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# 1 Orthogonality and determinant

## Orthogonality

To prove:  $\vec{v}_1 \perp \vec{v}_2 \Leftrightarrow \vec{v}_1 * \vec{v}_2 = 0$  :

Be  $\vec{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = |\vec{v}_1| \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}$ ,  $\vec{v}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = |\vec{v}_2| \begin{pmatrix} \cos \beta \\ \sin \beta \end{pmatrix}$  with  $\alpha = \angle \vec{v}_1$ ,  $\beta = \angle \vec{v}_2$

$$\vec{v}_1 * \vec{v}_2 = |\vec{v}_1| \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} |\vec{v}_2| \begin{pmatrix} \cos \beta \\ \sin \beta \end{pmatrix} = |\vec{v}_1| |\vec{v}_2| (\cos \alpha \cos \beta + \sin \alpha \sin \beta) = \underline{\underline{|\vec{v}_1| |\vec{v}_2| \cos(\alpha - \beta)}}.$$

Be  $\vec{v}_1 \perp \vec{v}_2 \Leftrightarrow \alpha = \beta + \frac{\pi}{2}$ , w.l.o.g.

$$\Rightarrow |\vec{v}_1| |\vec{v}_2| \cos(\alpha - \beta) = |\vec{v}_1| |\vec{v}_2| \cos(\beta + \frac{\pi}{2} - \beta) = 0$$

□

## Determinant

Let's choose  $\sin()$  instead of  $\cos()$  in the above equation:

$$\begin{aligned} |\vec{v}_1| |\vec{v}_2| \sin(\alpha - \beta) &= |\vec{v}_1| |\vec{v}_2| (\sin \alpha \cos \beta - \cos \alpha \sin \beta) \\ &= |\vec{v}_1| |\vec{v}_2| \begin{pmatrix} \sin \alpha \\ \cos \alpha \end{pmatrix} \begin{pmatrix} \cos \beta \\ -\sin \beta \end{pmatrix} = \begin{pmatrix} |\vec{v}_1| \sin \alpha \\ |\vec{v}_1| \cos \alpha \end{pmatrix} \begin{pmatrix} |\vec{v}_2| \cos \beta \\ -|\vec{v}_2| \sin \beta \end{pmatrix} = \begin{pmatrix} y_1 \\ x_1 \end{pmatrix} \begin{pmatrix} x_2 \\ -y_2 \end{pmatrix} \\ &= \det \begin{pmatrix} x_2 & x_1 \\ y_2 & y_1 \end{pmatrix} = \underline{\underline{\det \begin{pmatrix} \vec{v}_2 & \vec{v}_1 \end{pmatrix}}}. \end{aligned}$$

If  $\alpha = \beta + k * \pi$ , w.l.o.g,  $k \in \mathbb{N}$

$$\Rightarrow \det \begin{pmatrix} \vec{v}_2 & \vec{v}_1 \end{pmatrix} = |\vec{v}_1| |\vec{v}_2| \sin(\alpha - \beta) = |\vec{v}_1| |\vec{v}_2| \sin(\beta + k * \pi - \beta) = 0$$

$\Rightarrow \vec{v}_1$  and  $\vec{v}_2$  are linear dependent and the determinant in general determines linear dependency.

## 2 Law of cosines

To prove:  $c^2 = a^2 + b^2 - 2ab \cos \gamma$

$\gamma \geq \frac{\pi}{2}$  :

$$d^2 = b^2 - e^2 \tag{1}$$

$$e = b \sin(\pi - \gamma) \tag{2}$$

$$1 = \sin^2 \gamma + \cos^2 \gamma \tag{3}$$

$$c^2 = (a + d)^2 + e^2$$

$$= (a + \sqrt{b^2 - e^2})^2 + e^2$$

$$= a^2 + 2a\sqrt{b^2 - e^2} + b^2$$

$$= a^2 + 2a\sqrt{b^2 - b^2 \sin^2(\pi - \gamma)} + b^2$$

$$= a^2 + 2ab\sqrt{1 - \sin^2(\pi - \gamma)} + b^2$$

$$= a^2 + 2ab\sqrt{\cos^2(\pi - \gamma)} + b^2$$

$$= a^2 + b^2 + 2ab \cos(\pi - \gamma)$$

$$= a^2 + b^2 - 2ab \cos \gamma$$

□

### 3 Polynomial derivation

To prove:  $(x^n)' = nx^{n-1}$

$$\begin{aligned}
& \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
&= \frac{(x + \Delta x)^n - x^n}{\Delta x} \\
&= \frac{\sum_{k=0}^n \binom{n}{k} (x^{n-k} \Delta x^k) - x^n}{\Delta x} \\
&= \frac{x^n \Delta x^0 + \sum_{k=1}^n \binom{n}{k} (x^{n-k} \Delta x^k) - x^n}{\Delta x} \\
&= \frac{\sum_{k=1}^n \binom{n}{k} (x^{n-k} \Delta x^k)}{\Delta x} \\
&= \frac{\Delta x \sum_{k=1}^n \binom{n}{k} (x^{n-k} \Delta x^{k-1})}{\Delta x} \\
&= \sum_{k=1}^n \binom{n}{k} (x^{n-k} \Delta x^{k-1}) \\
&= \sum_{k=1}^n \frac{n!}{k!(n-k)!} (x^{n-k} \Delta x^{k-1}) \\
&= nx^{n-1} + \sum_{k=2}^n \binom{n}{k} (x^{n-k} \Delta x^k) \\
&\lim_{\Delta x \rightarrow 0} \left( nx^{n-1} + \sum_{k=2}^n \binom{n}{k} (x^{n-k} \Delta x^k) \right) = nx^{n-1}
\end{aligned}$$

□

## 4 Binomial theorem

To prove:  $(x + y)^n = \sum_{k=0}^n \binom{n}{k} (x^{n-k} y^k)$

Induction:

$n = 0$ :

$$(x + y)^0 = 1 = \sum_{k=0}^0 \binom{0}{k} (x^{0-k} y^k)$$

$n \rightarrow n + 1$ :

$$\begin{aligned} (x + y)^{n+1} &= (x + y)^n (x + y) \stackrel{\text{i.p.}}{=} \sum_{k=0}^n \binom{n}{k} (x^{n-k} y^k) (x + y) \\ &= \sum_{k=0}^n \binom{n}{k} (x^{n+1-k} y^k) + \sum_{k=0}^n \binom{n}{k} (x^{n-k} y^{k+1}) \\ &= \sum_{k=0}^n \binom{n}{k} (x^{n+1-k} y^k) + \sum_{k=1}^{n+1} \binom{n}{k-1} (x^{n+1-k} y^k) \\ &= \binom{n}{0} x^{n+1} y^0 + \sum_{k=1}^n \binom{n}{k} (x^{n+1-k} y^k) + \binom{n}{n} x^0 y^{n+1} + \sum_{k=1}^n \binom{n}{k-1} (x^{n+1-k} y^k) \\ &= \binom{n+1}{0} x^{n+1} y^0 + \sum_{k=1}^n \binom{n}{k} (x^{n+1-k} y^k) + \binom{n+1}{n+1} x^0 y^{n+1} + \sum_{k=1}^n \binom{n}{k-1} (x^{n+1-k} y^k) \\ &= \binom{n+1}{0} x^{n+1} y^0 + \sum_{k=1}^n \binom{n}{k-1} (x^{n+1-k} y^k) + \sum_{k=1}^n \binom{n}{k} (x^{n+1-k} y^k) + \binom{n+1}{n+1} x^0 y^{n+1} \\ &=^* \binom{n+1}{0} x^{n+1} y^0 + \sum_{k=1}^n \left( \binom{n}{k-1} + \binom{n}{k} \right) (x^{n+1-k} y^k) + \binom{n+1}{n+1} x^0 y^{n+1} \\ &= \binom{n+1}{0} x^{n+1} y^0 + \sum_{k=1}^n \binom{n+1}{k} (x^{n+1-k} y^k) + \binom{n+1}{n+1} x^0 y^{n+1} \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} (x^{n+1-k} y^k) \end{aligned}$$

□

\*

$$\begin{aligned}
 \binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} \\
 &= \frac{n!k}{k!(n-k+1)!} + \frac{n!(n-k+1)}{k!(n-k+1)!} \\
 &= \frac{n!k + n!(n-k+1)}{k!(n+1-k)!} \\
 &= \frac{n!(k+n+1-k)}{k!(n+1-k)!} \\
 &= \frac{(n+1)!}{k!(n+1-k)!} \\
 &= \underline{\underline{\binom{n+1}{k}}}
 \end{aligned}$$

## 5 Linear regression

measured values  $x_i, y_i \mid 1 \leq i \leq n, n \in \mathbb{N}$

regression line:  $y = mx + b$

minimize error by calculating least squares

$$S = \sum_{i=1}^n (y_i - (mx_i + b))^2 = \sum_{i=1}^n (y_i - mx_i - b)^2$$

set  $\frac{\partial S}{\partial m} = 0$  :

$$\frac{\partial S}{\partial m} = -2 \sum_{i=1}^n (y_i - mx_i - b)x_i = 0$$

$$\iff 0 = \sum_{i=1}^n (y_i - mx_i - b)x_i = \sum_{i=1}^n (x_i y_i - mx_i x_i - bx_i)$$

$$= \sum_{i=1}^n x_i y_i - m \sum_{i=1}^n x_i x_i - b \sum_{i=1}^n x_i \quad : \text{I}$$

set  $\frac{\partial S}{\partial b} = 0$  :

$$\frac{\partial S}{\partial b} = -2 \sum_{i=1}^n (y_i - mx_i - b) = 0$$

$$\iff 0 = \sum_{i=1}^n (y_i - mx_i - b) = \sum_{i=1}^n (y_i - mx_i - b) = \sum_{i=1}^n y_i - m \sum_{i=1}^n x_i - nb$$

$$\iff b = \frac{1}{n} \sum_{i=1}^n y_i - m \frac{1}{n} \sum_{i=1}^n x_i = \underline{\underline{y_M - mx_M}} \quad : \text{II}$$

II in I:

$$0 = \sum_{i=1}^n x_i y_i - m \sum_{i=1}^n x_i x_i - b \sum_{i=1}^n x_i = \sum_{i=1}^n x_i y_i - m \sum_{i=1}^n x_i x_i - (y_M - m x_M) \sum_{i=1}^n x_i$$

$$= \sum_{i=1}^n x_i y_i - m \sum_{i=1}^n x_i x_i - y_M \sum_{i=1}^n x_i + m x_M \sum_{i=1}^n x_i$$

$$\Longleftrightarrow m \left( \sum_{i=1}^n x_i x_i - x_M \sum_{i=1}^n x_i \right) = \sum_{i=1}^n x_i y_i - y_M \sum_{i=1}^n x_i$$

$$\Longleftrightarrow m = \frac{\sum_{i=1}^n x_i y_i - y_M \sum_{i=1}^n x_i}{\sum_{i=1}^n x_i x_i - x_M \sum_{i=1}^n x_i}$$

$$= \frac{\frac{1}{n} \sum_{i=1}^n x_i y_i - x_M y_M}{\frac{1}{n} \sum_{i=1}^n x_i x_i - x_M x_M}$$

$$=^* \frac{Cov(x, y)}{\underline{\underline{Var(x)}}}$$

$$\implies y = mx + n$$

$$= \frac{Cov(x, y)}{Var(x)} x + (y_M - \frac{Cov(x, y)}{Var(x)} x_M)$$

$$= \underline{\underline{\frac{Cov(x, y)}{Var(x)} (x - x_M) + y_M}}$$



\*

$$\begin{aligned}
Cov(x, y) &:= \frac{1}{n} \sum_{i=1}^n (x_i - x_M)(y_i - y_M) \\
Var(x) &:= \frac{1}{n} \sum_{i=1}^n (x_i - x_M)^2 = Cov(x, x) \\
Cov(x, y) &:= \frac{1}{n} \sum_{i=1}^n (x_i - x_M)(y_i - y_M) \\
&= \frac{1}{n} \sum_{i=1}^n (x_i y_i - x_i y_M - x_M y_i + x_M y_M) \\
&= \frac{1}{n} \left( \sum_{i=1}^n x_i y_i - y_M \sum_{i=1}^n x_i - x_M \sum_{i=1}^n y_i + n x_M y_M \right) \\
&= \frac{1}{n} \sum_{i=1}^n x_i y_i - y_M x_M - x_M y_M + x_M y_M \\
&= \underline{\underline{\frac{1}{n} \sum_{i=1}^n x_i y_i - x_M y_M}}
\end{aligned}$$

## 6 linear interpolation of discrete vector field

Be  $\vec{g}: \mathbb{Z}^n \rightarrow \mathbb{R}^m$

We define  $\vec{f}: \mathbb{R}^n \rightarrow \mathbb{R}^m$  as:

$$\vec{f}(\vec{v}) := \sum_{p \in \mathcal{P}(\dim)} \prod_{i=1}^n \left( (1 - d_{v_i})^{1-\delta_p(i)} d_{v_i}^{\delta_p(i)} \right) \vec{g}(\vec{v}_p)$$

with

$$\dim := \{x \in \mathbb{N} \mid 1 \leq x \leq n\}, \quad p \in \mathcal{P}(\dim)$$

$$\delta_p: \mathbb{N} \rightarrow \{0, 1\}$$

$$\delta_p(x) := \begin{cases} 1 & x \in p \\ 0 & x \notin p \end{cases}$$

$$\vec{v}_p: v_{p_i} := \begin{cases} \lceil v_i \rceil & i \in p \\ \lfloor v_i \rfloor & i \notin p \end{cases}$$

$$\vec{d}_{\vec{v}} := \vec{v} - \vec{v}_{\emptyset}$$

.

Example:  $n = 2, \quad x, y \in \mathbb{Z}, \quad d_x, d_y \in \{z \in \mathbb{R} \mid 0 \leq z < 1\}$

$$\begin{aligned} \vec{f}(\vec{v}) &= \vec{f}(\vec{v}_{\emptyset} + \vec{d}_{\vec{v}}) = \dots \\ &= (1 - d_x)(1 - d_y)\vec{g}(\vec{v}_{\emptyset}) + (1 - d_y)d_x\vec{g}(\vec{v}_{\{1\}}) + (1 - d_x)d_y\vec{g}(\vec{v}_{\{2\}}) + d_xd_y\vec{g}(\vec{v}_{\{1,2\}}) \end{aligned}$$