# Contents

# 1 Orthogonality and determinant

## Orthogonality

To proove: $\vec{v_1} \perp \vec{v_2} <=> \vec{v_1} * \vec{v_2} = 0$ :

Be $\vec{v_1} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = |\vec{v_1}| \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$, $\vec{v_2} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = |\vec{v_2}| \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix}$ with $\alpha = \measuredangle\vec{v_1}$, $\beta = \measuredangle\vec{v_2}$

$$\vec{v_1} * \vec{v_2} = |\vec{v_1}| \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} |\vec{v_2}| \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix} = |\vec{v_1}||\vec{v_2}|(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = \underline{\underline{|\vec{v_1}||\vec{v_2}|\cos(\alpha - \beta)}}.$$

Be $\vec{v_1} \perp \vec{v_2} <=> \alpha = \beta + \frac{\pi}{2}$, w.l.o.g.

$=> |\vec{v_1}||\vec{v_2}|\cos(\alpha - \beta) = |\vec{v_1}||\vec{v_2}|\cos(\beta + \frac{\pi}{2} - \beta) = 0$

$\square$

## Determinant

Let's choose sin() instead of cos() in the above equation:

$|\vec{v_1}||\vec{v_2}|\sin(\alpha - \beta) = |\vec{v_1}||\vec{v_2}|(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$

$$= |\vec{v_1}||\vec{v_2}| \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta \\ -\sin\beta \end{pmatrix} = \begin{pmatrix} |\vec{v_1}|\sin\alpha \\ |\vec{v_1}|\cos\alpha \end{pmatrix} \begin{pmatrix} |\vec{v_2}|\cos\beta \\ -|\vec{v_2}|\sin\beta \end{pmatrix} = \begin{pmatrix} y_1 \\ x_1 \end{pmatrix} \begin{pmatrix} x_2 \\ -y_2 \end{pmatrix}$$
$$= \det \begin{pmatrix} x_2 & x_1 \\ y_2 & y_1 \end{pmatrix} = \underline{\underline{\det \begin{pmatrix} \vec{v_2} & \vec{v_1} \end{pmatrix}}}.$$

If $\alpha = \beta + k * \pi$, w.l.o.g, $k \in \mathbb{N}$

$=> \det \begin{pmatrix} \vec{v_2} & \vec{v_1} \end{pmatrix} = |\vec{v_1}||\vec{v_2}|\sin(\alpha - \beta) = |\vec{v_1}||\vec{v_2}|\sin(\beta + k * \pi - \beta) = 0$

$=> \vec{v_1}$ and $\vec{v_2}$ are linear dependent and the determinant in general determines linear dependency.

## 2 Polynomial derivation

To proove: $(x^n)' = nx^{n-1}$

$$\frac{f(x+\triangle x)-f(x)}{\triangle x}$$

$$=\frac{(x+\triangle x)^n - x^n}{\triangle x}$$

$$=\frac{\sum_{k=0}^{n}\binom{n}{k}(x^{n-k}\triangle x^k) - x^n}{\triangle x}$$

$$=\frac{x^n\Delta x^0 + \sum_{k=1}^{n}\binom{n}{k}(x^{n-k}\Delta x^k) - x^n}{\Delta x}$$

$$=\frac{\sum_{k=1}^{n}\binom{n}{k}(x^{n-k}\Delta x^k)}{\Delta x}$$

$$=\frac{\Delta x\sum_{k-1}^{n}\binom{n}{k}(x^{n-k}\Delta x^{k-1})}{\Delta x}$$

$$=\sum_{k-1}^{n}\binom{n}{k}(x^{n-k}\Delta x^{k-1})$$

$$=\sum_{k=1}^{n}\frac{n!}{k!(n-k)!}(x^{n-k}\Delta x^{k-1})$$

$$= nx^{n-1} + \sum_{k=2}^{n}\binom{n}{k}(x^{n-k}\Delta x^k)$$

$$\lim_{\Delta x\to 0}\left(nx^{n-1} + \sum_{k=2}^{n}\binom{n}{k}(x^{n-k}\Delta x^k)\right) = nx^{n-1}$$

□

## 3 Binomial theorem

To proove: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k}(x^{n-k}y^k)$

Induction:
$n = 0$:

$$(x+y)^0 = 1 = \sum_{k=0}^{0} \binom{0}{k}(x^{0-k}y^k)$$

$n \to n+1$:

$$(x+y)^{n+1} = (x+y)^n(x+y) \overset{\text{i.p.}}{=} \sum_{k=0}^{n} \binom{n}{k}(x^{n-k}y^k)(x+y)$$

$$= \sum_{k=0}^{n} \binom{n}{k}(x^{n+1-k}y^k) + \sum_{k=0}^{n} \binom{n}{k}(x^{n-k}y^{k+1})$$

$$= \sum_{k=0}^{n} \binom{n}{k}(x^{n+1-k}y^k) + \sum_{k=1}^{n+1} \binom{n}{k-1}(x^{n+1-k}y^k)$$

$$= \binom{n}{0}x^{n+1}y^0 + \sum_{k=1}^{n} \binom{n}{k}(x^{n+1-k}y^k) + \binom{n}{n}x^0y^{n+1} + \sum_{k=1}^{n} \binom{n}{k-1}(x^{n+1-k}y^k)$$

$$= \binom{n+1}{0}x^{n+1}y^0 + \sum_{k=1}^{n} \binom{n}{k}(x^{n+1-k}y^k) + \binom{n+1}{n+1}x^0y^{n+1} + \sum_{k=1}^{n} \binom{n}{k-1}(x^{n+1-k}y^k)$$

$$= \binom{n+1}{0}x^{n+1}y^0 + \sum_{k=1}^{n} \binom{n}{k-1}(x^{n+1-k}y^k) + \sum_{k=1}^{n} \binom{n}{k}(x^{n+1-k}y^k) + \binom{n+1}{n+1}x^0y^{n+1}$$

$$=^* \binom{n+1}{0}x^{n+1}y^0 + \sum_{k=1}^{n} \left(\binom{n}{k-1} + \binom{n}{k}\right)(x^{n+1-k}y^k) + \binom{n+1}{n+1}x^0y^{n+1}$$

$$= \binom{n+1}{0}x^{n+1}y^0 + \sum_{k=1}^{n} \binom{n+1}{k}(x^{n+1-k}y^k) + \binom{n+1}{n+1}x^0y^{n+1}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k}(x^{n+1-k}y^k)$$

□

*

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!}$$

$$= \frac{n!k}{k!(n-k+1)!} + \frac{n!(n-k+1)}{k!(n-k+1)!}$$

$$= \frac{n!k + n!(n-k+1)}{k!(n+1-k)!}$$

$$= \frac{n!(k+n+1-k)}{k!(n+1-k)!}$$

$$= \frac{(n+1)!}{k!(n+1-k)!}$$

$$= \underline{\underline{\binom{n+1}{k}}}$$

# 4 Linear regression

measured values $x_i, y_i \mid 1 <= i <= n, n \in \mathbb{N}$

regression line: $y = mx + b$
minimize error by calculating least squares

$$S = \sum_{i=1}^{n}(y_i - (mx_i + b))^2 = \sum_{i=1}^{n}(y_i - mx_i - b)^2$$

set $\frac{\partial S}{\partial m} = 0$ :

$$\frac{\partial S}{\partial m} = -2\sum_{i=1}^{n}(y_i - mx_i - b)x_i = 0$$

$$\iff \quad 0 = \sum_{i=1}^{n}(y_i - mx_i - b)x_i = \sum_{i=1}^{n}(x_iy_i - mx_ix_i - bx_i)$$

$$= \sum_{i=1}^{n} x_iy_i - m\sum_{i=1}^{n} x_ix_i - b\sum_{i=1}^{n} x_i \quad \textbf{:} \text{ I}$$

set $\frac{\partial S}{\partial b} = 0$ :

$$\frac{\partial S}{\partial b} = -2\sum_{i=1}^{n}(y_i - mx_i - b) = 0$$

$$\iff \quad 0 = \sum_{i=1}^{n}(y_i - mx_i - b) = \sum_{i=1}^{n}(y_i - mx_i - b) = \sum_{i=1}^{n} y_i - m\sum_{i=1}^{n} x_i - nb$$

$$\iff \quad b = \frac{1}{n}\sum_{i=1}^{n} y_i - m\frac{1}{n}\sum_{i=1}^{n} x_i = \underline{\underline{y_M - mx_M}} \quad \textbf{:} \text{ II}$$

II in I:

$$0 = \sum_{i=1}^{n} x_i y_i - m \sum_{i=1}^{n} x_i x_i - b \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i - m \sum_{i=1}^{n} x_i x_i - (y_M - m x_M) \sum_{i=1}^{n} x_i$$

$$= \sum_{i=1}^{n} x_i y_i - m \sum_{i=1}^{n} x_i x_i - y_M \sum_{i=1}^{n} x_i + m x_M \sum_{i=1}^{n} x_i$$

$$\iff \quad m \left( \sum_{i=1}^{n} x_i x_i - x_M \sum_{i=1}^{n} x_i \right) = \sum_{i=1}^{n} x_i y_i - y_M \sum_{i=1}^{n} x_i$$

$$\iff \quad m = \frac{\sum_{i=1}^{n} x_i y_i - y_M \sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i x_i - x_M \sum_{i=1}^{n} x_i}$$

$$= \frac{\frac{1}{n} \sum_{i=1}^{n} x_i y_i - x_M y_M}{\frac{1}{n} \sum_{i=1}^{n} x_i x_i - x_M x_M}$$

$$=^* \underline{\underline{\frac{Cov(x,y)}{Var(x)}}}$$

$$\implies y = mx + n$$

$$= \frac{Cov(x,y)}{Var(x)} x + (y_M - \frac{Cov(x,y)}{Var(x)} x_M)$$

$$= \underline{\underline{\frac{Cov(x,y)}{Var(x)} (x - x_M) + y_M}}$$

*

$$Cov(x,y) := \frac{1}{n}\sum_{i=1}^{n}(x_i - x_M)(y_i - y_M)$$

$$Var(x) := \frac{1}{n}\sum_{i=1}^{n}(x_i - x_M)^2 = Cov(x,x)$$

$$Cov(x,y) := \frac{1}{n}\sum_{i=1}^{n}(x_i - x_M)(y_i - y_M)$$

$$= \frac{1}{n}\sum_{i=1}^{n}(x_i y_i - x_i y_M - x_M y_i + x_M y_M)$$

$$= \frac{1}{n}\left(\sum_{i=1}^{n} x_i y_i - y_M \sum_{i=1}^{n} x_i - x_M \sum_{i=1}^{n} y_i + n x_M y_M\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i y_i - y_M x_M - x_M y_M + x_M y_M$$

$$\underline{\underline{= \frac{1}{n}\sum_{i=1}^{n} x_i y_i - x_M y_M}}$$