Contents

1 Orthogonality and determinant

Orthogonality

To proove: $\vec{v_1} \perp \vec{v_2} <=> \vec{v_1} * \vec{v_2} = 0$:

Be $\vec{v_1} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = |\vec{v_1}| \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$, $\vec{v_2} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = |\vec{v_2}| \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix}$ with $\alpha = \measuredangle\vec{v_1}$, $\beta = \measuredangle\vec{v_2}$

$$\vec{v_1} * \vec{v_2} = |\vec{v_1}| \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} |\vec{v_2}| \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix} = |\vec{v_1}||\vec{v_2}|(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = \underline{\underline{|\vec{v_1}||\vec{v_2}|\cos(\alpha - \beta)}}.$$

Be $\vec{v_1} \perp \vec{v_2} <=> \alpha = \beta + \frac{\pi}{2}$, w.l.o.g.

$=> |\vec{v_1}||\vec{v_2}|\cos(\alpha - \beta) = |\vec{v_1}||\vec{v_2}|\cos(\beta + \frac{\pi}{2} - \beta) = 0$

$\Box$

Determinant

Let's choose sin() instead of cos() in the above equation:

$|\vec{v_1}||\vec{v_2}|\sin(\alpha - \beta) = |\vec{v_1}||\vec{v_2}|(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$

$$= |\vec{v_1}||\vec{v_2}| \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta \\ -\sin\beta \end{pmatrix} = \begin{pmatrix} |\vec{v_1}|\sin\alpha \\ |\vec{v_1}|\cos\alpha \end{pmatrix} \begin{pmatrix} |\vec{v_2}|\cos\beta \\ -|\vec{v_2}|\sin\beta \end{pmatrix} = \begin{pmatrix} y_1 \\ x_1 \end{pmatrix} \begin{pmatrix} x_2 \\ -y_2 \end{pmatrix}$$
$$= \det \begin{pmatrix} x_2 & x_1 \\ y_2 & y_1 \end{pmatrix} = \underline{\underline{\det \begin{pmatrix} \vec{v_2} & \vec{v_1} \end{pmatrix}}}.$$

If $\alpha = \beta + k * \pi$, w.l.o.g, $k \in \mathbb{N}$

$=> \det \begin{pmatrix} \vec{v_2} & \vec{v_1} \end{pmatrix} = |\vec{v_1}||\vec{v_2}|\sin(\alpha - \beta) = |\vec{v_1}||\vec{v_2}|\sin(\beta + k * \pi - \beta) = 0$

$=>$ $\vec{v_1}$ and $\vec{v_2}$ are linear dependent and the determinant in general determines linear dependency.

2 Law of cosines

To proove: $c^2 = a^2 + b^2 - 2ab\cos\gamma$

$\gamma >= \frac{\pi}{2}$:

$$d^2 = b^2 - e^2 \tag{1}$$
$$e = b\sin(\pi - \gamma) \tag{2}$$
$$1 = \sin^2\gamma + \cos^2\gamma \tag{3}$$

$$\begin{aligned}
c^2 &= (a+d)^2 + e^2 \\
&= (a + \sqrt{b^2 - e^2})^2 + e^2 \\
&= a^2 + 2a\sqrt{b^2 - e^2} + b^2 \\
&= a^2 + 2a\sqrt{b^2 - b^2\sin^2(\pi - \gamma)} + b^2 \\
&= a^2 + 2ab\sqrt{1 - \sin^2(\pi - \gamma)} + b^2 \\
&= a^2 + 2ab\sqrt{\cos^2(\pi - \gamma)} + b^2 \\
&= a^2 + b^2 + 2ab\cos(\pi - \gamma) \\
&= a^2 + b^2 - 2ab\cos\gamma
\end{aligned}$$

$\square$

3 Polynomial derivation

To proove: $(x^n)' = nx^{n-1}$

$$\frac{f(x+\triangle x) - f(x)}{\triangle x}$$

$$= \frac{(x+\triangle x)^n - x^n}{\triangle x}$$

$$= \frac{\sum_{k=0}^{n} \binom{n}{k}(x^{n-k} \triangle x^k) - x^n}{\triangle x}$$

$$= \frac{x^n \Delta x^0 + \sum_{k=1}^{n} \binom{n}{k}(x^{n-k}\Delta x^k) - x^n}{\Delta x}$$

$$= \frac{\sum_{k=1}^{n} \binom{n}{k}(x^{n-k}\Delta x^k)}{\Delta x}$$

$$= \frac{\Delta x \sum_{k-1}^{n} \binom{n}{k}(x^{n-k}\Delta x^{k-1})}{\Delta x}$$

$$= \sum_{k-1}^{n} \binom{n}{k}(x^{n-k}\Delta x^{k-1})$$

$$= \sum_{k=1}^{n} \frac{n!}{k!(n-k)!}(x^{n-k}\Delta x^{k-1})$$

$$= nx^{n-1} + \sum_{k=2}^{n} \binom{n}{k}(x^{n-k}\Delta x^k)$$

$$\lim_{\Delta x \to 0} \left(nx^{n-1} + \sum_{k=2}^{n} \binom{n}{k}(x^{n-k}\Delta x^k) \right) = nx^{n-1}$$

□

4 Binomial theorem

To proove: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} (x^{n-k} y^k)$

Induction:
$n = 0$:

$$(x+y)^0 = 1 = \sum_{k=0}^{0} \binom{0}{k} (x^{0-k} y^k)$$

$n \to n+1$:

$$(x+y)^{n+1} = (x+y)^n (x+y) \overset{\text{i.p.}}{=} \sum_{k=0}^{n} \binom{n}{k} (x^{n-k} y^k)(x+y)$$

$$= \sum_{k=0}^{n} \binom{n}{k} (x^{n+1-k} y^k) + \sum_{k=0}^{n} \binom{n}{k} (x^{n-k} y^{k+1})$$

$$= \sum_{k=0}^{n} \binom{n}{k} (x^{n+1-k} y^k) + \sum_{k=1}^{n+1} \binom{n}{k-1} (x^{n+1-k} y^k)$$

$$= \binom{n}{0} x^{n+1} y^0 + \sum_{k=1}^{n} \binom{n}{k} (x^{n+1-k} y^k) + \binom{n}{n} x^0 y^{n+1} + \sum_{k=1}^{n} \binom{n}{k-1} (x^{n+1-k} y^k)$$

$$= \binom{n+1}{0} x^{n+1} y^0 + \sum_{k=1}^{n} \binom{n}{k} (x^{n+1-k} y^k) + \binom{n+1}{n+1} x^0 y^{n+1} + \sum_{k=1}^{n} \binom{n}{k-1} (x^{n+1-k} y^k)$$

$$= \binom{n+1}{0} x^{n+1} y^0 + \sum_{k=1}^{n} \binom{n}{k-1} (x^{n+1-k} y^k) + \sum_{k=1}^{n} \binom{n}{k} (x^{n+1-k} y^k) + \binom{n+1}{n+1} x^0 y^{n+1}$$

$$=^* \binom{n+1}{0} x^{n+1} y^0 + \sum_{k=1}^{n} \left(\binom{n}{k-1} + \binom{n}{k} \right) (x^{n+1-k} y^k) + \binom{n+1}{n+1} x^0 y^{n+1}$$

$$= \binom{n+1}{0} x^{n+1} y^0 + \sum_{k=1}^{n} \binom{n+1}{k} (x^{n+1-k} y^k) + \binom{n+1}{n+1} x^0 y^{n+1}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} (x^{n+1-k} y^k)$$

$\square$

*

$$\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} \\
&= \frac{n!k}{k!(n-k+1)!} + \frac{n!(n-k+1)}{k!(n-k+1)!} \\
&= \frac{n!k + n!(n-k+1)}{k!(n+1-k)!} \\
&= \frac{n!(k+n+1-k)}{k!(n+1-k)!} \\
&= \frac{(n+1)!}{k!(n+1-k)!} \\
&= \underline{\underline{\binom{n+1}{k}}}
\end{aligned}$$

5 Linear regression

measured values $x_i, y_i \mid 1 <= i <= n, n \in \mathbb{N}$

regression line: $y = mx + b$
minimize error by calculating least squares

$$S = \sum_{i=1}^{n}(y_i - (mx_i + b))^2 = \sum_{i=1}^{n}(y_i - mx_i - b)^2$$

set $\frac{\partial S}{\partial m} = 0$:

$$\frac{\partial S}{\partial m} = -2\sum_{i=1}^{n}(y_i - mx_i - b)x_i = 0$$

$$\iff \quad 0 = \sum_{i=1}^{n}(y_i - mx_i - b)x_i = \sum_{i=1}^{n}(x_i y_i - mx_i x_i - bx_i)$$

$$= \sum_{i=1}^{n} x_i y_i - m\sum_{i=1}^{n} x_i x_i - b\sum_{i=1}^{n} x_i \quad \textbf{:}\ \mathrm{I}$$

set $\frac{\partial S}{\partial b} = 0$:

$$\frac{\partial S}{\partial b} = -2\sum_{i=1}^{n}(y_i - mx_i - b) = 0$$

$$\iff \quad 0 = \sum_{i=1}^{n}(y_i - mx_i - b) = \sum_{i=1}^{n}(y_i - mx_i - b) = \sum_{i=1}^{n} y_i - m\sum_{i=1}^{n} x_i - nb$$

$$\iff \quad b = \frac{1}{n}\sum_{i=1}^{n} y_i - m\frac{1}{n}\sum_{i=1}^{n} x_i = \underline{\underline{y_M - mx_M}} \quad \textbf{:}\ \mathrm{II}$$

II in I:

$$0 = \sum_{i=1}^{n} x_i y_i - m \sum_{i=1}^{n} x_i x_i - b \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i - m \sum_{i=1}^{n} x_i x_i - (y_M - m x_M) \sum_{i=1}^{n} x_i$$

$$= \sum_{i=1}^{n} x_i y_i - m \sum_{i=1}^{n} x_i x_i - y_M \sum_{i=1}^{n} x_i + m x_M \sum_{i=1}^{n} x_i$$

$$\iff \quad m \left(\sum_{i=1}^{n} x_i x_i - x_M \sum_{i=1}^{n} x_i \right) = \sum_{i=1}^{n} x_i y_i - y_M \sum_{i=1}^{n} x_i$$

$$\iff \quad m = \frac{\sum_{i=1}^{n} x_i y_i - y_M \sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i x_i - x_M \sum_{i=1}^{n} x_i}$$

$$= \frac{\frac{1}{n} \sum_{i=1}^{n} x_i y_i - x_M y_M}{\frac{1}{n} \sum_{i=1}^{n} x_i x_i - x_M x_M}$$

$$=^* \underline{\underline{\frac{Cov(x,y)}{Var(x)}}}$$

$$\implies y = mx + n$$

$$= \frac{Cov(x,y)}{Var(x)} x + (y_M - \frac{Cov(x,y)}{Var(x)} x_M)$$

$$= \underline{\underline{\frac{Cov(x,y)}{Var(x)} (x - x_M) + y_M}}$$

*

$$Cov(x,y) := \frac{1}{n}\sum_{i=1}^{n}(x_i - x_M)(y_i - y_M)$$

$$Var(x) := \frac{1}{n}\sum_{i=1}^{n}(x_i - x_M)^2 = Cov(x,x)$$

$$Cov(x,y) := \frac{1}{n}\sum_{i=1}^{n}(x_i - x_M)(y_i - y_M)$$

$$= \frac{1}{n}\sum_{i=1}^{n}(x_i y_i - x_i y_M - x_M y_i + x_M y_M)$$

$$= \frac{1}{n}\left(\sum_{i=1}^{n} x_i y_i - y_M \sum_{i=1}^{n} x_i - x_M \sum_{i=1}^{n} y_i + n x_M y_M\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i y_i - y_M x_M - x_M y_M + x_M y_M$$

$$\underline{\underline{= \frac{1}{n}\sum_{i=1}^{n} x_i y_i - x_M y_M}}$$

6 linear interpolation of discrete vector field

Be $\vec{g} : \mathbb{Z}^n \to \mathbb{R}^m$

We define $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$ as:

$$\vec{f}(\vec{v}) := \sum_{p \in \mathcal{P}(dim)} \prod_{i=1}^{n} \left((1 - d_{v_i})^{1-\delta_p(i)} d_{v_i}^{\delta_p(i)} \right) \vec{g}(\vec{v}_p)$$

with

$$dim := \{x \in \mathbb{N} \mid 1 \leq x \leq n\}, \quad p \in \mathcal{P}(dim)$$

$$\delta_p : \mathbb{N} \to \{0, 1\}$$

$$\delta_p(x) := \begin{cases} 1 & x \in p \\ 0 & x \notin p \end{cases}$$

$$\vec{v}_p : v_{p_i} := \begin{cases} \lceil v_i \rceil & i \in p \\ \lfloor v_i \rfloor & i \notin p \end{cases}$$

$$\vec{d}_{\vec{v}} := \vec{v} - \vec{v}_{\emptyset}$$