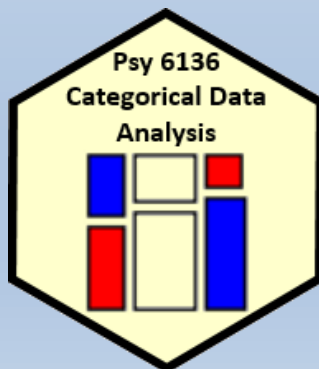


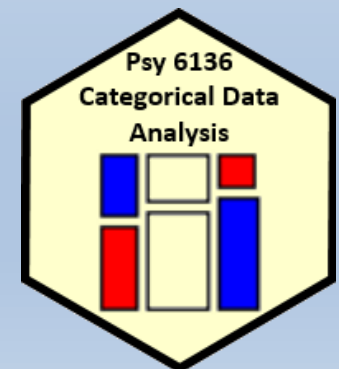
# Correspondence analysis



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Psych 6136

<http://friendly.github.io/psy6136>



# Correspondence analysis: Basic ideas

## Analog of PCA for frequency data

- Account for **maximum** % of  $\chi^2$  in few (2-3) dimensions
- Finds scores for row ( $x_{im}$ ) and col ( $y_{jm}$ ) categories on these dimensions
- Uses Singular Value Decomposition of residuals from independence,

$$d_{ij} = (n_{ij} - \hat{m}_{ij}) / \sqrt{\hat{m}_{ij}} \quad \longrightarrow \quad d_{ij} = \sqrt{n} \sum_{m=1}^M \lambda_m x_{im} y_{jm} \quad \leftrightarrow \quad \mathbf{D} = \mathbf{X} \mathbf{\Lambda} \mathbf{Y}^T$$

- **Optimal scaling**: each pair of scores for rows ( $x_{im}$ ) and col ( $y_{jm}$ ) have highest possible correlation ( $= \lambda_m$ )
- Plots of the row and column scores show associations
  - Row point ( $x_{im}$ ) near col point ( $y_{jm}$ )  $\rightarrow$  positive association  $d_{ij} > 0$

# CA software for R

- `ca` package
  - `ca()` – two-way tables; `plot(ca())` for graphs
  - `mjca()` – multiple & joint CA
- `FactoMineR` & `factoextra` packages
  - `CA()` – many options for graphical displays, but quirky

# Example: Hair color, eye color

```
> library(ca)
> haireye <- margin.table(HairEyeColor, 1:2)
> (haireye.ca <- ca(haireye))
```

Principal inertias (eigenvalues):

	1	2	3
Value	0.208773	0.022227	0.002598
Percentage	89.37%	9.52%	1.11%

$\chi^2$  % for dimensions

Rows:

	Black	Brown	Red	Blond
Mass	0.1824	0.4831	0.1199	0.215
ChiDist	0.5512	0.1595	0.3548	0.838
Inertia	0.0554	0.0123	0.0151	0.151
Dim. 1	-1.1043	-0.3245	-0.2835	1.828
Dim. 2	1.4409	-0.2191	-2.1440	0.467

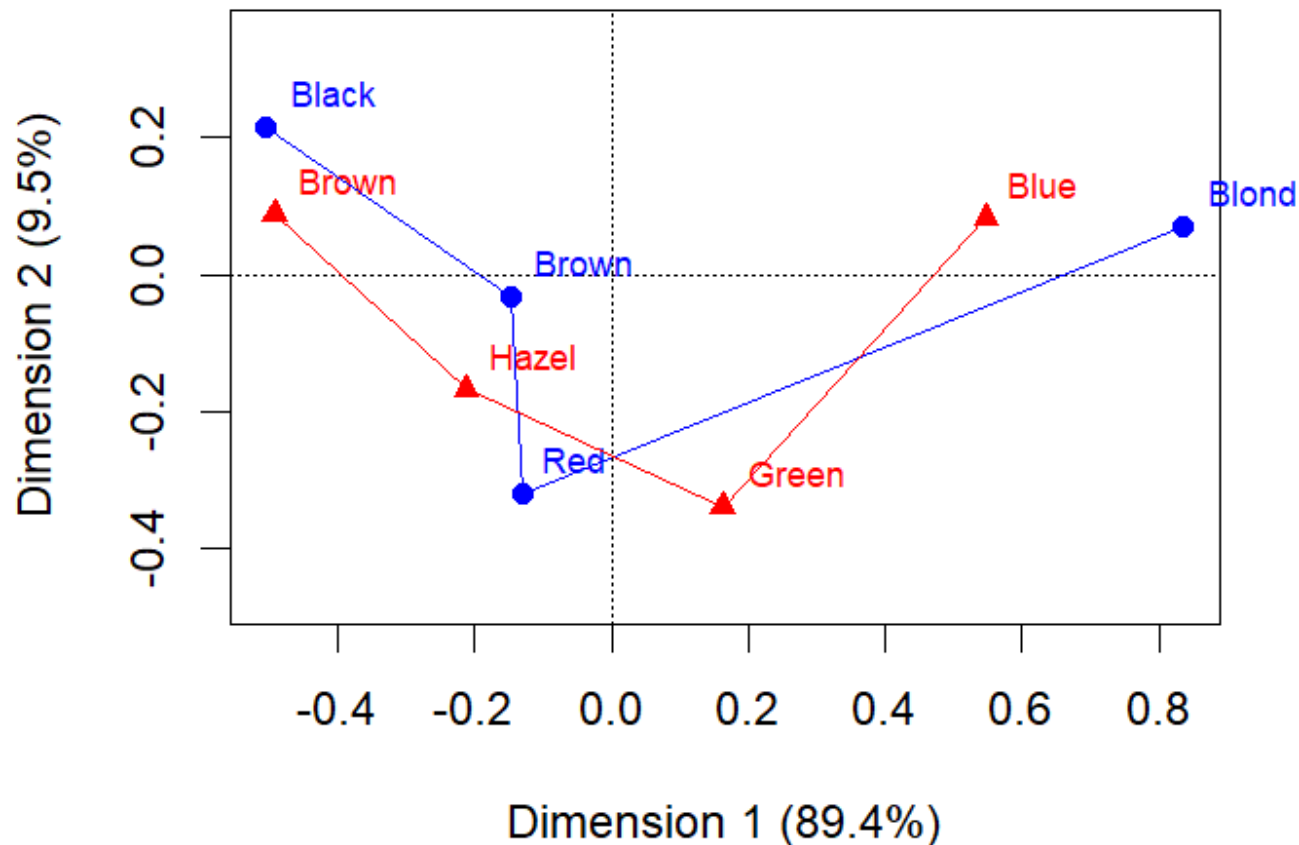
} Hair category scores, Dim1-2

Columns:

	Brown	Blue	Hazel	Green
Mass	0.3716	0.363	0.1571	0.1081
ChiDist	0.5005	0.554	0.2887	0.3857
Inertia	0.0931	0.111	0.0131	0.0161
Dim. 1	-1.0771	1.198	-0.4653	0.3540
Dim. 2	0.5924	0.556	-1.1228	-2.2741

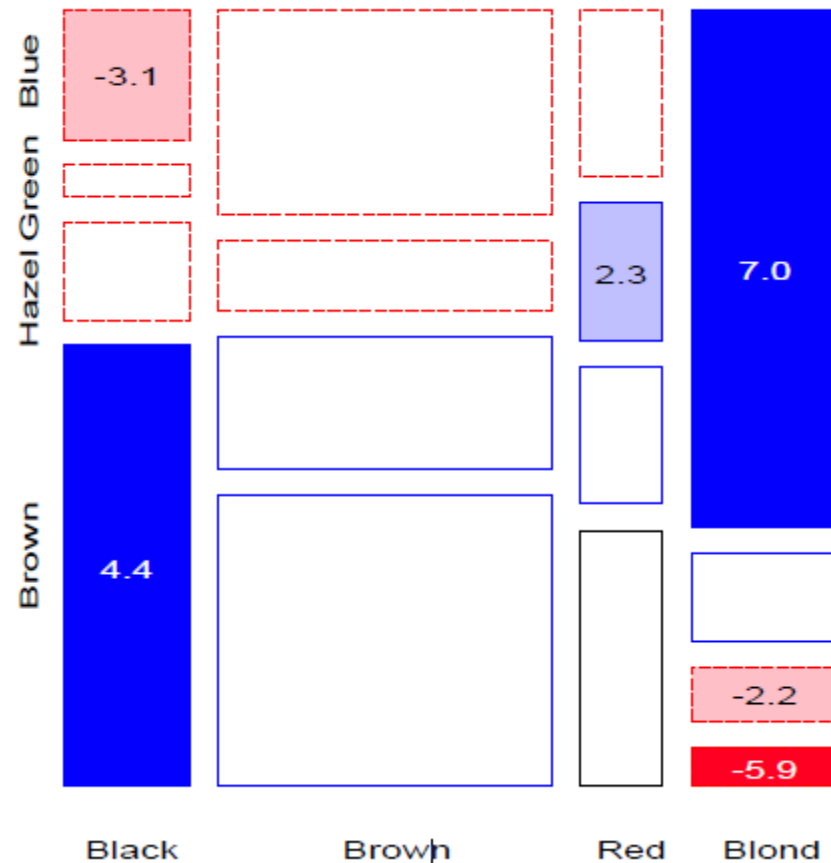
} Eye category scores, Dim1-2

```
plot(haireye.ca, lines=TRUE)
```



- Rough interpretation: row/col points “near” each other are positively associated (independence residuals  $d_{ij} \gg 0$ )
- Dim 1: 89.4% of  $\chi^2$  (dark  $\rightarrow$  light)
- Dim 2: 9.5% of  $\chi^2$  (Red/Green vs. others)

## Hair color, Eye color data: Compare with mosaic display



- The main dark–light dimension is reflected in the opposite-corner pattern of residuals
- The 2nd dimension is reflected in deviations from this pattern (e.g., Red hair–Green eyes)
- CA is “accounting for” residuals (deviations) from independence

# Row & column profiles

- For a two-way table, row profiles & column profiles give relative proportions of the categories
- An association is present to the extent that the row/col profiles differ
- Profiles add to 1.0 (100%), and can be visualized in profile space

## Example: Toothpaste purchases by region

120 people in three regions were asked which of four brands of toothpaste, A–D, they had most recently purchased. Is there a difference among regions?

```
toothpaste
```

##		Region			
##	Brand		R1	R2	R3
##	Brand A		5	5	30
##	Brand B		5	25	5
##	Brand C		15	5	5
##	Brand D		15	5	0

- Row profiles pertain to the differences among **brand** preference
- Column profiles pertain to the differences among **regions**

Region				
Brand	R1	R2	R3	Sum
Brand A	12.5	12.5	75.0	100
Brand B	14.3	71.4	14.3	100
Brand C	60.0	20.0	20.0	100
Brand D	75.0	25.0	0.0	100

Region			
Brand	R1	R2	R3
Brand A	12.5	12.5	75.0
Brand B	12.5	62.5	12.5
Brand C	37.5	12.5	12.5
Brand D	37.5	12.5	0.0
Sum	100.0	100.0	100.0

There is clearly an association: the row (& column) profiles differ

```
> chisq.test(toothpaste)
```

```
Pearson's Chi-squared test
```

```
data: toothpaste
```

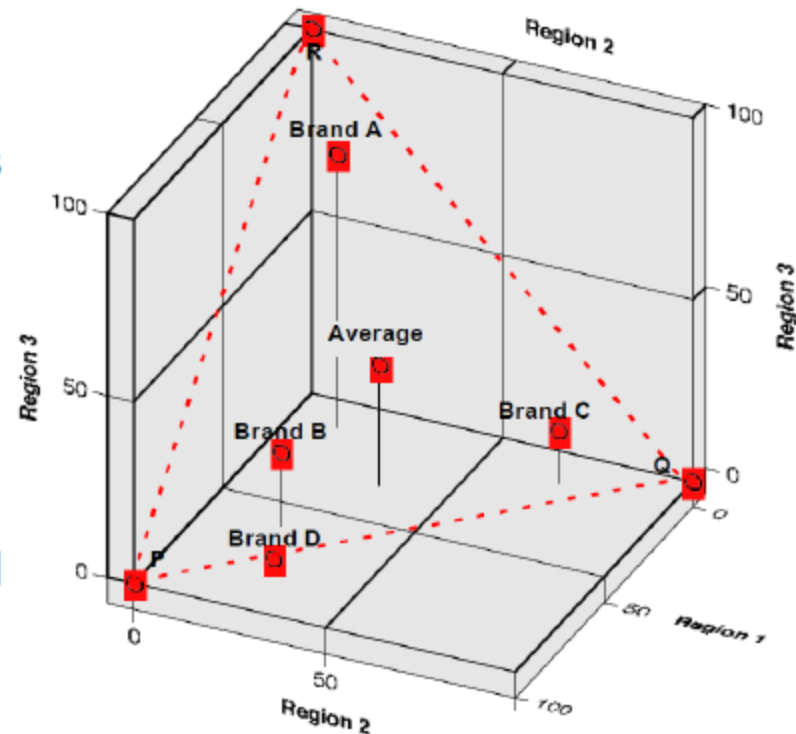
```
X-squared = 79.6, df = 6, p-value = 4.3e-15
```



# Plotting profiles

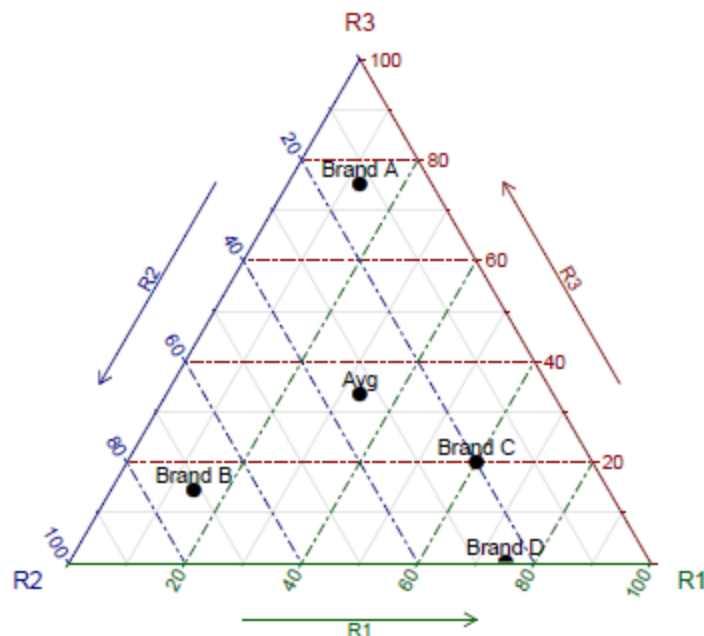
In this simple example we can plot the row profiles as points in 3D space, with axes corresponding to regions, R1, R2, R3

- Each brand is positioned in this space according to its proportions for the regions
- Because proportions sum to 100%, all points lie in the dashed plane PQR
- The Average profile is at the (weighted) centroid
- If no association, all brands would appear at the centroid



# Plotting profiles

Analogous 2D plot is a **trilinear plot** that automatically scales the R1–R3 values so they sum to 100%



- The Avg profile has coordinates of 33.3% for each region
- Brand preferences by region can be seen by their positions wrt the R1–R3 axes
- This suggests that differences among brands can be measured by their (squared) distances from the centroid, weighted by their row margins (**mass**)
- Physical analogy suggests the term **inertia** for this weighted variation

# CA solution

The CA solution has at most  $\min(r - 1, c - 1)$  dimensions

The 2D solution here is exact, i.e., accounts for 100% of Pearson  $\chi^2$

```
> tp.ca <- ca(toothpaste)
> summary(tp.ca, rows=FALSE, columns=FALSE)
```

Principal inertias (eigenvalues):

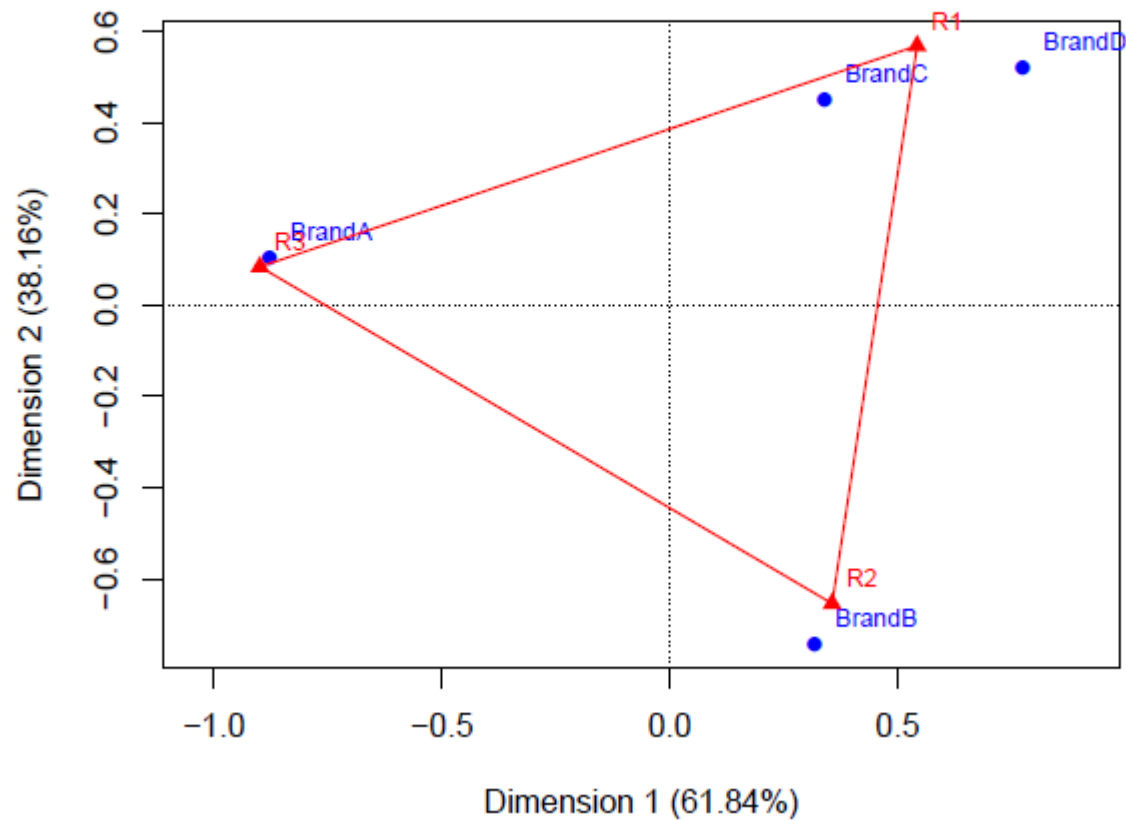
dim	value	%	cum%	scree plot
1	0.410259	61.8	61.8	*****
2	0.253134	38.2	100.0	*****
<hr/>				
Total:	0.663393	100.0		

Pearson  $\chi^2 = \sum \lambda^2 / N$

```
> # reproduce chi-square
> sum(tp.ca$sv^2) * sum(toothpaste)
[1] 79.607
```

# CA solution

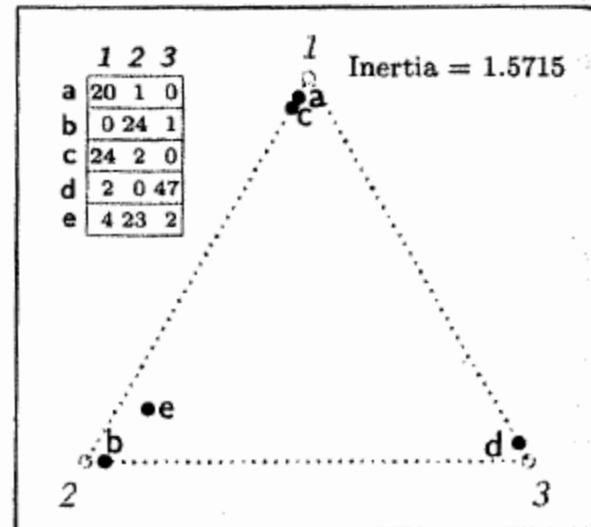
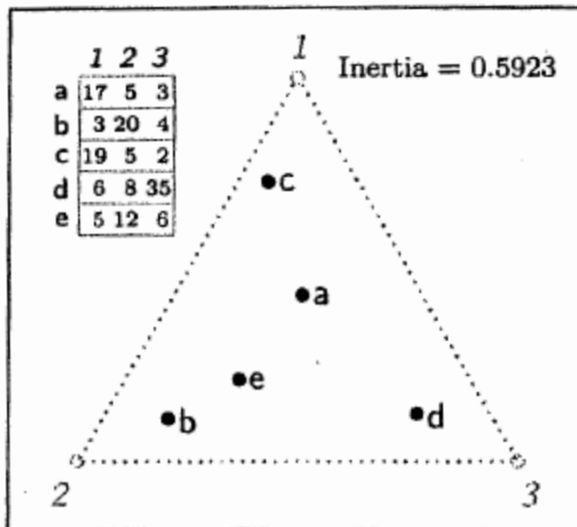
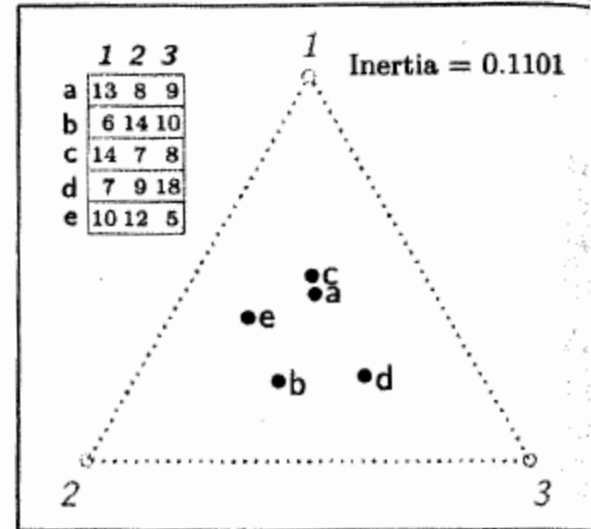
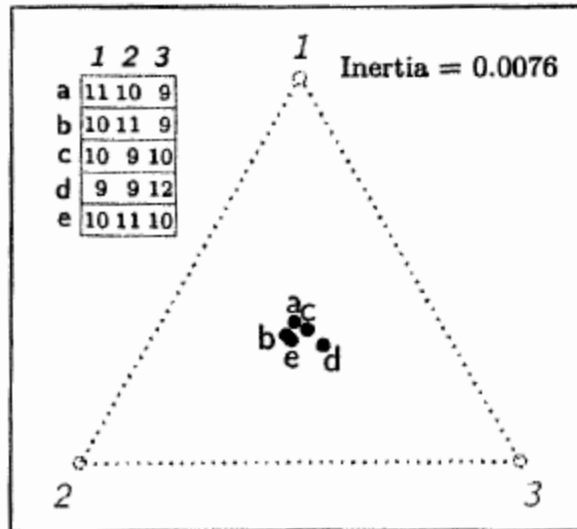
```
res <- plot(tp.ca)  
polygon(res$cols, border="red", lwd=2)
```



# Profiles & inertia

**Exhibit 4.2:**

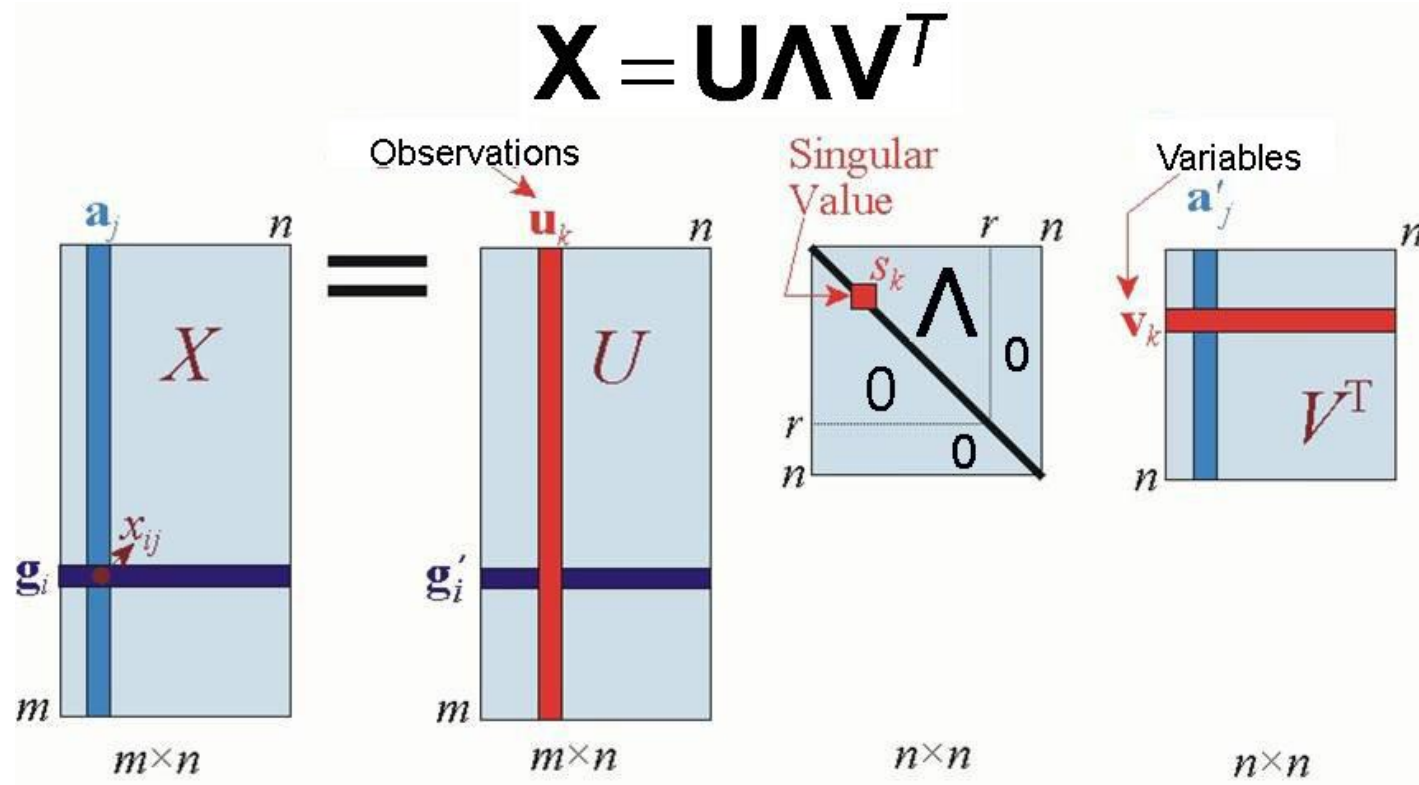
A series of data tables with increasing total inertia. The higher the total inertia, the greater is the association between the rows and columns, displayed by the higher dispersion of the profile points in the profile space. The values in these tables have been chosen specifically so that the column sums are all equal, so the weights in the  $\chi^2$ -distance formulation are the same, and hence distances we observe in these maps are true  $\chi^2$ -distances.



# Singular value decomposition

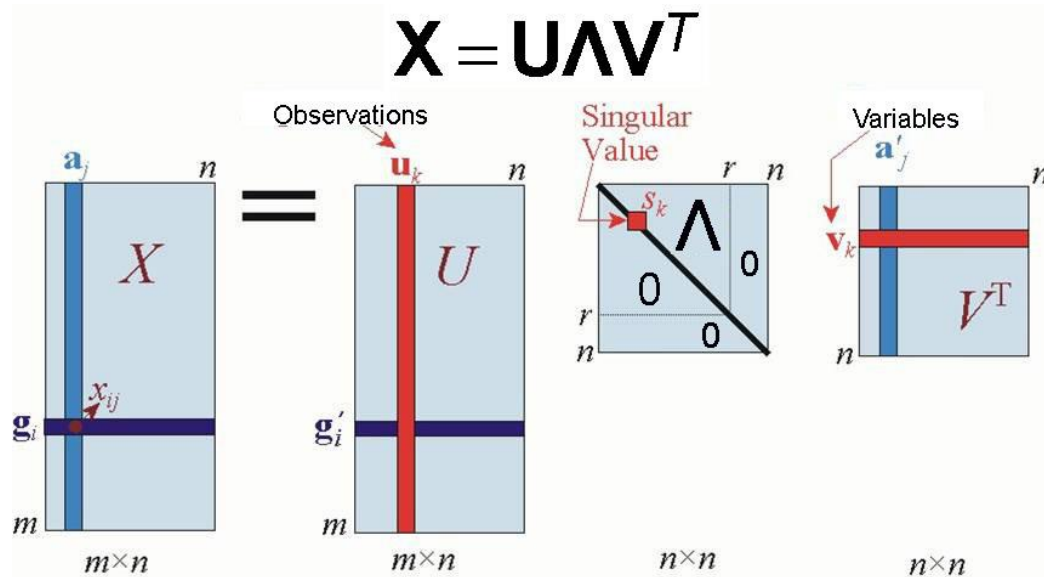
The singular value decomposition (SVD) is a basic technique for factoring a matrix and for matrix approximation

For an  $m \times n$  matrix  $\mathbf{X}$  of rank  $r \leq \min(m, n)$  the SVD of  $\mathbf{X}$  is:



# Properties of the SVD

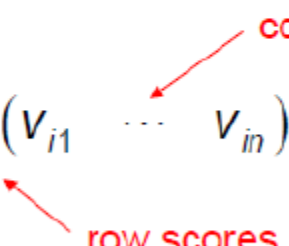
- **U**: columns are **eigenvectors** of  $\mathbf{X}\mathbf{X}^T$  and form an orthonormal basis for observation profiles such that  $\mathbf{U}^T\mathbf{U} = \mathbf{I}$
- **$\Lambda$** : diagonal,  $r$  singular values = sqrt eigenvalues of both  $\mathbf{X}\mathbf{X}^T$  and  $\mathbf{X}^T\mathbf{X}$
- **V**: columns are eigenvectors of  $\mathbf{X}^T\mathbf{X}$ , orthonormal:  $\mathbf{V}^T\mathbf{V} = \mathbf{I}$



# SVD: Matrix approximation

- Let  $\mathbf{X}$  be an  $m \times n$  matrix such that  $\text{rank}(\mathbf{X}) = r$
- If  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r$  are the singular values of  $\mathbf{X}$ , then  $\hat{\mathbf{X}}$ , the rank  $q$  approximation of  $\mathbf{X}$  that minimizes  $\|\mathbf{X} - \hat{\mathbf{X}}\|$ , is

$$\hat{\mathbf{X}}_{m \times n} = \sum_{i=1}^q \lambda_i \begin{pmatrix} u_{i1} \\ \vdots \\ u_{im} \end{pmatrix} (v_{i1} \ \dots \ v_{in}) = \lambda_1 u_1 v_1^T + \dots + \lambda_q u_q v_q^T$$



a sum of  $q$  rank=1 (outer) products. The variance in  $\mathbf{X}$  accounted for each term is  $\lambda_1^2$



# CA notation & terminology

Notation:

- Contingency table:  $\mathbf{N} = \{n_{ij}\}$
- Correspondence matrix (cell probabilities):  $\mathbf{P} = \{p_{ij}\} = \mathbf{N}/n$
- Row/column masses (marginal probabilities):  $\mathbf{r} = \sum_j p_{ij}$  and  $\mathbf{c} = \sum_i p_{ij}$
- Diagonal weight matrices:  $\mathbf{D}_r = \text{diag}(\mathbf{r})$  and  $\mathbf{D}_c = \text{diag}(\mathbf{c})$

The SVD is then applied to the correspondence matrix of cell probabilities as:

$$\mathbf{P} = \mathbf{A}\mathbf{D}_\lambda\mathbf{B}^\top$$

where

- Singular values:  $\mathbf{D}_\lambda = \text{diag}(\lambda)$  is the diagonal matrix of singular values  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_M$
- Row scores:  $\mathbf{A}_{I \times M}$ , normalized so that  $\mathbf{A}\mathbf{D}_r^{-1}\mathbf{A}^\top = \mathbf{I}$
- Column scores:  $\mathbf{B}_{J \times M}$ , normalized so that  $\mathbf{B}\mathbf{D}_c^{-1}\mathbf{B}^\top = \mathbf{I}$

# Principal & standard coordinates

Two types of coordinates are used in CA, based on re-scalings of A and B.  
Principal coordinates are most commonly used in plotting CA solutions.

## Principal coordinates

Coordinates of the row (**F**) and column (**G**) profiles *wrt* their own principal axes

$$\begin{aligned}\mathbf{F} &= \mathbf{D}_r^{-1} \mathbf{A} \mathbf{D}_\lambda \quad \text{scaled so that} \quad \mathbf{F}^\top \mathbf{D}_r \mathbf{F} = \mathbf{D}_\lambda \\ \mathbf{G} &= \mathbf{D}_c^{-1} \mathbf{B} \mathbf{D}_\lambda \quad \text{scaled so that} \quad \mathbf{G}^\top \mathbf{D}_c \mathbf{G} = \mathbf{D}_\lambda\end{aligned}$$

- Defined so that the *inertia* along each axis is the corresponding singular value,  $\lambda_i$ ,
- i.e., weighted average of squared principal coordinates =  $\lambda_i$  on dim.  $i$
- The joint plot in principal coordinates, **F** and **G**, is called the *symmetric map* because both row and column profiles are overlaid in the same coordinate system.

# Standard coordinates

## Standard coordinates

The standard coordinates  $(\Phi, \Gamma)$  are a rescaling of the principal coordinates to **unit inertia** along each axis,

$$\Phi = D_r^{-1} \mathbf{A} \quad \text{scaled so that} \quad \Phi^T D_r \Phi = I$$

$$\Gamma = D_c^{-1} \mathbf{B} \quad \text{scaled so that} \quad \Gamma^T D_c \Gamma = I$$

- The weighted average of squared standard coordinates = 1 on each dimension
- An **asymmetric map** shows one set of points (say, the rows) in principal coordinates and the other set in standard coordinates.
-

# Geometric & statistical properties

- **Nested solutions:** CA solutions are **nested**, meaning that the first two dimensions of a 3D solution will be identical to the 2D solution (similar to PCA)
- **Centroids at origin:** In both principal coordinates and standard coordinates the points representing the row and column profiles have their centroids (weighted averages) at the origin.
  - The origin represents the (weighted) average row profile and column profile.
- **Chi-square distances:** In principal coordinates, distances between two row profiles,  $r_i$  and  $r_{i'}$ , are  $\chi^2$  distances
  - The squared difference  $(r_{ij} - r_{i'j})^2$  between two row profiles is inversely weighted by the column frequency, to account for the different relative frequency of the column categories.
- **Plotting:** For distances to be interpretable, it's crucial to scale the axes equally, so  $1^{\text{cm}}$  is the same on both axes (**aspect ratio = 1**). This is standard in most packages.

# The `ca` package in R

`ca()` calculates CA solutions, returning a “`ca`” object with all the details

```
> names(haireye.ca)
[1] "sv"          "nd"          "rownames"    "rowmass"     "rowdist"
[6] "rowinertia"  "rowcoord"    "rowsup"      "colnames"    "colmass"
[11] "coldist"     "colinertia"  "colcoord"    "colsup"      "N"
[16] "call"
```

The result contains the standard row coordinates (`rowcoord`:  $\Phi$ ) and column coordinates (`colcoord`:  $\Gamma$ ) used in plotting

```
> haireye.ca$rowcoord
      Dim1    Dim2    Dim3
Black -1.104  1.441 -1.089
Brown -0.324 -0.219  0.957
Red    -0.283 -2.144 -1.631
Blond  1.828  0.467 -0.318
```

```
> haireye.ca$colcoord
      Dim1    Dim2    Dim3
Brown -1.077  0.592 -0.4240
Blue   1.198  0.556  0.0924
Hazel  -0.465 -1.123  1.9719
Green  0.354 -2.274 -1.7184
```

# ca plots

The `plot()` method provides a wide variety of scalings (`map=` ), with different interpretative properties. Some of these:

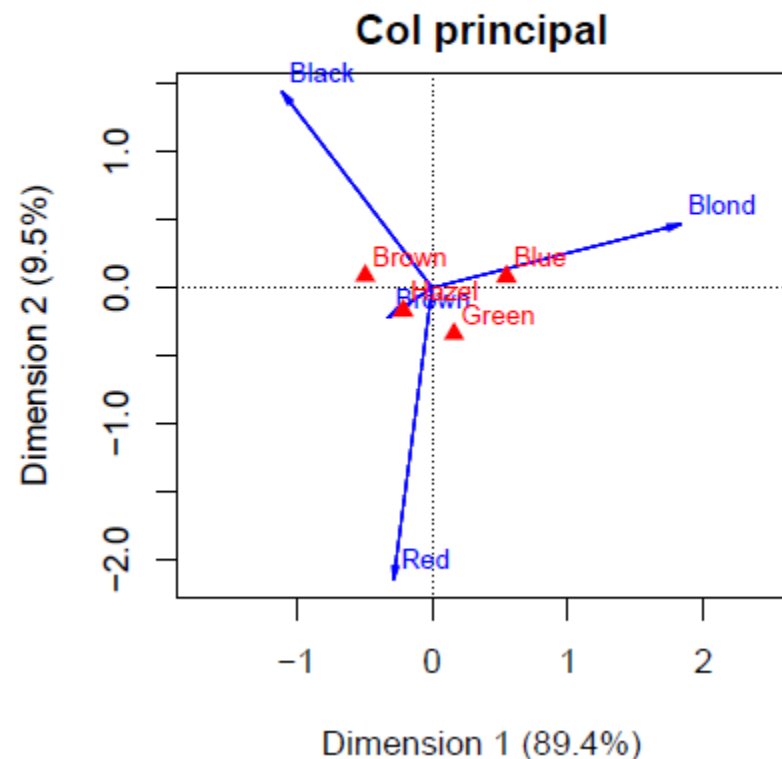
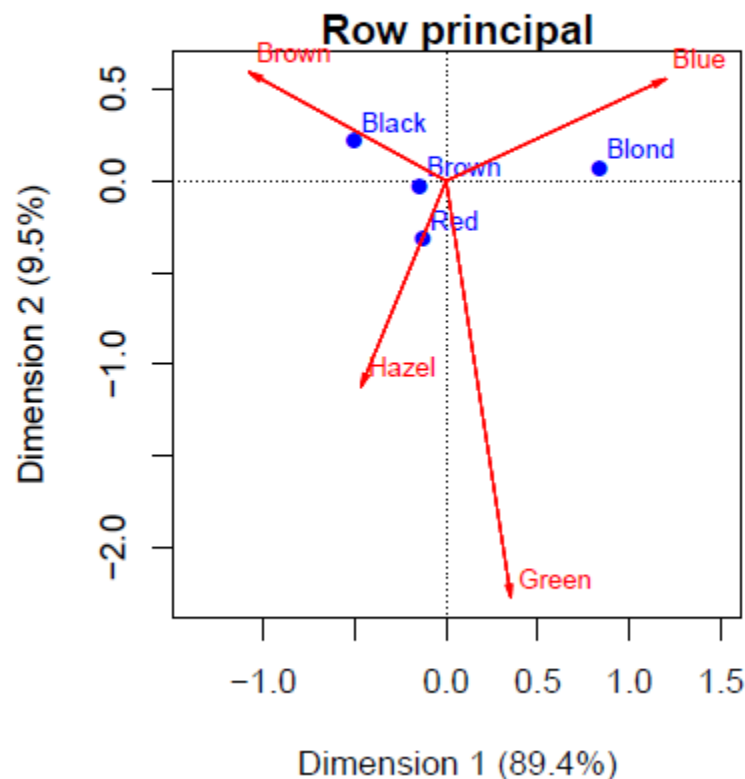
- `"symmetric"` – both rows & cols in principal coordinates (default)
- `"rowprincipal"` or `"colprincipal"` – asymmetric maps with rows in principal coordinates and cols in std coordinates, or vice versa
- `"symbiplot"` – scales both rows and cols to have variances equal to the singular value

The `mjca()` function is used for multiple correspondence analysis (MCA) for 3+ way tables. Has analogous `print()`, `summary()` and `plot()` methods

- `vcdExtra::mcaplot()` does a nicer job of plotting MCA solutions

Asymmetric row/col principal plots are **biplots** – can interpret the projection of points on vectors for the other variable

```
plot(haireye.ca, map="rowprincipal", arrows=c(FALSE, TRUE))  
plot(haireye.ca, map="colprincipal", arrows=c(TRUE, FALSE))
```



# Optimal category scores

- CA has a close relation to **canonical correlation analysis**, applied to dummy variables representing the categories
- The singular values,  $\lambda_i$ , are the **correlations** between the category scores
  - Assign Dim 1 scores, **X1** and **Y1** to the row/column categories: → Max. possible correlation,  $\lambda_1$
  - Assign Dim 2 scores, **X2** and **Y2** to the row/column categories: → Max. possible correlation,  $\lambda_2$ , but uncorrelated with **X1**, **Y1**
  - All association between row/col categories is captured by the scores
- This optimal scaling interpretation can be used to quantify categorical variables, particularly if Dim 1 is large



# Optimal category scores

```
> haireye.ca <- ca(haireye)
> round(haireye.ca$sv, 3)
[1] 0.457 0.149 0.051
```

The singular values  $\lambda_i$  = canonical correlations

To demonstrate category scores, extract row/col coordinates to a data frame

```
HE.df <- as.data.frame(haireye)

RC <- haireye.ca$rowcoord # row coordinates
CC <- haireye.ca$colcoord # col coordinates

Y1 <- RC[match(HE.df$Hair, haireye.ca$rownames), 1] # Dim 1
X1 <- CC[match(HE.df$Eye, haireye.ca$colnames), 1]
Y2 <- RC[match(HE.df$Hair, haireye.ca$rownames), 2] # Dim 2
X2 <- CC[match(HE.df$Eye, haireye.ca$colnames), 2]

HE.df <- cbind(HE.df, X1, Y1, X2, Y2)
```

# Optimal category scores

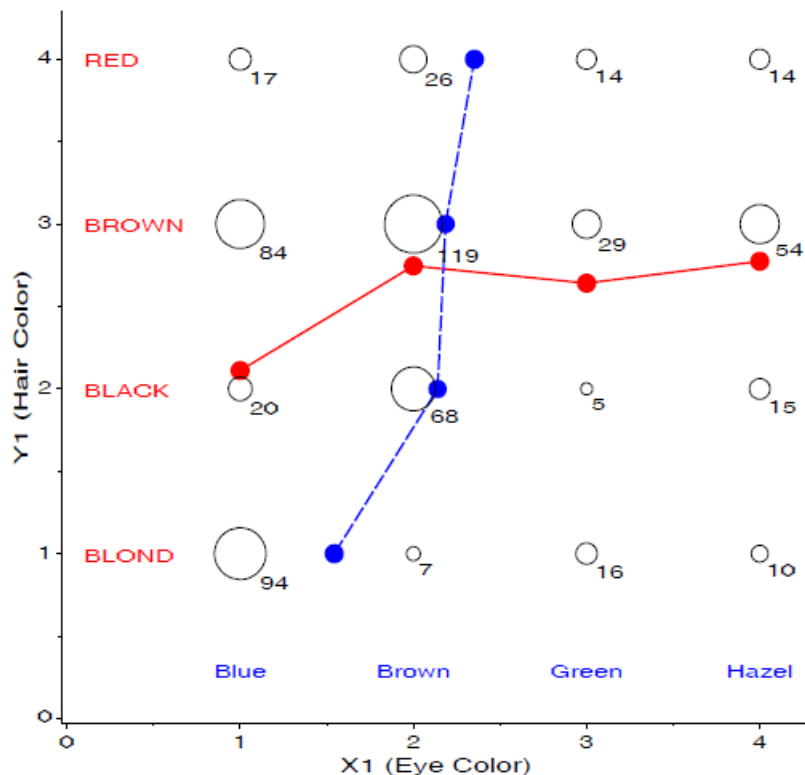
```
> HE.df <- cbind(HE.df, X1, Y1, X2, Y2)
> print(HE.df, digits=3)
   Hair  Eye Freq    X1    Y1    X2    Y2
1 Black Brown   68 -1.077 -1.104  0.592  1.441
2 Brown Brown  119 -1.077 -0.324  0.592 -0.219
3  Red Brown   26 -1.077 -0.283  0.592 -2.144
4 Blond Brown    7 -1.077  1.828  0.592  0.467
. . .
13 Black Green    5  0.354 -1.104 -2.274  1.441
14 Brown Green   29  0.354 -0.324 -2.274 -0.219
15  Red Green   14  0.354 -0.283 -2.274 -2.144
16 Blond Green   16  0.354  1.828 -2.274  0.467
```

Calculate Freq-weighted correlations. All are zero except  $r(X1, Y1) = \lambda_1$  &  $r(X2, Y2) = \lambda_2$

```
> corr <- cov.wt(HE.df[,4:7], wt=HE.df$Freq, cor=TRUE)$cor
> round(zapsmall(corr), 3)
      X1    Y1    X2    Y2
X1 1.000 0.457 0.000 0.000
Y1 0.457 1.000 0.000 0.000
X2 0.000 0.000 1.000 0.149
Y2 0.000 0.000 0.149 1.000
```

# Simultaneous linear regression

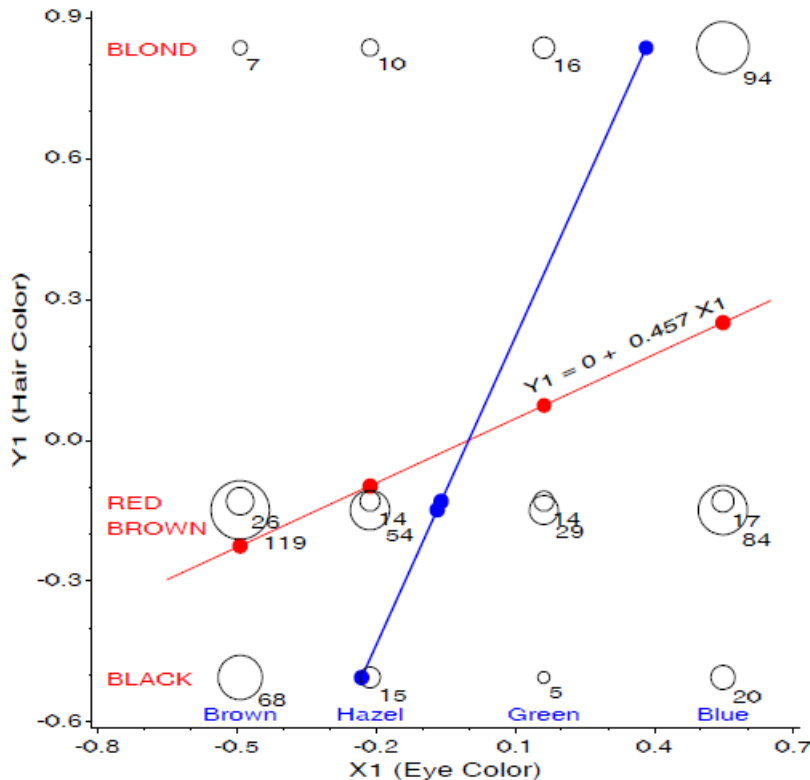
Assign linear scores (1-4) X1 to eye color and Y1 to hair color



- Lines connecting the weighted (conditional) means of  $Y1 | X1$  and  $X1 | Y1$  are not-linear
- The scatterplot uses bubble symbols showing frequency in each cell
- Is it possible to assign row and column scores so that both regressions are linear?

# Simultaneous linear regressions

Yes, use CA scores on the first dimension



- The regression of  $Y_1$  on  $X_1$  is linear, with slope  $\lambda_1$
- The regression of  $X_1$  on  $Y_1$  is linear, with slope  $1/\lambda_1$
- $\lambda_1$  is the (canonical) correlation between  $X_1$  and  $Y_1$
- The angle between the two lines would be 0 if perfect correlation
- The conditional means (dots) are the principal coordinates

# Example: Mental impairment & parent' SES

Data on mental health status of 1660 young NYC residents, by parents' SES, a 6 x 4 table. Is higher SES associated with better kids' mental health?

```
> data("Mental", package="vcdExtra")
> str(Mental)
'data.frame':      24 obs. of  3 variables:
 $ ses      : Ord.factor w/ 6 levels "1"<"2"<"3"<"4"<..: 1 1 1 1 2 2 2 2 3 3 ...
 $ mental   : Ord.factor w/ 4 levels "Well"<"Mild"<..: 1 2 3 4 1 2 3 4 1 2 ...
 $ Freq     : int   64 94 58 46 57 94 54 40 57 105 ...
```

Both ses and mental are **ordered** factors in a frequency data frame

- For `ca()`, convert this to a table using `xtabs()`

```
> (mental.tab <- xtabs(Freq ~ ses + mental, data=Mental))
      mental
ses Well Mild Moderate Impaired
  1    64   94         58        46
  2    57   94         54        40
  3    57  105         65        60
  4    72  141         77        94
  5    36   97         54        78
  6    21   71         54        71
```

# Mental data: CA solution

```
> mental.ca <- ca(mental.tab)
> summary(mental.ca, rows=FALSE, columns=FALSE)
```

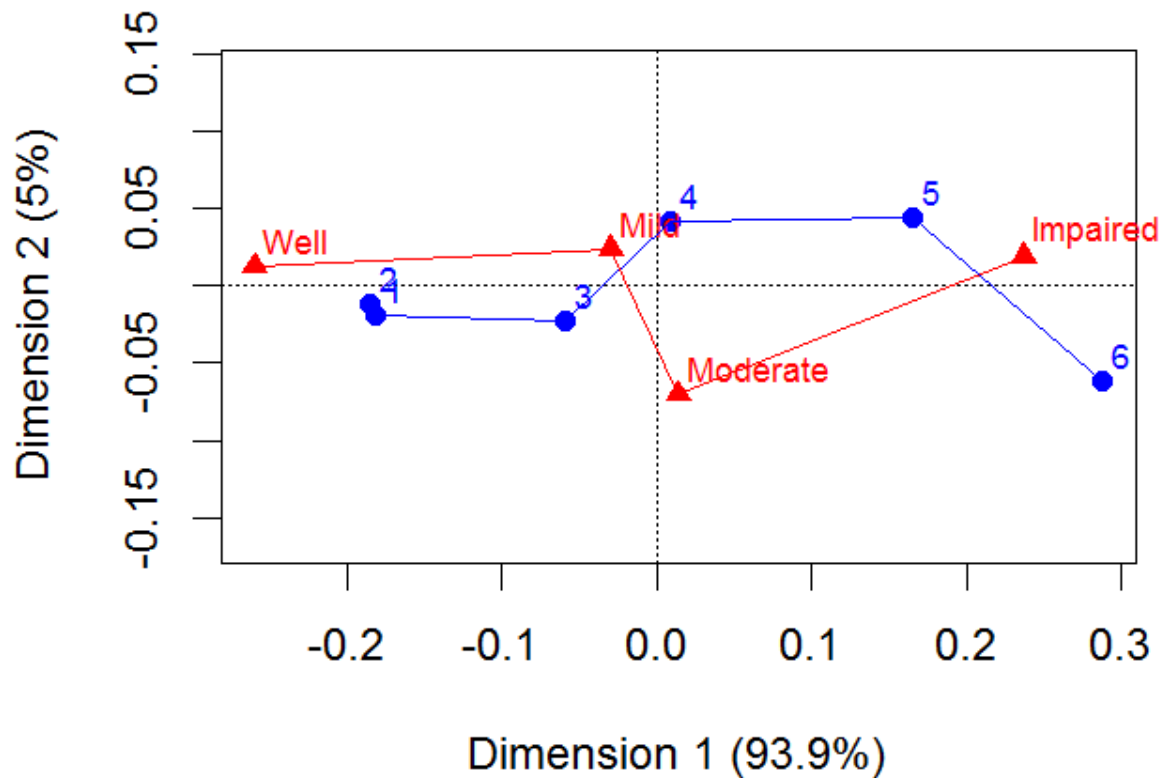
Principal inertias (eigenvalues):

dim	value	%	cum%	scree plot
1	0.026025	93.9	93.9	*****
2	0.001379	5.0	98.9	*
3	0.000298	1.1	100.0	
-----				
Total:	0.027702	100.0		

- The exact CA solution requires  $\min(r-1, c-1) = 3$  dimensions
- Total Pearson  $\chi^2$  is  $n \sum \lambda_i^2 = 1660 \times 0.0277 = 45.98$  with 15 df
- Of this, 93.9% is accounted for by the 1<sup>st</sup> dimension

# Mental data: CA plot

```
plot(mental.ca, lines = TRUE)
```



Category spacing:

SES: perhaps collapse categories (1,2) ??

Mental: Smaller diff betw. Mild, Moderate ??

# Looking ahead

- CA is largely an exploratory method — row/column scores are not parameters of a statistical model; no confidence intervals
- Only rough tests for the number of CA dimensions
- Can't test a hypothesis that the row/column scores are have some particular spacing (e.g., are `mental` and `ses` equally spaced?)
- These questions can be answered with specialized loglinear models
- Nevertheless, `plot(ca(table))` gives an excellent quick view of associations



# Multi-way tables

Correspondence analysis can be extended to  $n$ -way tables in several ways:

## Stacking approach

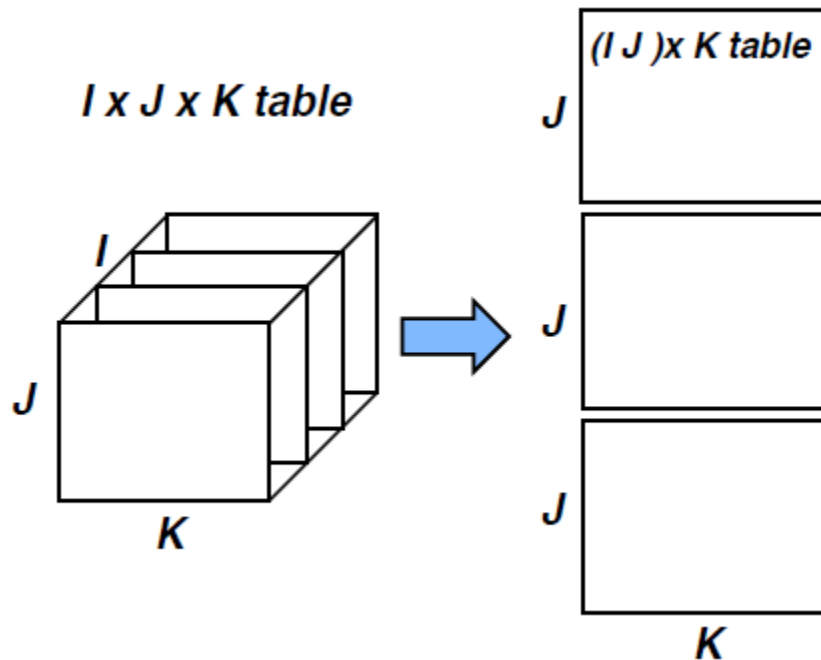
- $n$ -way table flattened to a 2-way table, combining several variables “interactively”
- Each way of stacking corresponds to a *loglinear model*
- Ordinary CA of the flattened table → visualization of that model
- Associations among stacked variables are *not visualized*

## Multiple correspondence analysis (MCA)

- Extends CA to  $n$ -way tables
- Analyzes all *pairwise bivariate* associations
- Can plot all factors in a single plot
- An extension, *joint correspondence analysis*, gives a better account of inertia for each dimension

# Multi-way tables: Stacking

A 3-way table of size  $I \times J \times K$  can be sliced and stacked as a two-way table in several ways



- The variables combined are treated “interactively”
- Each way of stacking corresponds to a loglinear model
  - $(I \times J) \times K \rightarrow [AB][C]$
  - $I \times (J \times K) \rightarrow [A][BC]$
  - $J \times (I \times K) \rightarrow [B][AC]$
- Only the associations in separate  $[ ]$  terms are analyzed and displayed
- The stacked table is analyzed with ordinary CA of the two-way stacked table

# Interactive coding in R

- Data in table or array form: use

`as.matrix(structable(rows ~ cols))`

```
mat1 <- as.matrix(structable(A + B ~ C, data=mytable))      # [A B] [C]
mat2 <- as.matrix(structable(A + C ~ B + D, data=mytable))  # [A C] [B D]
ca(mat2)
```

- Data as frequency data frame: use `interaction()` or `paste()` followed by `xtabs()`

```
mydf$AB <- interaction(mydf$A, mydf$B, sep='.')             # levels: A.B
mydf$AB <- paste(mydf$A, mydf$B, sep=':')                  # levels: A:B
...
mytab <- xtabs(Freq ~ AB + C, data=mydf)                   # [A B] [C]
```

# Example: suicide rates in Germany

- **vcd::Suicide** gives a 2 x 5 x 8 table of sex by age.group by method for 53,158 suicides in Germany, in a frequency data frame
- Use `paste()` to join age.group and sex → age\_sex in the form '10-20 M'

```
> Suicide <- within(Suicide, {  
  age_sex <- paste(age.group, toupper(substr(sex,1,1)))  
})  
> head(Suicide)
```

	Freq	sex	method	age	age.group	method2	age_sex
1	4	male	poison	10	10-20	poison	10-20 M
2	0	male	cookgas	10	10-20	gas	10-20 M
3	0	male	toxicgas	10	10-20	gas	10-20 M
4	247	male	hang	10	10-20	hang	10-20 M
5	1	male	drown	10	10-20	drown	10-20 M
6	17	male	gun	10	10-20	gun	10-20 M

# Suicide rates in Germany

```
> suicide.tab <- xtabs(Freq ~ age_sex + method2, data=Suicide)
```

```
> suicide.tab
```

		method2							
age_sex		poison	gas	hang	drown	gun	knife	jump	other
10-20	F	921	40	212	30	25	11	131	100
10-20	M	1160	335	1524	67	512	47	189	464
25-35	F	1672	113	575	139	64	41	276	263
25-35	M	2823	883	2751	213	852	139	366	775
40-50	F	2224	91	1481	354	52	80	327	305
40-50	M	2465	625	3936	247	875	183	244	534
55-65	F	2283	45	2014	679	29	103	388	296
55-65	M	1531	201	3581	207	477	154	273	294
70-90	F	1548	29	1355	501	3	74	383	106
70-90	M	938	45	2948	212	229	105	268	147

- The CA analysis will be that of the loglinear model [\[Age Sex\]](#) [\[Method\]](#)
- It will show associations between the age–sex combinations and method of suicide
- Associations between age and sex will [not be shown](#) in this analysis

# Suicide rates in Germany

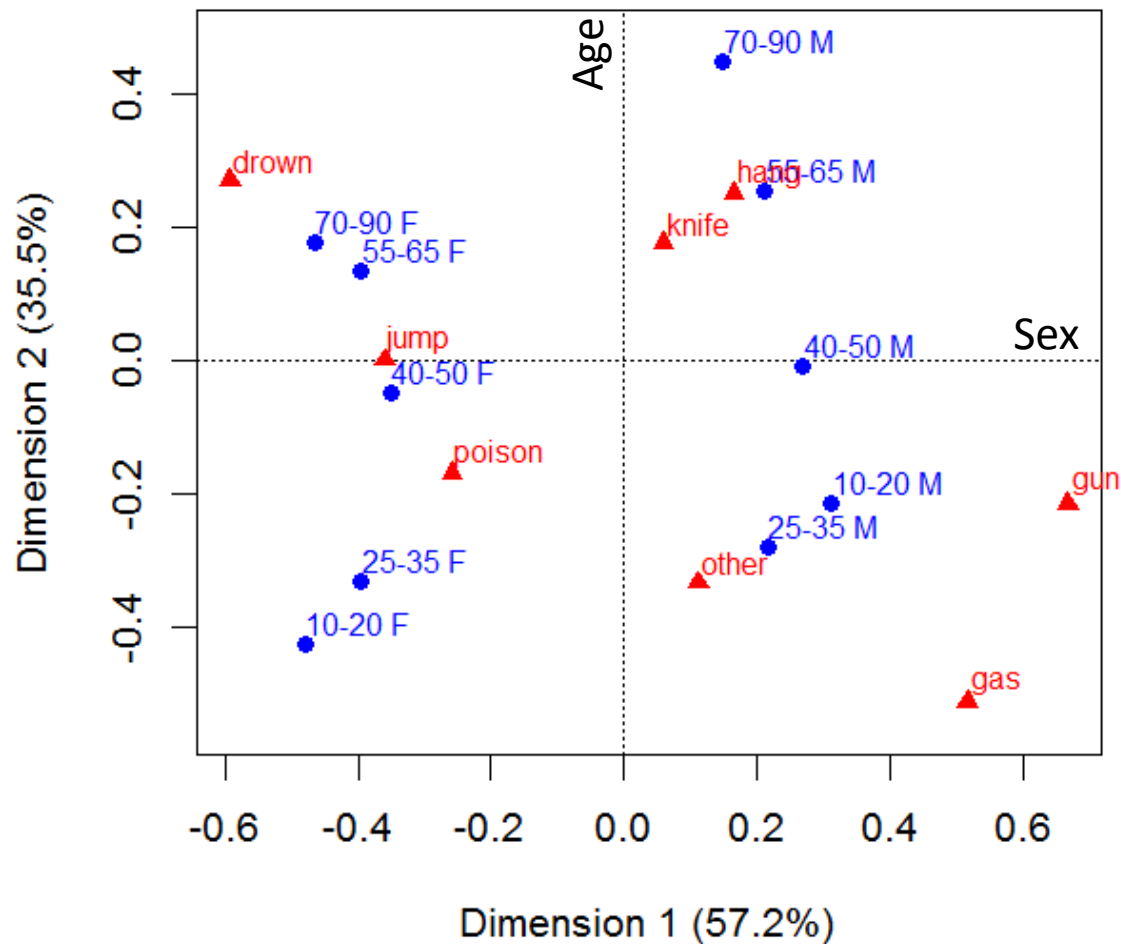
```
> suicide.ca <- ca(suicide.tab)
> summary(suicide.ca, rows=FALSE, columns = FALSE)
```

Principal inertias (eigenvalues):

dim	value	%	cum%	scree plot
1	0.096151	57.2	57.2	*****
2	0.059692	35.5	92.6	*****
3	0.008183	4.9	97.5	*
4	0.002158	1.3	98.8	
5	0.001399	0.8	99.6	
6	0.000557	0.3	100.0	
7	6.7e-050	0.0	100.0	
-----				
Total:	0.168207	100.0		

For this table  $\chi^2(63) = 8946$ . Of this, 92.6% is accounted for in the first two dimensions

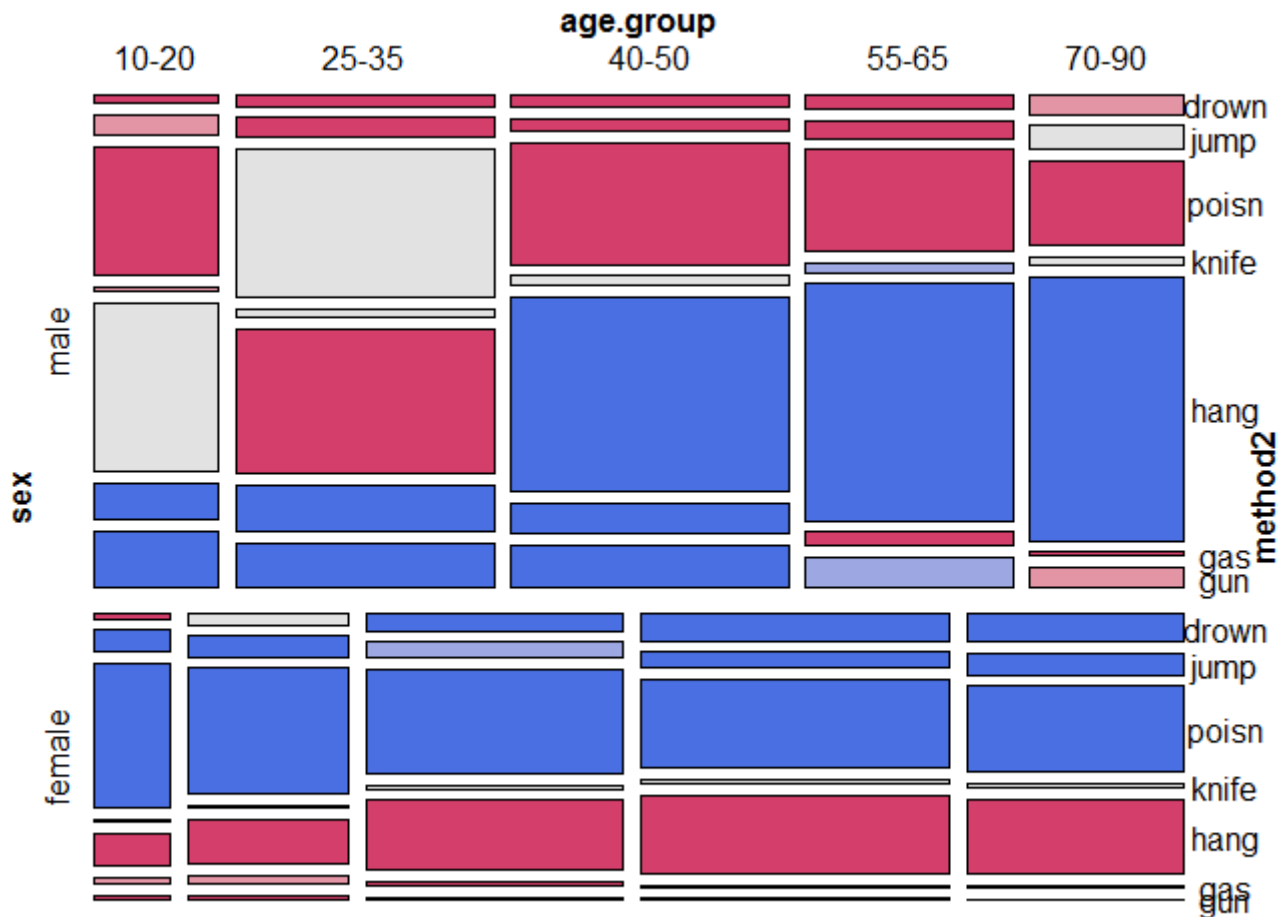
```
> plot(suicide.ca)
```



- Dim 1: Sex
- Dim 2: Age
- Can interpret method use by age-sex combination
  - young M: gas, gun,
  - young F: poison

Compare with a mosaic plot, also fitting the model [Age Sex][Method]

```
suicide.tab3 <- xtabs(Freq ~ sex + age.group + method2, data=Suicide)
mosaic(suicide.tab3, shade=TRUE, legend=FALSE,
       expected=~age.group*sex + method2, ... )
```



DDAR Fig 6.7, p 238

(I permuted methods by CA Dim1 & deleted "Other")



# Marginal tables & supplementary variables

- Supplementary variables provide a way to include more info in CA
  - An  $n$ -way table is collapsed to a marginal table by ignoring factors
  - Omitted variables can be included by treating them as supplementary
  - These are projected into the space of the marginal CA
- E.g., age by method, ignoring sex as the main analysis

```
> suicide.tab2 <- xtabs(Freq ~ age.group + method2, data=Suicide)
> suicide.tab2
```

	method2							
age.group	poison	gas	hang	drown	gun	knife	jump	other
10-20	2081	375	1736	97	537	58	320	564
25-35	4495	996	3326	352	916	180	642	1038
40-50	4689	716	5417	601	927	263	571	839
55-65	3814	246	5595	886	506	257	661	590
70-90	2486	74	4303	713	232	179	651	253

## Also have data on relation of sex and method

```
> (suicide.sup <- xtabs(Freq ~ sex + method2, data=Suicide))
      method2
sex      poison    gas  hang drown   gun knife  jump other
male      8917   2089 14740    946  2945    628  1340   2214
female    8648    318  5637   1703   173    309  1505   1070
> suicide.tab2s <- rbind(suicide.tab2, suicide.sup)
```

	method2								
<u>age.group</u>	poison	gas	hang	drown	gun	knife	jump	other	
10-20	2081	375	1736	97	537	58	320	564	
25-35	4495	996	3326	352	916	180	642	1038	
40-50	4689	716	5417	601	927	263	571	839	
55-65	3814	246	5595	886	506	257	661	590	
70-90	2486	74	4303	713	232	179	651	253	
sex	poison	gas	hang	drown	gun	knife	jump	other	
male	8917	2089	14740	946	2945	628	1340	2214	
female	8648	318	5637	1703	173	309	1505	1070	

Main analysis table

Supplementary rows

# Supplementary variables

Call `ca(table, suprow = )` to treat some rows as supplementary variables

```
> suicide.ca2s <- ca(suicide.tab2s, suprow=6:7)  
> summary(suicide.ca2s, rows=FALSE, columns = FALSE)
```

Principal inertias (eigenvalues):

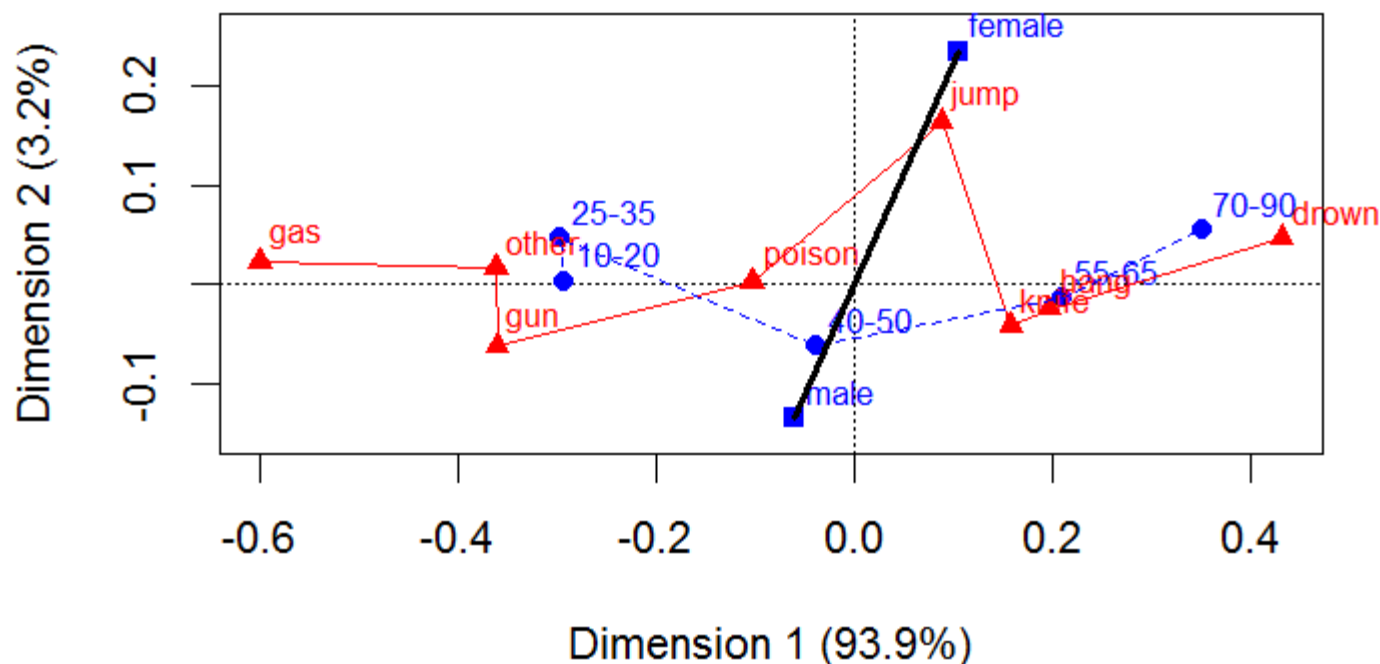
dim	value	%	cum%	scree plot
1	0.060429	93.9	93.9	*****
2	0.002090	3.2	97.1	*
3	0.001479	2.3	99.4	*
4	0.000356	0.6	100.0	
-----				
Total:	0.064354	100.0		

The relation of age and method is now essentially 1 dimensional

The inertia of Dim 1 here (0.604) is nearly the same as that of Dim 2 (0.596) for age in the stacked table

```
res <- plot(suicide.ca2s,
            pch=c(16, 15, 17, 24),
            lines = c(FALSE, TRUE))
lines(res$rows[1:5,], col = "blue", lty=2)
lines(res$rows[6:7,], col = "black", lwd=3)
```

Plotting the solution  
shows points for row, col  
& supplementary rows

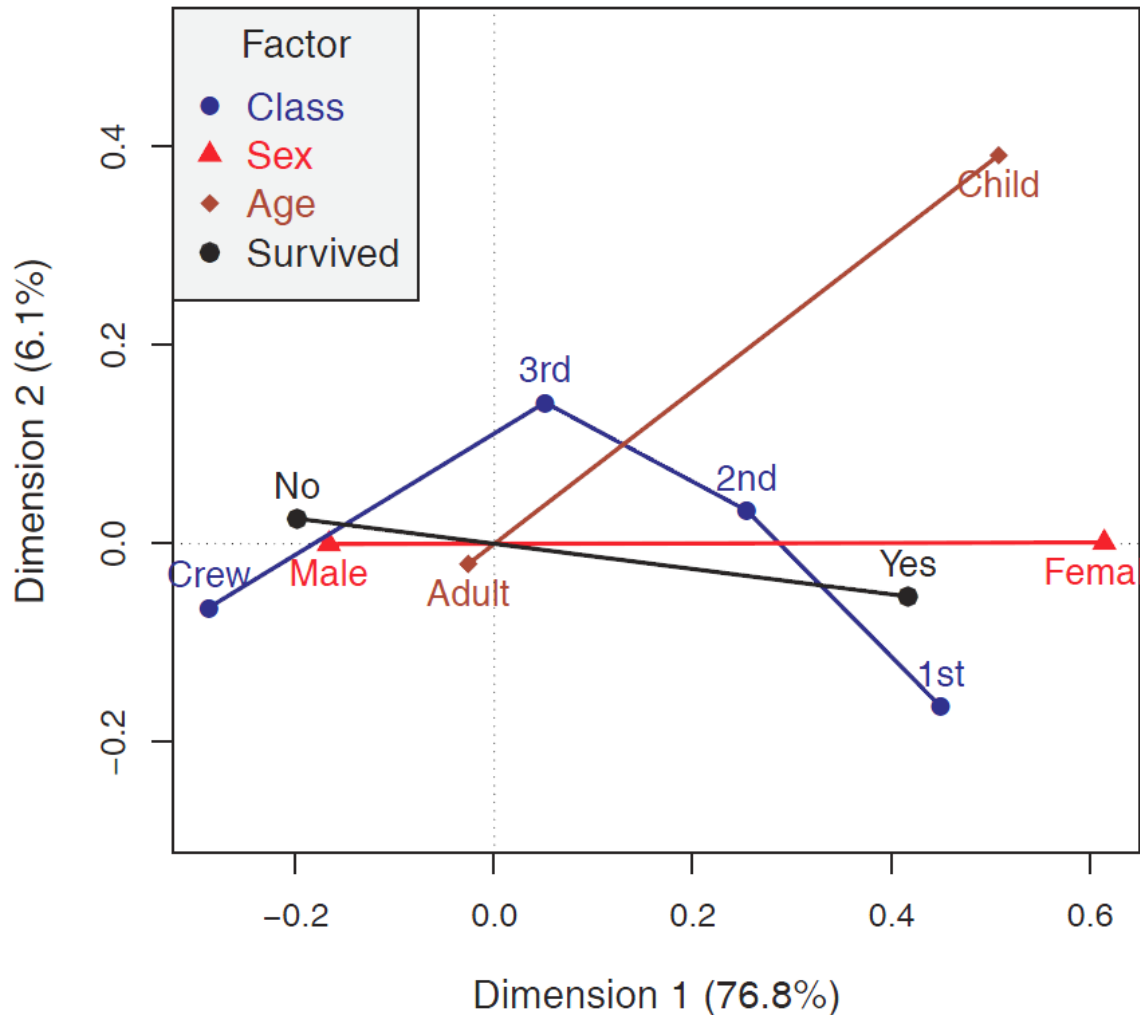


Ignoring Sex has collapsed Sim 1 (Sex) of the [Age Sex][Method] analysis  
Supp. points for Sex show the association of Method with Sex in this space

# Multiple correspondence analysis

- Extends CA to  $n$ -way tables
- Useful when simpler stacking approach doesn't work well, e.g., 10 categorical attitude items
- Analyzes all **pairwise bivariate** associations. Analogous to:
  - Correlation matrix (numbers)
  - Scatterplot matrix (graphs)
  - All pairwise  $\chi^2$  tests (numbers)
  - Mosaic matrix (graphs)
- Provides an **optimal scaling** of the category scores for each variable
- Can plot all factors in a single plot
- An extension, **joint correspondence analysis**, gives a better account of inertia for each dimension

# Example: Titanic data



Plot of MCA for the Titanic data

All 4 variables represented in a single plot

Dim 1: Sex

Dim 2: Class & Age

# CA $\rightarrow$ MCA: Indicator & Burt

Two ways to think about MCA:

## Indicator matrix (dummy variables)

- A given categorical variable,  $q$ , can be represented by an indicator matrix  $\mathbf{Z}(n \times J_q)$  of dummy variables,  $z_{ij} = 1$  if case  $i$  is in category  $j$
- Let  $\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_Q$  be the indicator matrices for  $Q$  variables
- MCA is then a simple CA applied to the partitioned matrix  $\mathbf{Z} = [\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_Q]$

## Burt matrix

- The Burt matrix is the product of the indicator matrix  $\mathbf{Z}$  and its transpose

$$\mathbf{B} = \mathbf{Z}^T \mathbf{Z}$$

- MCA can be defined using the SVD of  $\mathbf{B}$ , giving category scores for all variables accounting for the largest proportion of all bivariate associations.

# Indicator matrix: Hair Eye color

- For the hair-eye data, the indicator matrix **Z** has  $n=592$  rows (observations) and  $4 + 4 = 8$  columns (categories).
  - Shown below in frequency form: h1 — h4 indicators for hair color, e1—e4 for eye color
  - E.g., 1<sup>st</sup> row represents 68 observations with black hair and brown eyes

	Hair	Eye	Freq	h1	h2	h3	h4	e1	e2	e3	e4
1	Black	Brown	68	1	0	0	0	1	0	0	0
2	Brown	Brown	119	0	1	0	0	1	0	0	0
3	Red	Brown	26	0	0	1	0	1	0	0	0
4	Blond	Brown	7	0	0	0	1	1	0	0	0
5	Black	Blue	20	1	0	0	0	0	1	0	0
6	Brown	Blue	84	0	1	0	0	0	1	0	0
7	Red	Blue	17	0	0	1	0	0	1	0	0
8	Blond	Blue	94	0	0	0	1	0	1	0	0
•	•	•									



## Expand this to case form to get **Z** (592 x 8)

```
> Z <- expand.dft(haireye.df)[, -(1:2)]
> vnames <- c(levels(haireye.df$Hair), levels(haireye.df$Eye))
> colnames(Z) <- vnames
> dim(Z)
[1] 592      8
```

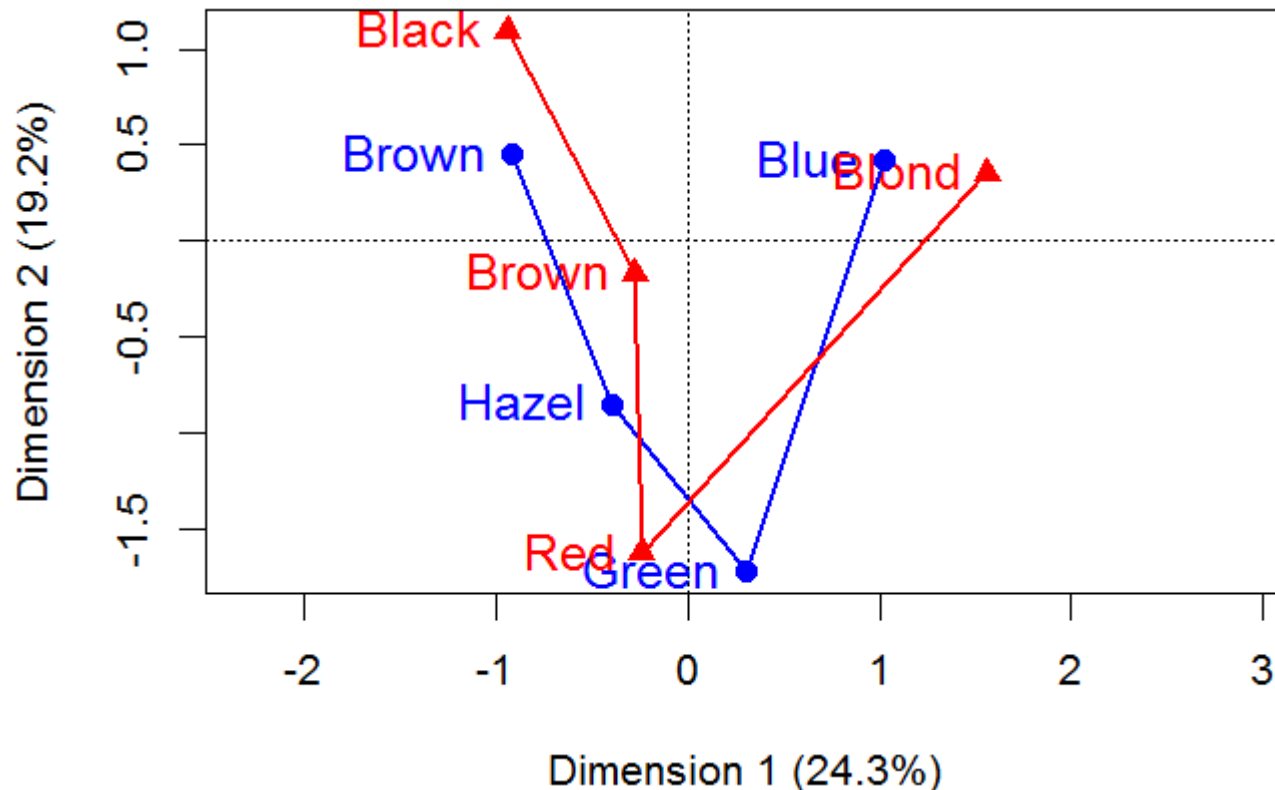
If the indicator matrix is partitioned as  $\mathbf{Z} = [\mathbf{Z}_1; \mathbf{Z}_2]$ , corresponding to the hair, eye categories, then the contingency table is given by  $\mathbf{N} = \mathbf{Z}_1^T \mathbf{Z}_2$ .

```
> Z1 <- as.matrix(Z[, 1:4])
> Z2 <- as.matrix(Z[, 5:8])
> (N <- t(Z1) %*% Z2)
```

	Brown	Blue	Hazel	Green
Black	68	20	15	5
Brown	119	84	54	29
Red	26	17	14	14
Blond	7	94	10	16

- We can then use ordinary CA on the indicator matrix,  $\mathbf{Z}$
- Except for scaling, this is the same as the CA of  $\mathbf{N}$
- The inertia contributions differ, and this is handled better by MCA

```
Z.ca <- ca(Z)
res <- plot(Z.ca, what=c("none", "all")) # plus customization
```



# The Burt matrix

For two categorical variables, the Burt matrix is

$$\mathbf{B} = \mathbf{Z}^T \mathbf{Z} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N} \\ \mathbf{N}^T & \mathbf{N}_2 \end{bmatrix} .$$

- $\mathbf{N}_1$  and  $\mathbf{N}_2$  are diagonal matrices containing the marginal frequencies of the two variables
- The contingency table,  $\mathbf{N}$  appears in the off-diagonal block

A similar analysis to that of the indicator matrix  $\mathbf{Z}$  is produced by:

```
Burt <- t(as.matrix(Z)) %*% as.matrix(Z)
rownames(Burt) <- colnames(Burt) <- vnames
Burt.ca <- ca(Burt)
plot(Burt.ca)
```

- Standard coords are the same
- Singular values of  $\mathbf{B}$  are the squares of those of  $\mathbf{Z}$

# Multivariate MCA

For  $Q$  categorical variables, the Burt matrix is

$$\mathbf{B} = \mathbf{Z}^T \mathbf{Z} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_{[12]} & \cdots & \mathbf{N}_{[1Q]} \\ \mathbf{N}_{[21]} & \mathbf{N}_2 & \cdots & \mathbf{N}_{[2Q]} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{N}_{[Q1]} & \mathbf{N}_{[Q2]} & \cdots & \mathbf{N}_Q \end{bmatrix} .$$

- The diagonal blocks  $\mathbf{N}_i$  contain the one-way marginal frequencies
- The off-diagonal blocks  $\mathbf{N}_{[ij]}$  contain the bivariate contingency tables for each pair  $(i, j)$  of variables.
- Classical MCA can be defined as a SVD of the matrix  $\mathbf{B}$
- It produces scores for the categories of *all* variables accounting for the greatest proportion of the bivariate associations in off-diagonal blocks in a small number of dimensions.

# MCA properties

- The **inertia** contributed by a given variable increases with the number of response categories:
  - $\text{inertia}(Z_q) = J_q - 1$
- The **centroid** of the categories for each variable is at the **origin** of the display.
- For a given variable, the inertia contributed by a given category increases as the marginal frequency in that category decreases.
  - Low frequency points therefore appear further from the origin.
- The category points for a **binary** variable lie on a line through the origin.

# MCA example: pre- and extramarital sex

- PreSex data: the  $2 \times 2 \times 2 \times 2$  table of gender, premarital sex, extramatrial sex and marital status (divorced, still married)
- The function `mjca()` provides several scalings for the singular values
- Here I use `lambda="Burt"`

```
data("PreSex", package="vcd")
PreSex <- aperm(PreSex, 4:1)           # order variables G, P, E, M
presex.mca <- mjca(PreSex, lambda="Burt")
summary(presex.mca, rows=FALSE, columns = FALSE)
```

Principal inertias (eigenvalues):

dim	value	%	cum%	scree plot
1	0.149930	53.6	53.6	*****
2	0.067201	24.0	77.6	*****
3	0.035396	12.6	90.2	***
4	0.027365	9.8	100.0	**
-----				
Total:	0.279892	100.0		

# MCA example: pre- and extramarital sex

```
vcdExtra::mcaplot(presex.mca,  
                  legend=TRUE, legend.pos = "bottomright")
```

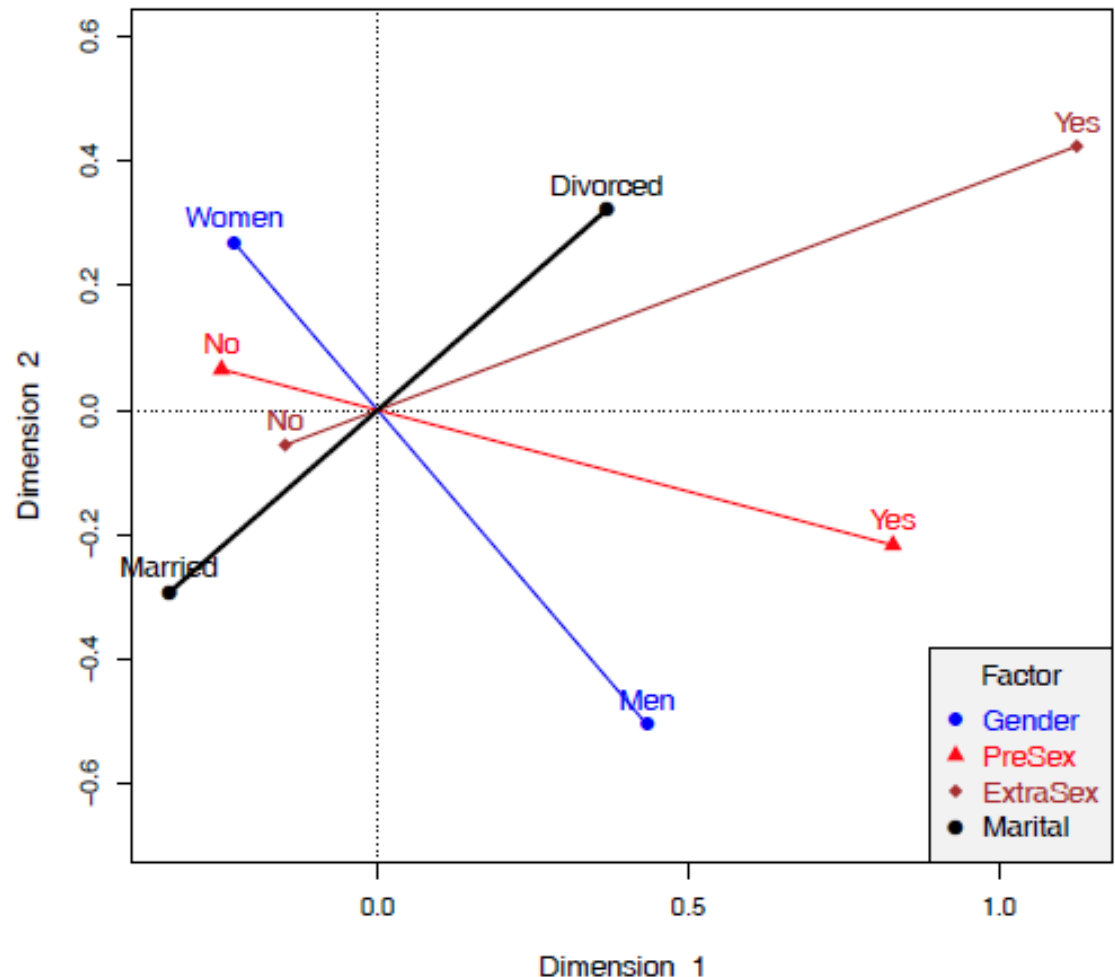
Accounts for 76% of total inertia

Women less likely to report pre- and/or extra-marital sex

Divorced associated with pre- and extra- sex

Gender  $\perp$  Marital

NB: This only analyzes **bivariate** associations, i.e., no 3-way associations



# Inertia in MCA

- In simple CA, **total inertia**  $= \sum \lambda_i^2 = \chi^2/n$
- $\implies$  sensible to consider % inertia for each dimension

Not so straight-forward in MCA:

- For a given indicator matrix,  $\mathbf{Z}_q$ , the inertia is  $J_q - 1$
- For all variables, with  $J = \sum J_q$  categories, the total inertia of  $\mathbf{Z} = [\mathbf{Z}_1, \dots, \mathbf{Z}_Q]$  is the average of the inertias of the sub-tables

$$inertia(\mathbf{Z}) = \frac{1}{Q} \sum_q inertia(\mathbf{Z}_q) = \frac{1}{Q} \sum_q (J_q - 1) = \frac{J - Q}{Q}$$

- The average inertia per dimension is therefore  $1/Q$
- $\implies$  Interpret dimensions with inertia  $> 1/Q$  (as in PCA:  $\lambda > 1$ )
- In analysis of the Burt matrix, average inertia is inflated by the diagonal blocks



# Inertia in MCA

Two solutions:

## Adjusted inertia

- Ignores the diagonal blocks in the Burt matrix
- Calculates adjusted inertia as

$$(\lambda_i^*)^2 = \left[ \frac{Q}{Q-1} (\lambda_i^Z - \frac{1}{Q}) \right]^2$$

- Express contributions of dimensions as  $(\lambda_i^*)^2 / \sum (\lambda_i^*)^2$ , with summation over only dimensions with  $(\lambda_i^Z)^2 > 1/Q$ .

## Joint correspondence analysis

- Start with MCA analysis of the Burt matrix
- Replace diagonal blocks with values estimated from that solution
- Repeat until solution converges, improving the fit to off-diagonal blocks

NB: JCA solutions aren't nested. I generally use adjusted inertia

# MCA example: Survival on the *Titanic*

Analyse the Titanic data using `ca::mcja()`

- The default inertia method is `lambda = "adjusted"`
- Other methods: "indicator", "Burt", "JCA"

```
data(Titanic)
titanic.mca <- mcja(Titanic)
summary(titanic.mca, columns = FALSE)
```

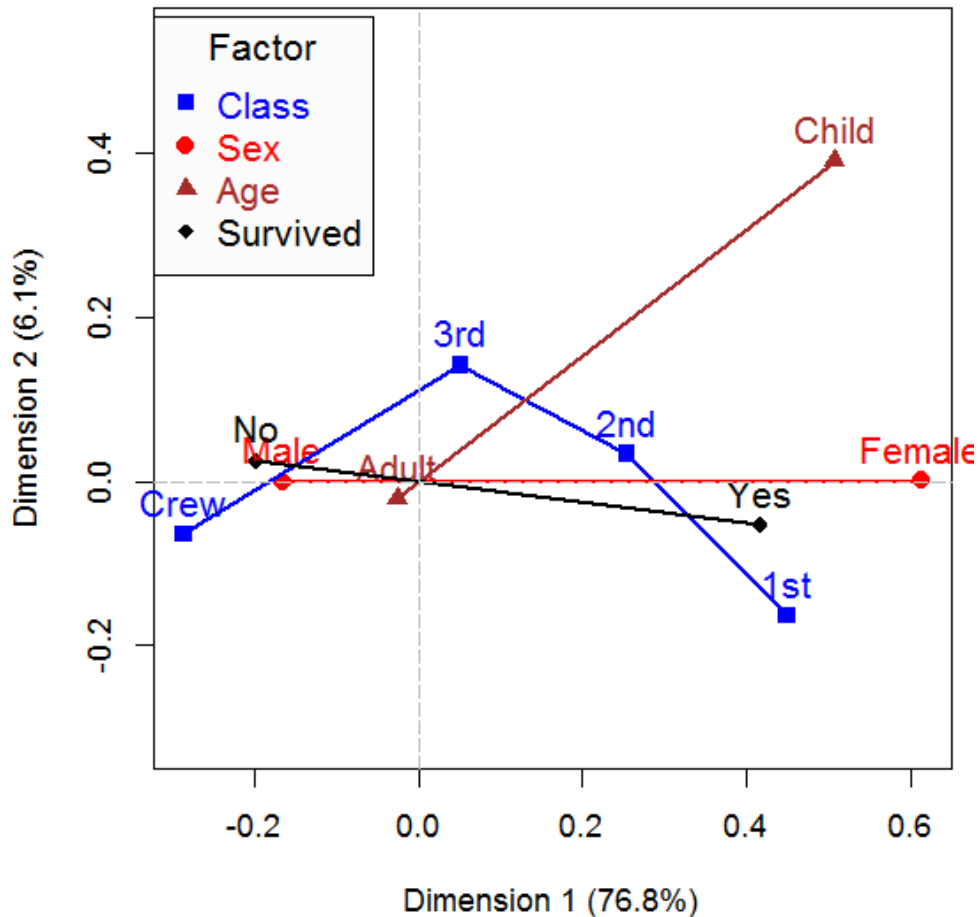
Principal inertias (eigenvalues):

dim	value	%	cum%	scree plot
1	0.067655	76.8	76.8	*****
2	0.005386	6.1	82.9	**
3	0.000000	0.0	82.9	
	-----	-----		
Total:	0.088118			

Using adjusted inertia, the 2D solution accounts for ~ 83% of total, bivariate association.

## Plot the solution with `vcdExtra::mcaplot()`

```
mcaplot(titanic.mca, legend=TRUE, legend.pos = "topleft")
```



Dim 1 perfectly aligned with Sex  
Also strongly aligned w/ survival  
& class

Dim 2: reflects class & age

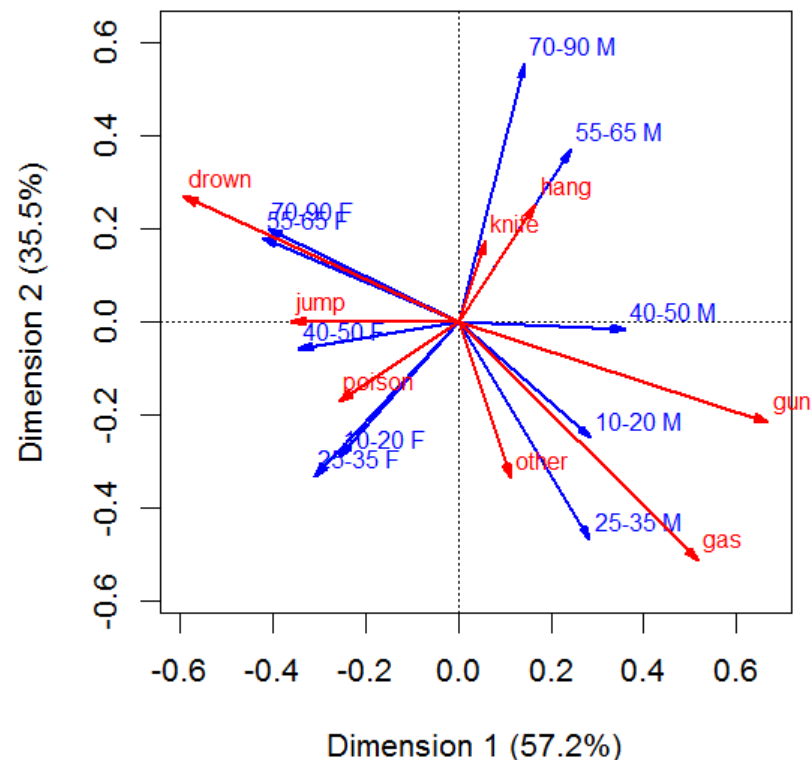
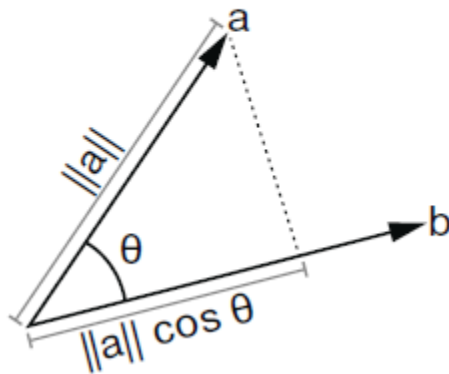
→ Survival associated with  
Female, 1<sup>st</sup> vs 3<sup>rd</sup> class, child

# Biplots for contingency tables

The biplot is a related visualization that also uses the SVD to give a low-rank (2D) approximation.

- In CA, the weighted  $\chi^2$  distances between row (column) points reflect the differences among row (column) profiles
- In the biplot, rows (columns) are represented by vectors from the origin, with an inner-product (projection) interpretation – row point  $\mathbf{a}_i$  is fit by projection on col point  $\mathbf{b}_j$

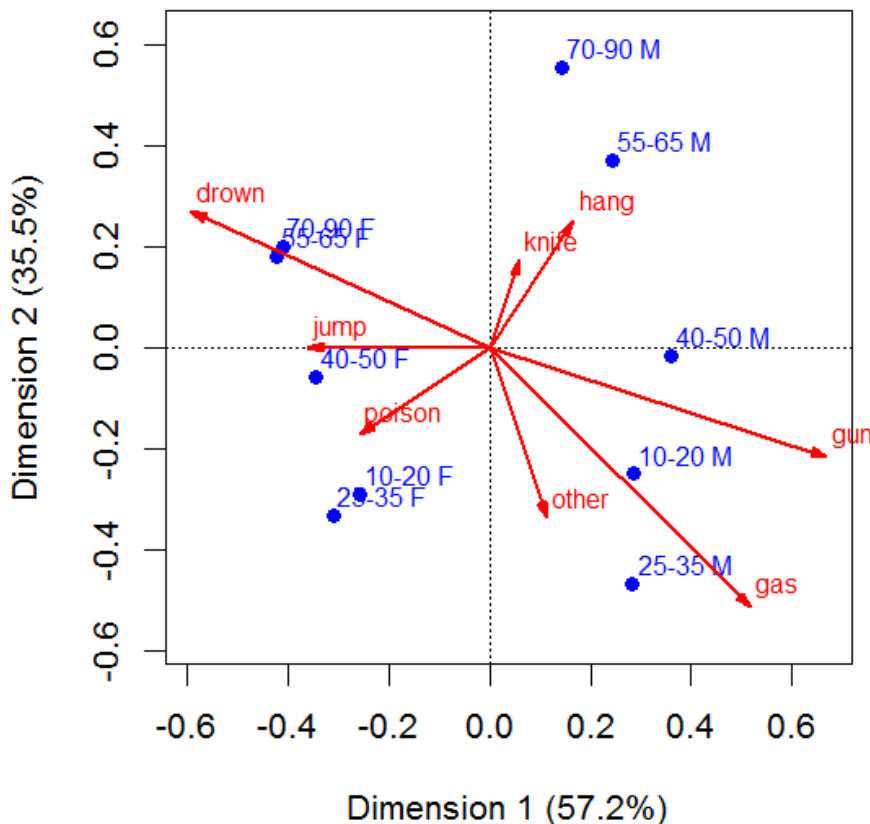
$$Y \approx AB^T \iff y_{ij} \approx \mathbf{a}_i^T \mathbf{b}_j$$



# Example: Suicide rates

There are different scalings for CA biplots. Here I use the 'contribution' biplot  
I find the plot less messy to plot arrows for only rows or cols and imagine the projection

```
plot(suicide.ca, map="colgreen", arrows=c(FALSE, TRUE), lwd=2)
```



Associations between age-sex categories and suicide methods can be read as projections of the points on the vectors

Lengths of vectors for suicide reflect their contributions to this 2D plot

# Summary

- CA is an exploratory method designed to account for association (Pearson  $\chi^2$ ) in a small number of dimensions
  - Row and column scores provide an **optimal scaling** of the category levels
  - Plots of these can suggest an explanation for association
- CA uses the **singular value decomposition** to approximate the matrix of residuals from independence
- Standard and principal coordinates have different geometric properties, but are essentially re-scalings of each other
- Multi-way tables can be handled by:
  - Stacking approach— collapse some dimensions interactively to a 2-way table
  - Each way of stacking  $\rightarrow$  a loglinear model
  - MCA analyzes the full  $n$  – way table using an indicator matrix or the **Burt** matrix

Given a new 2-way table, my first thought is nearly always: `plot(ca(mytable))`