Contents

Part I
Proofs

1 Orthogonality and determinant

Be $\vec{v_1} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = |\vec{v_1}| \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$, $\vec{v_2} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = |\vec{v_2}| \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix}$ with $\alpha = \measuredangle\vec{v_1}$, $\beta = \measuredangle\vec{v_2}$

Orthogonality

To proove: $\vec{v_1} \perp \vec{v_2} <=> \vec{v_1} * \vec{v_2} = 0$:

$\vec{v_1} * \vec{v_2} = |\vec{v_1}| \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} |\vec{v_2}| \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix} = |\vec{v_1}||\vec{v_2}|(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = \underline{\underline{|\vec{v_1}||\vec{v_2}|\cos(\alpha-\beta)}}.$

Be $\vec{v_1} \perp \vec{v_2} <=> \alpha = \beta + \frac{\pi}{2}$, w.l.o.g.

$=> |\vec{v_1}||\vec{v_2}|\cos(\alpha-\beta) = |\vec{v_1}||\vec{v_2}|\cos(\beta + \frac{\pi}{2} - \beta) = 0$

$\square$

Determinant

Let's choose sin() instead of cos() in the above equation:

$|\vec{v_1}||\vec{v_2}|\sin(\alpha-\beta) = |\vec{v_1}||\vec{v_2}|(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$

$= |\vec{v_1}||\vec{v_2}| \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta \\ -\sin\beta \end{pmatrix} = \begin{pmatrix} |\vec{v_1}|\sin\alpha \\ |\vec{v_1}|\cos\alpha \end{pmatrix} \begin{pmatrix} |\vec{v_2}|\cos\beta \\ -|\vec{v_2}|\sin\beta \end{pmatrix} = \begin{pmatrix} y_1 \\ x_1 \end{pmatrix} \begin{pmatrix} x_2 \\ -y_2 \end{pmatrix}$
$= \det \begin{pmatrix} x_2 & x_1 \\ y_2 & y_1 \end{pmatrix} = \underline{\underline{\det \begin{pmatrix} \vec{v_2} & \vec{v_1} \end{pmatrix}}}.$

If $\alpha = \beta + k * \pi$, w.l.o.g, $k \in \mathbb{N}$

$=> \det \begin{pmatrix} \vec{v_2} & \vec{v_1} \end{pmatrix} = |\vec{v_1}||\vec{v_2}|\sin(\alpha-\beta) = |\vec{v_1}||\vec{v_2}|\sin(\beta + k * \pi - \beta) = 0$

$=>$ $\vec{v_1}$ and $\vec{v_2}$ are linear dependent and the determinant in general determines linear dependency.

2 Law of cosine

To proove: $c^2 = a^2 + b^2 - 2ab\cos\gamma$

$\gamma >= \frac{\pi}{2}$:

$$d^2 = b^2 - e^2 \tag{1}$$
$$e = b\sin(\pi - \gamma) \tag{2}$$
$$1 = \sin^2\gamma + \cos^2\gamma \tag{3}$$

$$c^2 = (a+d)^2 + e^2$$

$$= (a + \sqrt{b^2 - e^2})^2 + e^2$$

$$= a^2 + 2a\sqrt{b^2 - e^2} + b^2$$

$$= a^2 + 2a\sqrt{b^2 - b^2\sin^2(\pi - \gamma)} + b^2$$

$$= a^2 + 2ab\sqrt{1 - \sin^2(\pi - \gamma)} + b^2$$

$$= a^2 + 2ab\sqrt{\cos^2(\pi - \gamma)} + b^2$$

$$= a^2 + b^2 + 2ab\cos(\pi - \gamma)$$

$$= a^2 + b^2 - 2ab\cos\gamma$$

$\square$

3 Polynomial derivation

To proove: $(x^n)' = nx^{n-1}$

$$\frac{f(x+\triangle x)-f(x)}{\triangle x}$$

$$=\frac{(x+\triangle x)^n - x^n}{\triangle x}$$

$$=\frac{\sum_{k=0}^{n}\binom{n}{k}(x^{n-k}\triangle x^k) - x^n}{\triangle x}$$

$$=\frac{x^n\Delta x^0 + \sum_{k=1}^{n}\binom{n}{k}(x^{n-k}\Delta x^k) - x^n}{\Delta x}$$

$$=\frac{\sum_{k=1}^{n}\binom{n}{k}(x^{n-k}\Delta x^k)}{\Delta x}$$

$$=\frac{\Delta x\sum_{k-1}^{n}\binom{n}{k}(x^{n-k}\Delta x^{k-1})}{\Delta x}$$

$$=\sum_{k-1}^{n}\binom{n}{k}(x^{n-k}\Delta x^{k-1})$$

$$=\sum_{k=1}^{n}\frac{n!}{k!(n-k)!}(x^{n-k}\Delta x^{k-1})$$

$$= nx^{n-1} + \sum_{k=2}^{n}\binom{n}{k}(x^{n-k}\Delta x^k)$$

$$\lim_{\Delta x\to 0}\left(nx^{n-1} + \sum_{k=2}^{n}\binom{n}{k}(x^{n-k}\Delta x^k)\right) = nx^{n-1}$$

$\square$

4 Binomial theorem

To proove: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k}(x^{n-k}y^k)$

Induction:
$n = 0$:

$$(x+y)^0 = 1 = \sum_{k=0}^{0} \binom{0}{k}(x^{0-k}y^k)$$

$n \to n+1$:

$$(x+y)^{n+1} = (x+y)^n(x+y) \overset{\text{i.p.}}{=} \sum_{k=0}^{n} \binom{n}{k}(x^{n-k}y^k)(x+y)$$

$$= \sum_{k=0}^{n} \binom{n}{k}(x^{n+1-k}y^k) + \sum_{k=0}^{n} \binom{n}{k}(x^{n-k}y^{k+1})$$

$$= \sum_{k=0}^{n} \binom{n}{k}(x^{n+1-k}y^k) + \sum_{k=1}^{n+1} \binom{n}{k-1}(x^{n+1-k}y^k)$$

$$= \binom{n}{0}x^{n+1}y^0 + \sum_{k=1}^{n} \binom{n}{k}(x^{n+1-k}y^k) + \binom{n}{n}x^0y^{n+1} + \sum_{k=1}^{n} \binom{n}{k-1}(x^{n+1-k}y^k)$$

$$= \binom{n+1}{0}x^{n+1}y^0 + \sum_{k=1}^{n} \binom{n}{k}(x^{n+1-k}y^k) + \binom{n+1}{n+1}x^0y^{n+1} + \sum_{k=1}^{n} \binom{n}{k-1}(x^{n+1-k}y^k)$$

$$= \binom{n+1}{0}x^{n+1}y^0 + \sum_{k=1}^{n} \binom{n}{k-1}(x^{n+1-k}y^k) + \sum_{k=1}^{n} \binom{n}{k}(x^{n+1-k}y^k) + \binom{n+1}{n+1}x^0y^{n+1}$$

$$=^* \binom{n+1}{0}x^{n+1}y^0 + \sum_{k=1}^{n} \left(\binom{n}{k-1} + \binom{n}{k}\right)(x^{n+1-k}y^k) + \binom{n+1}{n+1}x^0y^{n+1}$$

$$= \binom{n+1}{0}x^{n+1}y^0 + \sum_{k=1}^{n} \binom{n+1}{k}(x^{n+1-k}y^k) + \binom{n+1}{n+1}x^0y^{n+1}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k}(x^{n+1-k}y^k)$$

$\square$

*

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!}$$

$$= \frac{n!k}{k!(n-k+1)!} + \frac{n!(n-k+1)}{k!(n-k+1)!}$$

$$= \frac{n!k + n!(n-k+1)}{k!(n+1-k)!}$$

$$= \frac{n!(k+n+1-k)}{k!(n+1-k)!}$$

$$= \frac{(n+1)!}{k!(n+1-k)!}$$

$$= \underline{\underline{\binom{n+1}{k}}}$$

5 Linear regression

measured values $x_i, y_i \mid 1 <= i <= n, n \in \mathbb{N}$

regression line: $y = mx + b$
minimize error by calculating least squares

$$S = \sum_{i=1}^{n}(y_i - (mx_i + b))^2 = \sum_{i=1}^{n}(y_i - mx_i - b)^2$$

set $\frac{\partial S}{\partial m} = 0$:

$$\frac{\partial S}{\partial m} = -2\sum_{i=1}^{n}(y_i - mx_i - b)x_i = 0$$

$$\iff \quad 0 = \sum_{i=1}^{n}(y_i - mx_i - b)x_i = \sum_{i=1}^{n}(x_iy_i - mx_ix_i - bx_i)$$

$$= \sum_{i=1}^{n} x_iy_i - m\sum_{i=1}^{n} x_ix_i - b\sum_{i=1}^{n} x_i \quad \textbf{:}\ \mathrm{I}$$

set $\frac{\partial S}{\partial b} = 0$:

$$\frac{\partial S}{\partial b} = -2\sum_{i=1}^{n}(y_i - mx_i - b) = 0$$

$$\iff \quad 0 = \sum_{i=1}^{n}(y_i - mx_i - b) = \sum_{i=1}^{n}(y_i - mx_i - b) = \sum_{i=1}^{n} y_i - m\sum_{i=1}^{n} x_i - nb$$

$$\iff \quad b = \frac{1}{n}\sum_{i=1}^{n} y_i - m\frac{1}{n}\sum_{i=1}^{n} x_i = \underline{\underline{y_M - mx_M}} \quad \textbf{:}\ \mathrm{II}$$

II in I:

$$0 = \sum_{i=1}^{n} x_i y_i - m \sum_{i=1}^{n} x_i x_i - b \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i - m \sum_{i=1}^{n} x_i x_i - (y_M - m x_M) \sum_{i=1}^{n} x_i$$

$$= \sum_{i=1}^{n} x_i y_i - m \sum_{i=1}^{n} x_i x_i - y_M \sum_{i=1}^{n} x_i + m x_M \sum_{i=1}^{n} x_i$$

$$\iff \quad m \left(\sum_{i=1}^{n} x_i x_i - x_M \sum_{i=1}^{n} x_i \right) = \sum_{i=1}^{n} x_i y_i - y_M \sum_{i=1}^{n} x_i$$

$$\iff \quad m = \frac{\sum_{i=1}^{n} x_i y_i - y_M \sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i x_i - x_M \sum_{i=1}^{n} x_i}$$

$$= \frac{\frac{1}{n} \sum_{i=1}^{n} x_i y_i - x_M y_M}{\frac{1}{n} \sum_{i=1}^{n} x_i x_i - x_M x_M}$$

$$=^{*} \underline{\underline{\frac{Cov(x,y)}{Var(x)}}}$$

$$\implies y = mx + n$$

$$= \frac{Cov(x,y)}{Var(x)} x + (y_M - \frac{Cov(x,y)}{Var(x)} x_M)$$

$$= \underline{\underline{\frac{Cov(x,y)}{Var(x)} (x - x_M) + y_M}}$$

*

$$Cov(x,y) := \frac{1}{n}\sum_{i=1}^{n}(x_i - x_M)(y_i - y_M)$$

$$Var(x) := \frac{1}{n}\sum_{i=1}^{n}(x_i - x_M)^2 = Cov(x,x)$$

$$Cov(x,y) := \frac{1}{n}\sum_{i=1}^{n}(x_i - x_M)(y_i - y_M)$$

$$= \frac{1}{n}\sum_{i=1}^{n}(x_i y_i - x_i y_M - x_M y_i + x_M y_M)$$

$$= \frac{1}{n}\left(\sum_{i=1}^{n} x_i y_i - y_M \sum_{i=1}^{n} x_i - x_M \sum_{i=1}^{n} y_i + n x_M y_M\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i y_i - y_M x_M - x_M y_M + x_M y_M$$

$$\underline{\underline{= \frac{1}{n}\sum_{i=1}^{n} x_i y_i - x_M y_M}}$$

6 Number theory - irrational roots

To proove: if a root of a natural number is not integer then it is irrational
formal: $\forall x \in \mathbb{N} : (\sqrt{x} \notin \mathbb{Z}) \Rightarrow (\nexists a, b \in \mathbb{Z} : \frac{a}{b} = \sqrt{x})$

assume $\exists a, b \in \mathbb{Z} : \frac{a}{b} = \sqrt{x}$, a, b have no common divisors

$$\Rightarrow \frac{a^2}{b^2} = x$$
$$\iff a^2 = x * b^2 = (x * b) * b$$

be

$$A = \{p_i | p_i \text{ is i-th of n primefactors of } a\}$$
$$X = \{p_j | p_j \text{ is j-th of m primefactors of } x\}$$
$$B = \{p_k | p_k \text{ is k-th of l primefactors of } b\}$$

$$\Rightarrow a = \prod_{p_i \in A} p_i, \;\; a^2 = \prod_{p_i \in A} {p_i}^2, \;\; x = \prod_{p_j \in X} p_j, \;\; b = \prod_{p_k \in B} p_k$$
$$\Rightarrow a^2 = \prod_{p_j \in X} p_j * \prod_{p_k \in B} {p_k}^2 = \prod_{p_i \in A} {p_i}^2$$

since integer factorization is unique $\Rightarrow B \subseteq A$

<u>case I:</u> $B = \emptyset$
$\Rightarrow b = 1 \Rightarrow \sqrt{x} = a$ is integer

<u>case II:</u> $B \neq \emptyset$
$\Rightarrow B$ is a set of common divisors of a and b $\Rightarrow\Leftarrow$ to assumption

$\square$

Part II
Computation

7 linear interpolation of discrete vector field

Be $\vec{g} : \mathbb{Z}^n \to \mathbb{R}^m$

We define $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$ as:

$$\vec{f}(\vec{v}) := \sum_{p \in \mathcal{P}(S)} \prod_{i=1}^{n} \left((1 - \vec{\delta}(\vec{v})_i)^{1-\xi_p(i)} \vec{\delta}(\vec{v})_i^{\xi_p(i)} \right) \vec{g}(\vec{h}_p(\vec{v}))$$

with

$$S := \{x \in \mathbb{N} \mid 1 \leq x \leq n\}, \quad p \in \mathcal{P}(S)$$

$$\begin{aligned}
&\text{indicator function:}\\
\xi_A : \mathbb{N} &\to \{0, 1\}, \quad A \subseteq \mathbb{N}\\
\xi_A(x) &:= \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}\\
\\
&\text{discretization function:}\\
\vec{h}_p &: \mathbb{R}^n \to \mathbb{Z}^n\\
h_i &:= \begin{cases} \lceil v_i \rceil & i \in p \\ \lfloor v_i \rfloor & i \notin p \end{cases}\\
&\text{with} \quad \vec{h} := \vec{h}_p(\vec{v})\\
\\
&\text{delta function:}\\
\vec{\delta} &: \mathbb{R}^n \to \{z \in \mathbb{R} \mid 0 \leq z < 1\}^n\\
\vec{\delta}(\vec{v}) &:= \vec{v} - \vec{h}_\emptyset(\vec{v})
\end{aligned}$$

.

<u>Example</u>: $n = 2, \quad d_1 := \vec{\delta}(\vec{v})_1, d_2 := \vec{\delta}(\vec{v})_2$

$$\begin{aligned}
\vec{f}(\vec{v}) = \vec{f}(\vec{h_\emptyset}(\vec{v}) + \vec{\delta}(\vec{v})) = \vec{f}(\begin{pmatrix} \lfloor v_1 \rfloor \\ \lfloor v_2 \rfloor \end{pmatrix} + \begin{pmatrix} d_1 \\ d_2 \end{pmatrix})\\
= (1 - d_1)(1 - d_2)\vec{g}(\vec{h}_\emptyset(\vec{v})) + (1 - d_2)d_1\vec{g}(\vec{h}_{\{1\}}(\vec{v})) + (1 - d_1)d_2\vec{g}(\vec{h}_{\{2\}}(\vec{v})) + d_1 d_2 \vec{g}(\vec{h}_{\{1,2\}}(\vec{v}))
\end{aligned}$$

8 partial velocity vector

In order to calculate the momentum between two colliding masses it is necessary to determine the partial velocity vector of a moving mass in direction to the second mass. Let's assume a mass m_1 moves with velocity $\vec{v}$ and would hit mass m_2. For simplicity m_2 remains stationary and the shapes of both masses are spherical. Even when the direction of $\vec{v}$ does not point directly to m_2, it behaves as if m_1 hits m_2 with a (partial) velocity $\vec{v_{m_2}}$ which points to the direction of m_2. The calculation holds for any dimension.

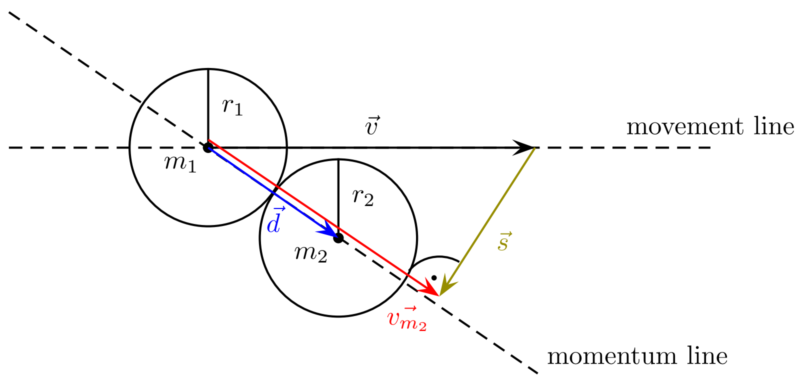

Figure 1: collision

We need to calculate $\vec{v_{m_2}}$. We define $\vec{d} = \vec{m_2} - \vec{m_1}$, where $\vec{m_1}$ and $\vec{m_2}$ are the positional vectors for m_1 resp. m_2. We require that the masses have a positive expansion ($r_1 > 0$, $r_2 > 0$), thus $\vec{d} \neq \vec{0}$ (collision takes place if $|\vec{d}| = r_1 + r_2$). Let $\vec{s}$ be a (the) vector with $\vec{v_{m_2}} = \vec{v} + \vec{s}$. Then $\vec{s}$ must be orthogonal to $\vec{v_{m_2}}$ and thus to $\vec{d}$. That is because $\vec{v_{m_2}}$ is a partial vector of $\vec{v}$ and points in the same direction as $\vec{d}$ which resides on the momentum line. See figure 1.

Be $\lambda \in \mathbb{R}$, then the following equations hold:

$$\vec{v_{m_2}} = \vec{v} + \vec{s} = \lambda \vec{d} \tag{4}$$

$$\vec{s} \cdot \vec{d} = 0 \tag{5}$$

This can easily be solved:

$$(4), (5) => (\lambda \vec{d} - \vec{v})\vec{d} = 0 \tag{6}$$

$$<=> \lambda \vec{d}^2 - \vec{v}\vec{d} = 0 \tag{7}$$

$$<=> \lambda = \frac{\vec{v}\vec{d}}{\vec{d}^2} \tag{8}$$

As a result we get:

$$\vec{v_{m_2}} = \frac{\vec{v}\vec{d}}{\vec{d}^2}\vec{d}$$

□

9 physical line adjustment

Points $P1, P2$ with $\vec{d} := \vec{P}2 - \vec{P}1, l :=$ required length of line

We obtain $P1', P2'$ with $|\vec{P}2' - \vec{P}1'| = l$ by calculating $\vec{P}2' = \vec{P}2 - \vec{\Delta}d$ and $\vec{P}1' = \vec{P}1 + \vec{\Delta}d$ with $|\vec{\Delta}d| = \frac{|\vec{d}|-l}{2}$:

$\vec{\Delta}d = \frac{\vec{d}}{|\vec{d}|} * |\vec{\Delta}d| = \frac{|\vec{d}|-l}{2|\vec{d}|}\vec{d}.$

10 n-dimensional polar coordinates

cartesian to polar

$(x_1, \ldots, x_n) \rightarrow (r, \alpha_1, \ldots, \alpha_{n-1})$

$\alpha_1 = \text{arctan}_2(\frac{x_2}{x_1})$

$\alpha_i = \text{arctan}_2(\frac{x_{i+1}}{\sqrt{\sum_{j=1}^{i} x_n^2}})$

polar to cartesion

$(r, \alpha_1, \ldots, \alpha_{n-1}) \rightarrow (x_1, \ldots, x_n)$

$x_1 = r \prod_{j=1}^{n_\alpha} \cos \alpha_j$
$x_i = r \sin \alpha_{i-1} \prod_{j=i}^{n_\alpha} \cos \alpha_j \quad \forall \;\; 1 < i <= n$