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1 Orthogonality and determinant

Orthogonality

To prove: $\vec{v}_1 \perp \vec{v}_2 \Leftrightarrow \vec{v}_1 * \vec{v}_2 = 0$:

Be $\vec{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = |\vec{v}_1| \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = |\vec{v}_2| \begin{pmatrix} \cos \beta \\ \sin \beta \end{pmatrix}$ with $\alpha = \angle \vec{v}_1$, $\beta = \angle \vec{v}_2$

$$\vec{v}_1 * \vec{v}_2 = |\vec{v}_1| \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} |\vec{v}_2| \begin{pmatrix} \cos \beta \\ \sin \beta \end{pmatrix} = |\vec{v}_1| |\vec{v}_2| (\cos \alpha \cos \beta + \sin \alpha \sin \beta) = \underline{\underline{|\vec{v}_1| |\vec{v}_2| \cos(\alpha - \beta)}}.$$

Be $\vec{v}_1 \perp \vec{v}_2 \Leftrightarrow \alpha = \beta + \frac{\pi}{2}$, w.l.o.g.

$$\Rightarrow |\vec{v}_1| |\vec{v}_2| \cos(\alpha - \beta) = |\vec{v}_1| |\vec{v}_2| \cos(\beta + \frac{\pi}{2} - \beta) = 0$$

□

Determinant

Let's choose $\sin()$ instead of $\cos()$ in the above equation:

$$\begin{aligned} |\vec{v}_1| |\vec{v}_2| \sin(\alpha - \beta) &= |\vec{v}_1| |\vec{v}_2| (\sin \alpha \cos \beta - \cos \alpha \sin \beta) \\ &= |\vec{v}_1| |\vec{v}_2| \begin{pmatrix} \sin \alpha \\ \cos \alpha \end{pmatrix} \begin{pmatrix} \cos \beta \\ -\sin \beta \end{pmatrix} = \begin{pmatrix} |\vec{v}_1| \sin \alpha \\ |\vec{v}_1| \cos \alpha \end{pmatrix} \begin{pmatrix} |\vec{v}_2| \cos \beta \\ -|\vec{v}_2| \sin \beta \end{pmatrix} = \begin{pmatrix} y_1 \\ x_1 \end{pmatrix} \begin{pmatrix} x_2 \\ -y_2 \end{pmatrix} \\ &= \det \begin{pmatrix} x_2 & x_1 \\ y_2 & y_1 \end{pmatrix} = \underline{\underline{\det \begin{pmatrix} \vec{v}_2 & \vec{v}_1 \end{pmatrix}}}. \end{aligned}$$

If $\alpha = \beta + k * \pi$, w.l.o.g, $k \in \mathbb{N}$

$$\Rightarrow \det \begin{pmatrix} \vec{v}_2 & \vec{v}_1 \end{pmatrix} = |\vec{v}_1| |\vec{v}_2| \sin(\alpha - \beta) = |\vec{v}_1| |\vec{v}_2| \sin(\beta + k * \pi - \beta) = 0$$

$\Rightarrow \vec{v}_1$ and \vec{v}_2 are linear dependent and the determinant in general determines linear dependency.

2 Polynomial derivation

To prove: $(x^n)' = nx^{n-1}$

$$\begin{aligned}
& \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
&= \frac{(x + \Delta x)^n - x^n}{\Delta x} \\
&= \frac{\sum_{k=0}^n \binom{n}{k} (x^{n-k} \Delta x^k) - x^n}{\Delta x} \\
&= \frac{x^n \Delta x^0 + \sum_{k=1}^n \binom{n}{k} (x^{n-k} \Delta x^k) - x^n}{\Delta x} \\
&= \frac{\sum_{k=1}^n \binom{n}{k} (x^{n-k} \Delta x^k)}{\Delta x} \\
&= \frac{\Delta x \sum_{k=1}^n \binom{n}{k} (x^{n-k} \Delta x^{k-1})}{\Delta x} \\
&= \sum_{k=1}^n \binom{n}{k} (x^{n-k} \Delta x^{k-1}) \\
&= \sum_{k=1}^n \frac{n!}{k!(n-k)!} (x^{n-k} \Delta x^{k-1}) \\
&= nx^{n-1} + \sum_{k=2}^n \binom{n}{k} (x^{n-k} \Delta x^k) \\
&\lim_{\Delta x \rightarrow 0} \left(nx^{n-1} + \sum_{k=2}^n \binom{n}{k} (x^{n-k} \Delta x^k) \right) = nx^{n-1}
\end{aligned}$$

□

3 Binomial theorem

To proof: $(x + y)^n = \sum_{k=0}^n \binom{n}{k} (x^{n-k} y^k)$

Induction:

$n = 0$:

$$(x + y)^0 = 1 = \sum_{k=0}^0 \binom{0}{k} (x^{0-k} y^k)$$

$n \rightarrow n + 1$:

$$\begin{aligned} (x + y)^{n+1} &= (x + y)^n (x + y) \stackrel{\text{i.p.}}{=} \sum_{k=0}^n \binom{n}{k} (x^{n-k} y^k) (x + y) \\ &= \sum_{k=0}^n \binom{n}{k} (x^{n+1-k} y^k) + \sum_{k=0}^n \binom{n}{k} (x^{n-k} y^{k+1}) \\ &= \sum_{k=0}^n \binom{n}{k} (x^{n+1-k} y^k) + \sum_{k=1}^{n+1} \binom{n}{k-1} (x^{n+1-k} y^k) \\ &= \binom{n}{0} x^{n+1} y^0 + \sum_{k=1}^n \binom{n}{k} (x^{n+1-k} y^k) + \binom{n}{n} x^0 y^{n+1} + \sum_{k=1}^n \binom{n}{k-1} (x^{n+1-k} y^k) \\ &= \binom{n+1}{0} x^{n+1} y^0 + \sum_{k=1}^n \binom{n}{k} (x^{n+1-k} y^k) + \binom{n+1}{n+1} x^0 y^{n+1} + \sum_{k=1}^n \binom{n}{k-1} (x^{n+1-k} y^k) \\ &= \binom{n+1}{0} x^{n+1} y^0 + \sum_{k=1}^n \binom{n}{k-1} (x^{n+1-k} y^k) + \sum_{k=1}^n \binom{n}{k} (x^{n+1-k} y^k) + \binom{n+1}{n+1} x^0 y^{n+1} \\ &= \binom{n+1}{0} x^{n+1} y^0 + \sum_{k=1}^n \left(\binom{n}{k-1} + \binom{n}{k} \right) (x^{n+1-k} y^k) + \binom{n+1}{n+1} x^0 y^{n+1} \\ &= \binom{n+1}{0} x^{n+1} y^0 + \sum_{k=1}^n \binom{n+1}{k} (x^{n+1-k} y^k) + \binom{n+1}{n+1} x^0 y^{n+1} \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} (x^{n+1-k} y^k) \end{aligned}$$

□

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$$\begin{aligned}
 \binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} \\
 &= \frac{n!k}{k!(n-k+1)!} + \frac{n!(n-k+1)}{k!(n-k+1)!} \\
 &= \frac{n!k + n!(n-k+1)}{k!(n+1-k)!} \\
 &= \frac{n!(k+n+1-k)}{k!(n+1-k)!} \\
 &= \frac{(n+1)!}{k!(n+1-k)!} \\
 &= \underline{\underline{\binom{n+1}{k}}}
 \end{aligned}$$