GRUNDLAGEN DER ANALYSIS

IV Integralrechnung

Zu einer gegebenen Funktion f wird eine Funktion F gesucht, deren erste Ableitung Funktion f ist.

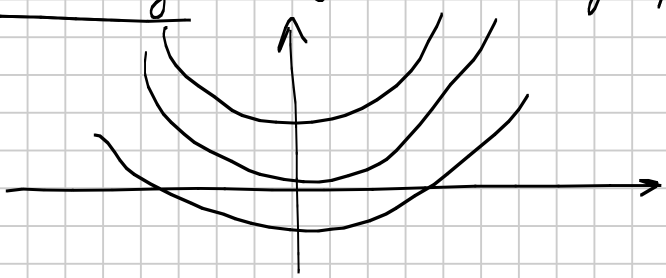
$f(x) = F'(x)$	$F(x)$
e^x	e^x
$2x$	x^2
$\sin x$	$-\cos x$
$\frac{1}{x}$	$\ln x$

→ Stammfunktion zu $f(x)$

Satz Sei $F(x)$ Stammfunktion zu $f(x) = F'(x)$ $\Rightarrow$ $F(x) + C$, $C \in \mathbb{R}$ ebenfalls Stammfunktion.

Beweis $[F(x) + C]' = F'(x) + C' = F'(x) = f(x)$

Bemerkung $F(x) + C$: graphisch Kurvenschar



$f(x) = 2x$

$F(x) = x^2 + C$;

Definition Die Menge aller Stammfunktionen zu einer Funktion f heißt unbestimmtes Integral,

$$\int f(x)\,dx = F(x) + C, \quad C \in \mathbb{R}$$

Die Funktion f ist der Integrand und die Konstante c wird Integrationskonstante genannt.

Beispiele $\int 1\,dx = x + C$; $\int x\,dx = \frac{1}{2}x^2 + C$

Bemerkung $F(x)$ existiert, falls $f(x)$ stückweise stetig und beschränkt ist.

Komplizierte Integrale werden (nach Möglichkeit) auf einfache Integrale zurückgeführt.

IV. 2. Grundintegrale

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \; ; \quad n \neq -1 \; ;$$

$$\int \frac{dx}{x} = \ln|x| + C \; ; \quad x \neq 0 \; ;$$

$$\int e^x \, dx = e^x + C \; ;$$

$$\int a^x \, dx = \frac{1}{\ln a} a^x + C \; ;$$

$$\int \sin x \, dx = -\cos x + C \; ;$$

$$\int \cos x \, dx = \sin x + C \; ;$$

$$\int \frac{1}{\cos^2 x} \, dx = \tan x + C \; ; \qquad x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$$

$$\int \frac{1}{\sin^2 x} \, dx = -\cot x + C \; ; \qquad x \neq 0, \pm\pi, \pm 2\pi, \ldots$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C = -\arccos x + k, \qquad (|x| < 1)$$

$$\int \frac{1}{1+x^2} \, dx = \arctan x + C = -\operatorname{arccot} x + k$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C \; ;$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{a} \operatorname{arctanh}\left(\frac{x}{a}\right) + C \; ;$$

IV. 3. Integrationsregeln

IV. 3. 1 Faktorregel, Summenregel, partielle Integration

Seien $f(x), g(x)$ integrierbar (stückweise stetig)

Faktorregel : $\int a f(x) \, dx = a \int f(x) \, dx$

Summenregel: $\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

Beispiel 1) $\int \tan^2 x\,dx = \int \frac{\sin^2 x}{\cos^2 x}\,dx = \int \frac{1-\cos^2 x}{\cos^2 x}\,dx =$

$= \int \frac{1}{\cos^2 x}\,dx - \int 1\,dx = \underline{\underline{\tan x - x + C}}$

2) $\int \left(x^3 + 4x + \frac{1}{x^2} - 3\right)dx = \int x^3\,dx + 4\cdot\int x\,dx + \int x^{-2}\,dx - 3\int dx =$

$= \underline{\underline{\frac{1}{4}x^4 + 2x^2 - \frac{1}{x} - 3x + C}};$

3) $\int \left(\sqrt{x} - \sqrt[3]{x} - \frac{1}{\sqrt[4]{x}} + \sqrt[5]{\frac{1}{x^3}}\right)dx = \int x^{1/2}\,dx - \int x^{1/3}\,dx - \int x^{-\frac{1}{4}}\,dx +$

$+ \int x^{-3/5}\,dx = \frac{x^{3/2}}{3/2} - \frac{x^{4/3}}{4/3} - \frac{x^{3/4}}{3/4} + \frac{x^{2/5}}{2/5} + C =$

$= \frac{2}{3}\sqrt{x^3} - \frac{3}{4}\sqrt[3]{x^4} - \frac{4}{3}\sqrt[4]{x^3} + \frac{5}{2}\sqrt[5]{x^2} + C;$

4) $\int \frac{1 - x\cdot e^{\alpha + x}}{x}\,dx = \int \frac{1}{x}\,dx - \int e^{\alpha+x}\,dx =$

$= \ln|x| - e^{\alpha}\cdot\int e^x\,dx = \underline{\underline{\ln|x| - e^{\alpha+x} + C}};$

Satz (Partielle Integration) Seien $v(x), u(x)$ integrierbar, sowie mindestens einmal stetig differenzierbar. Dann gilt:

$$\int u(x)\cdot v'(x)\,dx = u(x)\cdot v(x) - \int u'(x)\cdot v(x)\,dx$$

Bemerkung

$$u(x)\cdot v(x) = u\cdot v$$

$$(u\cdot v)' = u'\cdot v + v' u$$

$$\int (u\cdot v)'\,dx = \int u' v\,dx + \int u v'\,dx$$

$$\Rightarrow \int u\cdot v'\cdot dx = uv - \int u'\cdot v\cdot dx$$

Beispiel 1) $I_1 = \int \overset{u\,\cdot\,v'}{x\cdot\cos x}\cdot dx$

$u(x) = x;\quad v'(x) = \cos x;\qquad u' = 1;\quad v = \sin x;$

$$I_1 = uv - \int u' \cdot v \, dx = x \cdot \sin x - \int \sin x =$$

$$= \underline{\underline{x \cdot \sin x + \cos x + C}};$$

2) $I_2 = \int \overset{u}{(3x-7)} \overset{v'}{e^x} \, dx$

$u = 3x - 7; \qquad u' = 3$

$v' = e^x; \qquad v = e^x$

$$I_2 = (3x-7)e^x - \int 3 \cdot e^x \, dx = (3x-7)e^x - 3e^x + C =$$

$$= \underline{\underline{(3x-10)e^x + C}};$$

3) $I_3 = \int 1 \cdot \ln x \cdot dx$

$u = \ln x; \quad v' = 1$

$u' = \frac{1}{x}; \quad v = x$

$$I_3 = x \cdot \ln x - \int \frac{1}{x} \cdot x \cdot dx = x \cdot \ln x - \int dx = \underline{\underline{x \ln x - x + C}}$$

4) $I_4 = \int x^2 e^x \, dx$

$u = x^2; \qquad v' = e^x$

$u' = 2x; \qquad v = e^x$

$$I_4 = x^2 \cdot e^x - 2\underbrace{\int x \cdot e^x \, dx}_{I}$$

$u = x; \quad v' = e^x$

$u' = 1; \quad v = e^x$

$$I = x \cdot e^x - \int 1 \cdot e^x \, dx = x \cdot e^x - e^x + C_1;$$

$$I_4 = x^2 e^x - 2(xe^x - e^x + C_1) = (x^2 - 2x + 2) \cdot e^x + 2C_1 =$$

$$= \underline{\underline{(x^2 - 2x + 2) \cdot e^x + C}};$$

5) $I_5 = \int x \cdot \sin x \, dx$

$u = x \qquad v' = \sin x$

$u' = 1 \qquad v = -\cos x$

$$I_5 = -x \cdot \cos x + \int \cos x \, dx = \underline{\underline{-x \cdot \cos x + \sin x + C}}.$$

6) $I_6 = \int \sin^2 x \, dx$

$u = \sin x \qquad v' = \sin x$

$u' = \cos x \qquad v = -\cos x$

$$I_6 = -\sin x \cdot \cos x + \int \cos x \cdot \cos x \, dx = -\sin x \cdot \cos x + \int \cos^2 x \, dx =$$

$$= -\sin x \cdot \cos x + \int [1 - \sin^2 x] \, dx = -\sin x \cdot \cos x + x - \int \sin^2 x \, dx =$$

$$= -\sin x \cdot \cos x + x - I_6 + C;$$

$$\Rightarrow \quad 2 \cdot I_6 = x - \sin x \cdot \cos x + C;$$

$$I_6 = \frac{x - \sin x \cdot \cos x + C}{2}$$

IV.3.2. Integration durch Substitution

1. Typ: $\int f(ax+b) \, dx$

Subst. $ax + b = t; \qquad a \neq 0$

$$\frac{dt}{dx} = a \quad \Rightarrow \quad dx = \frac{dt}{a};$$

$$\boxed{\int f(ax+b) \, dx = \frac{1}{a} \int f(t) \, dt}$$

Beispiel 1) $\int (5x-7)^4 dx = \frac{1}{5} \int t^4 dt = \frac{1}{5} \cdot \frac{t^5}{5} + C =$

$$= \frac{1}{25} (5x-7)^5 + C;$$

2) $\int \frac{dx}{\sqrt[3]{1-2x}} =$ $\quad 1 - 2x = t; \quad \frac{dt}{dx} = -2; \quad dx = -\frac{dt}{2}$

$$= -\frac{1}{2} \int \frac{dt}{\sqrt[3]{t}} = -\frac{1}{2} \int t^{-1/3} dt = -\frac{1}{2} \cdot \frac{3}{2} \cdot t^{2/3} + C = -\frac{3}{4} \sqrt[3]{(1-2x)^2} + C;$$

3) $\int \sin(4x-3) \, dx =$ $\quad 4x - 3 = t; \quad \frac{dt}{dx} = 4; \quad dx = \frac{1}{4} dt$

$$= \frac{1}{4} \int \sin t \, dt = -\frac{1}{4} \cos t \, dt + C = -\frac{1}{4} \cos(4x-3) + C;$$

2. Typ: $\int f(\varphi(x)) \cdot \varphi'(x)\, dx$

Subst. $\varphi(x) = t$

$\varphi'(x)\, dx = dt$;

$$\boxed{\int f(\varphi(x))\, \varphi'(x)\, dx = \int f(t)\, dt}$$

Beispiel 1) $\int \sin^2 x \cdot \cos x\, dx = I$

Subst. $\sin x = t$;

$\cos x\, dx = dt$

$$I = \int t^2\, dt = \frac{t^3}{3} + C = \underline{\underline{\frac{1}{3} \sin^3 x + C}};$$

2) $\int \frac{\sqrt{\ln x}}{x}\, dx \overset{\bigcirc}{=}$ $\ln x = t$; $\frac{dt}{dx} = \frac{1}{x}$;

$dt = \frac{1}{x} \cdot dx$

$$\overset{\bigcirc}{=} \int \sqrt{t}\, dt = \frac{t^{3/2}}{3/2} + C = 2/3 \sqrt{\ln^3 x} + C;$$

3) $\int e^{\cos x} \cdot \sin x \cdot dx \overset{\bigcirc}{=}$ $\cos x = t$
$-\sin x\, dx = dt$

$$\overset{\bigcirc}{=} -\int e^t\, dt = -e^t + C = \underline{\underline{-e^{\cos x} + C}};$$

3 Typ.: logarithmische Integration

$$\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + C;$$

Beispiel 1) $\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx = -\int \frac{-\sin x}{\cos x}\, dx =$

$= -\ln|\cos x| + C;$

2) $\int \frac{1}{x \ln x}\, dx = \int \frac{\frac{1}{x}}{\ln x}\, dx = \ln|\ln(x)| + C;$

3) $\int \frac{10x}{x^2 + 4}\, dx = 5 \cdot \int \frac{2x}{x^2 + 4}\, dx = 5 \cdot \ln|x^2 + 4| + C;$