Orthogonality and determinant

Orthogonality

To proove: $\vec{v_1} \perp \vec{v_2} <=> \vec{v_1} * \vec{v_2} = 0$:

Be $\vec{v_1} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = |\vec{v_1}| \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$, $\vec{v_2} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = |\vec{v_2}| \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix}$ with $\alpha = \measuredangle\vec{v_1}$, $\beta = \measuredangle\vec{v_2}$

$$\vec{v_1} * \vec{v_2} = |\vec{v_1}| \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} |\vec{v_2}| \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix} = |\vec{v_1}||\vec{v_2}|(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = \underline{\underline{|\vec{v_1}||\vec{v_2}|\cos(\alpha - \beta)}}.$$

Be $\vec{v_1} \perp \vec{v_2} <=> \alpha = \beta + \frac{\pi}{2}$, w.l.o.g.

$=> |\vec{v_1}||\vec{v_2}|\cos(\alpha - \beta) = |\vec{v_1}||\vec{v_2}|\cos(\beta + \frac{\pi}{2} - \beta) = 0$

$\square$

Determinant

Let's choose sin() instead of cos() in the above equation:

$|\vec{v_1}||\vec{v_2}|\sin(\alpha - \beta) = |\vec{v_1}||\vec{v_2}|(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$

$$= |\vec{v_1}||\vec{v_2}| \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta \\ -\sin\beta \end{pmatrix} = \begin{pmatrix} |\vec{v_1}|\sin\alpha \\ |\vec{v_1}|\cos\alpha \end{pmatrix} \begin{pmatrix} |\vec{v_2}|\cos\beta \\ -|\vec{v_2}|\sin\beta \end{pmatrix} = \begin{pmatrix} y_1 \\ x_1 \end{pmatrix} \begin{pmatrix} x_2 \\ -y_2 \end{pmatrix}$$
$$= \det \begin{pmatrix} x_2 & x_1 \\ y_2 & y_1 \end{pmatrix} = \underline{\underline{\det \begin{pmatrix} \vec{v_2} & \vec{v_1} \end{pmatrix}}}.$$

If $\alpha = \beta + k * \pi$, w.l.o.g, $k \in \mathbb{N}$

$=> \det \begin{pmatrix} \vec{v_2} & \vec{v_1} \end{pmatrix} = |\vec{v_1}||\vec{v_2}|\sin(\alpha - \beta) = |\vec{v_1}||\vec{v_2}|\sin(\beta + k * \pi - \beta) = 0$

$=> \vec{v_1}$ and $\vec{v_2}$ are linear dependent and the determinant in general determines linear dependency.

Polynomial derivation

To proove: $(x^n)' = nx^{n-1}$

$$\frac{f(x+\triangle x)-f(x)}{\triangle x}$$

$$=\frac{(x+\triangle x)^n-x^n}{\triangle x}$$

$$=\frac{\sum_{k=0}^{n}\binom{n}{k}(x^{n-k}\triangle x^k)-x^n}{\triangle x}$$

$$=\frac{x^n\Delta x^0+\sum_{k=1}^{n}\binom{n}{k}(x^{n-k}\Delta x^k)-x^n}{\Delta x}$$

$$=\frac{\sum_{k=1}^{n}\binom{n}{k}(x^{n-k}\Delta x^k)}{\Delta x}$$

$$=\frac{\Delta x\sum_{k-1}^{n}\binom{n}{k}(x^{n-k}\Delta x^{k-1})}{\Delta x}$$

$$=\sum_{k-1}^{n}\binom{n}{k}(x^{n-k}\Delta x^{k-1})$$

$$=\sum_{k=1}^{n}\frac{n!}{k!(n-k)!}(x^{n-k}\Delta x^{k-1})$$

$$=nx^{n-1}+\sum_{k=2}^{n}\binom{n}{k}(x^{n-k}\Delta x^k)$$

$$\lim_{\Delta x\to 0}(nx^{n-1}+\sum_{k=2}^{n}\binom{n}{k}(x^{n-k}\Delta x^k))=nx^{n-1}$$

□

Binomial theorem

To proof: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} (x^{n-k} y^k)$

Induction:

$n = 0$:

$$(x+y)^0 = 1 = \sum_{k=0}^{0} \binom{0}{k} (x^{0-k} y^k)$$

$n \to n+1$:

$$(x+y)^{n+1} = (x+y)^n (x+y) \stackrel{\text{i.p.}}{=} \sum_{k=0}^{n} \binom{n}{k} (x^{n-k} y^k)(x+y)$$

$$= \sum_{k=0}^{n} \binom{n}{k} (x^{n+1-k} y^k) + \sum_{k=0}^{n} \binom{n}{k} (x^{n-k} y^{k+1})$$

$$= \sum_{k=0}^{n} \binom{n}{k} (x^{n+1-k} y^k) + \sum_{k=1}^{n+1} \binom{n}{k-1} (x^{n+1-k} y^k)$$