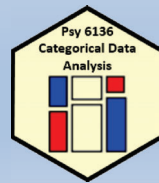


Discrete distributions



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Discrete distributions: Basic ideas

- Quantitative data: often assumed Normal (μ, σ^2) – unreasonable for CDA
- Binomial, Poisson, Negative binomial, ... are the building blocks for CDA
- Form the basis for modeling techniques
 - logistic regression, generalized linear models, Poisson regression
- Data:
 - outcome variable ($k = 0, 1, 2, \dots$)
 - counts of occurrences (n_k): accidents, words in text, males in families of size k

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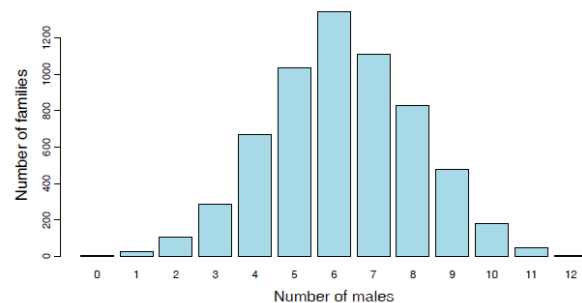
Examples: binomial

Human sex ratio (Geissler, 1889): Is there evidence that $\Pr(\text{male}) = 0.5$?

Saxony families

Saxony families with 12 children having $k = 0, 1, \dots, 12$ sons.

k	0	1	2	3	4	5	6	7	8	9	10	11	12
n_k	3	24	104	286	670	1033	1343	1112	829	478	181	45	7



3

Examples: count data

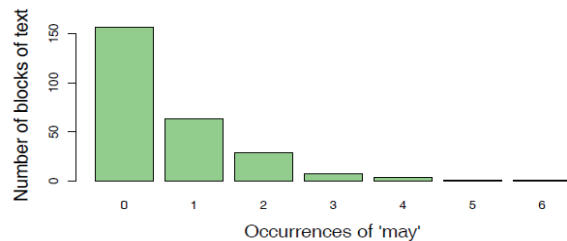
Federalist papers: Disputed authorship

- 77 essays by Alexander Hamilton, John Jay, James Madison to persuade voters to ratify the US constitution, all signed with pseudonym “Publius”
 - Who wrote each?
 - 65 known, 12 disputed (H & M both claimed sole authorship)
- Mosteller & Wallace (1984): analysis of frequency distⁿs of key “marker” words: from, **may**, whilst, ...
- e.g., blocks of 200 words: occurrences (k) of “may” in how many blocks (n_k)

```
> data(Federalist, package = "vcd")
> Federalist
nMay
  0   1   2   3   4   5   6
156 63 29  8  4  1  1
```

4

Count data: models



For each word (“from”, “may”, “whilst”, ...)

- Fit a probability model (Poisson, NegBin)
- Estimate parameters (λ , θ)
- → Calculate log Odds (Hamilton vs. Madison)
- → All 12 disputed papers most likely written by **Madison**

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Example: Type-token distributions

- Basic count, k : number of “types”; frequency, n_k : number of instances observed
 - Frequencies of distinct words in a book or literary corpus
 - Number of subjects listing words as members of the semantic category “fruit”
 - Distinct species of animals caught in traps
- Differs from other distributions in that the frequency for $k = 0$ is *unobserved*
- Distribution is often extremely skewed (J-shaped)

Table: Number of butterfly species n_k for which k individuals were collected

Individuals (k)	1	2	3	4	5	6	7	8	9	10	11	12	
Species (n_k)	118	74	44	24	29	22	20	19	20	15	12	14	
Individuals (k)	13	14	15	16	17	18	19	20	21	22	23	24	Sum
Species (n_k)	6	12	6	9	9	6	10	10	11	5	3	3	5

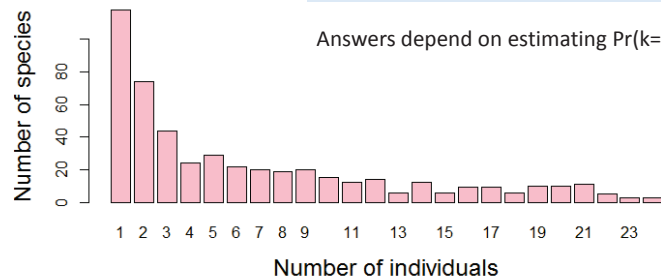
6

```
data(Butterfly, package="vcd")
barplot(Butterfly,
  xlab = "Number of individuals",
  ylab = "Number of species",
  col = "pink",
  cex.lab = 1.5)
```

Questions:

What is the total pop. of butterflies in Malaysia?
How many wolves remain in Canada NWT?
How many words did Shakespeare know?

Answers depend on estimating $\Pr(k=0)$



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Discrete distributions: Questions

- General questions
 - What process gave rise to the distribution?
 - What is the form: uniform, binomial, Poisson, negative binomial, ... ?
 - → Fit & estimate parameters
 - Visualize goodness of fit
 - → Use in some larger context to tell a story
- Examples
 - *Families in Saxony*: might expect $\text{Bin}(n=12, p)$; $p=0.5$?
 - *Federalist papers*: Perhaps $\text{Poisson}(\lambda)$
 - *Butterfly data*: Perhaps a log-series distribution?

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Fitting discrete distributions

Lack of fit:

- Lack of fit tells us something about the **process** giving rise to the data
- Poisson: assumes constant small probability of the basic event
- Binomial: assumes constant probability and independent trials
- Negative binomial: allows for **overdispersion**, relative to Poisson

Motivation:

- Models for more complex categorical data use these basic discrete distributions
- Binomial (with predictors) → logistic regression
- Poisson (with predictors) → poisson regression, loglinear models
- ⇒ many of these are special cases of **generalized linear models**

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Common discrete distributions

Discrete distributions are characterized by a probability function, $\Pr(X = k) \equiv p(k)$, that the random variable X has value k .

- Common discrete distributions have the following forms:

Discrete distribution	Probability function, $p(k)$	Parameters
Binomial	$\binom{n}{k} p^k (1-p)^{n-k}$	$p = \Pr(\text{success})$; $n = \# \text{ trials}$
Poisson	$e^{-\lambda} \lambda^k / k!$	$\lambda = \text{mean}$
Negative binomial	$\binom{n+k-1}{k} p^n (1-p)^k$	p ; $n = \# \text{ successful trials}$
Geometric	$p(1-p)^k$	p
Logarithmic series	$\theta^k / [-k \log(1-\theta)]$	θ

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Discrete distributions: R

R functions: {d, p, q, r}

- **d**___ density function, $\Pr(X=k) = p(k)$
- **p**___ cumulative probability, $F(k) = \sum_{X \leq k} p(k)$
- **q**___ quantile function, find $k = F^{-1}(p)$, smallest value such that $F(k) \geq p$
- **r**___ random number generator

Discrete distribution	Density (pmf) function	Cumulative (CDF)	Quantile CDF^{-1}	Random # generator
Binomial	<code>dbinom()</code>	<code>pbinom()</code>	<code>qbinom()</code>	<code>rbinom()</code>
Poisson	<code>dpois()</code>	<code>ppois()</code>	<code>qpois()</code>	<code>rpois()</code>
Negative binomial	<code>dnbinom()</code>	<code>pnbinom()</code>	<code>qnbinom()</code>	<code>rnbinom()</code>
Geometric	<code>dgeom()</code>	<code>pgeom()</code>	<code>qgeom()</code>	<code>rgeom()</code>
Logarithmic series	<code>dlogseries()</code>	<code>plogseries()</code>	<code>qlogseries()</code>	<code>rlogseries()</code>

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Binomial distribution

The binomial distribution, $\text{Bin}(n, p)$,

$$\text{Bin}(n, p) : \Pr\{X = k\} \equiv p(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad k = 0, 1, \dots, n, \quad (1)$$

\swarrow # ways to get k out of n
 \swarrow $\Pr(k \text{ events})$
 \swarrow $\Pr(n-k \text{ non-events})$

arises as the distribution of the number of events of interest ("successes") which occur in n **independent trials** when the probability of the event on any one trial is the **constant** value $p = \Pr(\text{event})$.

Examples

- Toss 10 fair coins— how many heads? $\text{Bin}(10, 1/2)$
- Toss 12 fair dice— how many 5s or 6s? $\text{Bin}(12, 1/3)$

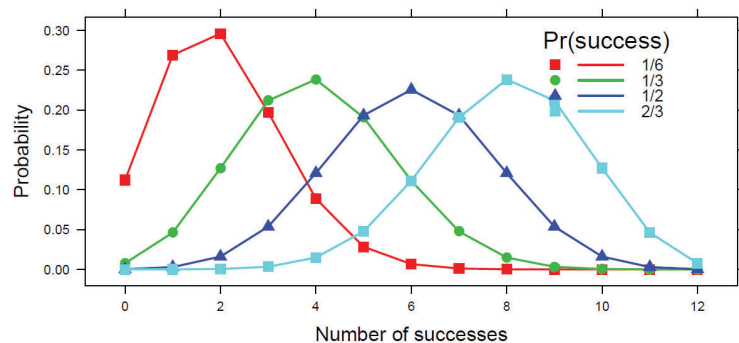
Mean, variance, skewness:

$$\begin{aligned} \text{Mean}[X] &= np & \text{MLE from data: } \hat{p} &= \frac{\bar{x}}{n} = \frac{\sum_k k \times n k / \sum_k n_k}{n} \\ \text{Var}[X] &= np(1-p) = npq \\ \text{Skew}[X] &= npq(q-p) \end{aligned}$$

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Binomial distribution

Binomial distributions for $k = 0, 1, 2, \dots, 12$ successes in $n=12$ trials, for 4 values of p



- Mean = $n p$
- Variance is maximum when $p = \frac{1}{2}$
- Skewed when $p \neq \frac{1}{2}$

DDAR Fig 3.9, pp 76-77

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Poisson distribution

The Poisson distribution, $\text{Pois}(\lambda)$,

$$\text{Pois}(\lambda) : \Pr\{X = k\} \equiv p(k) = \frac{e^{-\lambda} \lambda^k}{k!} \quad k = 0, 1, \dots \quad (2)$$

gives the probability of an event occurring $k = 0, 1, 2, \dots$ times over a *large number of independent* trials, when the probability, p , that the event occurs on any one trial (in time or space) is *small and constant*.

Examples:

- Number of highway accidents at some given location
- Defects in a manufacturing process
- Number of goals scored in soccer games

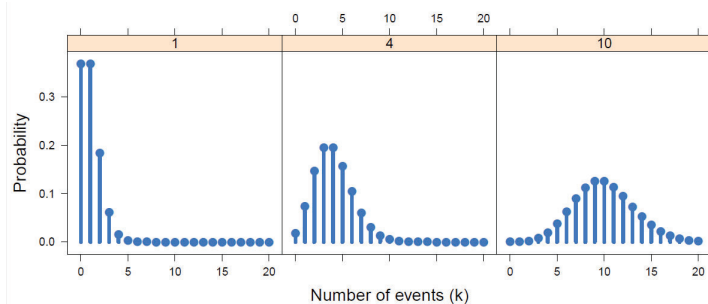
Table: Total goals scored in 380 games in the Premier Football League, 1995/95 season

Total goals	0	1	2	3	4	5	6	7
Number of games	27	88	91	73	49	31	18	3

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Poisson distribution

Poisson distributions for $\lambda = 1, 4, 10$



DDAR Fig 3.10, p 81

Mean, variance, skewness:

$$\text{Mean}[X] = \lambda$$

$$\text{Var}[X] = \lambda$$

$$\text{Skew}[X] = \lambda^{-1/2}$$

$$\text{MLE: } \hat{\lambda} = \bar{x}$$

Properties:

Sum of $\text{Pois}(\lambda_1, \lambda_2, \lambda_3, \dots) = \text{Pois}(\sum \lambda_i)$

Approaches $N(\lambda, \lambda)$ as $n \rightarrow \infty$

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Negative binomial distribution

The Negative binomial distribution, $\text{NBin}(n, p)$,

$$\text{NBin}(n, p) : \Pr\{X = k\} \equiv p(k) = \binom{n+k-1}{k} p^n (1-p)^k \quad k = 0, 1, \dots, \infty$$

is a *waiting time* distribution. It arises when n trials are observed with constant probability p of some event, and we ask how many *non-events* (failures), k , it takes to observe n *successful events*.

Example: Toss a coin; what is probability of getting $k = 0, 1, 2, \dots$ tails before $n = 3$ heads?

This distribution is often used as an alternative to the Poisson when

- constant probability p or *independence* are violated
- variance is greater than the mean (*overdispersion*: $\text{Var}[X] > \text{Mean}[X]$)

$$\text{Mean}(X) = nq/p = \mu$$

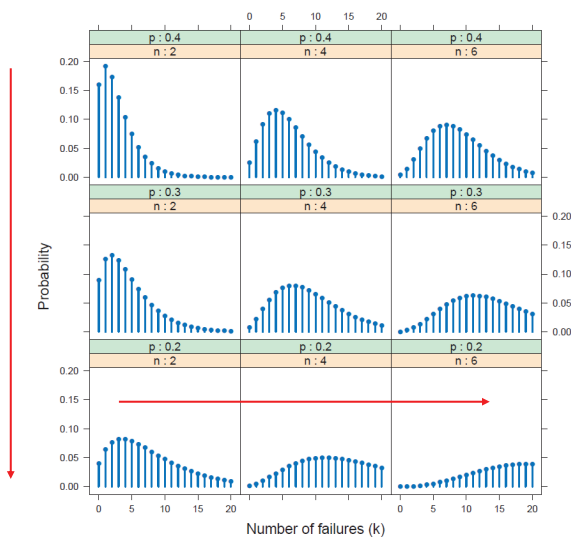
$$\text{Var}(X) = nq/p^2$$

$$\text{Skew}(X) = \frac{2-p}{\sqrt{nq}}$$

$$\text{Mean}(X) = \mu = \frac{n(1-p)}{p} \implies p = \frac{n}{n+\mu}$$

$$\text{Var}(X) = \frac{n(1-p)}{p^2} \implies \text{Var}(X) = \mu + \frac{\mu^2}{n}$$

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Negative binomial distributions for $n = 2, 4, 6$
 $p = 0.2, 0.3, 0.4$

Mean:
 Increases with n
 Decreases with p

DDAR Fig 3.13, p 85

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Fitting discrete distributions

Fitting a discrete distribution involves the following steps:

1. Estimate the parameter(s) from the data, e.g., p for binomial, λ for Poisson, etc. Typically done using maximum likelihood, but some distributions have simple expressions:
 - Binomial, $\hat{p} = \sum k n_k / (n \sum n_k) = \text{mean} / n$
 - Poisson, $\hat{\lambda} = \sum k n_k / \sum n_k = \text{mean}$
2. Calculate fitted probabilities, $\hat{p}(k)$ for the distribution, and then fitted frequencies, $N\hat{p}(k)$.
3. Assess Goodness of fit: Pearson χ^2 or likelihood-ratio G^2

$$\chi^2 = \sum_{k=1}^K \frac{(n_k - N\hat{p}_k)^2}{N\hat{p}_k} \quad G^2 = \sum_{k=1}^K n_k \log\left(\frac{n_k}{N\hat{p}_k}\right)$$

Both have asymptotic chisquare distributions, χ^2_{K-s} with s estimated parameters, under the hypothesis that the data follows the chosen distribution.

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Fitting & graphing discrete distributions

In R, the **vcd** and **vcdExtra** packages provide functions to fit, visualize and diagnose discrete distributions

- **Fitting:** `goodfit()` fits uniform, binomial, Poisson, neg bin, geometric, logseries, ...
- **Graphing:** `rootogram()` assess departure between observed, fitted counts
- **Ord plot:** `Ordplot()` diagnose form of a discrete distribution
- **Robust plots:** `distplot()` handle problems with discrepant counts

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Example: Saxony families

```
> data(Saxony, package="vcd")
> Saxony
nMales
  0    1    2    3    4    5    6    7    8    9   10   11   12
3   24  104  286  670 1033 1343 1112  829  478  181   45    7
```

Use `goodfit()` to fit the binomial; test with `summary()`

```
> Sax.fit <- goodfit(Saxony, type = "binomial", par=list(size=12))
> summary(Sax.fit)

Goodness-of-fit test for binomial distribution

      X^2 df P(> X^2)
Likelihood Ratio  97 11 6.98e-16
```

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Example: Saxony families

The `print()` method for `goodfit` objects shows the details

```
> Sax.fit # print
```

Observed and fitted values for binomial distribution with parameters estimated by 'ML'

count	observed	fitted	pearson	residual
0	3	0.933		2.140
1	24	12.089		3.426
2	104	71.803		3.800
3	286	258.475		1.712
4	670	628.055		1.674
5	1033	1085.211		-1.585
6	1343	1367.279		-0.657
7	1112	1265.630		-4.318
8	829	854.247		-0.864
9	478	410.013		3.358
10	181	132.836		4.179
11	45	26.082		3.704
12	7	2.347		3.037

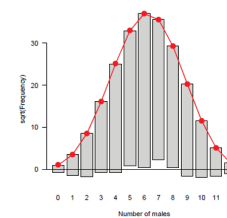
Pay attention to the **pattern** & magnitudes of residuals, d_k

$$\text{Pearson } \chi^2 = \sum d_k^2$$

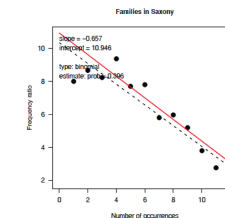
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Graphing discrete distributions

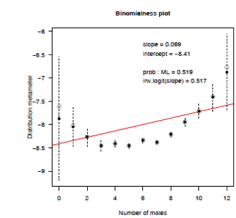
Rootograms



Ord plots



Robust distribution plots



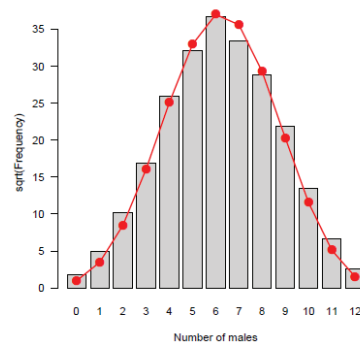
22

What's wrong with simple histograms?

Discrete distributions are often graphed as histograms, with a theoretical fitted distribution superimposed

The `plot()` method for `goodfit` objects provides some alternatives

```
> plot(Sax.fit, type = "standing", xlab = "Number of males")
```



Problems:

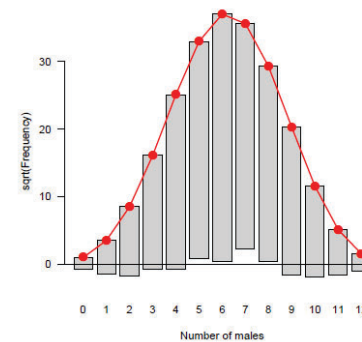
- Largest frequencies dominate
- Must assess deviations vs. the fitted curve

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Hanging rootograms

```
> plot(Sax.fit, type = "hanging", xlab = "Number of males")
```

default



Tukey (1972, 1977):

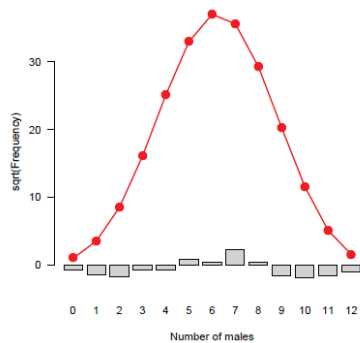
- shift histogram bars to the fitted curve
- → judge deviations vs. horizontal line.
- plot $\sqrt{\text{freq}}$ → smaller frequencies are emphasized.

We can now see clearly **where** the binomial doesn't fit

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Deviation rootograms

```
> plot(Sax.fit, type = "deviation", xlab = "Number of males")
```



Deviation rootogram:

- emphasize differences between observed and fitted frequencies
- bars now show the residuals (gaps) directly

There are more families with very low or very high number of sons than the binomial predicts.

Q: Why is this so much better than the lack-of-fit test?

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Example: Federalist papers

```
> data(Federalist, package="vcd")
> Federalist
nMay
 0  1  2  3  4  5  6
156 63 29  8  4  1  1
```

Fit the Poisson distribution

```
> Fed.fit0 <- goodfit(Federalist, type="poisson")
> summary(Fed.fit0)
```

Goodness-of-fit test for poisson distribution

	X ²	df	P(> X ²)
Likelihood Ratio	25.2	5	0.000125

This fits very poorly!

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Example: Federalist papers

Try the Negative binomial distribution

```
> Fed.fit1 <- goodfit(Federalist, type="nbinomial")
> summary(Fed.fit1)
```

Goodness-of-fit test for nbinomial distribution

	X ²	df	P(> X ²)
Likelihood Ratio	1.96	4	0.742

This now fits very well, indeed! Why?

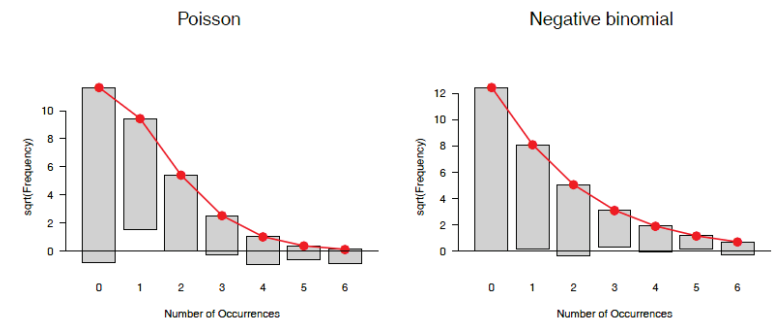
- Poisson assumes that the probability of a given word ("may") is constant across all blocks of text.
- Negative binomial allows the rate parameter λ to vary over blocks of text

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Federalist papers: Rootograms

Hanging rootograms for the Federalist papers data, comparing Poisson and Negative binomial

```
> plot(Fed.fit0, main = "Poisson")
> plot(Fed.fit1, main = "Negative binomial")
```

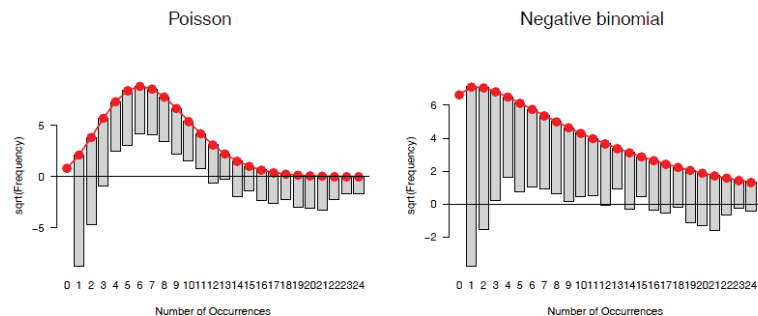


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Butterfly data

Both Poisson and Negative binomial are terrible fits! What to do??

```
But.fit1 <- goodfit(Butterfly, type="poisson")
But.fit2 <- goodfit(Butterfly, type="nbinomial")
plot(But.fit1, main="Poisson")
plot(But.fit2, main="Negative binomial")
```



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Ord plots: Diagnose form of distribution

How to tell which discrete distributions are likely candidates?

- Ord (1967): for each of Poisson, Binomial, Negative binomial, and Logarithmic series distributions,
 - plot of $k p_k / p_{k-1}$ against k is linear
 - signs of intercept and slope → determine the form, give rough estimates of parameters

Slope (b)	Intercept (a)	Distribution (parameter)	Parameter estimate
0	+	Poisson (λ)	$\lambda = a$
-	+	Binomial (n, p)	$p = b / (b - 1)$
+	+	Neg. binomial (n, p)	$p = 1 - b$
+	-	Log. series (θ)	$\theta = b$ $\theta = -a$

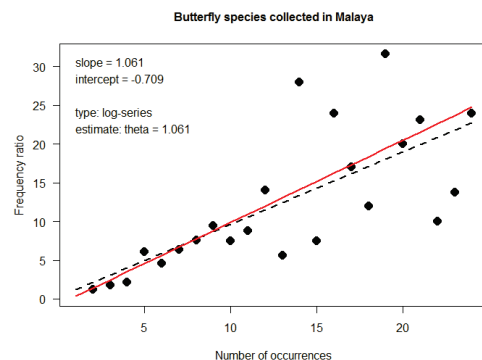
- Fit line by WLS, using $\sqrt{n_k - 1}$ as weights
- A heuristic method: doesn't always work, but often a good start.

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Ord plot: Examples

Butterfly data: The slope and intercept correctly diagnoses the log-series distribution

```
> Ord_plot(Butterfly,
  main = "Butterfly species collected in Malaya",
  gp=gpar(cex=1), pch=16)
```



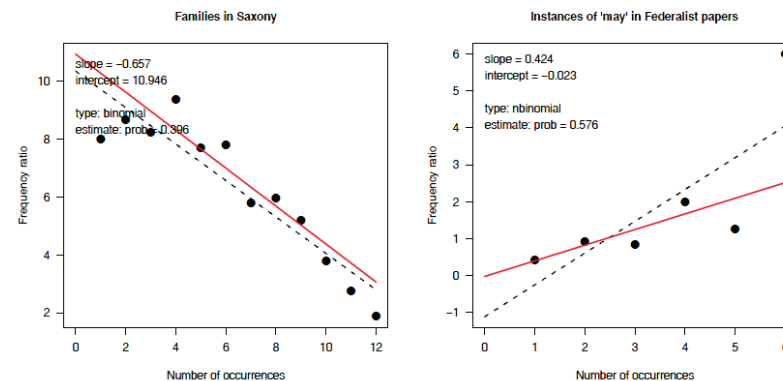
+ slope
- intercept
→ log-series

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Ord plots: Examples

Ord plots for the Saxony and Federalist data

```
> Ord_plot(Saxony, main = "Families in Saxony", gp=gpar(cex=1), pch=16)
> Ord_plot(Federalist, main = "Instances of 'may' in Federalist papers", gp=gpar(cex=1), pch=16)
```

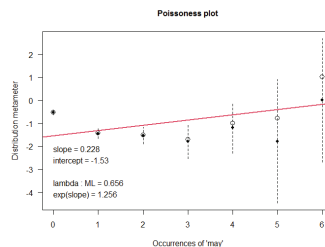


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Robust distribution plots

- Ord plots lack robustness
 - one discrepant frequency, n_k affects points for both k and $k + 1$
 - the use of WLS to fit the line is a small attempt to minimize this
- Robust plots for Poisson distribution (Hoaglin and Tukey, 1985)
 - For Poisson, plot **count metameter** $= \phi(n_k) = \log_e(k! n_k / N)$ vs. k
 - Linear relation \Rightarrow Poisson, slope gives $\hat{\lambda}$
 - CI for points, diagnostic (influence) plot
 - Implemented in `distplot()` in the `vcd` package

For the Poisson distribution, this is called a “poissonness plot”



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Poissonness plot: Details

- If the distribution of n_k is $\text{Poisson}(\lambda)$ for some fixed λ , then each observed frequency, $n_k \approx m_k = Np_k$.
- Then, setting $n_k = Np_k = e^{-\lambda} \lambda^k / k!$, and taking logs of both sides gives

$$\log(n_k) = \log N - \lambda + k \log \lambda - \log k!$$

which can be rearranged to

$$\phi(n_k) \equiv \log\left(\frac{k! n_k}{N}\right) = -\lambda + (\log \lambda) k$$

- \Rightarrow if the distribution is Poisson, plotting $\phi(n_k)$ vs. k should give a line with
 - intercept $= -\lambda$
 - slope $= \log \lambda$
- Nonlinear relation \rightarrow distribution is *not* Poisson
- Hoaglin and Tukey (1985) give details on calculation of confidence intervals and influence measures.

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Other distributions

This idea extends readily to other discrete data distributions:

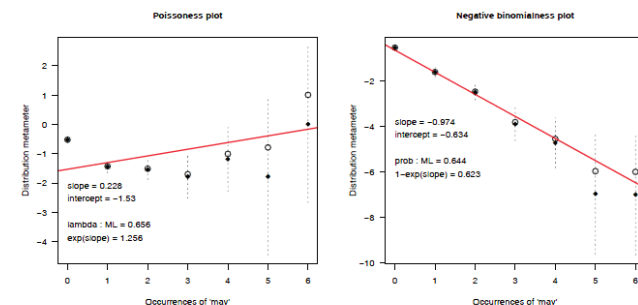
- The binomial, Poisson, negative binomial, geometric and logseries distributions are all members of a general **power series family** of discrete distributions. See: *DDAR*, Table 3.10 for details.
- This allows all of these to be represented in a plot of a suitable count metameter, $\phi(n_k)$ vs. k . See: *DDAR*, Table 3.12 for details.
- In these plots, a straight line confirms that the data follow the given distribution.
- Confidence intervals around the points indicate **uncertainty** for the count metameter.
- The slope and intercept of the line give **estimates** of the distribution parameters.

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distplot: Federalist

Try both Poisson & Negative binomial

```
distplot(Federalist, type="poisson", xlab="Occurrences of 'may'")
distplot(Federalist, type="nbinomial", xlab="Occurrences of 'may'")
```



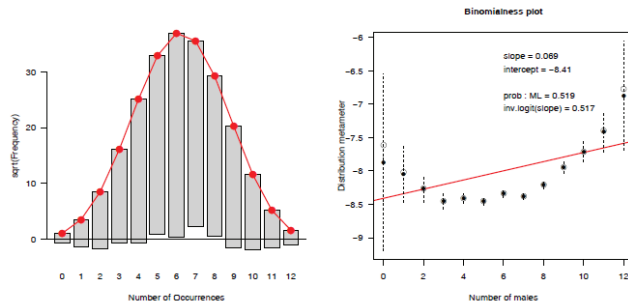
Again, the Poisson distribution is seen not to fit, while the Negative binomial appears reasonable.

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distplot: Saxony

For purported binomial distributions, the result is a “binomialness” plot

```
plot(goodfit(Saxony, type="binomial", par=list(size=12)))
distplot(Saxony, type="binomial", size=12, xlab="Number of males")
```



Both plots show heavier tails than the binomial distribution. `distplot()` is more sensitive in diagnosing this

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What have we learned?

Main points:

- Discrete distributions involve basic *counts* of occurrences of some event occurring with varying *frequency*.
- The ideas and methods for one-way tables are building blocks for analysis of more complex data.
- Commonly used discrete distributions include the binomial, Poisson, negative binomial, and logarithmic series distributions, all members of a *power series* family.
- Fitting observed data to a distribution → fitted frequencies, $N\hat{p}_k$, → goodness-of-fit tests (Pearson X^2 , LR G^2)
- R: `goodfit()` provides `print()`, `summary()` and `plot()` methods.
- Plotting with rootograms, Ord plots and generalized distribution plots can reveal *how* or *where* a distribution does not fit.

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What have we learned?

Some explanations:

- The Saxony data were part of a much larger data set from Geissler (1889) (Geissler in `vcdExtra`).
 - For the binomial, with families of size $n = 12$, our analyses give $\hat{p} = \Pr(\text{male}) = 0.52$.
 - Other analyses (using more complex models) conclude that p varies among families with the same size.
 - One explanation is that family decisions to have another child are influenced by the boy–girl ratio in earlier children.
- As suggested earlier, the lack of fit of the Poisson distribution for words in the Federalist papers can be explained by *context* of the writing:
 - Given “marker” words appear more or less often over time and subject than predicted by constant rates (λ) for a given author (Madison or Hamilton)
 - The negative binomial distribution fit much better.
 - The estimated parameters for these texts allowed assigning all 12 disputed papers to Madison.

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Looking ahead: PhdPubs data

Example 3.24 in DDAR gives data on the number of publications by PhD candidates in the last 3 years of study

```
data("PhdPubs", package = "vcdExtra")
table(PhdPubs$articles)
```

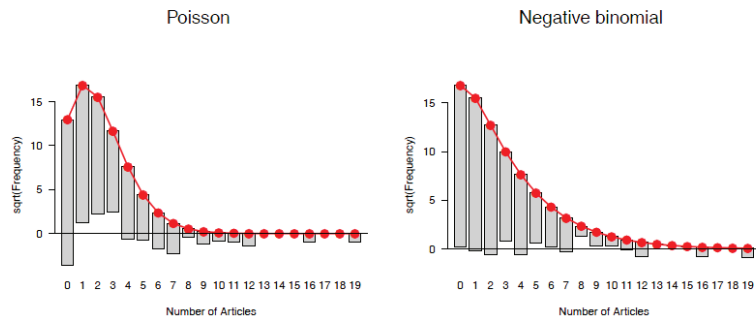
```
##
##  0  1  2  3  4  5  6  7  8  9 10 11 12 16 19
## 275 246 178 84 67 27 17 12 1 2 1 1 2 1 1
```

- There are predictors: gender, marital status, number of children, prestige of dept., # pubs by student’s mentor
- We fit such models with `glm()`, but need to specify the form of the distribution
- Ignoring the predictors for now, a baseline model could be `glm(articles ~ 1, data=PhdPubs, family = "poisson")`

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Looking ahead: PhdPubs

```
plot(goodfit(PhdPubs$articles), xlab = "Number of Articles",
     main = "Poisson")
plot(goodfit(PhdPubs$articles, type = "nbinomial"),
     xlab = "Number of Articles", main = "Negative binomial")
```



Poisson doesn't fit: Need to account for excess 0s (some never published)
Neg binomial: Sort of OK, but should take predictors into account

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Looking ahead: Count data models

Count data regression models (DDAR Ch 11)

- Include predictors
- Allow different distributions for unexplained variation
- Provide tests of one model vs. another
- Special models handle the problems of excess zeros: `zeroInfl()`, `hurdle()`

```
# predictors: female, married, kid5, phdprestige, mentor
phd.pois <- glm(articles ~ ., data=PhdPubs, family=poisson)
phd.nbin <- glm.nb(articles ~ ., data=PhdPubs)

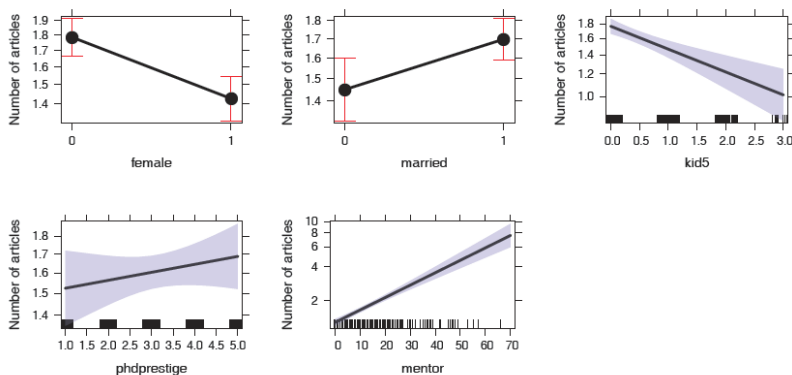
lrstats(phd.pois, phd.nbin)

## Likelihood summary table:
##           AIC   BIC LR Chisq Df Pr(>Chisq)
## phd.pois 3313 3342    1634  909   <2e-16 ***
## phd.nbin 3135 3169    1004  909    0.015 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

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Looking ahead: Effect plots

Effect plots show the predicted values for each term in a model, averaging over all other factors.



These are better visual summaries for a model than a table of coefficients.

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Summary

- Discrete distributions are the building blocks for categorical data analysis
 - Typically consist of basic counts of occurrences, with varying frequencies
 - Most common: binomial, Poisson, negative binomial
 - Others: geometric, log-series
- Fit with `goodfit()`; plot with `rootogram()`
 - Diagnostic plots: `Ord_plot()`, `distplot()`
- Models with predictors
 - Binomial → logistic regression
 - Poisson → poisson regression; loglinear models
 - These are special cases of generalized linear models

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