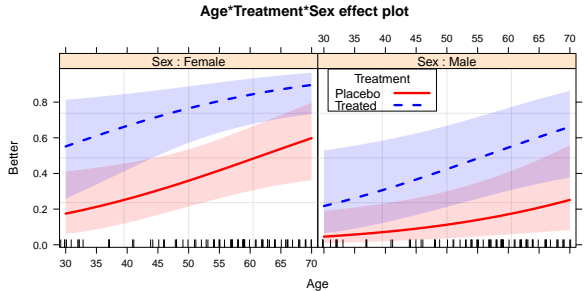
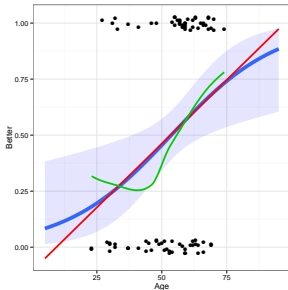


Logistic Regression

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Psych 6136

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Model-based methods: Overview

Structure

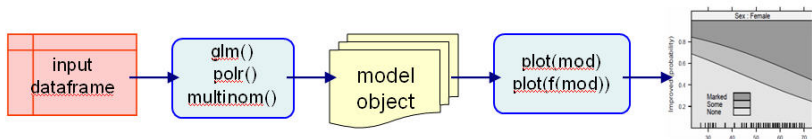
- Explicitly assume some probability distribution for the data, e.g., binomial, Poisson, ...
- Distinguish between the **systematic** component— explained by the model— and a **random** component, which is not
- Allow a compact summary of the data in terms of a (hopefully) small number of parameters

Advantages

- Inferences: hypothesis tests **and confidence intervals**
- Methods for model selection: adjust balance between goodness-of-fit and parsimony
- Predicted values give **model-smoothed** summaries for plotting
- \implies Interpret the fitted model graphically

Fitting and graphing: Overview

Object-oriented approach in R:



- Fit model (`obj <- glm(...)`) → a **model object**
- `print(obj)` and `summary(obj)` → numerical results
- `anova(obj)` and `Anova(obj)` → tests for model terms
- `update(obj)`, `add1(obj)`, `drop1(obj)` for model selection

Plot methods:

- `plot(obj)` often gives diagnostic plots
- Other plot methods:
 - Mosaic plots: `mosaic(obj)` for "loglm" and "glm" objects
 - Effect plots: `plot(Effect(obj))` for nearly all linear models
 - Influence plots (`car`): `influencePlot(obj)` for "glm" objects

Objects and methods

How this works:

- Model objects have a "class" attribute:
 - `loglm()`: "loglm"
 - `glm()`: `c("glm", "lm")` — inherits also from `lm()`
- Class-specific methods have names like `method.class`, e.g., `plot.glm()`, `mosaic.loglm()`
- Generic functions (`print()`, `summary()`, `plot()` ...) call the appropriate method for the class

```
arth.mod <- glm(Better ~ Age + Sex + Treatment, data=Arthritis)
class(arth.mod)
```

```
## [1] "glm" "lm"
```

Objects and methods

Methods for "glm" objects

```
library(MASS); library(vcdExtra)
methods(class="glm")
```

```
## [1] add1.glm*          addterm.glm*
## [3] anova.glm*         Anova.glm*
## [5] asGlm.glm*         assoc.glm
## [7] avPlot.glm*        Boot.glm*
## [9] bootCase.glm*      ceresPlot.glm*
## [11] coeftest.glm*      confidenceEllipse.glm*
## [13] confint.glm*       cooks.distance.glm*
## [15] deviance.glm*      drop1.glm*
## [17] dropterm.glm*      effects.glm*
## [19] extractAIC.glm*    family.glm*
## [21] formula.glm*       gamma.shape.glm*
## [23] influence.glm*     leveragePlot.glm*
## [25] linearHypothesis.glm* logLik.glm*
## [27] mmp.glm*           model.frame.glm*
## [29] modFit.glm         mosaic.glm
## [31] ncvTest.glm*       nobs.glm*
## [33] predict.glm        print.glm*
## [35] profile.glm*       qqPlot.glm*
## [37] residualPlot.glm*  residualPlots.glm*
## [39] residuals.glm      rstandard.glm*
## [41] rstudent.glm*      sieve.glm
## [43] sigmaHat.glm*      summarise.glm*
## [45] summary.glm        vcov.glm*
```

Objects and methods

Some available `plot()` methods:

```
methods("plot")
```

```
## [1] plot.acf*          plot.ACF*
## [3] plot.augPred*      plot.coef.mer*
## [5] plot.compareFits*  plot.correspondence*
## [7] plot.data.frame*   plot.decomposed.ts*
## [9] plot.default       plot.dendrogram*
## [11] plot.density*      plot.ecdf
## [13] plot.eff*          plot.efflist*
## [15] plot.effpoly*      plot.factor*
## [17] plot.formula*      plot.function
## [19] plot.gam*          plot.ggplot*
## [21] plot.gls*          plot.gnm*
## [23] plot.goodfit*      plot.gtable*
## [25] plot.hclust*       plot.histogram*
## [27] plot.HLtest*       plot.HoltWinters*
## [29] plot.intervals.lmList* plot.isoreg*
## [31] plot.jam*          plot.lda*
## [33] plot.lm*           plot.lme*
## [35] plot.lmList*       plot.lmList.confint*
## [37] plot.loddsratio*   plot.loglm*
## [39] plot.mca*          plot.medpolish*
## [41] plot.merMod*       plot.mlm*
## [43] plot.mlm.efflist*  plot.nffGroupedData*
## [45] plot.nfnGroupedData* plot.nls*
## [47] plot.nmGroupedData* plot.oddsratio*
```

Modeling approaches: Overview

• Association models

• Loglinear models

(Contingency table form)

[Admit][GenderDept]

[AdmitDept][GenderDept]

[AdmitDept][AdmitGender][GenderDept]

• Poisson GLMs

(Frequency data frame)

$\text{Freq} \sim \text{Admit} + \text{Gender} * \text{Dept}$

$\text{Freq} \sim \text{Admit} * \text{Dept} + \text{Gender} * \text{Dept}$

$\text{Freq} \sim \text{Admit} * \text{Dept} + \text{Admit} * \text{Gender} + \text{Gender} * \text{Dept}$

• Ordered variables

$\text{Freq} \sim \text{right} + \text{left} + \text{Diag}(\text{right}, \text{left})$

$\text{Freq} \sim \text{right} + \text{left} + \text{Symm}(\text{right}, \text{left})$

• Response models

• Binary response

• Categorical predictors: Logit models

$\text{logit}(\text{Admit}) \sim 1$

$\text{logit}(\text{Admit}) \sim \text{Dept}$

$\text{logit}(\text{Admit}) \sim \text{Dept} + \text{Gender}$

• Continuous/mixed predictors:

Logistic regression models

$\text{Pr}(\text{Admit}) \sim \text{Dept} + \text{Age} + \text{GRE}$

• Polytomous response

• Ordinal: proportional odds model

$\text{Improve} \sim \text{Age} + \text{Sex} + \text{Treatment}$

• General: multinomial model

$\text{WomenWork} \sim \text{Kids} + \text{HusbandInc}$

Logistic regression models

Response variable

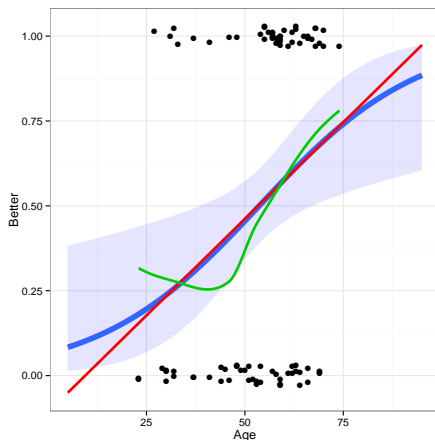
- Binary response: success/failure, vote: yes/no
- Binomial data: x successes in n trials (grouped data)
- Ordinal response: none < some < severe depression
- Polytomous response: vote Liberal, Tory, NDP, Green

Explanatory variables

- Quantitative regressors: age, dose
- Transformed regressors: $\sqrt{\text{age}}$, $\log(\text{dose})$
- Polynomial regressors: age^2 , age^3 , \dots (or better: splines)
- Categorical predictors: treatment, sex (dummy variables, contrasts)
- Interaction regressors: $\text{treatment} \times \text{age}$, $\text{sex} \times \text{age}$

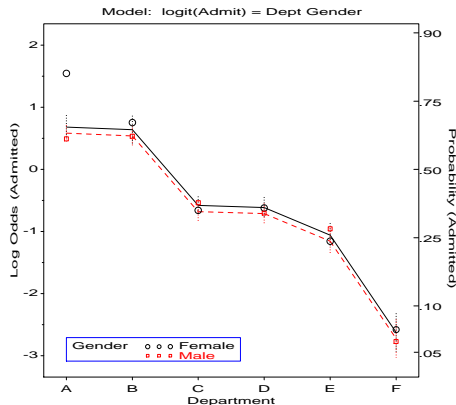
This is exactly the same as in classical ANOVA, regression models

Arthritis treatment data



- The response variable, `Improved` is **ordinal**: `"None" < "Some" < "Marked"`
- A binary logistic model can consider just `Better = (Improved>"None")`
- Other important predictors: Sex, Treatment
- Main Q: how does treatment affect outcome?
- How does this vary with Age and Sex?
- This plot shows the binary observations, with several model-based smoothings

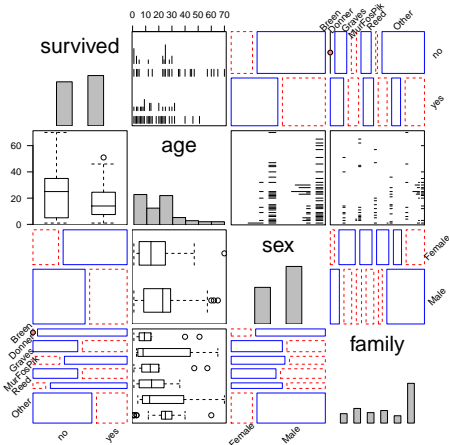
Berkeley admissions data



- Admit/Reject can be considered a **binomial response** for each Dept and Gender
- Logistic regression here is analogous to an ANOVA model, but for $\text{log odds}(\text{Admit})$
- (With categorical predictors, these are often called **logit** models)
- Every such model has an equivalent **loglinear** model form.
- This plot shows fitted logits for the main effects model, $\text{Dept} + \text{Gender}$

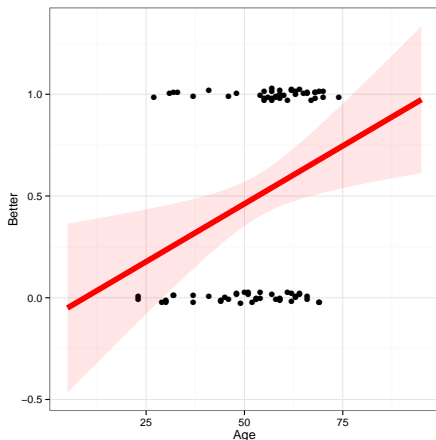
Survival in the Donner Party

- Binary response: **survived**
- Categorical predictors: **sex**, **family**
- Quantitative predictor: **age**
- Q: Is the effect of age linear?
- Q: Are there interactions among predictors?
- This is a **generalized pairs plot**, with different plots for each pair



Binary response: What's wrong with OLS?

- For a binary response, $Y \in (0, 1)$, want to predict $\pi = \Pr(Y = 1 | x)$
- A **linear probability model** uses classical linear regression (OLS)
- Problems:
 - Gives predicted values and CIs outside $0 \leq \pi \leq 1$
 - Homogeneity of variance is violated: $\mathcal{V}(\hat{\pi}) = \hat{\pi}(1 - \hat{\pi}) \neq \text{constant}$
 - Inferences, hypothesis tests are wrong!



OLS vs. Logistic regression

OLS regression:

- Assume $y|x \sim N(0, \sigma^2)$

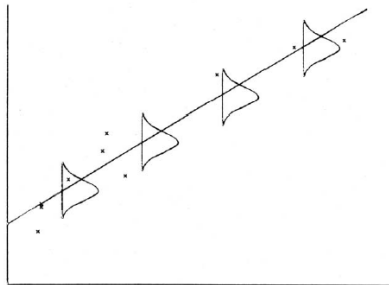


Fig. 2.1. Graphical representation of a simple linear normal regression.

Logistic regression:

- Assume $\Pr(y=1|x) \sim \text{binomial}(p)$

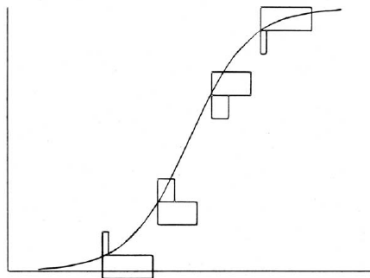
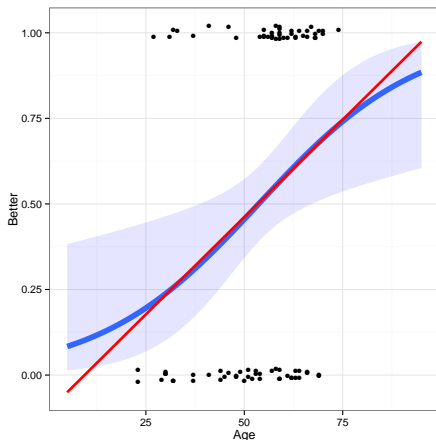


Fig. 2.2. Graphical representation of a simple linear logistic regression.

Logistic regression

- Logistic regression avoids these problems
- Models $\text{logit}(\pi_i) \equiv \log[\pi/(1 - \pi)]$
- logit is interpretable as “log odds” that $Y = 1$
- A related **probit** model gives very similar results, but is less interpretable
- For $0.2 \leq \pi \leq 0.8$ fitted values are close to those from linear regression.



Logistic regression: One predictor

For a single quantitative predictor, x , the simple **linear logistic regression model** posits a linear relation between the **log odds** (or **logit**) of $\Pr(Y = 1)$ and x ,

$$\text{logit}[\pi(x)] \equiv \log \left(\frac{\pi(x)}{1 - \pi(x)} \right) = \alpha + \beta x .$$

- When $\beta > 0$, $\pi(x)$ and the log odds increase as x increases; when $\beta < 0$ they decrease with x .
- This model can also be expressed as a model for the probabilities $\pi(x)$

$$\pi(x) = \text{logit}^{-1}[\pi(x)] = \frac{1}{1 + \exp[-(\alpha + \beta x)]}$$

Logistic regression: One predictor

The coefficients of this model have simple interpretations in terms of odds and log odds:

- The odds can be expressed as a **multiplicative** model

$$\text{odds}(Y = 1) \equiv \frac{\pi(x)}{1 - \pi(x)} = \exp(\alpha + \beta x) = e^{\alpha} (e^{\beta})^x . \quad (1)$$

Thus:

- β is the change in the log odds associated with a unit increase in x .
- The odds are multiplied by e^{β} for each unit increase in x .
- α is log odds at $x = 0$; e^{α} is the odds of a favorable response at this x -value.
- In R, use `exp(coef(obj))` to get these values.
- Another interpretation: In terms of probability, the slope of the logistic regression curve is $\beta\pi(1 - \pi)$
- This has the maximum value $\beta/4$ at $\pi = \frac{1}{2}$

Logistic regression models: Multiple predictors

- For a binary response, $Y \in (0, 1)$, let \mathbf{x} be a vector of p regressors, and π_i be the probability, $\Pr(Y = 1 \mid \mathbf{x})$.
- The logistic regression model is a linear model for the *log odds*, or *logit* that $Y = 1$, given the values in \mathbf{x} ,

$$\begin{aligned}\text{logit}(\pi_i) \equiv \log\left(\frac{\pi_i}{1 - \pi_i}\right) &= \alpha + \mathbf{x}_i^\top \boldsymbol{\beta} \\ &= \alpha + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip}\end{aligned}$$

- An equivalent (non-linear) form of the model may be specified for the probability, π_i , itself,

$$\pi_i = \{1 + \exp(-[\alpha + \mathbf{x}_i^\top \boldsymbol{\beta}])\}^{-1}$$

- The logistic model is also a *multiplicative* model for the odds of “success,”

$$\frac{\pi_i}{1 - \pi_i} = \exp(\alpha + \mathbf{x}_i^\top \boldsymbol{\beta}) = \exp(\alpha) \exp(\mathbf{x}_i^\top \boldsymbol{\beta})$$

Increasing x_{ij} by 1 increases $\text{logit}(\pi_i)$ by β_j , and multiplies the odds by e^{β_j} .

Fitting the logistic regression model

Logistic regression models are the special case of **generalized linear models**, fit in R using `glm(..., family=binomial)`
For this example, we define **Better** as any improvement at all:

```
data("Arthritis", package="vcd")  
Arthritis$Better <- as.numeric(Arthritis$Improved > "None")
```

Fit and print:

```
arth.logistic <- glm(Better ~ Age, data=Arthritis, family=binomial)  
arth.logistic  
  
##  
## Call:  glm(formula = Better ~ Age, family = binomial, data = Arthritis)  
##  
## Coefficients:  
## (Intercept)          Age  
##    -2.6421         0.0492  
##  
## Degrees of Freedom: 83 Total (i.e. Null);  82 Residual  
## Null Deviance:      116  
## Residual Deviance: 109  AIC: 113
```

The `summary()` method gives details:

```
summary(arth.logistic)

##
## Call:
## glm(formula = Better ~ Age, family = binomial, data = Arthritis)
##
## Deviance Residuals:
##      Min       1Q   Median       3Q      Max
## -1.5106  -1.1277   0.0794   1.0677   1.7611
##
## Coefficients:
##              Estimate Std. Error z value Pr(>|z|)
## (Intercept)  -2.6421     1.0732  -2.46    0.014 *
## Age           0.0492     0.0194   2.54    0.011 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for binomial family taken to be 1)
##
##      Null deviance: 116.45  on 83  degrees of freedom
## Residual deviance: 109.16  on 82  degrees of freedom
## AIC: 113.2
##
## Number of Fisher Scoring iterations: 4
```

Interpreting coefficients

```
coef(arth.logistic)
```

```
## (Intercept)      Age
##    -2.642071    0.049249
```

```
exp(coef(arth.logistic))
```

```
## (Intercept)      Age
##    0.071214    1.050482
```

```
exp(10*coef(arth.logistic)[2])
```

```
##      Age
##  1.6364
```

Interpretations:

- log odds(Better) increase by $\beta = 0.0492$ for each year of age
- odds(Better) multiplied by $e^\beta = 1.05$ for each year of age— a 5% increase
- over 10 years, odds(Better) are multiplied by $\exp(10 \times 0.0492) = 1.64$, a 64% increase.
- Pr(Better) increases by $\beta/4 = 0.0123$ for each year (near $\pi = \frac{1}{2}$)

Multiple predictors

The main interest here is the effect of Treatment. Sex and Age are **control variables**. Fit the **main effects** model (no interactions):

$$\text{logit}(\pi_i) = \alpha + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3}$$

where x_1 is *Age* and x_2 and x_3 are the factors representing *Sex* and *Treatment*, respectively. R uses dummy (0/1) variables for factors.

$$x_2 = \begin{cases} 0 & \text{if Female} \\ 1 & \text{if Male} \end{cases} \quad x_3 = \begin{cases} 0 & \text{if Placebo} \\ 1 & \text{if Treatment} \end{cases}$$

- α doesn't have a sensible interpretation here. Why?
- β_1 : increment in log odds(Better) for each year of age.
- β_2 : difference in log odds for male as compared to female.
- β_3 : difference in log odds for treated vs. the placebo group

Multiple predictors: Fitting

Fit the main effects model. Use $I(\text{Age}-50)$ to center Age, making α interpretable.

```
arth.logistic2 <- glm(Better ~ I(Age-50) + Sex + Treatment,
                      data=Arthritis, family=binomial)
```

`coeftest()` in `lmtest` gives just the tests of coefficients provided by `summary()`:

```
library(lmtest)
coeftest(arth.logistic2)

##
## z test of coefficients:
##
##              Estimate Std. Error z value Pr(>|z|)
## (Intercept)   -0.5781    0.3674   -1.57    0.116
## I(Age - 50)     0.0487    0.0207    2.36    0.018 *
## SexMale        -1.4878    0.5948   -2.50    0.012 *
## TreatmentTreated 1.7598    0.5365    3.28    0.001 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Interpreting coefficients

```
cbind(coef=coef(arth.logistic2),
      OddsRatio=exp(coef(arth.logistic2)), exp(confint(arth.logistic2)))
```

	coef	OddsRatio	2.5 %	97.5 %
## (Intercept)	-0.5781	0.561	0.2647	1.132
## I(Age - 50)	0.0487	1.050	1.0100	1.096
## SexMale	-1.4878	0.226	0.0652	0.689
## TreatmentTreated	1.7598	5.811	2.1187	17.727

- $\alpha = -0.578$: At age 50, females given placebo have odds(Better) of $e^{-0.578} = 0.56$.
- $\beta_1 = 0.0487$: Each year of age multiplies odds(Better) by $e^{0.0487} = 1.05$, a 5% increase.
- $\beta_2 = -1.49$: Males $e^{-1.49} = 0.26 \times$ less likely to show improvement as females. (Or, females $e^{1.49} = 4.437 \times$ more likely than males.)
- $\beta_3 = 1.76$: Treated $e^{1.76} = 5.81 \times$ more likely Better than Placebo

Hypothesis testing: Questions

- **Overall test:** How does my model, $\text{logit}(\pi) = \alpha + \mathbf{x}^T \beta$ compare with the null model, $\text{logit}(\pi) = \alpha$?

$$H_0 : \beta_1 = \beta_2 = \dots = \beta_p = 0$$

- **One predictor:** Does x_k significantly improve my model? Can it be dropped?

$$H_0 : \beta_k = 0 \quad \text{given other predictors retained}$$

- **Lack of fit:** How does my model compare with a perfect model ([saturated model](#))?

For ANOVA, regression, these tests are carried out using F -tests and t -tests. In logistic regression (fit by [maximum likelihood](#)) we use

- F -tests \rightarrow likelihood ratio G^2 tests
- t -tests \rightarrow Wald z or χ^2 tests

Maximum likelihood estimation

- Likelihood, $\mathcal{L} = \Pr(\text{data} \mid \text{model})$, as function of model parameters
- For case i ,

$$\mathcal{L}_i = \begin{cases} p_i & \text{if } Y = 1 \\ 1 - p_i & \text{if } Y = 0 \end{cases} = p_i^{Y_i} (1 - p_i^{1-Y_i}) \quad \text{where} \quad p_i = 1 / (1 + \exp(\mathbf{x}_i \boldsymbol{\beta}))$$

- Under independence, joint likelihood is the product over all cases

$$\mathcal{L} = \prod_i^n p_i^{Y_i} (1 - p_i^{1-Y_i})$$

- \implies Find estimates $\hat{\boldsymbol{\beta}}$ that maximize $\log \mathcal{L}$. Iterative, but this solves the “estimating equations”

$$\mathbf{X}^T \mathbf{y} = \mathbf{X}^T \hat{\mathbf{p}}$$

Overall test

■ Likelihood ratio test (G^2)

- Compare *nested* models, similar to incremental F tests in OLS

- Let \mathcal{L}_1 = maximized likelihood for **our** model

$$\text{logit}(\pi_i) = \beta_0 + \mathbf{x}_i^T \boldsymbol{\beta} \quad \text{w/ } k \text{ predictors}$$

- Let \mathcal{L}_0 = maximized likelihood for **null** model

$$\text{logit}(\pi_i) = \beta_0 \quad \text{under} \quad H_0 : \beta_1 = \beta_2 = \cdots = \beta_k = 0$$

- Likelihood-ratio test statistic:

$$G^2 = -2 \log \left(\frac{L_0}{L_1} \right) = 2(\log L_1 - \log L_0) \sim \chi_k^2$$

Wald tests and confidence intervals

- Analogous to t -tests in OLS
- $H_0: \beta_i = 0$

$$z = \frac{b_k}{s(b_k)} \sim \mathcal{N}(0,1) \quad \text{or} \quad z^2 \sim \chi_1^2$$

(Wald chi-square)

- Confidence interval:

$$b_k \pm z_{1-\alpha/2} s(b_k)$$

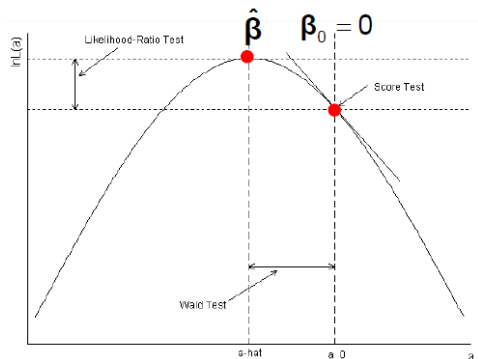
e.g.,

Analysis of Maximum Likelihood Estimates					
Parameter	DF	Estimate	Standard Error	Wald Chi-Square	Pr > ChiSq
Intercept	1	-4.5033	1.3074	11.8649	0.0006
sex Female	1	1.4878	0.5948	6.2576	0.0124
treat Treated	1	1.7598	0.5365	10.7596	0.0010
age	1	0.0487	0.0207	5.5655	0.0183

LR, Wald and score tests

Testing Global Null Hypothesis: BETA=0

Test	Chi-Square	DF	Pr > ChiSq
Likelihood Ratio	24.3859	3	<.0001
Score	22.0051	3	<.0001
Wald	17.5147	3	0.0006



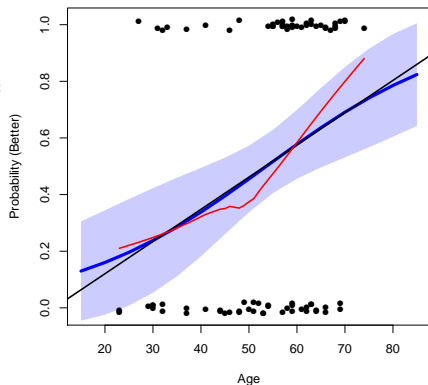
Different ways to measure departure from $H_0 : \beta = 0$

- LR test: diff in log L
- Wald test: $(\hat{\beta} - \beta_0)^2$
- Score test: slope at $\beta = 0$

Plotting logistic regression data

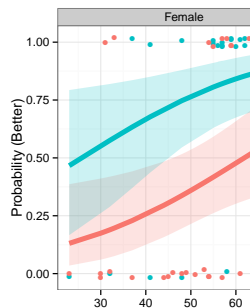
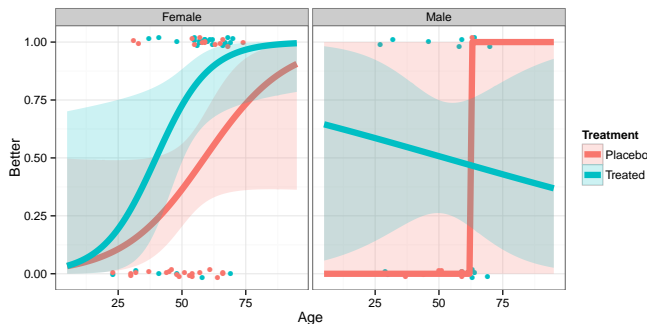
Plotting a binary response together with a fitted logistic model can be difficult because the 0/1 response leads to much overplotting.

- Need to **jitter** the points
- Useful to show the fitted logistic curve
- Confidence band gives a sense of uncertainty
- Adding a non-parametric (loess) smooth shows possible nonlinearity
- NB: Can plot either on the **response** scale (probability) or the **link** scale (logit) where effects are linear



Types of plots

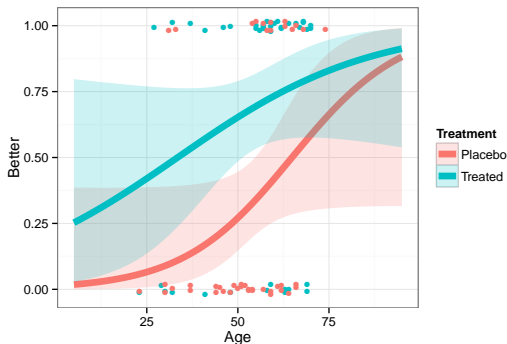
- **Conditional plots:** Stratified plot of Y or $\text{logit}(Y)$ vs. one X , conditioned by other predictors— only that subset is plotted for each
- **Full-model plots:** plots of fitted response surface, showing all effects; usually shown in separate panels
- **Effect plots:** plots of predicted effects for **terms** in the model, averaged over predictors not involved in a given term.



Conditional plots with ggplot2

Plot of Arthritis treatment data, by Treatment (ignoring Sex)

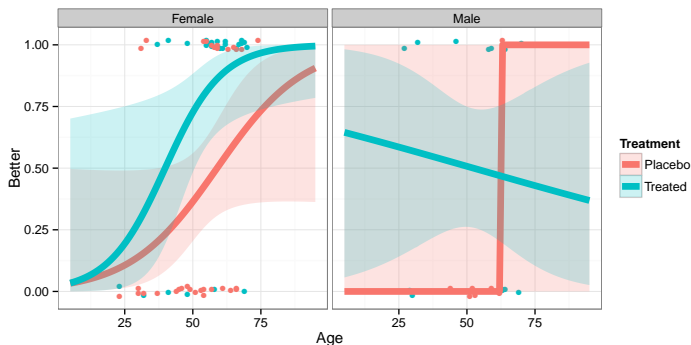
```
library(ggplot2)
gg <- ggplot(Arthritis, aes(Age, Better, color=Treatment)) +
  xlim(5, 95) + theme_bw() +
  geom_point(position = position_jitter(height = 0.02, width = 0)) +
  stat_smooth(method = "glm", family = binomial, alpha = 0.2,
             aes(fill=Treatment), size=2.5, fullrange=TRUE)
gg # show the plot
```



Conditional plots with ggplot2

Conditional plot, faceted by Sex

```
gg + facet_wrap(~ Sex)
```



The data is too thin for males to estimate each regression separately

Full-model plots

Full-model plots show the fitted values on the logit scale or on the response scale (probability), usually with confidence bands. This often requires a bit of custom programming.

Steps:

- Obtain fitted values with `predict(model, se.fit=TRUE)` — `type="link"` (logit) is the default
- Can use `type="response"` for probability scale
- Join this to your data (`cbind()`)
- Plot as you like: `plot()`, `ggplot()`, ...

```
arth.fit2 <- cbind(Arthritis,  
                  predict(arth.logistic2, se.fit = TRUE))  
head(arth.fit2[, -9], 4)
```

##	ID	Treatment	Sex	Age	Improved	Better	fit	se.fit
## 1	57	Treated	Male	27	Some	1	-1.43	0.758
## 2	46	Treated	Male	29	None	0	-1.33	0.728
## 3	77	Treated	Male	30	None	0	-1.28	0.713
## 4	17	Treated	Male	32	Marked	1	-1.18	0.684

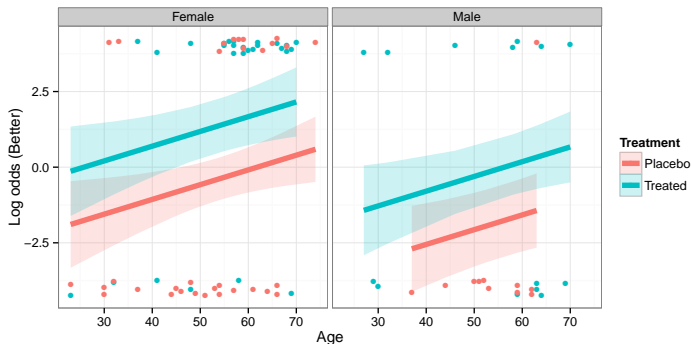
Plotting with ggplot2 package

```
arth.fit2$obs <- c(-4, 4)[1+arth.fit2$Better]

gg2 <- ggplot( arth.fit2, aes(x=Age, y=fit, color=Treatment)) +
  geom_line(size = 2) +
  geom_ribbon(aes(ymin = fit - 1.96 * se.fit,
                 ymax = fit + 1.96 * se.fit,
                 fill = Treatment), alpha = 0.2,
            color = "transparent") +
  labs(x = "Age", y = "Log odds (Better)") +
  geom_point(aes(y=obs), position=position_jitter(height=0.25, width=0))
gg2 + facet_wrap(~ Sex)
```

Full-model plots

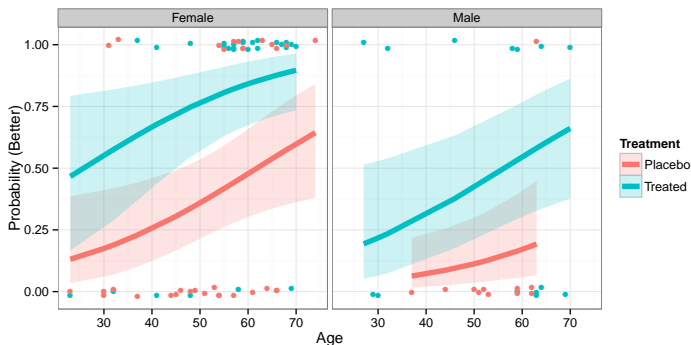
Plotting on the logit scale shows the additive effects of age, treatment and sex



These plots show the data (jittered) as well as model uncertainty (confidence bands)

Full-model plots

Plotting on the probability scale may be simpler to interpret



These plots show the data (jittered) as well as model uncertainty (confidence bands)

Models with interactions

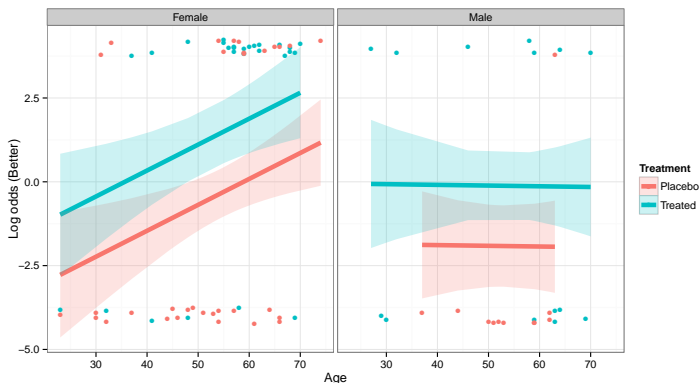
Allow an interaction of Age x Sex

```
arth.logistic4 <- update(arth.logistic2, . ~ . + Age:Sex)
Anova(arth.logistic4)

## Analysis of Deviance Table (Type II tests)
##
## Response: Better
##          LR Chisq Df Pr(>Chisq)
## I(Age - 50)          0
## Sex                6.98   1    0.00823 **
## Treatment          11.90   1    0.00056 ***
## Sex:Age             3.42   1    0.06430 .
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Interaction is NS, but we can plot it the model anyway

Models with interactions

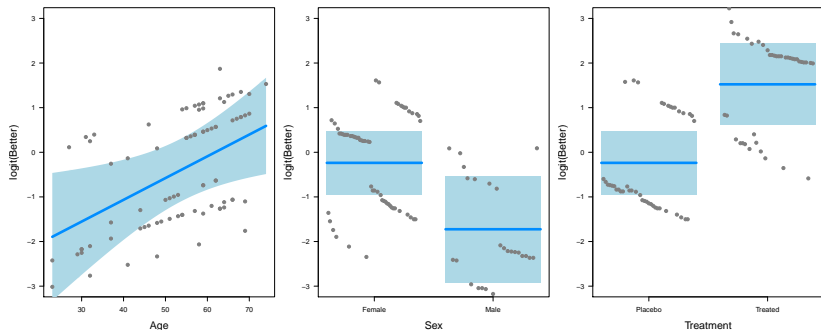


- Only the model changes
- `predict()` automatically incorporates the revised model terms
- Plotting steps remain the same
- This interpretation is quite different!

The visreg package

- Provides a more convenient way to plot model results from the model object
- A consistent interface for linear models, generalized linear models, robust regression, etc.
- Shows the data as **partial residuals** or **rug plots**
- Can plot on the response or logit scale
- Can produce plots with separate panels for conditioning variables

```
library(visreg)
visreg(arth.logistic2, ylab="logit(Better)", ...)
```



- One plot for each variable in the model
- Other variables: **continuous**—held fixed at median; **factors**—held fixed at most frequent value
- **Partial residuals** (r_j): the coefficient $\hat{\beta}_j$ in the full model is the slope of the simple fit of r_j on \mathbf{x}_j .

Effect plots: basic ideas

Show a given effect (and low-order relatives) controlling for other model effects.

Data

	x1	x2	sex	x1:x2	y	<u>yhat</u>
1	1	1	F	1	4.73	4.46
2	2	1	M	0	6.10	5.55
3	3	1	F	-1	4.32	4.34
4	1	1	F	1	4.84	4.46
5	2	1	F	0	4.73	4.40
...
29	2	2	M	0	6.10	6.15
30	3	2	F	1	6.71	7.14

• Fit data: $\mathbf{X}\hat{\boldsymbol{\beta}} \Rightarrow \hat{\mathbf{y}}$

• Score data $\mathbf{X}^* \hat{\boldsymbol{\beta}} \Rightarrow \hat{\mathbf{y}}^*$

- plot vars: vary over range
- control vars: fix at means

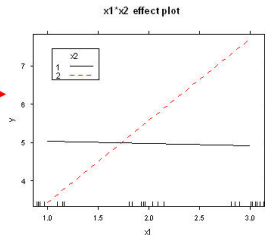
Score data

	x1	x2	sex	x1:x2	y	<u>yhat</u> *
31	1	1	0.5	1	NA	5.030
32	2	1	0.5	2	NA	4.971
33	3	1	0.5	3	NA	4.912
34	1	2	0.5	2	NA	3.437
35	2	2	0.5	4	NA	5.574
36	3	2	0.5	6	NA	7.710

plot vars

control vars

plot



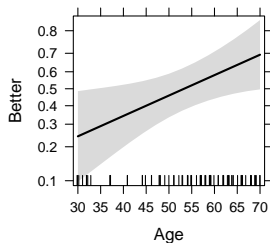
Effect plots for generalized linear models: Details

- For simple models, full model plots show the complete relation between response and *all predictors*.
- Fox(1987)— For complex models, often wish to plot a specific main effect or interaction (including lower-order relatives)— *controlling for other effects*
 - Fit full model to data with linear predictor (e.g., logit) $\eta = \mathbf{X}\beta$ and link function $g(\mu) = \eta \rightarrow$ estimate \mathbf{b} of β and covariance matrix $\widehat{V}(\mathbf{b})$ of \mathbf{b} .
 - Construct “score data”
 - Vary each predictor in the term over its’ range
 - Fix other predictors at “typical” values (mean, median, proportion in the data)
 - \rightarrow “effect model matrix,” \mathbf{X}^*
 - Use `predict()` on \mathbf{X}^*
 - Calculate fitted effect values, $\hat{\eta}^* = \mathbf{X}^* \mathbf{b}$.
 - Standard errors are square roots of $\text{diag } \mathbf{X}^* \widehat{V}(\mathbf{b}) \mathbf{X}^{*\top}$
 - Plot $\hat{\eta}^*$, or values transformed back to scale of response, $g^{-1}(\hat{\eta}^*)$.
- *Note*: This provides a general means to visualize interactions in *all* linear and generalized linear models.

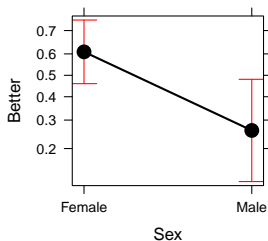
Plotting main effects:

```
library(effects)
arth.eff2 <- allEffects(arth.logistic2)
plot(arth.eff2, rows=1, cols=3)
```

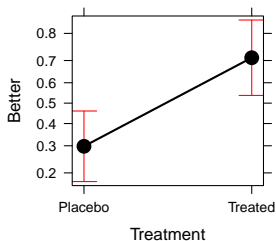
Age effect plot



Sex effect plot

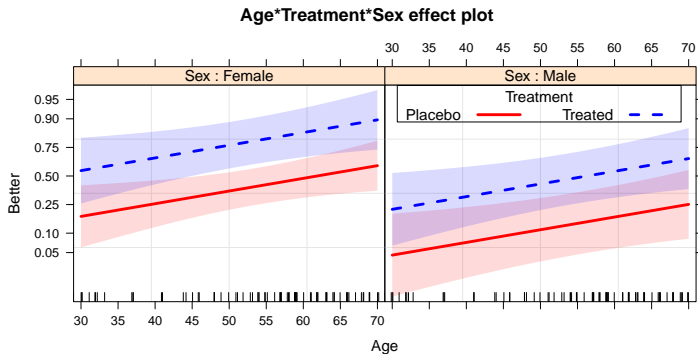


Treatment effect plot



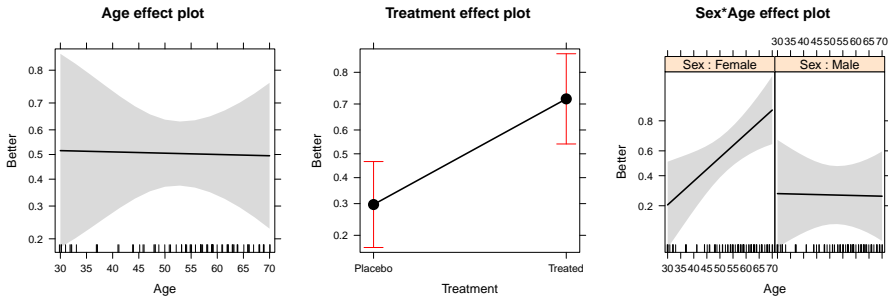
Full model plots:

```
arth.full <- Effect(c("Age", "Treatment", "Sex"), arth.logistic2)
plot(arth.full, multiline=TRUE, ci.style="bands", colors = c("red",
"blue"), lwd=3, ...)
```



Model with interaction of Age x Sex

```
plot(allEffects(arth.logistic4), rows=1, cols=3)
```



- Only the high-order terms for Treatment and Sex*Age need to be interpreted
- (How would you describe this?)
- The main effect of Age looks very different, averaged over Treatment and Sex

Case study: Arrests for Marijuana Possession

Context & background

- In Dec. 2002, the *Toronto Star* examined the issue of **racial profiling**, by analyzing a data base of 600,000+ arrest records from 1996-2002.
- They focused on a subset of arrests for which police action was **discretionary**, e.g., simple possession of small quantities of marijuana, where the police could:
 - Release the arrestee with a summons— like a parking ticket
 - Bring to police station, hold for bail, etc.— harsher treatment
- **Response** variable: `released` – Yes, No
- Main **predictor** of interest: `skin-colour` of arrestee (black, white)

The *Toronto Star* meets mosaic displays...

B SECTION > **TORONTO STAR** < WEDNESDAY, DECEMBER 11, 2002 ★ thestar.com

Race and Crime



VINCE TALOTTA/TORONTO STAR

York University professor Michael Friendly's expert statistical analysis provided confirmation for the *Toronto Star*'s series on racial profiling by city police.




Man behind the numbers

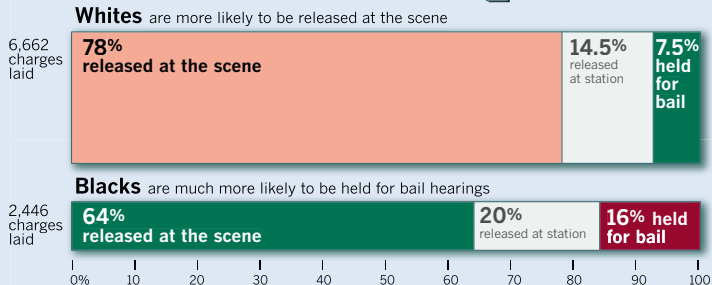
... Which got turned into this infographic:

Same charge, different treatment

Statistical analysis of single drug possession charges shows that blacks are much less likely to be released at the scene and much more likely to be held in custody for a bail hearing. Darker colours represent a stronger statistical link between skin colour and police treatment.

Degree of likelihood

-  *Much less* likely to occur
-  *Much more* likely to occur
-  *More likely* to occur



SOURCE: Toronto police arrest records 1996-2002

... Hey, they even spelled likelihood correctly!

Arrests for Marijuana Possession: Data

Data

Control variables:

- year, age, sex
- employed, citizen — Yes, No
- checks — Number of police data bases (previous arrests, previous convictions, parole status, etc.) in which the arrestee's name was found.

```
library(effects)      # for Arrests data
library(car)          # for Anova()
data(Arrests)
some(Arrests)
```

##	released	colour	year	age	sex	employed	citizen	checks
## 299	Yes	Black	2001	24	Male	Yes	Yes	3
## 766	Yes	White	2000	18	Male	Yes	Yes	2
## 1530	Yes	White	2000	14	Male	Yes	Yes	0
## 2367	Yes	White	1999	23	Male	Yes	Yes	0
## 2619	No	White	2001	22	Male	Yes	Yes	2
## 2664	No	Black	1998	38	Male	Yes	No	3
## 4202	No	White	2001	47	Female	No	Yes	1
## 4206	Yes	Black	1999	26	Male	No	Yes	0
## 4323	Yes	Black	1999	22	Male	No	Yes	6
## 5102	Yes	White	2000	19	Male	Yes	Yes	0

Arrests for Marijuana Possession: Model

To allow possibly non-linear effects of `year`, we treat it as a factor:

```
> Arrests$year <- as.factor(Arrests$year)
```

Logistic regression model with all main effects, plus interactions of `colour:year` and `colour:age`

```
> arrests.mod <- glm(released ~ employed + citizen + checks + colour *
+   year + colour * age, family = binomial, data = Arrests)
> Anova(arrests.mod)
```

Analysis of Deviance Table (Type II tests)

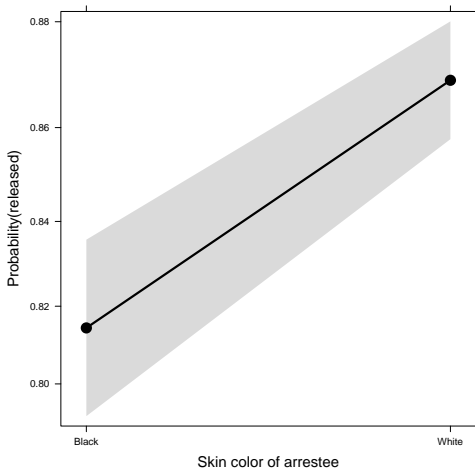
Response: released

	LR	Chisq	Df	Pr(>Chisq)
employed	72.673	1	< 2.2e-16	***
citizen	25.783	1	3.820e-07	***
checks	205.211	1	< 2.2e-16	***
colour	19.572	1	9.687e-06	***
year	6.087	5	0.2978477	
age	0.459	1	0.4982736	
colour:year	21.720	5	0.0005917	***
colour:age	13.886	1	0.0001942	***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Effect plots: colour

```
plot(Effect("colour", arrests.mod), ci.style="bands", ...)
```



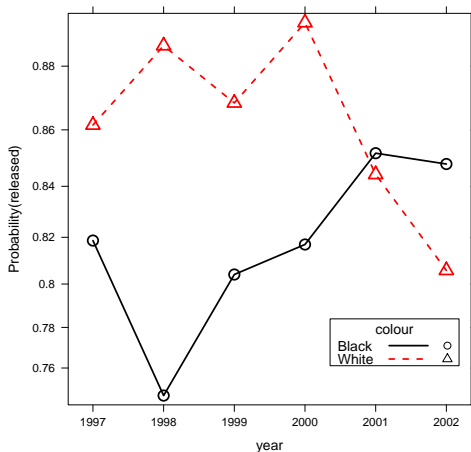
- Effect plot for colour shows average effect **controlling** (adjusting) for *all* other factors simultaneously
- (The *Star* analysis, controlled for these one at a time.)
- \Rightarrow Evidence for different treatment of blacks and whites ("racial profiling")
- (Even Frances Nunziata could understand this.)
- NB: Effects smaller than claimed by *the Star*

Effect plots: Interactions

The story turned out to be more nuanced than reported by the *Toronto Star*, as shown in effect plots for interactions with colour.

```
> plot(effect("colour:year", arrests.mod), multiline = TRUE, ...)
```

colour*year effect plot

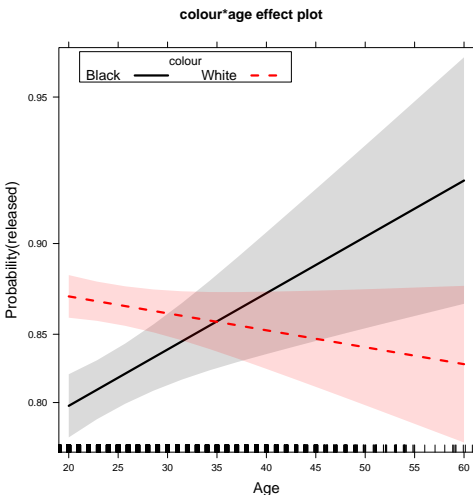


- Up to 2000, strong evidence for differential treatment of blacks and whites
- Also evidence to support Police claim of effect of training to reduce racial effects in treatment

Effect plots: Interactions

The story turned out to be more nuanced than reported by the *Toronto Star*, as shown in effect plots for interactions with colour.

```
> plot(effect("colour:age", arrests.mod), multiline = TRUE, ...)
```



A more surprising finding:

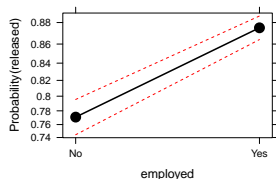
- Opposite age effects for blacks and whites—
- Young blacks treated *more* harshly than young whites
- Older blacks treated *less* harshly than older whites

Effect plots: allEffects

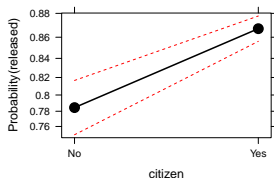
All model effects can be viewed together using `plot(allEffects(mod))`

```
> arrests.effects <- allEffects(arrests.mod, xlevels = list(age = seq(15,
+ 45, 5)))
> plot(arrests.effects, ylab = "Probability(released)")
```

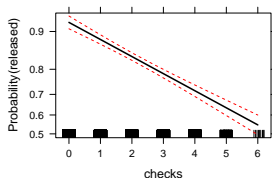
employed effect plot



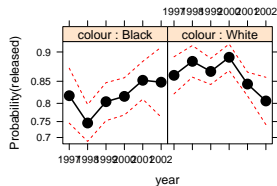
citizen effect plot



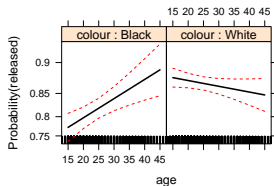
checks effect plot



colour*year effect plot



colour*age effect plot



Model diagnostics

As in regression and ANOVA, the validity of a logistic regression model is threatened when:

- Important predictors have been omitted from the model
- Predictors assumed to be **linear** have **non-linear** effects on $\Pr(Y = 1)$
- Important **interactions** have been omitted
- A few “wild” observations have a large impact on the fitted model or coefficients

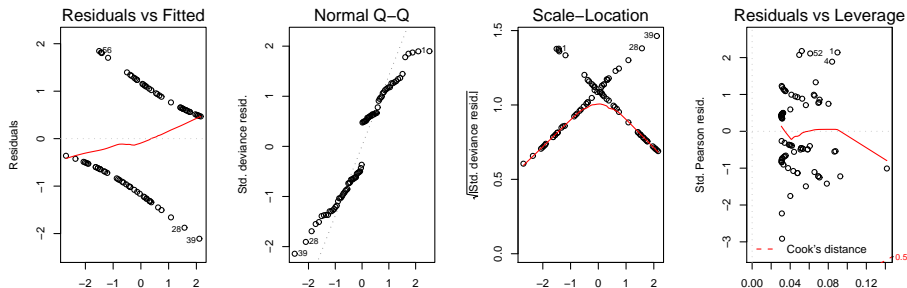
Model specification: Tools and techniques

- Use non-parametric smoothed curves to detect non-linearity
- Consider using polynomial terms (X^2, X^3, \dots) or **regression splines** (e.g., `ns(X, 3)`)
- Use `update(model, ...)` to test for interactions— formula: $. \sim .^2$

Diagnostic plots in R

In R, plotting a `glm` object gives the “regression quartet” — basic diagnostic plots

```
arth.mod1 <- glm(Better ~ Age + Sex + Treatment, data=Arthritis,
                 family='binomial')
plot(arth.mod1)
```



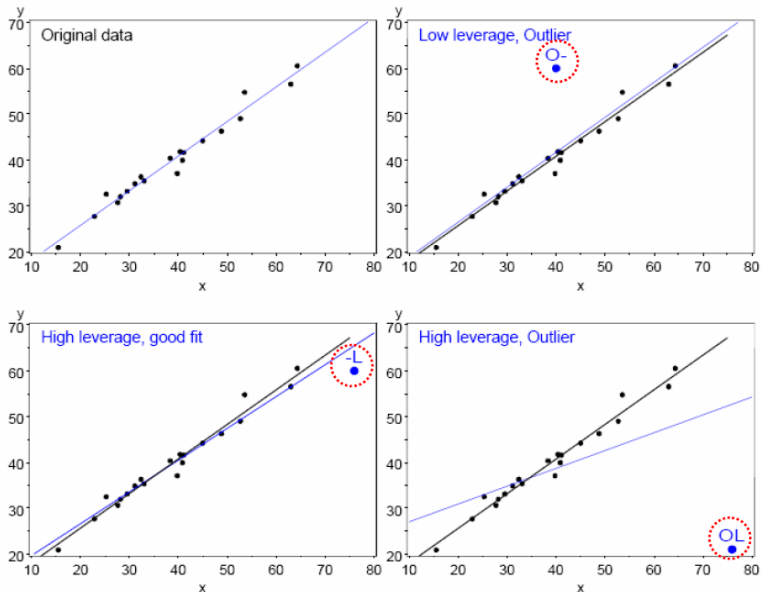
Better versions of these plots are available in the [car](#) package

Unusual data: Leverage and Influence

- “Unusual” observations can have dramatic effects on estimates in linear models
 - Can change the coefficients for the predictors
 - Can change the predicted values for **all** observations
- Three archetypal cases:
 - Typical X (low leverage), bad fit — Not much harm
 - Unusual X (high leverage), good fit — Not much harm
 - Unusual X (high leverage), bad fit — **BAD, BAD, BAD**
- Influential observations: unusual in **both** X and Y
- Heuristic formula:

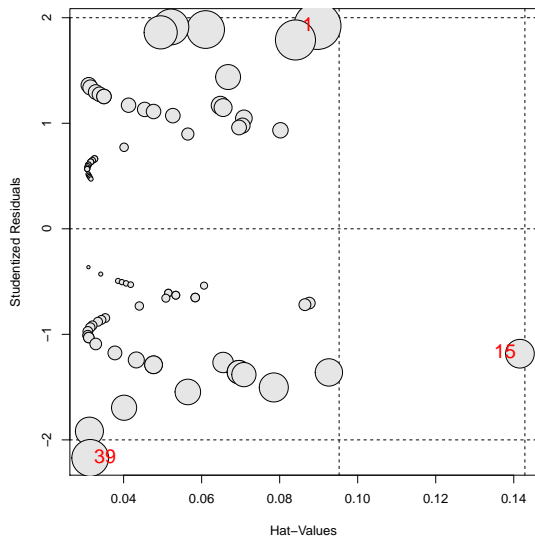
$$\text{Influence} = \text{Leverage}_X \times \text{Residual}_Y$$

Effect of adding one more point in simple linear regression (new point in blue)



Influence plots in R

```
library(car)  
influencePlot(arth.logistic2)
```



- X axis: Leverage ("hat values")
- Y axis: Studentized residuals
- Bubble size \sim Cook D (influence on coefficients)

Which cases are influential?

	ID	Treatment	Sex	Age	Better	StudRes	Hat	CookD
1	57	Treated	Male	27	1	1.922	0.08968	0.3358
15	66	Treated	Female	23	0	-1.183	0.14158	0.2049
39	11	Treated	Female	69	0	-2.171	0.03144	0.2626

