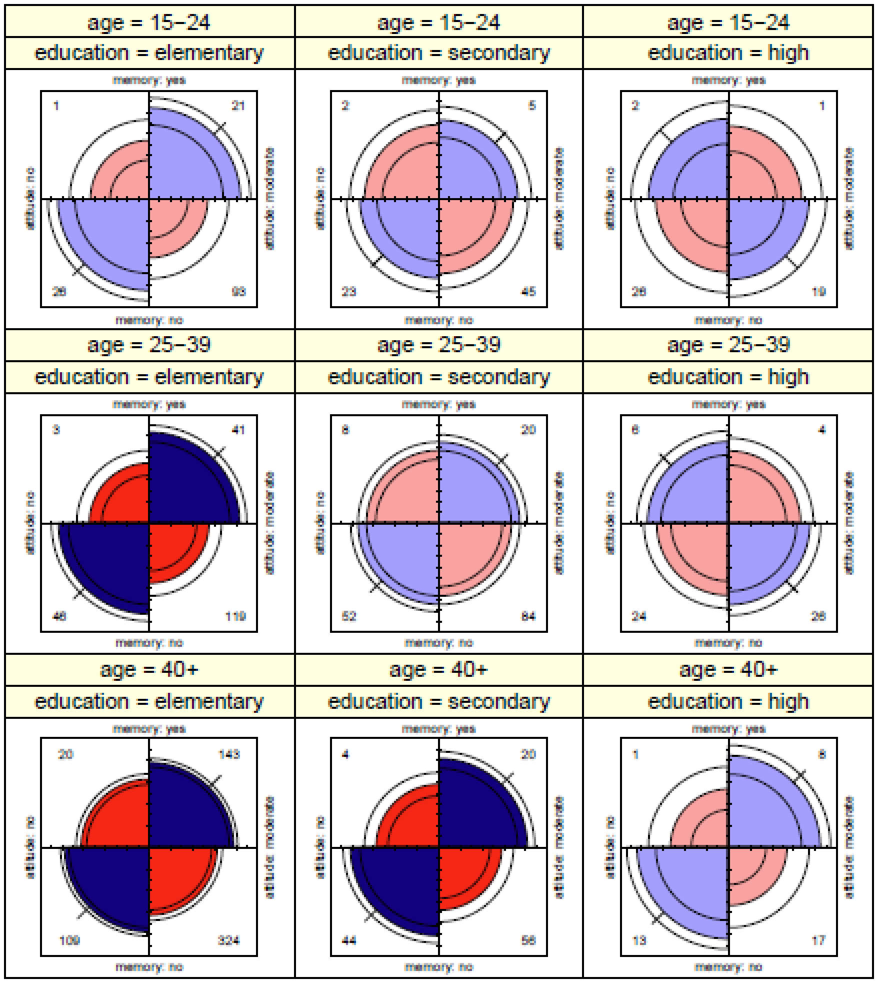
- Only Memory and Education have significant effects
- A more complex model with all two-way interactions showed no improvement

Attitude: Effect plots

- **Model plots**, showing fitted values for high-order terms in any model
- Other predictors averaged over in each plot
- Simple interpretation:
  - Those who remembered punishment as children more likely to approve
  - Approval decreases with education
  - No effect of age



Association of attitude with memory: Fourfold plots



How does the association of attitude and memory vary with education and age?

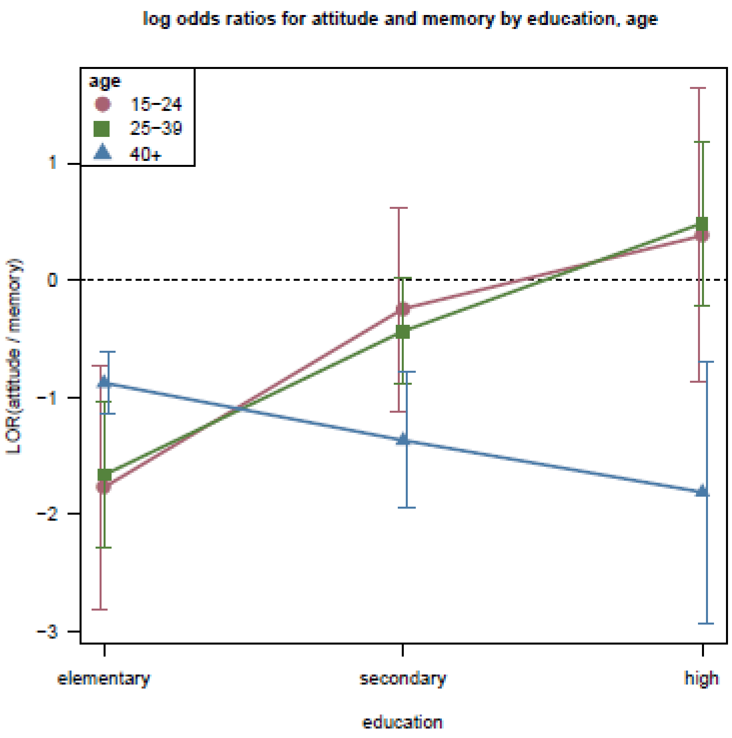
Each fourfold plot visualizes the log odds ratio between them

What's going on here?

Log odds ratio plot

```
(lor.pun <- loddsratio(punish))

## log odds ratios for memory and attitude by age, education
##
##        education
## age      elementary secondary    high
##   15-24     -1.7700   -0.2451  0.3795
##   25-39     -1.6645   -0.4367  0.4855
##   40+       -0.8777   -1.3683 -1.8112
```



- Structure now completely clear
- Little diff^{ce} between younger groups
- Opposite pattern for the 40+
- Fit an LOR model to confirm appearences (SEs large)!