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Part I

Proofs

1 Orthogonality and determinant

Be $\vec{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = |\vec{v}_1| \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = |\vec{v}_2| \begin{pmatrix} \cos \beta \\ \sin \beta \end{pmatrix}$ with $\alpha = \angle \vec{v}_1$, $\beta = \angle \vec{v}_2$

Orthogonality

To prove: $\vec{v}_1 \perp \vec{v}_2 \iff \vec{v}_1 * \vec{v}_2 = 0$:

$$\vec{v}_1 * \vec{v}_2 = |\vec{v}_1| \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} |\vec{v}_2| \begin{pmatrix} \cos \beta \\ \sin \beta \end{pmatrix} = |\vec{v}_1| |\vec{v}_2| (\cos \alpha \cos \beta + \sin \alpha \sin \beta) = \underline{\underline{|\vec{v}_1| |\vec{v}_2| \cos(\alpha - \beta)}}.$$

Be $\vec{v}_1 \perp \vec{v}_2 \iff \alpha = \beta + \frac{\pi}{2}$, w.l.o.g.

$$\implies |\vec{v}_1| |\vec{v}_2| \cos(\alpha - \beta) = |\vec{v}_1| |\vec{v}_2| \cos(\beta + \frac{\pi}{2} - \beta) = 0$$

□

Determinant

Let's choose $\sin()$ instead of $\cos()$ in the above equation:

$$\begin{aligned} |\vec{v}_1| |\vec{v}_2| \sin(\alpha - \beta) &= |\vec{v}_1| |\vec{v}_2| (\sin \alpha \cos \beta - \cos \alpha \sin \beta) \\ &= |\vec{v}_1| |\vec{v}_2| \begin{pmatrix} \sin \alpha \\ \cos \alpha \end{pmatrix} \begin{pmatrix} \cos \beta \\ -\sin \beta \end{pmatrix} = \begin{pmatrix} |\vec{v}_1| \sin \alpha \\ |\vec{v}_1| \cos \alpha \end{pmatrix} \begin{pmatrix} |\vec{v}_2| \cos \beta \\ -|\vec{v}_2| \sin \beta \end{pmatrix} = \begin{pmatrix} y_1 \\ x_1 \end{pmatrix} \begin{pmatrix} x_2 \\ -y_2 \end{pmatrix} \\ &= \det \begin{pmatrix} x_2 & x_1 \\ y_2 & y_1 \end{pmatrix} = \underline{\underline{\det \begin{pmatrix} \vec{v}_2 & \vec{v}_1 \end{pmatrix}}}. \end{aligned}$$

If $\alpha = \beta + k * \pi$, w.l.o.g, $k \in \mathbb{N}$

$$\implies \det \begin{pmatrix} \vec{v}_2 & \vec{v}_1 \end{pmatrix} = |\vec{v}_1| |\vec{v}_2| \sin(\alpha - \beta) = |\vec{v}_1| |\vec{v}_2| \sin(\beta + k * \pi - \beta) = 0$$

$\implies \vec{v}_1$ and \vec{v}_2 are linear dependent and the determinant in general determines linear dependency.

2 Law of cosines

To prove: $c^2 = a^2 + b^2 - 2ab \cos \gamma$

$\gamma \geq \frac{\pi}{2}$:

$$d^2 = b^2 - e^2 \tag{1}$$

$$e = b \sin(\pi - \gamma) \tag{2}$$

$$1 = \sin^2 \gamma + \cos^2 \gamma \tag{3}$$

$$c^2 = (a + d)^2 + e^2$$

$$= (a + \sqrt{b^2 - e^2})^2 + e^2$$

$$= a^2 + 2a\sqrt{b^2 - e^2} + b^2$$

$$= a^2 + 2a\sqrt{b^2 - b^2 \sin^2(\pi - \gamma)} + b^2$$

$$= a^2 + 2ab\sqrt{1 - \sin^2(\pi - \gamma)} + b^2$$

$$= a^2 + 2ab\sqrt{\cos^2(\pi - \gamma)} + b^2$$

$$= a^2 + b^2 + 2ab \cos(\pi - \gamma)$$

$$= a^2 + b^2 - 2ab \cos \gamma$$

□

3 Polynomial derivation

To prove: $(x^n)' = nx^{n-1}$

$$\begin{aligned}
& \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
&= \frac{(x + \Delta x)^n - x^n}{\Delta x} \\
&= \frac{\sum_{k=0}^n \binom{n}{k} (x^{n-k} \Delta x^k) - x^n}{\Delta x} \\
&= \frac{x^n \Delta x^0 + \sum_{k=1}^n \binom{n}{k} (x^{n-k} \Delta x^k) - x^n}{\Delta x} \\
&= \frac{\sum_{k=1}^n \binom{n}{k} (x^{n-k} \Delta x^k)}{\Delta x} \\
&= \frac{\Delta x \sum_{k=1}^n \binom{n}{k} (x^{n-k} \Delta x^{k-1})}{\Delta x} \\
&= \sum_{k=1}^n \binom{n}{k} (x^{n-k} \Delta x^{k-1}) \\
&= \sum_{k=1}^n \frac{n!}{k!(n-k)!} (x^{n-k} \Delta x^{k-1}) \\
&= nx^{n-1} + \sum_{k=2}^n \binom{n}{k} (x^{n-k} \Delta x^k) \\
&\lim_{\Delta x \rightarrow 0} \left(nx^{n-1} + \sum_{k=2}^n \binom{n}{k} (x^{n-k} \Delta x^k) \right) = nx^{n-1}
\end{aligned}$$

□

4 Binomial theorem

To prove: $(x + y)^n = \sum_{k=0}^n \binom{n}{k} (x^{n-k} y^k)$

Induction:

$n = 0$:

$$(x + y)^0 = 1 = \sum_{k=0}^0 \binom{0}{k} (x^{0-k} y^k)$$

$n \rightarrow n + 1$:

$$\begin{aligned} (x + y)^{n+1} &= (x + y)^n (x + y) \stackrel{\text{i.p.}}{=} \sum_{k=0}^n \binom{n}{k} (x^{n-k} y^k) (x + y) \\ &= \sum_{k=0}^n \binom{n}{k} (x^{n+1-k} y^k) + \sum_{k=0}^n \binom{n}{k} (x^{n-k} y^{k+1}) \\ &= \sum_{k=0}^n \binom{n}{k} (x^{n+1-k} y^k) + \sum_{k=1}^{n+1} \binom{n}{k-1} (x^{n+1-k} y^k) \\ &= \binom{n}{0} x^{n+1} y^0 + \sum_{k=1}^n \binom{n}{k} (x^{n+1-k} y^k) + \binom{n}{n} x^0 y^{n+1} + \sum_{k=1}^n \binom{n}{k-1} (x^{n+1-k} y^k) \\ &= \binom{n+1}{0} x^{n+1} y^0 + \sum_{k=1}^n \binom{n}{k} (x^{n+1-k} y^k) + \binom{n+1}{n+1} x^0 y^{n+1} + \sum_{k=1}^n \binom{n}{k-1} (x^{n+1-k} y^k) \\ &= \binom{n+1}{0} x^{n+1} y^0 + \sum_{k=1}^n \binom{n}{k-1} (x^{n+1-k} y^k) + \sum_{k=1}^n \binom{n}{k} (x^{n+1-k} y^k) + \binom{n+1}{n+1} x^0 y^{n+1} \\ &=^* \binom{n+1}{0} x^{n+1} y^0 + \sum_{k=1}^n \left(\binom{n}{k-1} + \binom{n}{k} \right) (x^{n+1-k} y^k) + \binom{n+1}{n+1} x^0 y^{n+1} \\ &= \binom{n+1}{0} x^{n+1} y^0 + \sum_{k=1}^n \binom{n+1}{k} (x^{n+1-k} y^k) + \binom{n+1}{n+1} x^0 y^{n+1} \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} (x^{n+1-k} y^k) \end{aligned}$$

□

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$$\begin{aligned}
 \binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} \\
 &= \frac{n!k}{k!(n-k+1)!} + \frac{n!(n-k+1)}{k!(n-k+1)!} \\
 &= \frac{n!k + n!(n-k+1)}{k!(n+1-k)!} \\
 &= \frac{n!(k+n+1-k)}{k!(n+1-k)!} \\
 &= \frac{(n+1)!}{k!(n+1-k)!} \\
 &= \underline{\underline{\binom{n+1}{k}}}
 \end{aligned}$$

5 Linear regression

measured values $x_i, y_i \mid 1 \leq i \leq n, n \in \mathbb{N}$

regression line: $y = mx + b$

minimize error by calculating least squares

$$S = \sum_{i=1}^n (y_i - (mx_i + b))^2 = \sum_{i=1}^n (y_i - mx_i - b)^2$$

set $\frac{\partial S}{\partial m} = 0$:

$$\frac{\partial S}{\partial m} = -2 \sum_{i=1}^n (y_i - mx_i - b)x_i = 0$$

$$\iff 0 = \sum_{i=1}^n (y_i - mx_i - b)x_i = \sum_{i=1}^n (x_i y_i - mx_i x_i - bx_i)$$

$$= \sum_{i=1}^n x_i y_i - m \sum_{i=1}^n x_i x_i - b \sum_{i=1}^n x_i \quad : \text{I}$$

set $\frac{\partial S}{\partial b} = 0$:

$$\frac{\partial S}{\partial b} = -2 \sum_{i=1}^n (y_i - mx_i - b) = 0$$

$$\iff 0 = \sum_{i=1}^n (y_i - mx_i - b) = \sum_{i=1}^n (y_i - mx_i - b) = \sum_{i=1}^n y_i - m \sum_{i=1}^n x_i - nb$$

$$\iff b = \frac{1}{n} \sum_{i=1}^n y_i - m \frac{1}{n} \sum_{i=1}^n x_i = \underline{\underline{y_M - mx_M}} \quad : \text{II}$$

II in I:

$$0 = \sum_{i=1}^n x_i y_i - m \sum_{i=1}^n x_i x_i - b \sum_{i=1}^n x_i = \sum_{i=1}^n x_i y_i - m \sum_{i=1}^n x_i x_i - (y_M - m x_M) \sum_{i=1}^n x_i$$

$$= \sum_{i=1}^n x_i y_i - m \sum_{i=1}^n x_i x_i - y_M \sum_{i=1}^n x_i + m x_M \sum_{i=1}^n x_i$$

$$\Longleftrightarrow m \left(\sum_{i=1}^n x_i x_i - x_M \sum_{i=1}^n x_i \right) = \sum_{i=1}^n x_i y_i - y_M \sum_{i=1}^n x_i$$

$$\Longleftrightarrow m = \frac{\sum_{i=1}^n x_i y_i - y_M \sum_{i=1}^n x_i}{\sum_{i=1}^n x_i x_i - x_M \sum_{i=1}^n x_i}$$

$$= \frac{\frac{1}{n} \sum_{i=1}^n x_i y_i - x_M y_M}{\frac{1}{n} \sum_{i=1}^n x_i x_i - x_M x_M}$$

$$=^* \frac{Cov(x, y)}{\underline{\underline{Var(x)}}}$$

$$\implies y = mx + n$$

$$= \frac{Cov(x, y)}{Var(x)} x + \left(y_M - \frac{Cov(x, y)}{Var(x)} x_M \right)$$

$$= \underline{\underline{\frac{Cov(x, y)}{Var(x)} (x - x_M) + y_M}}$$

*

$$\begin{aligned}
Cov(x, y) &:= \frac{1}{n} \sum_{i=1}^n (x_i - x_M)(y_i - y_M) \\
Var(x) &:= \frac{1}{n} \sum_{i=1}^n (x_i - x_M)^2 = Cov(x, x) \\
Cov(x, y) &:= \frac{1}{n} \sum_{i=1}^n (x_i - x_M)(y_i - y_M) \\
&= \frac{1}{n} \sum_{i=1}^n (x_i y_i - x_i y_M - x_M y_i + x_M y_M) \\
&= \frac{1}{n} \left(\sum_{i=1}^n x_i y_i - y_M \sum_{i=1}^n x_i - x_M \sum_{i=1}^n y_i + n x_M y_M \right) \\
&= \frac{1}{n} \sum_{i=1}^n x_i y_i - y_M x_M - x_M y_M + x_M y_M \\
&= \underline{\underline{\frac{1}{n} \sum_{i=1}^n x_i y_i - x_M y_M}}
\end{aligned}$$

Part II

Computation

6 linear interpolation of discrete vector field

Be $\vec{g}: \mathbb{Z}^n \rightarrow \mathbb{R}^m$

We define $\vec{f}: \mathbb{R}^n \rightarrow \mathbb{R}^m$ as:

$$\vec{f}(\vec{v}) := \sum_{p \in \mathcal{P}(S)} \prod_{i=1}^n \left((1 - \vec{\delta}(\vec{v})_i)^{1 - \xi_p(i)} \vec{\delta}(\vec{v})_i^{\xi_p(i)} \right) \vec{g}(\vec{h}_p(\vec{v}))$$

with

$$S := \{x \in \mathbb{N} \mid 1 \leq x \leq n\}, \quad p \in \mathcal{P}(S)$$

indicator function:

$$\xi_A: \mathbb{N} \rightarrow \{0, 1\}, \quad A \subseteq \mathbb{N}$$

$$\xi_A(x) := \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

discretization function:

$$\vec{h}_p: \mathbb{R}^n \rightarrow \mathbb{Z}^n$$

$$\vec{h}_p(\vec{v}) := \vec{h}, \quad h_i := \begin{cases} \lceil v_i \rceil & i \in p \\ \lfloor v_i \rfloor & i \notin p \end{cases}$$

delta function:

$$\vec{\delta}: \mathbb{R}^n \rightarrow \{z \in \mathbb{R} \mid 0 \leq z < 1\}^n$$

$$\vec{\delta}(\vec{v}) := \vec{v} - \vec{h}_\emptyset(\vec{v})$$

.

Example: $n = 2$, $d_1 := \vec{\delta}(\vec{v})_1, d_2 := \vec{\delta}(\vec{v})_2$

$$\begin{aligned} \vec{f}(\vec{v}) &= \vec{f}(\vec{h}_\emptyset(\vec{v}) + \vec{\delta}(\vec{v})) = \vec{f}\left(\begin{pmatrix} \lfloor v_1 \rfloor \\ \lfloor v_2 \rfloor \end{pmatrix} + \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}\right) \\ &= (1 - d_1)(1 - d_2)\vec{g}(\vec{h}_\emptyset(\vec{v})) + (1 - d_2)d_1\vec{g}(\vec{h}_{\{1\}}(\vec{v})) + (1 - d_1)d_2\vec{g}(\vec{h}_{\{2\}}(\vec{v})) + d_1d_2\vec{g}(\vec{h}_{\{1,2\}}(\vec{v})) \end{aligned}$$

7 partial velocity vector

In order to calculate the momentum between two colliding masses it is necessary to determine the partial velocity vector of a moving mass in direction to the second mass. Let's assume a mass m_1 moves with velocity \vec{v} and would hit mass m_2 . For simplicity m_2 remains stationary and the shapes of both masses are spherical. Even when the direction of \vec{v} does not point directly to m_2 , it behaves as if m_1 hits m_2 with a (partial) velocity \vec{v}_{m_2} which points to the direction of m_2 . The calculation holds for any dimension.

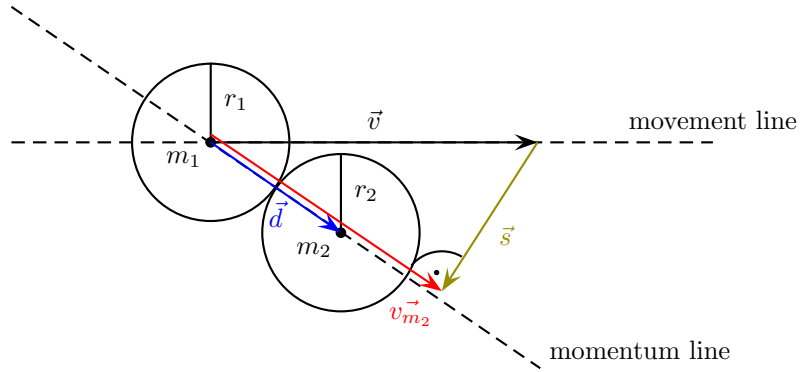


Figure 1: collision

We need to calculate \vec{v}_{m_2} . We define $\vec{d} = \vec{m}_2 - \vec{m}_1$, where \vec{m}_1 and \vec{m}_2 are the positional vectors for m_1 resp. m_2 . We require that the masses have a positive expansion ($r_1 > 0, r_2 > 0$), thus $\vec{d} \neq \vec{0}$ (collision takes place if $|\vec{d}| = r_1 + r_2$). Let \vec{s} be a (the) vector with $\vec{v}_{m_2} = \vec{v} + \vec{s}$. Then \vec{s} must be orthogonal to \vec{v}_{m_2} and thus to \vec{d} . That is because \vec{v}_{m_2} is a partial vector of \vec{v} and points in the same direction as \vec{d} which resides on the momentum line. See figure 1.

Be $\lambda \in \mathbb{R}$, then the following equations hold:

$$v_{m_2}^{\vec{}} = \vec{v} + \vec{s} = \lambda \vec{d} \quad (4)$$

$$\vec{s} \cdot \vec{d} = 0 \quad (5)$$

This can easily be solved:

$$(4), (5) \Rightarrow (\lambda \vec{d} - \vec{v}) \vec{d} = 0 \quad (6)$$

$$\Leftrightarrow \lambda \vec{d}^2 - \vec{v} \vec{d} = 0 \quad (7)$$

$$\Leftrightarrow \lambda = \frac{\vec{v} \vec{d}}{\vec{d}^2} \quad (8)$$

As a result we get:

$$v_{m_2}^{\vec{}} = \frac{\vec{v} \vec{d}}{\vec{d}^2} \vec{d}$$

□

8 physical line adjustment

Points $P1, P2$ with $\vec{d} := \vec{P2} - \vec{P1}, l :=$ required length of line

We obtain $P1', P2'$ with $|\vec{P2'} - \vec{P1'}| = l$ by calculating $\vec{P2'} = \vec{P2} - \vec{\Delta d}$ and $\vec{P1'} = \vec{P1} + \vec{\Delta d}$ with $|\vec{\Delta d}| = \frac{|\vec{d}| - l}{2}$:

$$\vec{\Delta d} = \frac{\vec{d}}{|\vec{d}|} * |\vec{\Delta d}| = \frac{|\vec{d}| - l}{2|\vec{d}|} \vec{d}.$$

9 n-dimensional polar coordinates

cartesian to polar

$$(x_1, \dots, x_n) \rightarrow (r, \alpha_1, \dots, \alpha_{n-1})$$

$$\alpha_1 = \arctan_2\left(\frac{x_2}{x_1}\right)$$

$$\alpha_i = \arctan_2\left(\frac{x_{i+1}}{\sqrt{\sum_{j=1}^i x_j^2}}\right)$$

polar to cartesian

$$(r, \alpha_1, \dots, \alpha_{n-1}) \rightarrow (x_1, \dots, x_n)$$

$$x_1 = r \prod_{j=1}^{n_\alpha} \cos \alpha_j$$

$$x_i = r \sin \alpha_{i-1} \prod_{j=i}^{n_\alpha} \cos \alpha_j \quad \forall \quad 1 < i \leq n$$