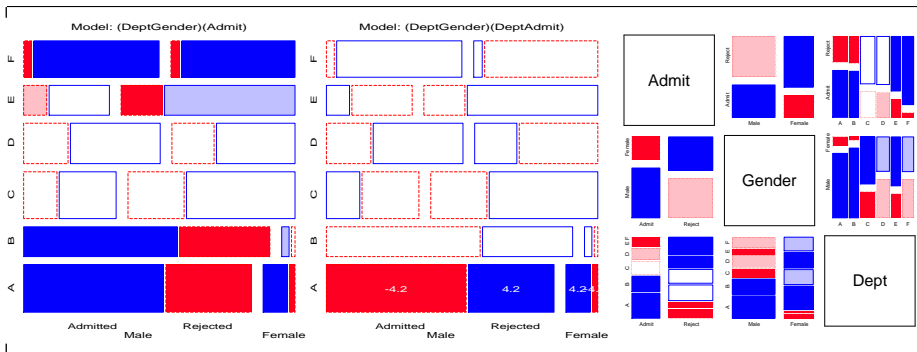# Loglinear Models and Mosaic Displays

Michael Friendly

Psych 6136

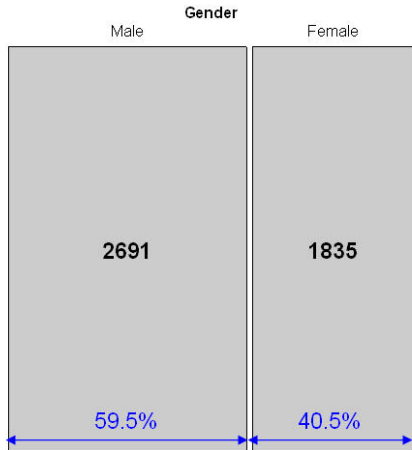## December 17, 2014

Mosaic displays: Basic ideas

Hartigan and Kleiner (1981), Friendly (1994, 1999)

- Area-proportional display of frequencies in an *n*-way table
- Tiles (cells): recursive splits of a unit square—
    - V1: width $\sim$ marginal frequencies, n_{i++}
    - V2: height $\sim$ relative frequencies $|$V1, n_{ij+}/n_{i++}
    - V3: width $\sim$ relative frequencies $|$ (V1, V2), n_{ijk}/n_{ij+}
    - $\cdots$
- $\Rightarrow$ area $\sim$ cell frequency, n_{ijk}

UCB Admissions: Observed frequencies

Mosaic displays: Basic ideas

Hartigan and Kleiner (1981), Friendly (1994, 1999)

- Area-proportional display of frequencies in an n-way table
- Tiles (cells): recursive splits of a unit square—
  - V1: width $\sim$ marginal frequencies, n_{i++}
  - V2: height $\sim$ relative frequencies $|$ V1, n_{ij+}/n_{i++}
  - V3: width $\sim$ relative frequencies $|$ (V1, V2), n_{ijk}/n_{ij+}
  - $\cdots$
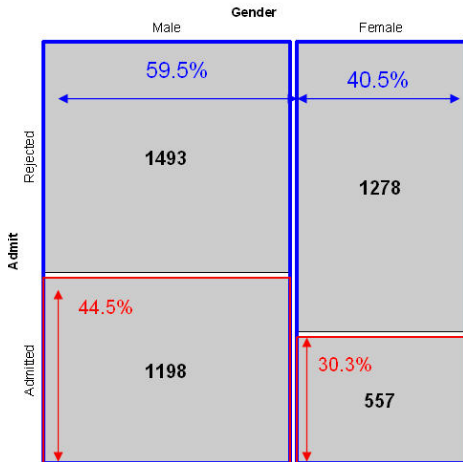- $\Rightarrow$ area $\sim$ cell frequency, n_{ijk}

UCB Admissions: Observed frequencies

Mosaic displays: Basic ideas

Hartigan and Kleiner (1981), Friendly (1994, 1999)

- Area-proportional display of frequencies in an *n*-way table
- Tiles (cells): recursive splits of a unit square—
    - V1: width $\sim$ marginal frequencies, n_{i++}
    - V2: height $\sim$ relative frequencies $|$V1, n_{ij+}/n_{i++}
    - V3: width $\sim$ relative frequencies $|$(V1, V2), n_{ijk}/n_{ij+}
    - $\cdots$
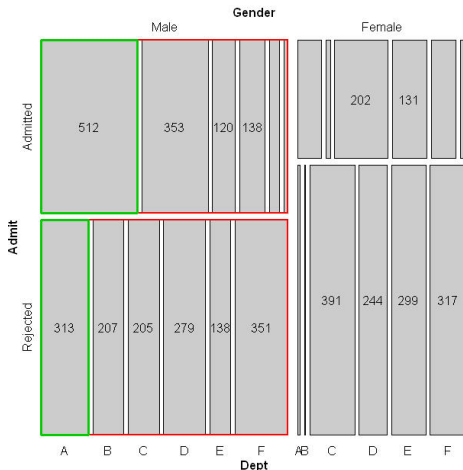- $\Rightarrow$ area $\sim$ cell frequency, n_{ijk}

3 way table: Gender x Admit x Dept
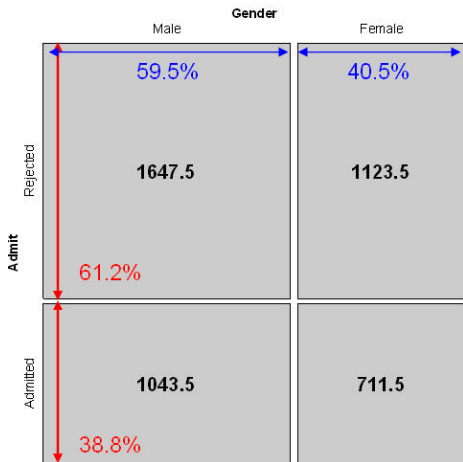
Mosaic displays: Basic ideas

- Independence: Two-way table
- Expected frequencies:

$$\widehat{m}_{ij} = \frac{n_{i+}n_{+j}}{n_{++}} = n_{++}\text{row \%col \%}$$

- $\Rightarrow$ rows & columns align when variables are independent
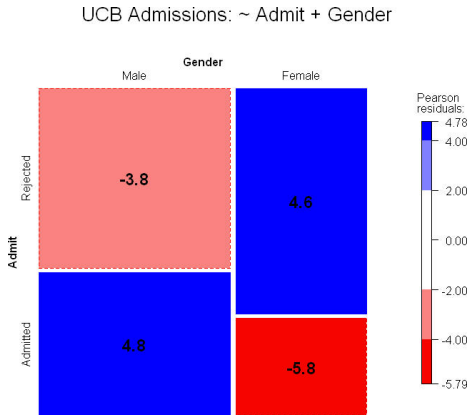

Independence: Expected frequencies

Mosaic displays: Residuals & shading

- Pearson residuals:

$$d_{ij} = \frac{n_{ij} - \widehat{m}_{ij}}{\sqrt{\widehat{m}_{ij}}}$$

- Pearson $\chi^2 = \Sigma\Sigma d_{ij}^2 = \Sigma\Sigma \frac{(n_{ij} - \hat{m}_{ij})^2}{\hat{m}_{ij}}$

- Other residuals: deviance (LR), Freeman-Tukey (FT), adjusted (ADJ), ...

- Shading:
  - Sign: − negative in red; + positive in blue
  - Magnitude: intensity of shading: $|d_{ij}| > 0, 2, 4, \ldots$

- ⇒ Independence: rows align, or cells are empty!

UCB Admissions: ~ Admit + Gender

Loglinear models: Perspectives I

Loglinear approach

Loglinear models were first developed as an analog of classical ANOVA models, where *multiplicative* relations (under independence) are re-expressed in *additive* form as models for log(frequency).

$$\log m_{ij} = \mu + \lambda_i^A + \lambda_j^B \equiv [A][B] \equiv \sim A + B$$

- This expresses the model of independence for a two-way table (no A*B association)
- The notations $[A][B] \equiv \sim A + B$ are shorthands

Loglinear models: Perspectives II

GLM approach

More generally, loglinear models are also generalized linear models (GLMs) for log(frequency), with a Poisson distribution for the cell counts.

$$\log \boldsymbol{m} = \boldsymbol{X\beta}$$

- This looks just like the general linear ANOVA, regression model, but for log frequency
- This approach allows quantitative predictors and special ways of treating ordinal factors

Loglinear models: Perspectives III

Logit models

When one table variable is a binary response, a logit model for that response is equivalent to a loglinear model (as discussed later).

$$\log(m_{1jk}/m_{2jk}) = \alpha + \beta_j^B + \beta_k^C \equiv [AB][AC][BC]$$

- $\log(m_{1jk}/m_{2jk})$ represents the log odds of response category 1 vs. 2
- The model formula includes only terms for the effects on A of variables B and C
- The equivalent loglinear model is [AB] [AC] [BC]
- The logit model assumes [BC] association, and [AB] $\rightarrow \beta_j^B$, [AC] $\rightarrow \beta_k^C$

Loglinear models: Overview

Two-way tables: Loglinear approach

For two discrete variables, *A* and *B*, suppose a multinomial sample of total size *n* over the *IJ* cells of a two-way $I \times J$ contingency table, with cell frequencies n_{ij}, and cell probabilities $\pi_{ij} = n_{ij}/n$.

- The table variables are statistically independent when the cell (joint) probability equals the product of the marginal probabilities, $\Pr(A = i \,\&\, B = j) = \Pr(A = i) \times \Pr(B = j)$, or,

$$\pi_{ij} = \pi_{i+}\pi_{+j} \ .$$

- An equivalent model in terms of expected frequencies, $m_{ij} = n\pi_{ij}$ is

$$m_{ij} = (1/n)\, m_{i+}\, m_{+j} \ .$$

- This multiplicative model can be expressed in additive form as a model for $\log m_{ij}$,

$$\log m_{ij} = -\log n + \log m_{i+} + \log m_{+j} \ . \tag{1}$$

Loglinear models: Overview

Independence model

By anology with ANOVA models, the independence model (1) can be expressed as

$$\log m_{ij} = \mu + \lambda_i^A + \lambda_j^B \ , \tag{2}$$

- μ is the grand mean of $\log m_{ij}$
- the parameters λ_i^A and λ_j^B express the marginal frequencies of variables A and B — "main effects"
- typically defined so that $\sum_i \lambda_i^A = \sum_j \lambda_j^B = 0$ as in ANOVA

Loglinear models: Overview

Saturated model

Dependence between the table variables is expressed by adding association parameters, λ_{ij}^{AB}, giving the *saturated model*,

$$\log m_{ij} = \mu + \lambda_i^A + \lambda_j^B + \lambda_{ij}^{AB} \equiv [AB] \equiv \sim A * B . \qquad (3)$$

- The saturated model fits the table perfectly ($\widehat{m}_{ij} = n_{ij}$): there are as many parameters as cell frequencies. Residual df = 0.
- A global test for association tests $H_0 : \boldsymbol{\lambda}_{ij}^{AB} = \mathbf{0}$.
- If reject H_0, which $\lambda_{ij}^{AB} \neq 0$?
- For ordinal variables, the λ_{ij}^{AB} may be structured more simply, giving tests for ordinal association.

Example: Independence

Generate a table of Education by Party preference, strictly independent

```
educ <- c(50, 100, 50)  # row marginal frequencies
names(educ) <- c("Low", "Med", "High")

party <- c(20, 50, 30)  # col marginal frequencies
names(party) <- c("NDP", "Liberal", "Cons")

table <- outer(educ, party) / sum(party)    # row x col / n
names(dimnames(table)) <- c("Education", "Party")
table

##           Party
## Education NDP Liberal Cons
##      Low   10      25   15
##      Med   20      50   30
##      High  10      25   15
```

Example: Independence

All row (and column) proportions are the same:

```
prop.table(table,1)

##           Party
## Education NDP Liberal Cons
##      Low  0.2     0.5  0.3
##      Med  0.2     0.5  0.3
##      High 0.2     0.5  0.3
```
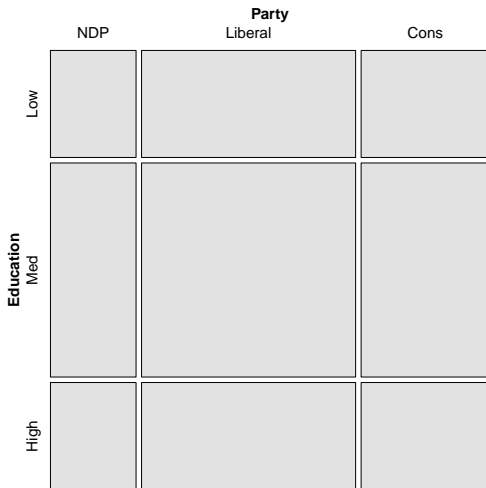
All statistics are 0:

```
vcd::assocstats(table)

##                      X^2 df P(> X^2)
## Likelihood Ratio       0  4        1
## Pearson                0  4        1
##
## Phi-Coefficient   : 0
## Contingency Coeff.: 0
## Cramer's V        : 0
```

Mosaic plot shows equal row and column proportions:

```
library(vcd)
mosaic(table, shade=TRUE, legend=FALSE)
```

Two-way tables: GLM approach

In the GLM approach, the vector of cell frequencies, $\boldsymbol{n} = \{n_{ij}\}$ is specified to have a Poisson distribution with means $\boldsymbol{m} = \{m_{ij}\}$ given by

$$\log \boldsymbol{m} = \boldsymbol{X}\boldsymbol{\beta}$$

- $\boldsymbol{X}$ is a known design (model) matrix, expressing the table factors
- $\boldsymbol{\beta}$ is a column vector containing the unknown λ parameters.
- This is the same as the familiar matrix formulation of ANOVA/regression, except that
  - The response, $\log \boldsymbol{m}$ makes multiplicative relations additive
  - The distribution is taken as Poisson rather than Gaussian (normal)

Example: 2 x 2 table

For a 2×2 table, the saturated model (3) with the usual zero-sum constraints can be represented as

$$\log \left(\begin{array}{c} m_{11} \\ m_{12} \\ m_{21} \\ m_{22} \end{array} \right) = \left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{array} \right] \left(\begin{array}{c} \mu \\ \lambda_1^A \\ \lambda_1^B \\ \lambda_{11}^{AB} \end{array} \right)$$

- only the linearly independent parameters are represented. $\lambda_2^A = -\lambda_1^A$, because $\lambda_1^A + \lambda_2^A = 0$, and so forth.
- association is represented by the parameter λ_{11}^{AB}
- can show that $\lambda_{11}^{AB} = \frac{1}{4} \log(\theta)$ (log odds ratio)
- Advantages of the GLM formulation: easier to express models with ordinal or quantitative variables, special terms, etc. Can also allow for *over-dispersion*.

Assessing goodness of fit

Goodness of fit of a specified model may be tested by the likelihood ratio G^2,

$$G^2 = 2 \sum_i n_i \log\left(\frac{n_i}{\widehat{m}_i}\right) \ , \tag{4}$$

or the Pearson X^2,

$$X^2 = \sum_i \frac{(n_i - \widehat{m}_i)^2}{\widehat{m}_i} \ , \tag{5}$$

with degrees of freedom df = # cells - # estimated parameters.

- E.g., for the model of independence, $[A][B]$, df = $IJ - [(I-1) - (J-1)] = (I-1)(J-1)$
- The terms summed in (4) and (5) are the squared *cell residuals*
- Other measures of balance goodness of fit against parsimony, e.g., *Akaike's Information Criterion* (smaller is better)

$$AIC = G^2 - 2df \text{ or } AIC = G^2 + 2 \text{ \# parameters}$$

R functions for loglinear models

- **`chisq.test()`** and vcd::**`assocstats()`** — only χ^2 tests for two-way tables, not a model (no parameters, no residuals)
- MASS::**`loglm()`** — general loglinear models for *n*-way tables

  ```
  loglm(formula, data, subset, na.action, ...)
  ```

- **`glm()`** — all generalized linear models; loglinear with `family=poisson`

  ```
  glm(formula, family = poisson, data, weights, subset, ...)
  ```

- Formulas have the form: `~ A + B + ...` (independence); `~ A*B + C` (allow A*B association)
- Both return an R object, with named components — use `names(object)`
- Both have **`print()`**, **`summary()`**, **`coef()`**, **`residuals()`**, **`plot()`** and other methods

Example: Arthritis treatment

Data on effects of treatment for rheumatoid arthritis (in case form)

```
data(Arthritis, package="vcd")
str(Arthritis)

## 'data.frame': 84 obs. of  5 variables:
##  $ ID       : int  57 46 77 17 36 23 75 39 33 55 ...
##  $ Treatment: Factor w/ 2 levels "Placebo","Treated": 2 2 2 2 2 2 2 2 2 2
##  $ Sex      : Factor w/ 2 levels "Female","Male": 2 2 2 2 2 2 2 2 2 2 .
##  $ Age      : int  27 29 30 32 46 58 59 59 63 63 ...
##  $ Improved : Ord.factor w/ 3 levels "None"<"Some"<..: 2 1 1 3 3 3 1 3
```

For now, examine the 2×3 table of `Treatment` and `Improved`

```
arth.tab <- with(Arthritis, table(Treatment, Improved))
arth.tab

##          Improved
## Treatment None Some Marked
##    Placebo  29    7      7
##    Treated  13    7     21
```

Example: Arthritis treatment

Fit the independence model, ˜ `Treatment + Improved`

```
library(MASS)
(arth.mod <- loglm(˜Treatment + Improved, data=arth.tab, fitted=TRUE))

## Call:
## loglm(formula = ˜Treatment + Improved, data = arth.tab, fitted = TRUE)
##
## Statistics:
##                    X^2 df  P(> X^2)
## Likelihood Ratio 13.530  2 0.0011536
## Pearson          13.055  2 0.0014626

residuals(arth.mod)

##           Improved
## Treatment      None      Some    Marked
##    Placebo  1.534955 -0.062501 -2.152482
##    Treated -1.776974  0.063501  1.837108

sum(residuals(arth.mod)^2)  # Pearson chisquare

## [1] 13.53
```
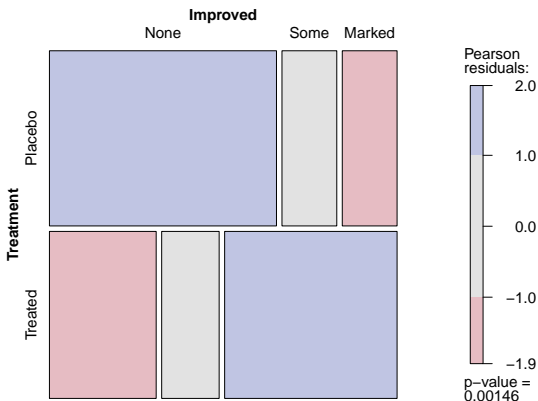
Example: Arthritis treatment
Visualize association: `mosaic()` or `plot()` the model *or* table

```
mosaic(arth.mod, shade=TRUE, gp_args=list(interpolate=1:4))
```

Example: Hair color and eye color

```
haireye <- margin.table(HairEyeColor, 1:2)
(HE.mod <- loglm(~ Hair + Eye, data=haireye))

## Call:
## loglm(formula = ~Hair + Eye, data = haireye)
##
## Statistics:
##                      X^2 df P(> X^2)
## Likelihood Ratio 146.44   9        0
## Pearson          138.29   9        0

round(residuals(HE.mod),2)

## Re-fitting to get frequencies and fitted values
##        Eye
## Hair    Brown  Blue Hazel Green
##    Black  4.00 -3.39 -0.49 -2.21
##    Brown  1.21 -2.02  1.31 -0.35
##    Red   -0.08 -1.85  0.82  2.04
##    Blond -7.33  6.17 -2.47  0.60
```
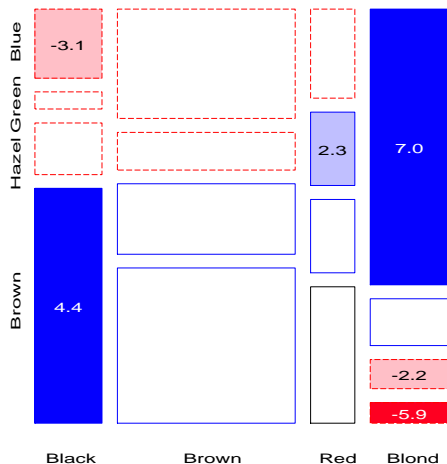
Mosaic displays: Hair color and eye color

We know that hair color and eye color are associated ($\chi^2(9) =$ 138.29). The question is how?

- Dark hair goes with dark eyes, light hair with light eyes
- Red hair, hazel eyes an exception?
- Effect ordering: Rows/cols permuted by CA Dimension 1
- $\Rightarrow$ Opposite corner pattern

Three-way tables

Saturated model

For a 3-way table, of size $I \times J \times K$ for variables A, B, C, the saturated loglinear model includes associations between all pairs of variables, as well as a 3-way association term, λ_{ijk}^{ABC}

$$
\begin{aligned}
\log m_{ijk} = \mu + \lambda_i^A + \lambda_j^B + \lambda_k^C \\
+ \lambda_{ij}^{AB} + \lambda_{ik}^{AC} + \lambda_{jk}^{BC} + \lambda_{ijk}^{ABC} \ .
\end{aligned}
\tag{6}
$$

- One-way terms $(\lambda_i^A, \lambda_j^B, \lambda_k^C)$: differences in the *marginal frequencies* of the table variables.
- Two-way terms $(\lambda_{ij}^{AB}, \lambda_{ik}^{AC}, \lambda_{jk}^{BC})$ pertain to the *partial association* for each pair of variables, *controlling* for the remaining variable.
- The three-way term, λ_{ijk}^{ABC} allows the partial association between any pair of variables to vary over the categories of the third variable.
- Fits perfectly, but doesn't *explain* anything, so we hope for a simpler model!

Three-way tables: Reduced models

Reduced models

- Loglinear models are usually hierarchical: a high-order term, such as $\lambda_{ijk}^{ABC} \rightarrow$ *all* low-order relatives are automatically included.
- Thus, a short-hand notation for a loglinear model lists only the high-order terms,
- i.e., the saturated model (6) $\equiv [ABC]$, and implies all two-way and one-way terms
- The usual goal is to fit the *smallest* model (fewest high-order terms) that is sufficient to explain/describe the observed frequencies.
- This is similar to ANOVA/regression models with all possible interactions

Three-way tables: Reduced models

Reduced models

- For a 3-way table there are a variety of models between the mutual independence model, [A][B][C], and the saturated model, [ABC]
- Each such model has an independence interpretation: $A \perp B$ means an hypothesis that A is independent of B.

Table: Log-linear Models for Three-Way Tables

Model	Model symbol	Interpretation
Mutual independence	[A][B][C]	$A \perp B \perp C$
Joint independence	[AB][C]	$(A\,B) \perp C$
Conditional independence	[AC][BC]	$(A \perp B) \mid C$
All two-way associations	[AB][AC][BC]	homogeneous assoc.
Saturated model	[ABC]	ABC interaction

Three-way tables: Model types

- **Joint independence**: $(AB) \perp C$, allows A*B association, but asserts no A*C and B*C associations

$$[AB][C] \equiv \log m_{ijk} = \mu + \lambda_i^A + \lambda_j^B + \lambda_k^C + \lambda_{ij}^{AB}$$

- **Conditional independence**: $A \perp B$, controlling for C

$$[AC][BC] \equiv \log m_{ijk} = \mu + \lambda_i^A + \lambda_j^B + \lambda_k^C + \lambda_{ik}^{AC} + \lambda_{jk}^{BC}$$

- **Homogeneous association**: All two-way, but each two-way is the *same* over the other factor

$$[AB][AC][BC] \equiv \log m_{ijk} = \mu + \lambda_i^A + \lambda_j^B + \lambda_k^C + \lambda_{ij}^{AB} + \lambda_{ik}^{AC} + \lambda_{jk}^{BC}$$

Response vs. Association models

- In association models, the interest is just on *which* variabels are associated, and *how*
    - Hair-eye data: [Hair Eye]? [Hair Sex]? [Eye Sex]
    - $\implies$ fit the homogeneous association model (or the saturated model)
    - Test the individual terms, delete those which are NS
- In response models, the interest is on which predictors are associated with the response
    - The minimal (null or baseline) model is the model of joint independence of the response (say, A) from all predictors, [A] [B C D . . .]
    - Associations among the predictors are fitted exactly (not analyzed)
    - Similar to regression, where predictors can be arbitrarily correlated
    - e.g., Berkeley data: fit the baseline model [Admit] [Gender Dept]
    - lack-of-fit $\implies$ associations [Admit Gender] and/or [Admit Dept]

Goodness of fit tests

As noted earlier, overall goodness of fit of a specified model may be tested by the likelihood ratio G^2, or the Pearson X^2,

$$G^2 = 2 \sum_i n_i \log \left(\frac{n_i}{\widehat{m}_i} \right) \qquad X^2 = \sum_i \frac{(n_i - \widehat{m}_i)^2}{\widehat{m}_i} \ ,$$

with residual degrees of freedom ν = # cells $-$ # estimated parameters.

- These measure the lack of fit of a given model— a large value $\implies$ a poor model
- Both are distributed as $\chi^2(\nu)$ (in large samples: all $\widehat{m}_i > 5$)
- $\mathcal{E}(\chi^2) = \nu$, so G^2/ν (or X^2/ν) measures lack of fit per degree of freedom (overdispersion)
- But: how to compare or test competing models?

Nested models and ANOVA-type tests

Nested models

Two models, M_1 and M_2 are nested when one (say, M_2) is a special case of the other

- Model M_2 (with ν_2 df) fits a subset of the parameters of M_1 (with ν_1 df)
- M_2 is more restrictive — cannot fit better than M_1: $G^2(M_2) \geq G^2(M_1)$
- The least restrictive model is the saturated model [ABC...] with $G^2 = 0$ and $\nu = 0$

Therefore, we can test the difference in G^2 as a specific test of the added restrictions in M_2 compared to M_1. This test has has a χ^2 distribution with df = $\nu_2 - \nu_1$.

$$\begin{aligned} \Delta G^2 \equiv G^2(M_2 \,|\, M_1) &= G^2(M_2) - G^2(M_1) \\ &= 2 \sum n_i \log(\widehat{m}_{i1}/\widehat{m}_{i2}) \end{aligned} \tag{7}$$

Example: Berkeley admissions data

For the UC Berkeley data, with table variables [A]dmit, [D]ept and [G]ender the following models form a nested chain

$$[A][D][G] \subset [A][DG] \subset [AD][AG][DG] \subset [ADG]$$

Table: Hierarchical G^2 tests for loglinear models fit to the UC Berkeley data

Type	LLM terms	G^2	df	$\Delta(G^2)$	$\Delta(df)$	$Pr(> \Delta(G^2))$
Mutual ind	[A][D][G]	2097.67	16			
Joint	[A][DG]	877.06	11	1220.62	5	0.0000
All 2-way	[AD][AG][DG]	20.20	5	1128.70	5	0.0000
Saturated	[ADG]	0.0	0	20.20	5	0.0011

- Only testing decrease in G^2 from one model to the next
- Here, each model is significantly better than the previous
- Joint vs. all 2-way: does Admit depend on Dept and/or Gender?
- Absolute fit of the all 2-way model is not terrible. Investigate further!

Fitting loglinear models in R

`loglm()` - data in contingency table form (MASS package)

```
1  data(UCBAdmissions)
2    ## conditional independence (AD, DG) in Berkeley data
3  mod.1 <- loglm(~ (Admit + Gender) * Dept, data=UCBAdmissions)
4    ## all two-way model (AD, DG, AG)
5  mod.2 <- loglm(~ (Admit + Gender + Dept)^2, data=UCBAdmissions)
```

`glm()` - data in frequency form

```
1  berkeley <- as.data.frame(UCBAdmissions)
2  mod.3 <- glm(Freq ~ (Admit + Gender) * Dept, data=berkeley,
3                      family='poisson')
```

- **`loglm()`** simpler for nominal variables
- **`glm()`** allows a wider class of models and quantitative predictors (covariates)
- **`gnm()`** fits models for structured association and generalized *non-linear* models
- vcdExtra package provides visualizations for all.

Example: Berkeley admission data

Fit the model of mutual independence using **loglm()**

```
data("UCBAdmissions")
library(MASS)
berk.loglm0 <- loglm(~ Dept + Gender + Admit, data=UCBAdmissions)
berk.loglm0

## Call:
## loglm(formula = ~Dept + Gender + Admit, data = UCBAdmissions)
##
## Statistics:
##                      X^2 df P(> X^2)
## Likelihood Ratio 2097.7 16        0
## Pearson          2000.3 16        0
```

Example: Berkeley admission data

Fit other models with **loglm()**

```
# conditional independence [AD] [AG]
berk.loglm1 <- loglm(˜ Admit * (Dept + Gender), data=UCBAdmissions)

# joint independence [A] [DG]
berk.loglm2 <- loglm(˜ Admit + (Dept * Gender), data=UCBAdmissions)
berk.loglm2

## Call:
## loglm(formula = ˜Admit + (Dept * Gender), data = UCBAdmissions)
##
## Statistics:
##                    X^2 df P(> X^2)
## Likelihood Ratio 877.06 11        0
## Pearson          797.70 11        0

# all two-way model [AD] [AG] [DG]
berk.loglm3 <-loglm(˜(Admit+Dept+Gender)^2, data=UCBAdmissions)
```

Example: Berkeley admission data

Compare nested models with **anova()**

```
anova(berk.loglm0, berk.loglm2, berk.loglm3, test="Chisq")

## LR tests for hierarchical log-linear models
##
## Model 1:
##  ˜Dept + Gender + Admit
## Model 2:
##  ˜Admit + (Dept * Gender)
## Model 3:
##  ˜(Admit + Dept + Gender)^2
##
##             Deviance df Delta(Dev) Delta(df) P(> Delta(Dev)
## Model 1    2097.671 16
## Model 2     877.056 11   1220.615         5        0.00000
## Model 3      20.204  5    856.852         6        0.00000
## Saturated    0.000  0     20.204         5        0.00114
```

Example: Berkeley admission data

LRStats() in vcdExtra gives one line summaries of a collection of models

```
LRstats(berk.loglm0, berk.loglm1, berk.loglm2, berk.loglm3)

## Likelihood summary table:
##              AIC  BIC LR Chisq Df Pr(>Chisq)
## berk.loglm0 2273 2282     2098 16     <2e-16 ***
## berk.loglm1 1336 1352     1149 10     <2e-16 ***
## berk.loglm2 1062 1077      877 11     <2e-16 ***
## berk.loglm3  217  240       20  5     0.0011 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

- AIC and BIC are GOF measures adjusted for model parsimony
- Not not significance tests, but smaller is better
- Also apply to non-nested models

$$AIC = G^2 + 2 \times \text{\# parameters}$$
$$BIC = G^2 + 2\log(n) \times \text{\# parameters}$$

Mosaic displays: Predictor variables

Berkeley data: Departments $\times$ Gender (ignoring Admit):

- Did departments differ in the total number of applicants?
- Did men and women apply differentially to departments?

Model: (Dept)(Gender)
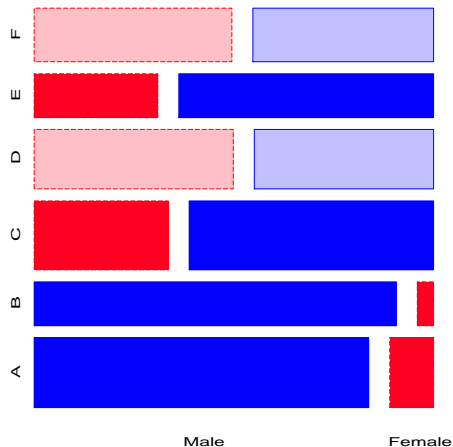
- Model [Dept] [Gender]: $G^2_{(5)} = 1220.6$.
- *Note*: Departments ordered A–F by overall rate of admission.
- Men more likely to apply to departments A,B; women more likely in depts C–F

Mosaic displays for multiway tables

- Generalizes to *n*-way tables: divide cells recursively
- Can fit *any* log-linear model (e.g., 2-way, 3-way, . . .),
  - For a 3-way table: [A][B][C], [AB][C], [AB][AC], . . . , [ABC]
- Each mosaics shows:
  - **DATA** (size of tiles)
  - (some) **marginal** frequencies (spacing $\rightarrow$ visual grouping)
  - **RESIDUALS** (shading) — what associations have been omitted?
- Visual fitting:
  - Pattern of lack-of-fit (residuals) $\rightarrow$ "better" model— smaller residuals
  - "cleaning the mosaic" $\rightarrow$ "better" model— empty cells
  - best done interactively!

Joint independence, [DG][A] (null model, Admit as response) [$G^2_{(11)}$ = 877.1]:

Model: (DeptGender)(Admit)
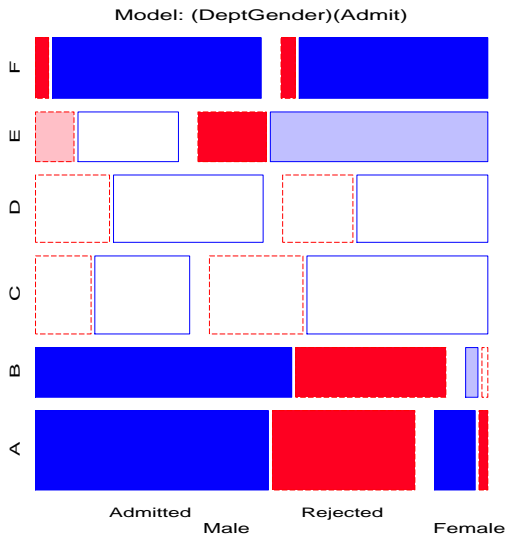
Mosaic displays for multiway tables

Conditional independence, [AD] [DG]:

Model: (DeptGender)(DeptAdmit)
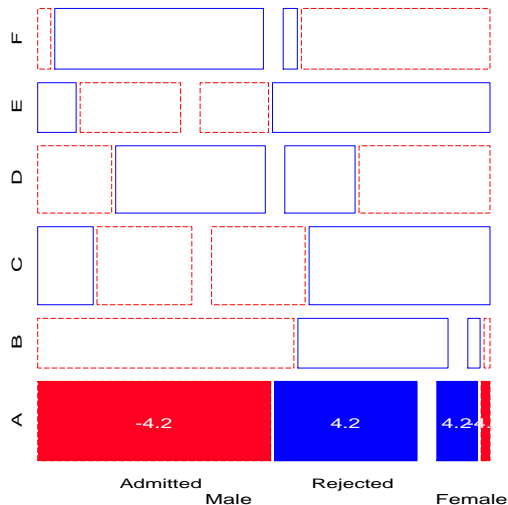
- E.g., Add [Admit Dept] association $\rightarrow$ Conditional independence:
  - Fits poorly: ($G^2_{(6)} = 21.74$)
  - But, only in Department A!
- GLM approach allows fitting a special term for Dept. A
- Note: These displays use *standardized residuals*: better statistical properties.

Other variations: Double decker plots

- Visualize dependence of one categorical (typically binary) variable on predictors
- Formally: mosaic plots with vertical splits for all predictor dimensions, highlighting the response by shading
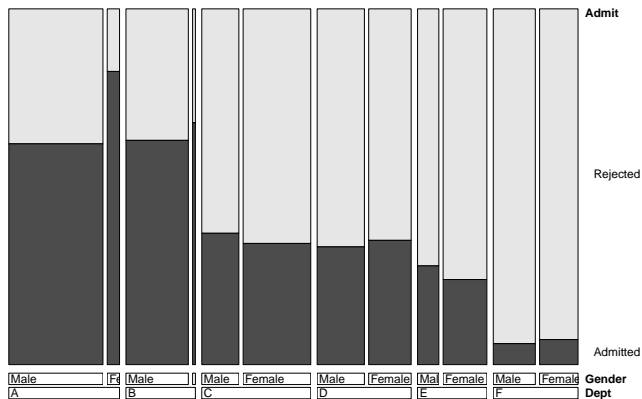
4-way example: Survival on the *Titanic*

Data on the fate of passengers and crew on the HMS *Titanic*, a $4 \times 2 \times 2 \times 2$ frequency table:

```
data(Titanic)
str(Titanic)

##  table [1:4, 1:2, 1:2, 1:2] 0 0 35 0 0 0 17 0 118 154 ...
##  - attr(*, "dimnames")=List of 4
##   ..$ Class   : chr [1:4] "1st" "2nd" "3rd" "Crew"
##   ..$ Sex     : chr [1:2] "Male" "Female"
##   ..$ Age     : chr [1:2] "Child" "Adult"
##   ..$ Survived: chr [1:2] "No" "Yes"
```

What proportion survived? Ans: 711 / 2201 = 32.3 %

```
margin.table(Titanic, 4)

## Survived
##   No  Yes
## 1490  711
```

Zero cells

```
structable(Titanic)

##              Sex      Male      Female
##              Survived  No Yes      No Yes
## Class Age
## 1st   Child             0   5       0   1
##       Adult           118  57       4 140
## 2nd   Child             0  11       0  13
##       Adult           154  14      13  80
## 3rd   Child            35  13      17  14
##       Adult           387  75      89  76
## Crew  Child             0   0       0   0
##       Adult           670 192       3  20
```

Two types of zero cells:

- structural zeros — could not occur (children in crew)
- sampling zeros — did not happen to occur (children in 1st & 2nd who died)
- Structural zeros can cause problems — loss of df; 0/0 = `NaN` in χ^2 tests

Exploratory plots

One-way doubledecker plots against survival show what might be expected:

```
doubledecker(Survived ˜ Sex, data=Titanic)
doubledecker(Survived ˜ Class, data=Titanic)
```
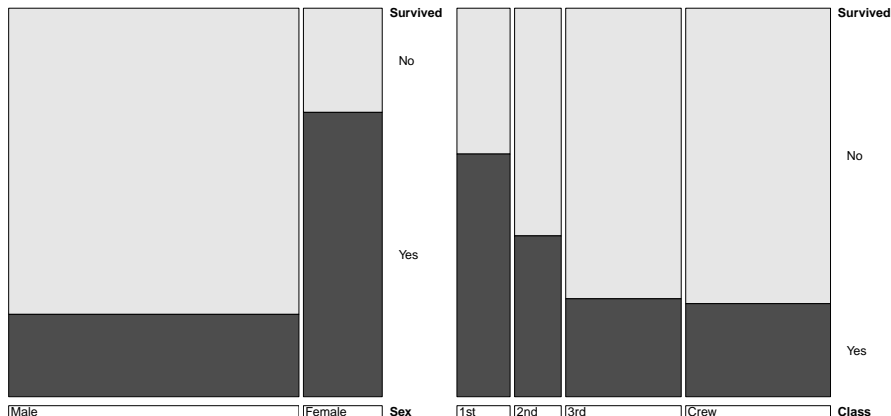
Exploratory plots

Two-way doubledecker plot against survival shows different effects of `Class` for men and women:

```
doubledecker(Survived ~ Sex + Class, data=Titanic)
```
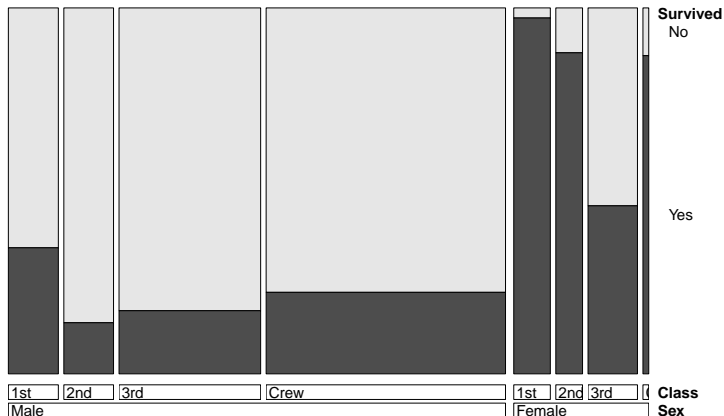
Fitting and visualizing models

In the model formulas for `loglm()`, I use the variable numbers 1–4, and letters *C*lass, *G*ender, *A*ge and *S*urvived

```
# mutual independence [C][G][A][S]
mod0 <- loglm(~ 1 + 2 + 3 + 4, data=Titanic)
# baseline (null) model
mod1 <- loglm(~ 1*2*3 + 4, data=Titanic)
mosaic(mod1, main="Model [CGA][S]")
```


Model [CGA][S]
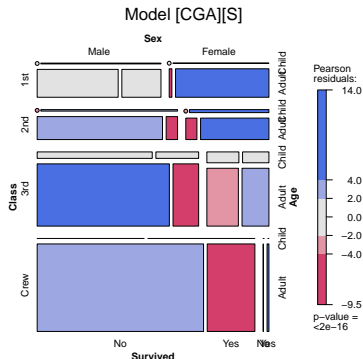
- With S as response, the baseline model includes all associations among [CGA]
- But this model asserts no associations of these with survival
- $G^2(15) = 671.96$, a very poor fit

Adding associations

```
# main effects of C, G, A on survival: [CGA][CS][GS][AS]
mod2 <- loglm(~ 1*2*3 + (1+2+3)*4, data=Titanic)
mosaic(mod2, main="Model [CGA][CS][GS][AS]")
```


Model [CGA][CS][GS][AS]
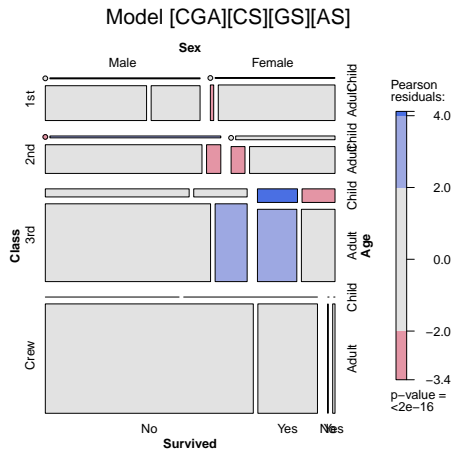
- This model allows associations of each of C, G, A with Survived
- $G^2(10) = 112.57$, still not good
- Pattern of residuals suggests 2-way interactions (3-way terms):
- "Women & children first": suggests a term [GAS]
- Allow interactions of Class with Gender [CGS] and Class with Age [CAS]

Final model

```
mod3 <- loglm(~ 1*2*3 + (1*2)*4 + (1*3)*4, data=Titanic)
mosaic(mod3, main="Model [CGA][CGS][CAS]")
```
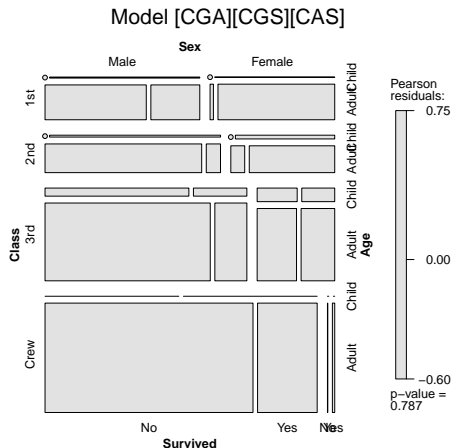
Comparing models

As usual, **anova()** (and **LRstats()**) give compact comparisons of a set of models

```
anova(mod0, mod1, mod2, mod3)

## LR tests for hierarchical log-linear models
##
## Model 1:
##  ~1 + 2 + 3 + 4
## Model 2:
##  ~1 * 2 * 3 + 4
## Model 3:
##  ~1 * 2 * 3 + (1 + 2 + 3) * 4
## Model 4:
##  ~1 * 2 * 3 + (1 * 2) * 4 + (1 * 3) * 4
##
##            Deviance df Delta(Dev) Delta(df) P(> Delta(Dev)
## Model 1    1243.6632 25
## Model 2     671.9622 15   571.7010        10        0.00000
## Model 3     112.5666 10   559.3956         5        0.00000
## Model 4       1.6855  4   110.8811         6        0.00000
## Saturated     0.0000  0     1.6855         4        0.79335
```

Model interpretation

- Regardless of Gender and Age, lower Class $\implies$ decreased survival
- Differences in survival by Class were moderated by both Gender and Age
- term [CGS]: Women in 3rd class did not have an advantage, while men in 1st did vs. other classes
- term [CAS]: No children in 1st or 2nd class died, but nearly 2/3 in 3rd class did
- Summary:
    - Not so much "women and children first" as
    - "women and chilren, ordered by class and 1st class men"
    -

Sequential plots and models

- Mosaic for an n-way table $\rightarrow$ hierarchical decomposition of association
- Joint cell probabilities are decomposed as

$$p_{ijk\ell\ldots} = \underbrace{\overbrace{p_i \times p_{j|i} \times p_{k|ij}}^{\{v_1 v_2\}} \times p_{\ell|ijk} \times \cdots \times p_{n|ijk\ldots}}_{\{v_1 v_2 v_3\}}$$

  - First 2 terms $\rightarrow$ mosaic for v_1 and v_2
  - First 3 terms $\rightarrow$ mosaic for v_1, v_2 and v_3
  - $\cdots$

- Roughly analogous to sequential fitting in regression: X_1, $X_2|X_1$, $X_3|X_1X_2$, $\cdots$
- The order of variables matters for interpretation

Sequential plots and models

Sequential models of *joint independence* $\rightarrow$ additive decomposition of the total association, $G^2_{[v_1][v_2]...[v_p]}$ (mutual independence),

$$G^2_{[v_1][v_2]...[v_p]} = G^2_{[v_1][v_2]} + G^2_{[v_1 v_2][v_3]} + G^2_{[v_1 v_2 v_3][v_4]} + \cdots + G^2_{[v_1...v_{p-1}][v_p]}$$

e.g., for Hair Eye color data

Model	Model symbol	df	G^2
Marginal	[Hair] [Eye]	9	146.44
Joint	[Hair, Eye] [Sex]	15	19.86
Mutual	[Hair] [Eye] [Sex]	24	166.30

Sequential plots and models: Example

- Hair color x Eye color marginal table (ignoring Sex)

(Hair)(Eye), G2 (9) = 146.44
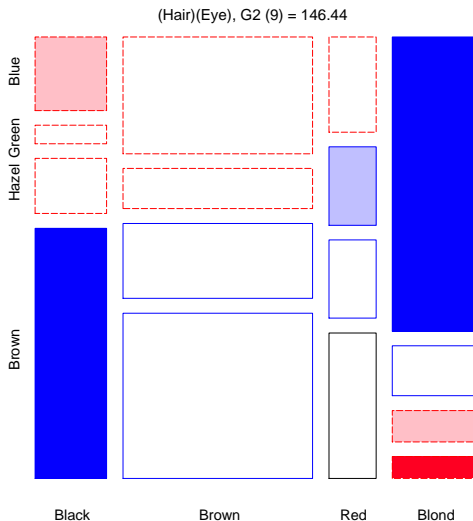
Sequential plots and models: Example

- 3-way table, Joint Independence Model [Hair Eye] [Sex]

(HairEye)(Sex), G2 (15) = 19.86
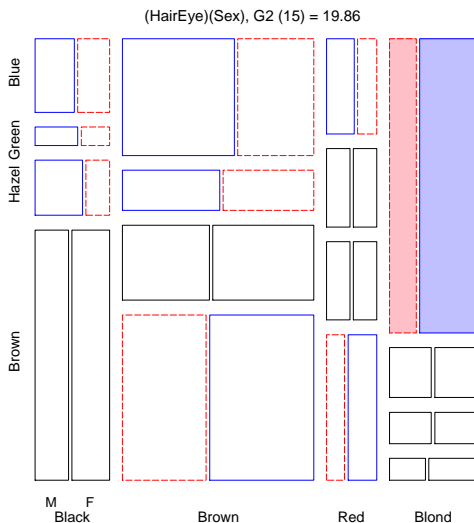
Sequential plots and models: Example

- 3-way table, Mutual Independence Model [Hair] [Eye] [Sex]

(Hair)(Eye)(Sex), G2 (24) = 166.30

Sequential plots and models: Example

Marginal

(Hair)(Eye), G2 (9) = 146.44

Joint

(HairEye)(Sex), G2 (15) = 19.86

Mutual (total)

(Hair)(Eye)(Sex), G2 (24) = 166.30

[Hair] [Eye]
$G^2_{(9)} = 146.44$

[Hair Eye] [Sex]
$G^2_{(15)} = 19.86$

[Hair] [Eye] [Sex]
$G^2_{(24)} = 166.30$
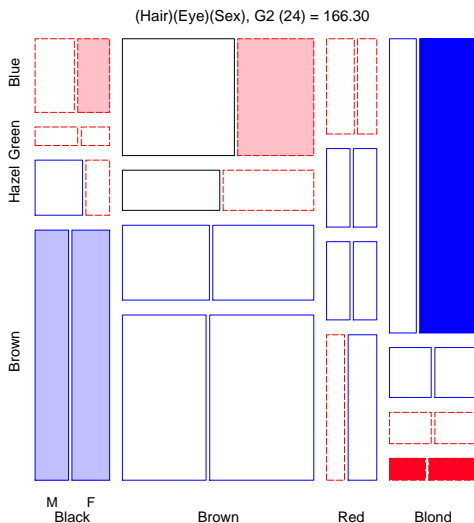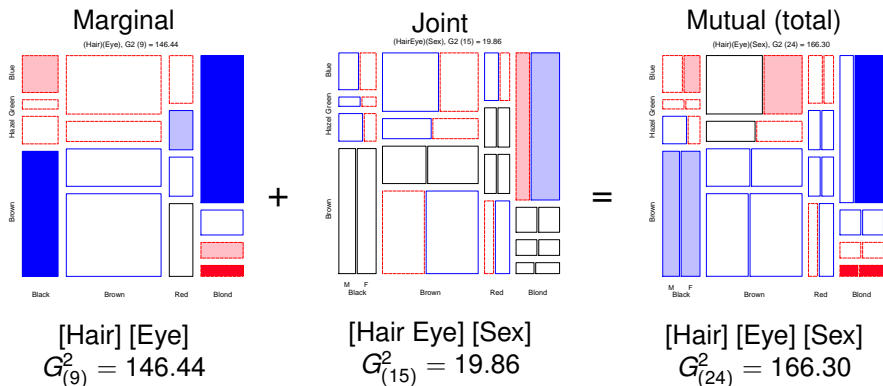
Applications

Response models

- When one variable, R, is a response and $E_1, E_2, \ldots$ are explantory, the baseline model is the model of joint independence, $[E_1, E_2, \ldots][R]$
- Sequential mosaics then show the associations among the predictors
- The last mosaic shows all associations with R
- Better-fitting models will need to add associations of the form $[E_i R], [E_i E_j R] \ldots$

Causal models

- Sometimes there is an assumed causal ordering of variables:

$$A \to B \to C \to D$$

- Each path of arrows: $A \to B$, $A \to B \to C$ is a sequential model of joint independence: [A][B], [AB] [C], [ABC] [D].
- Testing these decomposes all joint probabilities

Example: Marital status, pre- and extra-marital sex

Thornes and Collard (1979) studied divorce patterns in relation to premarital and extramarital sex, a 2^4 table, `PreSex` in vcd

```
data("PreSex", package="vcd")
structable(Gender+PremaritalSex+ExtramaritalSex ~ MaritalStatus, PreSex)

##                 Gender          Women             Men
##                 PremaritalSex   Yes      No       Yes      No
##                 ExtramaritalSex Yes No Yes  No Yes No Yes  No
## MaritalStatus
## Divorced                        17 54 36 214  28 60 17  68
## Married                          4 25  4 322  11 42  4 130
```

Sub-models:

- [*G*][*P*] : do men and women differ in pre-marital sex?
- [*GP*][*E*] : given G & P, are there differences in extra-marital sex?
- [*GPE*][*M*] : given G, P & E, are there differences in divorce?

Example: Marital status, pre- and extra-marital sex

- Order the table variables as $G \rightarrow P \rightarrow E \rightarrow M$

```
PreSex <- aperm(PreSex, 4:1)    # order variables G, P, E, M
```

- Fit each sequential model to the marginal sub-table

```
mod.1 <- loglm(~ Gender + PremaritalSex, data=PreSex)
mod.2 <- loglm(~ Gender * PremaritalSex + ExtramaritalSex, data=PreSex
...
```

Model	df	G^2
[G] [P]	1	75.259
[GP] [E]	3	48.929
[GPE] [M]	7	107.956
[G] [P] [E] [M]	11	232.142

Mosaic plots:

```
# (Gender Pre)
mosaic(margin.table(PreSex, 1:2), shade=TRUE,
       main = "Gender and Premarital Sex")
# (Gender Pre)(Extra)
mosaic(margin.table(PreSex, 1:3),
       expected = ˜Gender * PremaritalSex + ExtramaritalSex,
       main = "Gender*Pre + ExtramaritalSex")
```
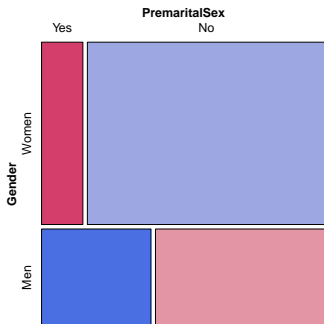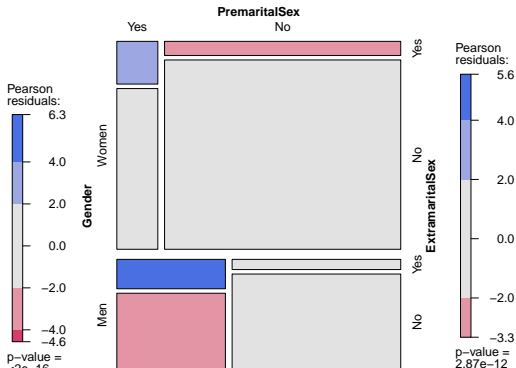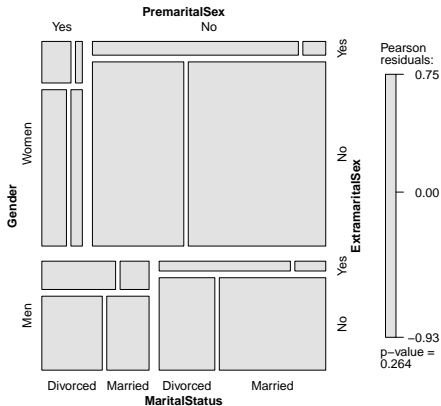
Mosaic plots:

```
mosaic(PreSex,
       expected = ~Gender*PremaritalSex*ExtramaritalSex
                   + MaritalStatus,
       main = "Gender*Pre*Extra + MaritalStatus")
# (GPE)(PEM)
mosaic(PreSex,
       expected = ~ Gender * PremaritalSex * ExtramaritalSex
                   + MaritalStatus * PremaritalSex * ExtramaritalSex,
       main = "G*P*E + P*E*M")
```
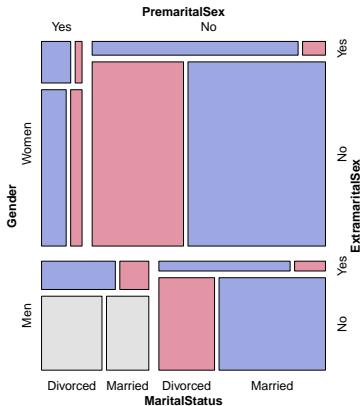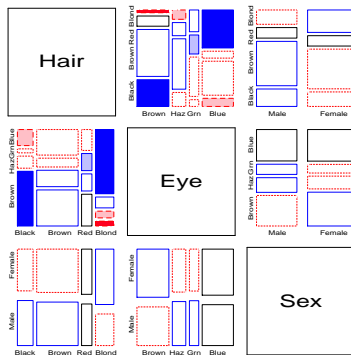
Mosaic matrices

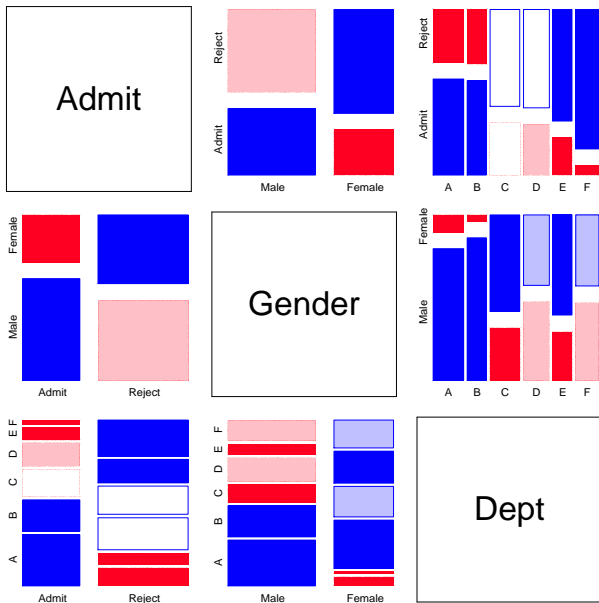- Analog of *scatterplot matrix* for categorical data (Friendly, 1999)
  - Shows all $p(p-1)$ pairwise views in a coherent display
  - Each pairwise mosaic shows bivariate (marginal) relation
  - Fit: marginal independence
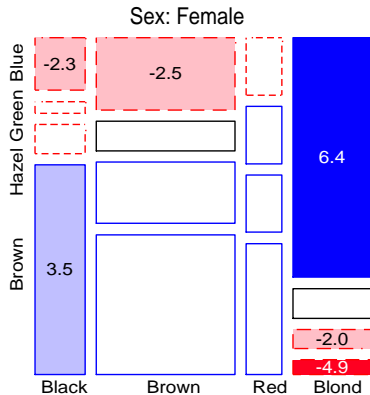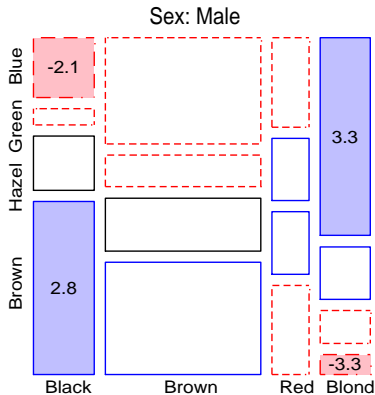 - Residuals: show marginal associations

Hair, Eye, Sex data:
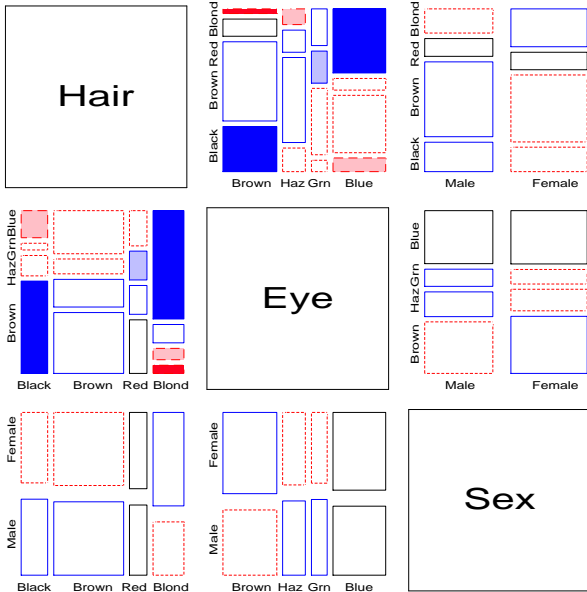
Berkeley data:

Partial association, Partial mosaics

- **Stratified analysis:**
  - How does the association between two (or more) variables vary over levels of other variables?
  - Mosaic plots for the main variables show *partial association* at each level of the other variables.
  - E.g., Hair color, Eye color *subset* by Sex

Partial association, Partial mosaics

Stratified analysis: conditional decomposition of G^2

- Fit models of partial (conditional) independence, $A \perp B \,|\, C_k$ at each level of (controlling for) C.
- $\Rightarrow$ partial G^2s add to the overall G^2 for conditional independence, $A \perp B \,|\, C$

$$G^2_{A \perp B \,|\, C} = \sum_k G^2_{A \perp B \,|\, C(k)}$$

Table: Partial and Overall conditional tests, *Hair* $\perp$ *Eye* | *Sex*

Model	df	G^2	*p*-value
[*Hair*][*Eye*] ∣ Male	9	44.445	0.000
[*Hair*][*Eye*] ∣ Female	9	112.233	0.000
[*Hair*][*Eye*] ∣ Sex	18	156.668	0.000

References I

Friendly, M. Mosaic displays for multi-way contingency tables. *Journal of the American Statistical Association*, 89:190–200, 1994.

Friendly, M. Extending mosaic displays: Marginal, conditional, and partial views of categorical data. *Journal of Computational and Graphical Statistics*, 8(3):373–395, 1999.

Hartigan, J. A. and Kleiner, B. Mosaics for contingency tables. In Eddy, W. F., editor, *Computer Science and Statistics: Proceedings of the 13th Symposium on the Interface*, pp. 268–273. Springer-Verlag, New York, NY, 1981.

Thornes, B. and Collard, J. *Who Divorces?* Routledge & Kegan, London, 1979.