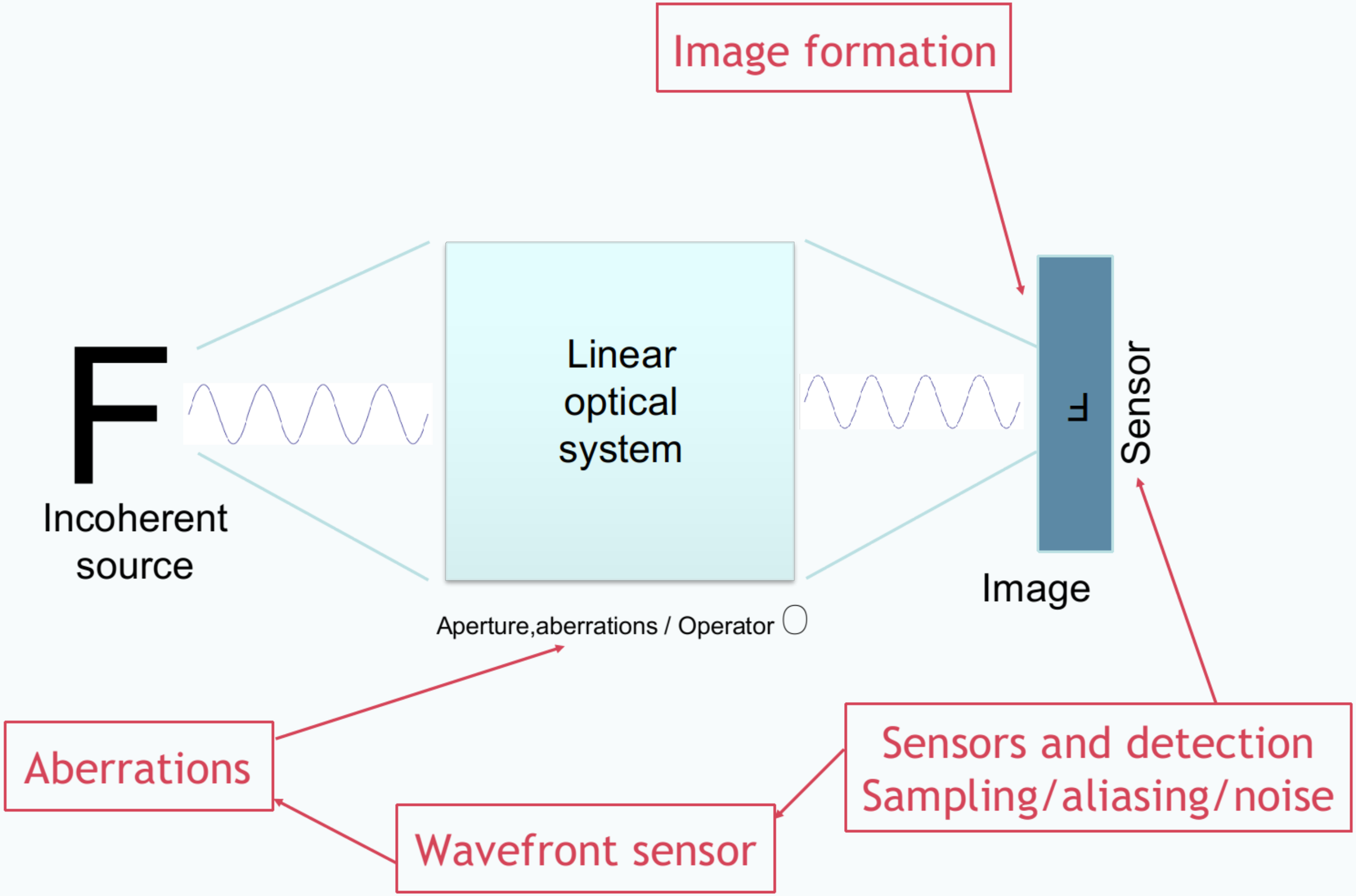
FOURIER OPTICS & FOURIER TRANSFORM

Prof François Rigaut
Research School of Astronomy & Astrophysics
The Australian National University

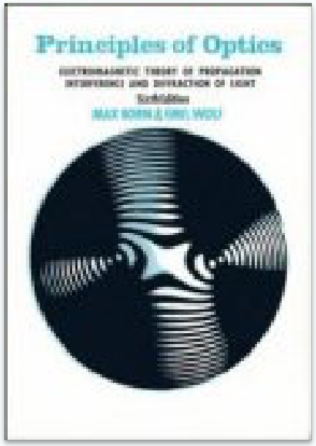
LINEAR OPTICAL SYSTEMS

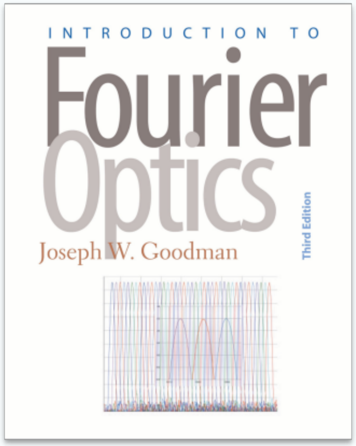


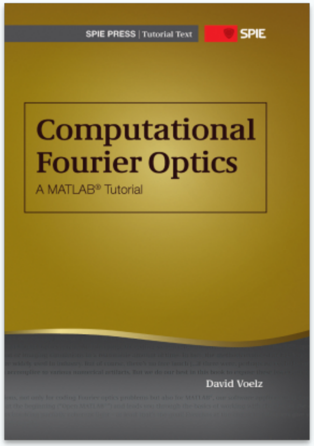
PREAMBLE

INTRODUCTION: SOURCES

- "**Introduction to Fourier Optics and coherence**", J.-M.Mariotti, in Diffraction limited imaging with very large telescope, editors D.M.Alloin and J.-M. Mariotti, 1988 (JMM),
- "**Introduction to Fourier Optics**", Joseph Goodman, 2004 (JG)
- "**Fundamental of Photonics**", B.E.A Saleh & M.C.Teich, 1991, mostly Chapter 4 (S&T),
- "**Principles of Optics**", Max Born and Emil Wolf, 1980 (B&W),
- "**Computational Fourier Optics: A MATLAB Tutorial**", David Voelz
- The web, wikipedia.







HISTORY OF FOURIER OPTICS

- 1660: First observation of diffraction by **Grimaldi**
- 1678: **Huygens** “Traité de la lumière” (published 1690): first wave theory of light. Require finite speed of light.
- 1803: **Thomas Young** two slits interferences experiment.
- 1818: **Fresnel** produces the first theory of diffraction.
- 1822: **Fourier** introduces his transform
- 1850-1950: **Krichhoff**, **Sommerfeld**, then Quantum Mechanics bring a firm mathematical foundation to the theory.

Grimaldi


Huygens


Fresnel


Fourier

- Of course **Newton** was involved too !

VALIDITY AND LIMIT CONDITIONS

- Previous lectures from PHYS3057 were **1D optics, coherent** (waveguides, lasers). The next lectures with me will be on **2D optics, incoherent sources** (imaging).
- We will consider **light as a scalar field** (B&W 8.4)
- We'll be focusing (pun intended) on **Fraunhofer diffraction**
- **Diffraction occurs with all waves**, including sound, water, electromagnetic (X through radio), elementary particles.
- We'll **browse through the maths**, it is there for reference and those who'd like to dig deeper

NEXT TWO LECTURES IN ONE SLIDE

- Basic understanding of the **Fourier transform** and its properties, **sampling and aliasing** issues
- In Fourier Optics, light is described by a scalar field $\Psi = A\exp^{i\varphi}$
- In **Fraunhofer diffraction**, the **far and near field** complex amplitudes are linked by a Fourier transform $\Psi(P) = \mathcal{F}(\Psi(M))$
- An optical system can be characterised by its **impulse function** H. The impulse function is $H = |\mathcal{F}\left(\Psi(x)\right)|^2$
- Object O and image $\mathcal{I}$ are linked by the relation $\mathcal{I} = O * H$
- The Optical Transfer Function of a system characterises its spatial frequencies filtering properties $\mathrm{OTF} = \mathcal{F}(H) = \Psi * \Psi^{\star}$

INTRODUCTION TO MODAL EXPANSION

MODAL EXPANSION

$$f = \sum_i a_i \mu_i$$

where:

- f is a discrete function/object
- μ_i are modes that you are going to use to represent
- a_i are the coefficients

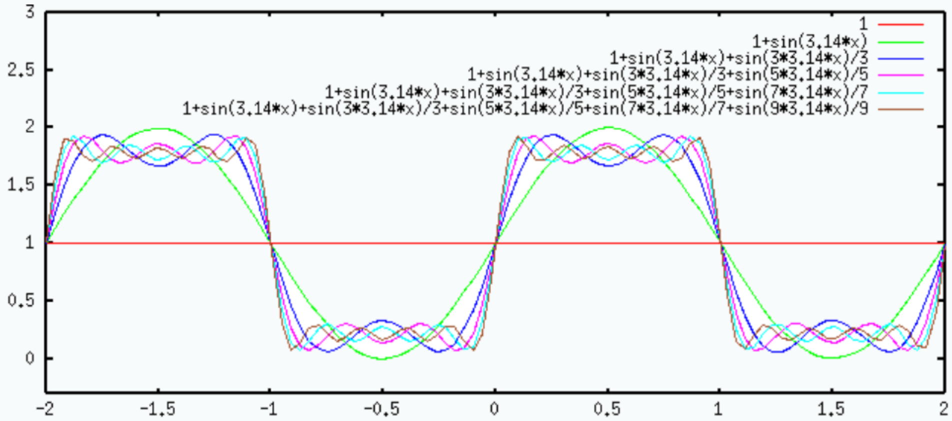
MODAL EXPANSION

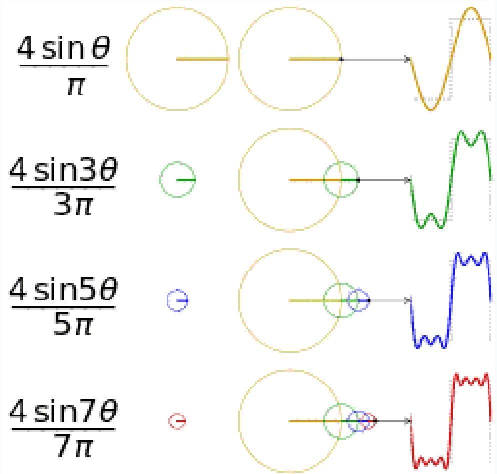
- The nature world is **continuous**
- Once measured, a **signal is discrete**.
  - Volt versus time
  - Elevation map
  - Image
- Chose the **modal basis** adapted to your problem.
- Goal is to try to **reduce the number of parameter** to describe function, and make use of convenient properties of this description
- Examples:
  - An optical phase using Zernike modes $\varphi = \sum_i a_i Z_i$
  - Finite Element Model analysis
  - Eigenvalues engenmodes
- Cyclic signals are naturally described by expanding on **sines and cosines** $\rightarrow f = \sum_i a_i \cos(i\theta) + b_i \sin(i\theta)$

THE FOURIER TRANSFORM

WHAT IS THE FOURIER TRANSFORM?

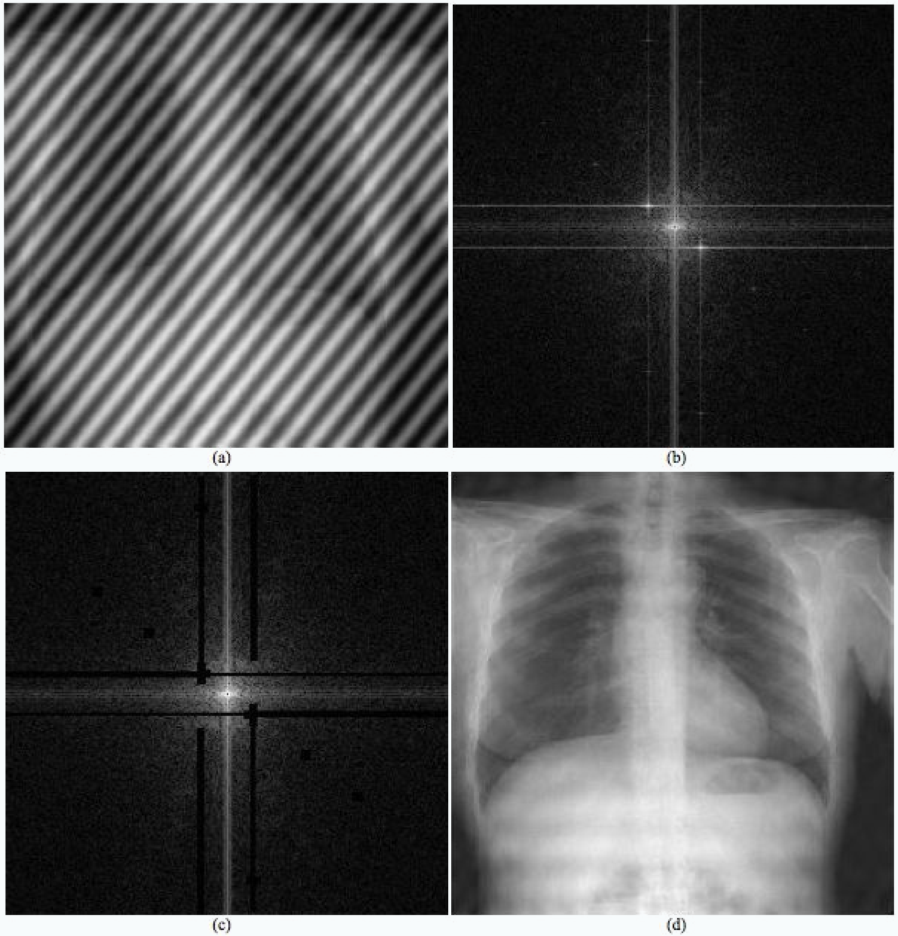
- The Fourier transform of a signal tells you **what frequencies are present in your signal and in what proportions**

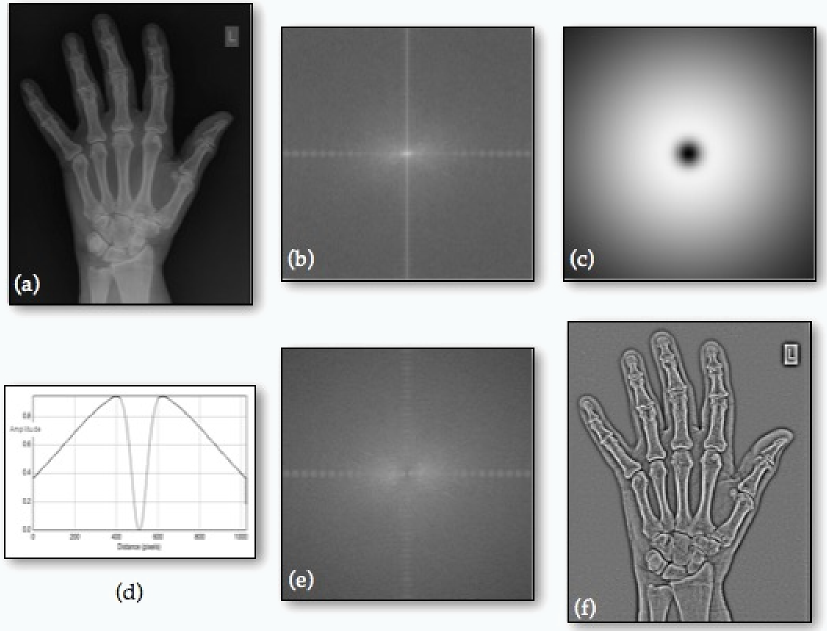




- IMHO, the **most useful mathematical tool for engineers and applied physicists**.
- It is used to:
  - **Characterise signals** (1D/2D..) and linear systems. Electronics, optics, acoustics, mechanics, civil engineering, etc, etc...
  - **Digitally process data/signals** (filtering, convolving, correlating, etc) in all above disciplines

FOURIER FILTERING





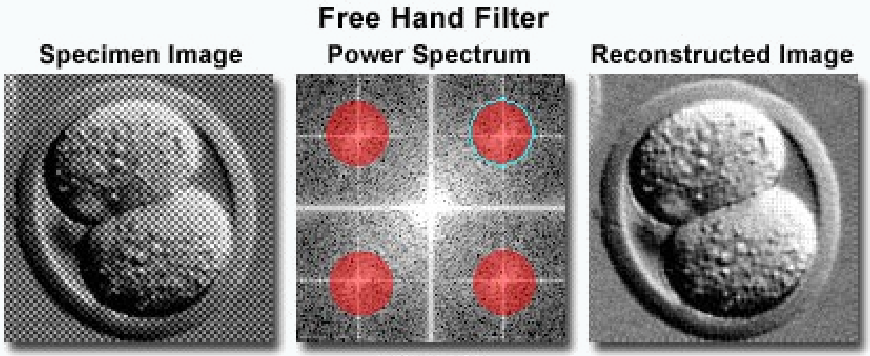


Figure 1

FOURIER TRANSFORM: DEFINITIONS

- We note $\hat{f}$ the **Fourier transform** of f

$$\hat{f}(u) = \int_{-\infty}^{+\infty} f(x) \exp^{-i2\pi ux} \, dx$$

- The **inverse** Fourier transform is $f(x) = \int_{-\infty}^{+\infty} \hat{f}(u) \exp^{+i2\pi ux} \, du$
- We will also use the **Fourier operator** $\mathcal{F}$: $\hat{f}(u) = \mathcal{F}[f(x)]$
- The Fourier transform is **cyclic**: $\mathcal{F}^{-1}[\mathcal{F}[f(x)]] = f(x)$
- To have a Fourier transform, a function must
  - Be absolutely integrable

$$\left| \int_{-\infty}^{+\infty} f(x) dx \right| < \infty$$

  - Not have any infinite discontinuity
  - Have only a finite number of discontinuities or extrema in any finite interval

FOURIER PAIRS

Function	Fourier Pair
$\exp(-\pi x^2)$	$\exp(-\pi u^2)$
$\text{sinc}(x)$	$\Pi(u)$
$\text{sinc}^2(x)$	$\Lambda(u)$
$\delta(x)$	1
$\text{III}(x)$	$\text{III}(u)$
$\sin(\pi x)$	$\frac{i}{2}\delta(u+\frac{1}{2}) - \frac{i}{2}\delta(u-\frac{1}{2})$

PROPERTIES

Property	Expression
Linearity	if $h(x) = af(x) + bg(x)$ then $\hat{h}(u) = a\hat{f}(u) + b\hat{g}(u)$
Similarity	$\mathcal{F}[f(ax)] = \frac{1}{\lvert a \rvert}\hat{f}\left(\frac{u}{a}\right)$
Shift	$\mathcal{F}[f(x-a)] = e^{-i2\pi au}\hat{f}(u)$
Convolution	$\mathcal{F}[f(x) * g(x)] = \mathcal{F}[f(x)] \times \mathcal{F}[g(x)] = \hat{f}(u) \times \hat{g}(u)$
Autocorrelation	$\mathcal{F}[f(x) * f(x)] = \lvert\hat{f}(u)\rvert^2$
Parseval	$\int_{-\infty}^{+\infty} f(x) \times g^*(x)dx = \int_{-\infty}^{+\infty} \hat{f}(u) \times \hat{g}^*(u)du$
Power	$\int_{-\infty}^{+\infty} \lvert f(x)\rvert^2 dx = \int_{-\infty}^{+\infty} \lvert\hat{f}(u)\rvert^2 du$
Derivative	$\mathcal{F}\left[\frac{d}{dx}f(x)\right] = i2\pi u\hat{f}(u)$

2D FT, DFT, FFT, PSD

- Acronyms:
  - FT: Fourier Transform
  - DFT: Discrete FT
  - FFT: Fast FT
  - PSD: Power Spectral Density (modulus square)

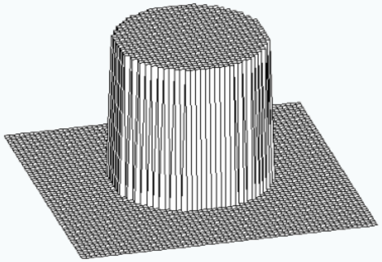
2D FOURIER TRANSFORM

Forward $$\hat{f}(u,v) = \iint_{-\infty}^{+\infty} f(x,y) e^{-i2\pi(ux+vy)} dxdy$$

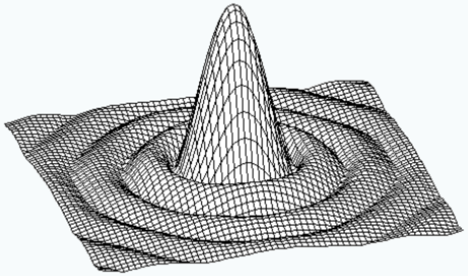
Reverse $$f(x,y) = \iint_{-\infty}^{+\infty} \hat{f}(u,v) e^{+i2\pi(ux+vy)} dudv$$

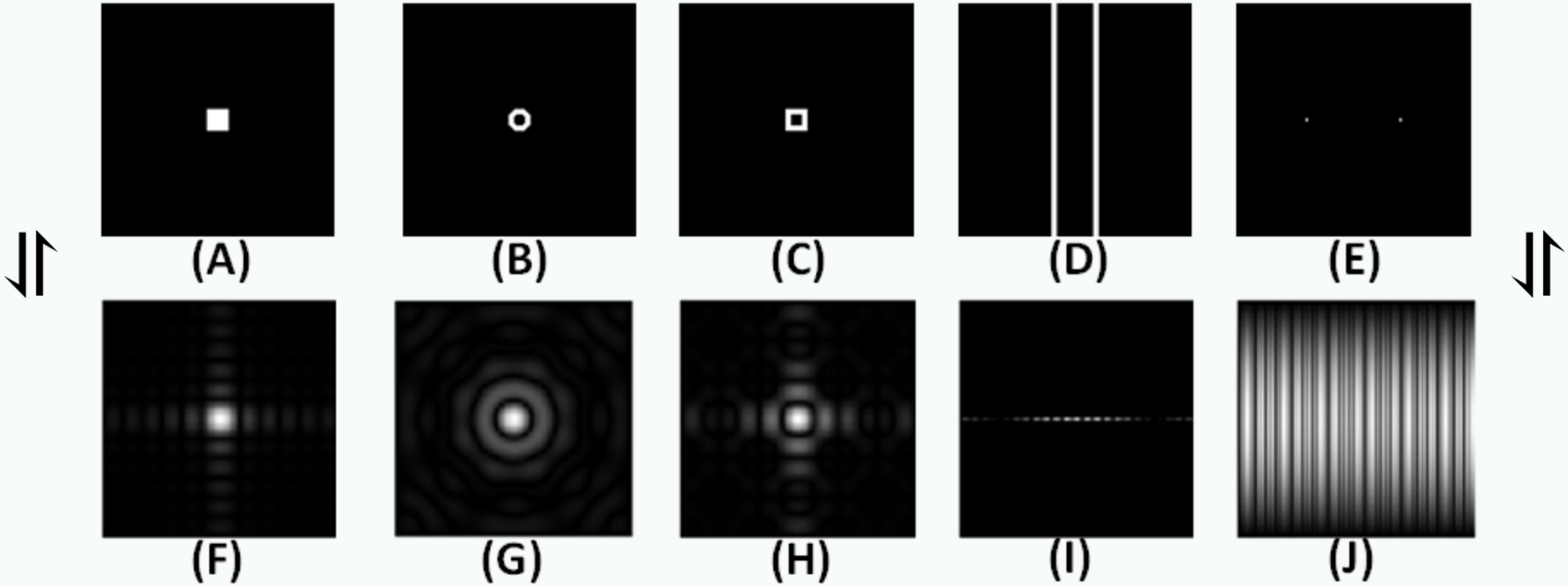
- Note that if f can be factorised (convenient) $f(x,y) = g(x).h(y)$ then $\hat{f}(u,x) = \hat{g}(u) \times \hat{h}(v)$
  - (but if $f(x,y) = g(r).h(\theta)$ the problem is more complicated ...)
- All other theorems apply as in 1D (linearity, similarity, power, etc)

SOME 2D FOURIER PAIRS



$$\Pi(\frac{r}{2a}) \rightleftharpoons \frac{a\ J_1(2\pi a\rho)}{\rho}$$





DISCRETE FT AND FAST FT

- The Fourier transform can be modified for **discrete datasets**, which is extremely useful to represent and analyse **sampled physical signals**. The discrete Fourier transform (DFT) is:

$$\hat{f}(\nu) = \frac{1}{N}\sum_{\tau=0}^{N-1} f(\tau) e^{-i2\pi\nu\tau/N}$$
$$f(\tau) = \sum_{\nu=0}^{N-1} \hat{f}(\nu) e^{+i2\pi\nu\tau/N}$$

- Both τ and ν are discrete variables. Both functions consist of sequences of N samples. All the basic theorems for the FT also apply to the DFT.
- The **Fast Fourier Transform (FFT)** is a DFT that uses a smart algorithm to drastically reduce the number of operations, from N^2 **down to** $N\log(N)$

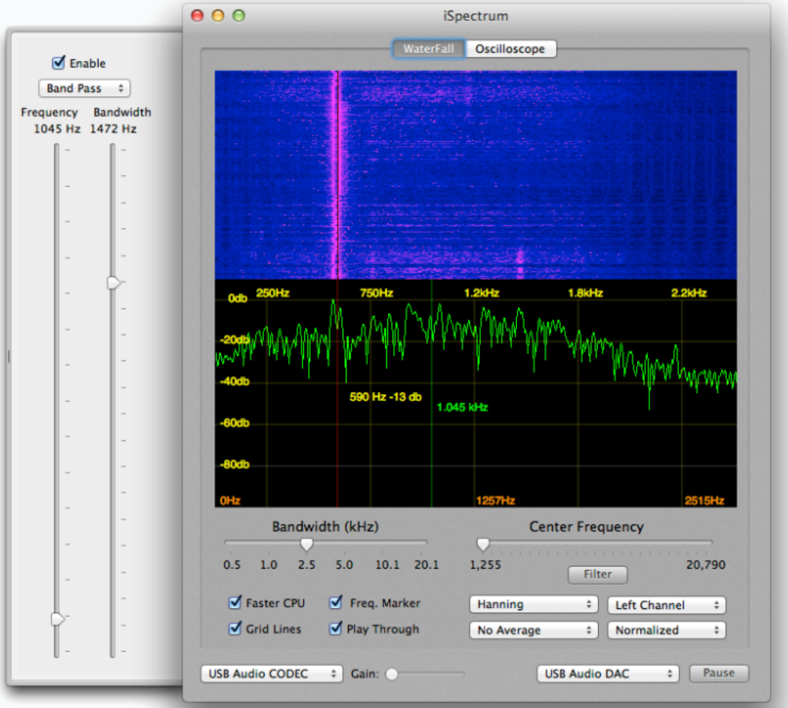
THE POWER SPECTRAL DENSITY (PSD)

- The square modulus of the Fourier transform of a signal

$$\mathrm{PSD}(f) = |\mathcal{F}(f(x))|^2$$

- PSD is insensitive to the phase of the input signal.
  - you get the power (intensity) per frequency bin over the frequency range 0 to cut off frequency
- In a DFT, assuming:
  - the units of x are seconds (s),
  - and the units of f, say, Volts (V)
  - then the PSD units are V^2/Hz.

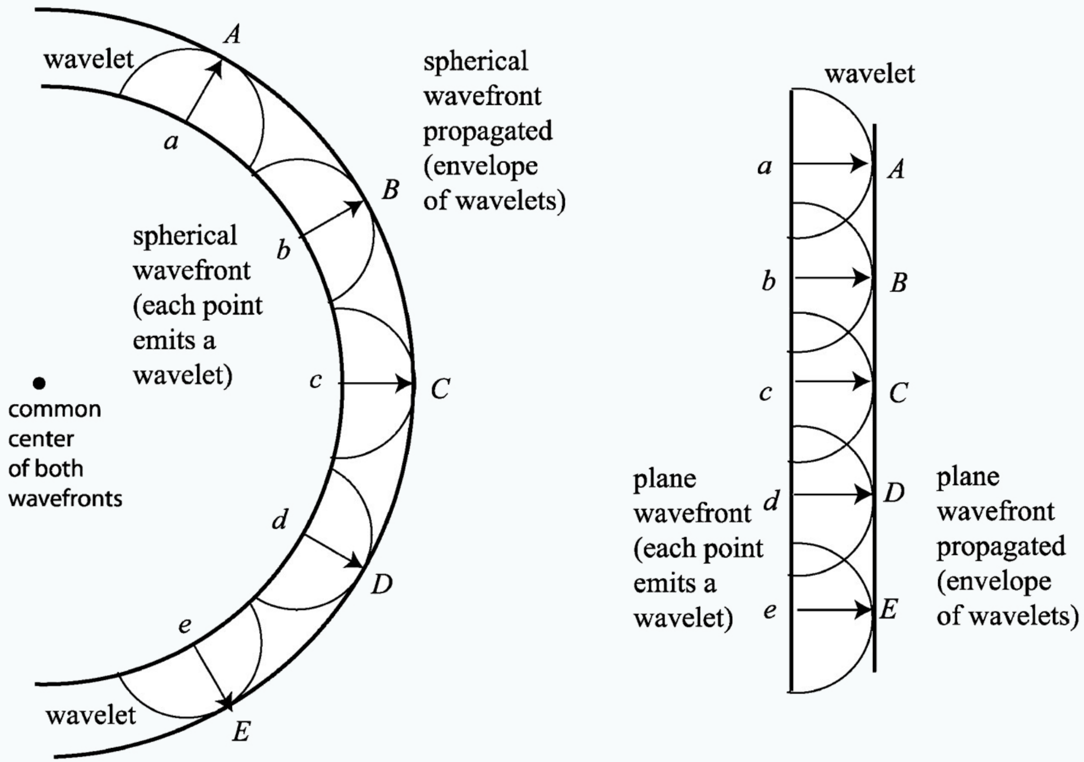
See Spectrum Density Analyser



DIFFRACTION THEORY

HUYGENS PRINCIPLE

- *"Every point on a wavefront may be considered a source of secondary spherical wavelets which spread out in the forward direction at the speed of light. The new wavefront is the tangential surface to all of these secondary wavelets."*



(NON) DERIVATION OF THE DIFFRACTED FIELD

- Fresnel, KrichHoff and Sommerfeld, within others, have worked out the math. It's messy, and requires a lot of approximations.
- Applying the Huygens principle and working out the field propagation from the point P_0 through the aperture W in plan M (near field), to the final plan P (far field), it can be demonstrated that the **field in P is the simple Fourier Transform of the field in M**:

$$\Psi(P) = \mathcal{F}(\Psi(M))$$

WAVEFRONT, PSF, OTF

THE IMPULSE FUNCTION

- Recalling the field in P: $\Psi(P) = \mathcal{F}(\Psi(M))$
- At visible wavelengths, it is extremely difficult to measure the complex field itself (for quantum noise reasons) - but we can measure the field intensity (irradiance), the square of the complex field. H is the image of a point, the impulse function, also called the **Point Spread Function (PSF)**:

$$H = \Psi(P).\Psi^*(P) = \mathcal{F}(\Psi(M)).\mathcal{F}^*(\Psi(M)) = |\mathcal{F}(\Psi(M))|^2$$

Remember that $\Psi = Ae^{i\varphi}$? So, in absence of aberrations ($\varphi \equiv 0$), we simply have:

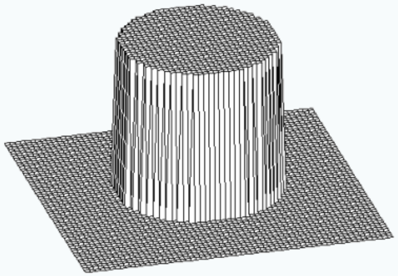
$$H = |\mathcal{F}(A)|^2$$

THE IMPULSE FUNCTION, CIRCULAR APERTURE

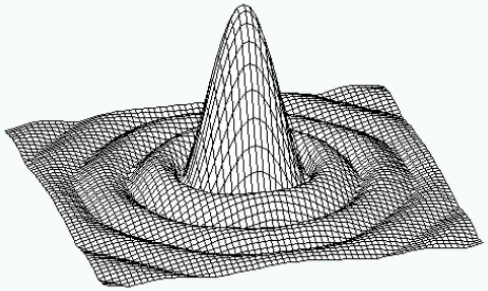
For a circular aperture:

$$\Psi(M) = \Psi(r,\theta) = \Pi\left(\frac{r}{2a}\right) = \begin{cases} 1 & \text{if } r \leq a \\ 0 & \text{if } r > a \end{cases}$$

$$H = |\mathcal{F}(\Psi(M))|^2 = |\mathcal{F}(\Psi(\Pi(r/2a)))|^2 = \left[a\frac{J_1(2\pi a\rho)}{\rho}\right]^2$$

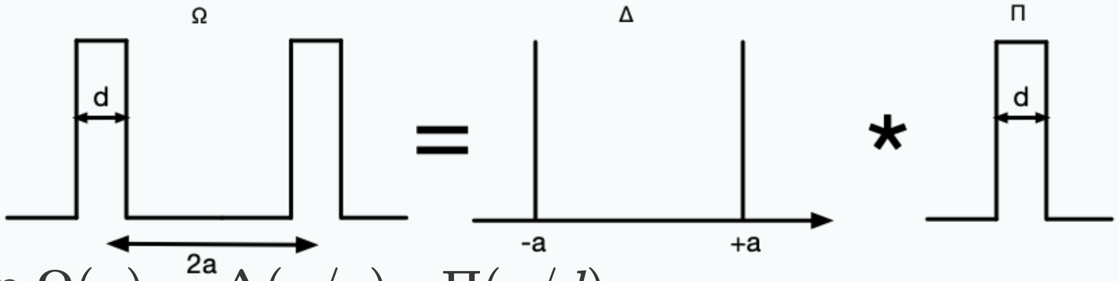


$$\Pi(\frac{r}{2a}) \rightleftharpoons \frac{a\ J_1(2\pi a\rho)}{\rho}$$



AN APPLICATION: YOUNG FRINGES

- Armed with this new mathematical description of diffraction, it is now trivial to find, e.g., the expression of the Young fringes.
- The slits can be described as a convolution:

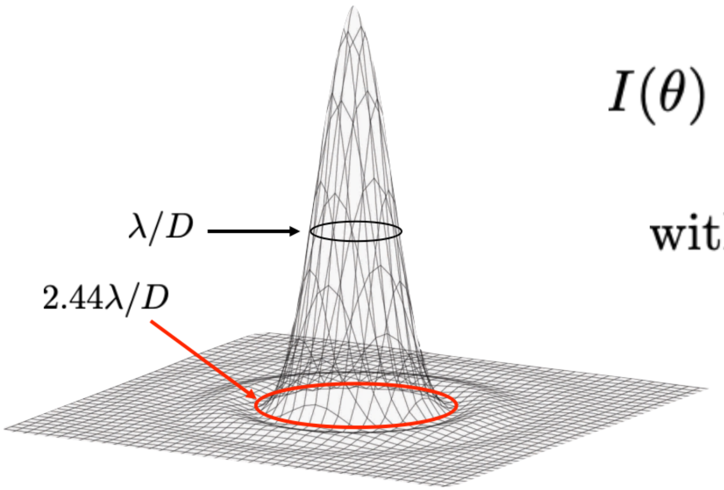


- The near field can be written $\Omega(x) = \Delta(x/a) * \Pi(x/d)$
- The far field is $\hat{\Omega}(u) = \mathcal{F}(\Delta(x/a) * \Pi(x/d)) = \mathcal{F}(\Delta(x/a)) \times \mathcal{F}(\Pi(x/d))$
- $\Delta(x/a) = \delta(x-a) + \delta(x+a)$ hence $\mathcal{F}(\Delta(x/a)) = e^{-i2\pi au} + e^{+i2\pi au} = 2\cos(2\pi au)$
- and $\mathcal{F}(\Pi(x)) = \mathrm{sinc}(u)$ hence $\mathcal{F}(\Pi(x/d)) = \mathrm{sinc}(ud)$

Thus $\hat{\Omega}(u) = 2\cos(2\pi au) \times \mathrm{sinc}(ud)$ and the intensity (measured)

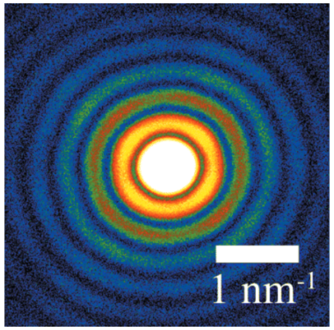
$$|\hat{\Omega}(u)|^2 = 4\cos^2(2\pi au) \times \mathrm{sinc}^2(ud)$$

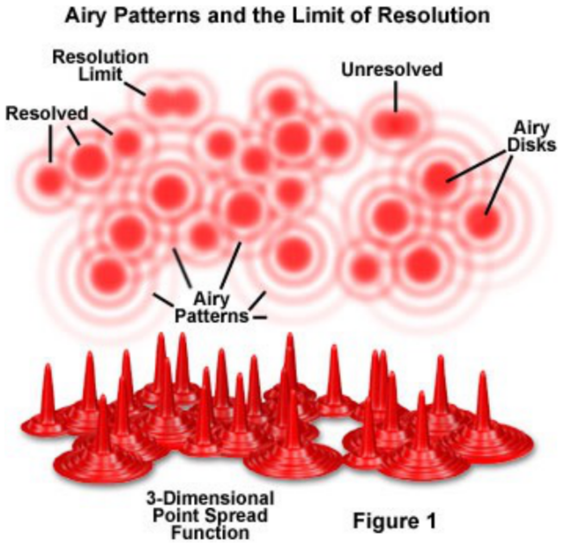
AIRY PATTERN, CIRCULAR APERTURE PSF

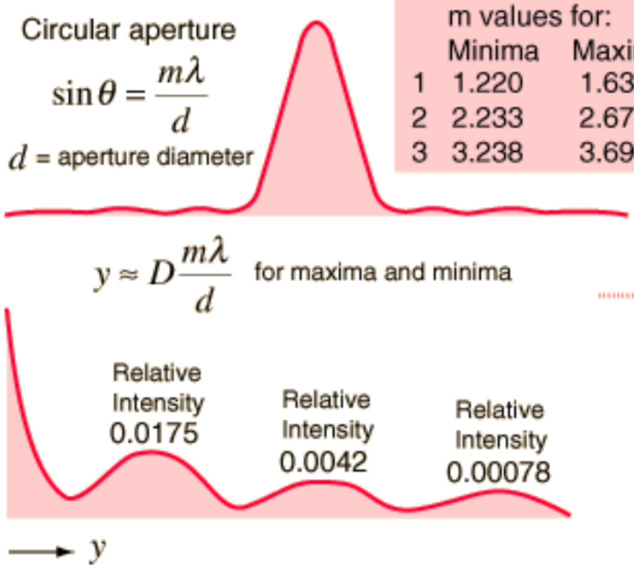


$$I(\theta) = I_0 \left(2 \frac{J_1(ka\sin(\theta))}{ka\sin(\theta))} \right)^2$$

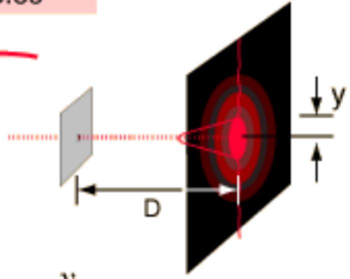
with $k = 2\pi/\lambda$ and $D = 2a$







m values for:	Minima	Maxima
1	1.220	1.635
2	2.233	2.679
3	3.238	3.69



$$\frac{y}{D} = \tan\theta \approx \sin\theta \approx \theta$$

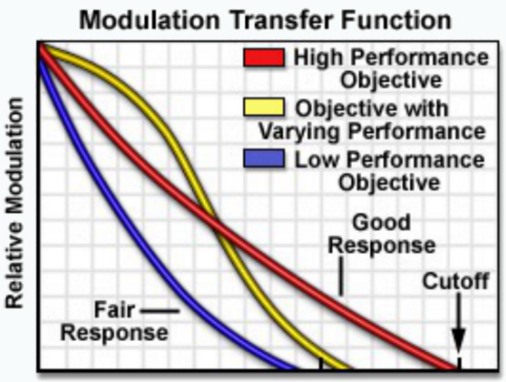
for small angles θ

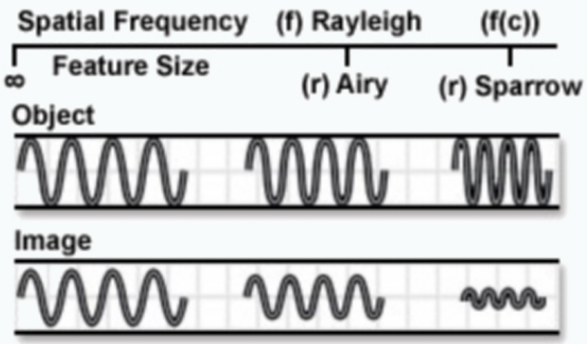
THE OPTICAL TRANSFER FUNCTION

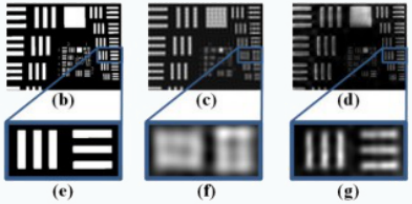
- Also called generically **Modulation Transfer Function (MTF)**

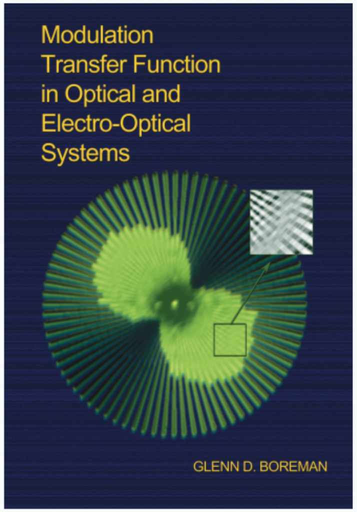
$$\mathrm{OTF} = |\mathcal{F}(H)| = |\Psi * \Psi^*|$$

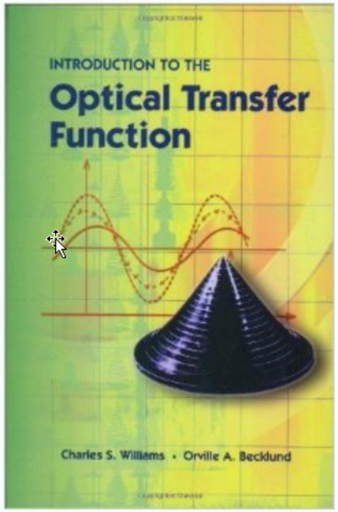
- Characterises the filtering properties of an optical system, including cut-off frequency
- For a circular aperture, the cut-off frequency is $f_c = D/\lambda$
- People have written books about it…





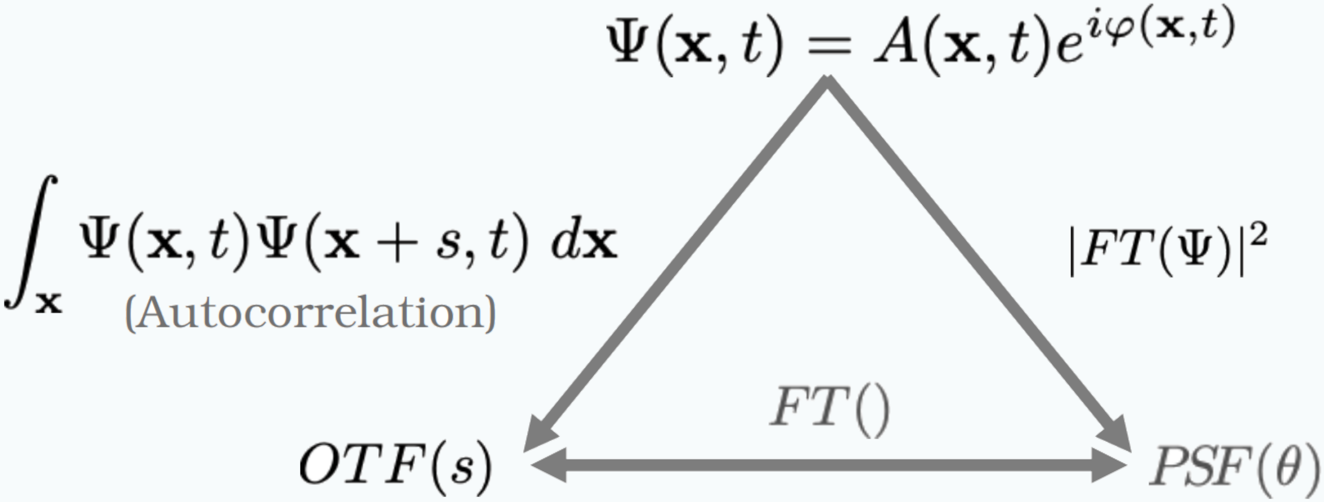






WAVEFRONT, PSF & OTF ARE LINKED

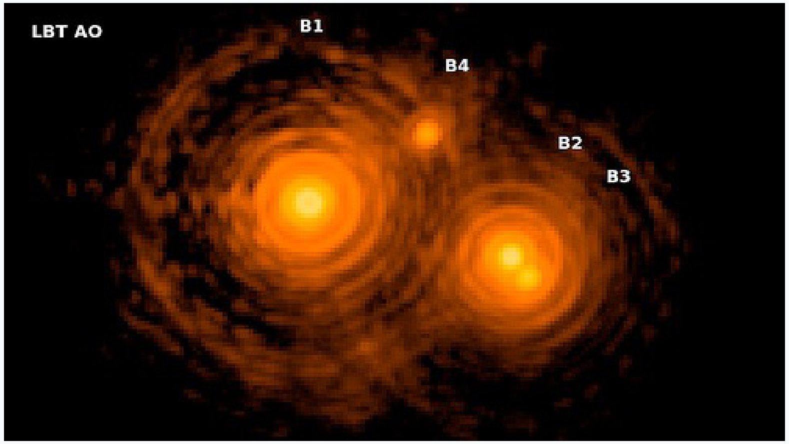
- The wavefront is $\Psi(x,y,t) = A(x,y,t)\exp(i\varphi(x,y,t))$
  - Ψ is the complex field defined by its amplitude and phase
  - A is the amplitude (e.g. pupil function)
  - φ is the phase
- The Optical Transfer Function (or MTF) is the spatial frequency response of the system.
- **Wavefront, PSF and OTF are linked**:



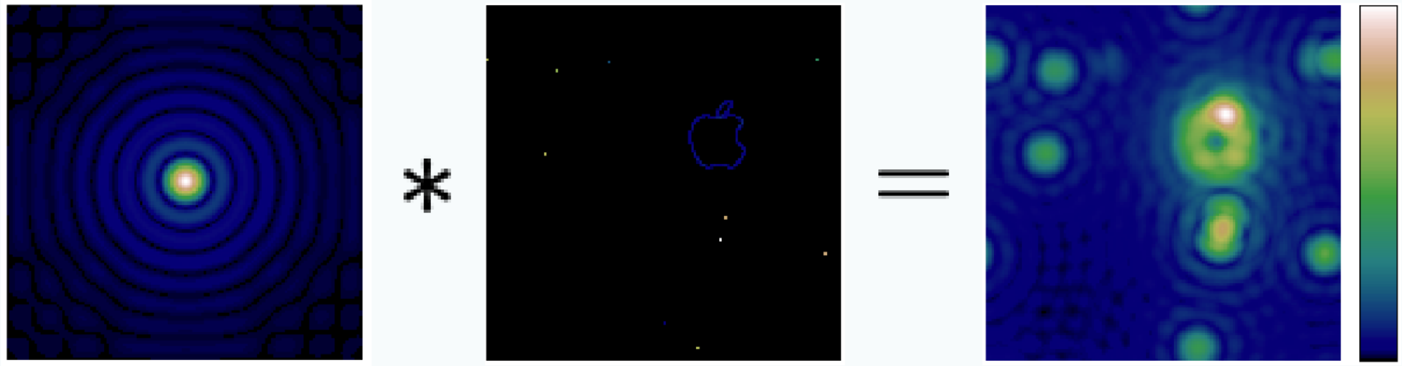
INTERFEROMETRY TO IMAGING ...AND BACK

IMAGE FORMATION FOR INCOHERENT SOURCES

- An object O can be decomposed into an infinite number of dirac function. In the case of an incoherent object (most objects in everyday's life, astronomical objects, medicine,etc), these points **do not interfere**, thus the resulting image is the convolution of the object and the impulse response (PSF)



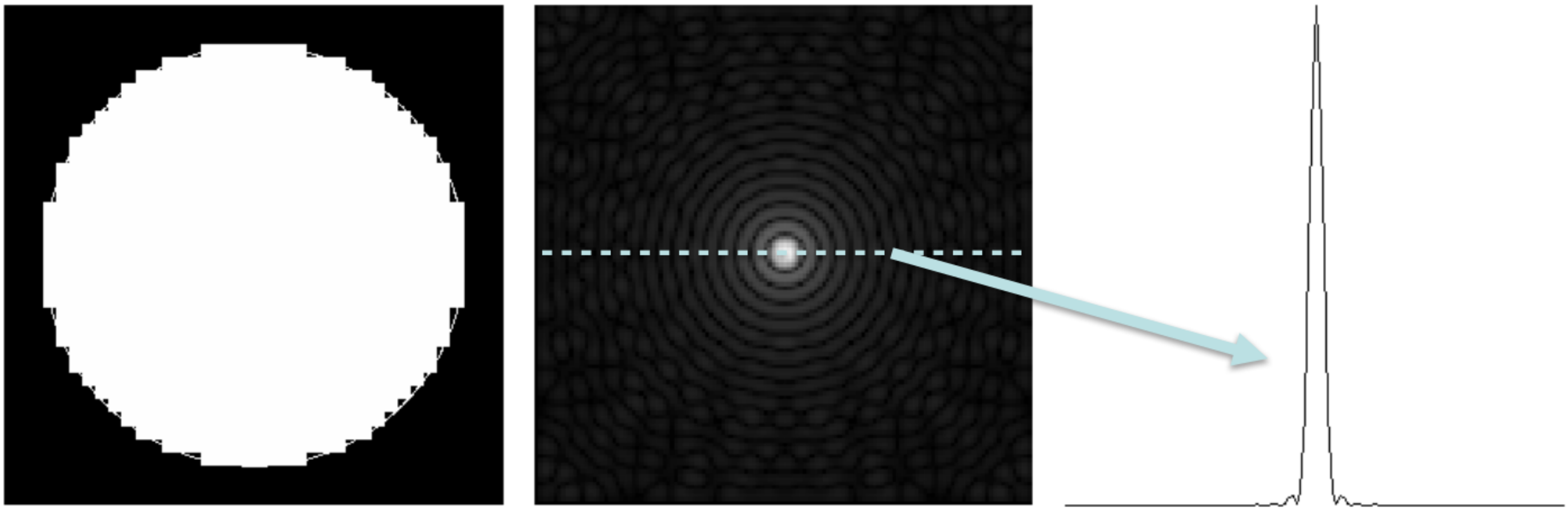
$$\mathcal{I} = O * H$$



Note that this assumes invariance of PSF with position in the field of view.

INTERFEROMETRY TO IMAGING...

- From "slit" to full aperture



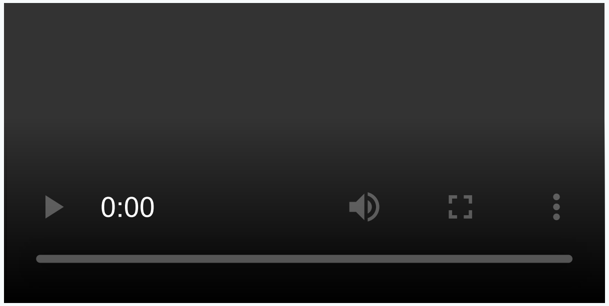
Near field
= Aperture
= Pupil

Far field
= Focal plane Image

Image cross section

INTERFEROMETRY TO IMAGING...

- From "slit" to full aperture

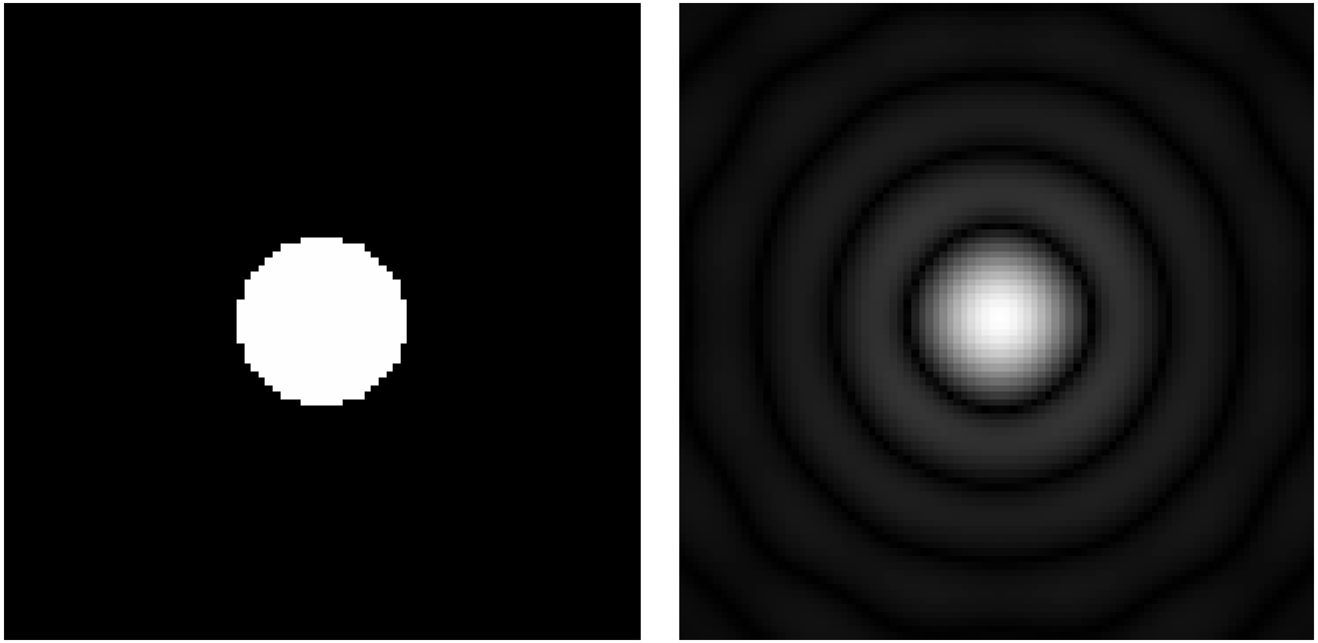


Near field
= Aperture
= Pupil

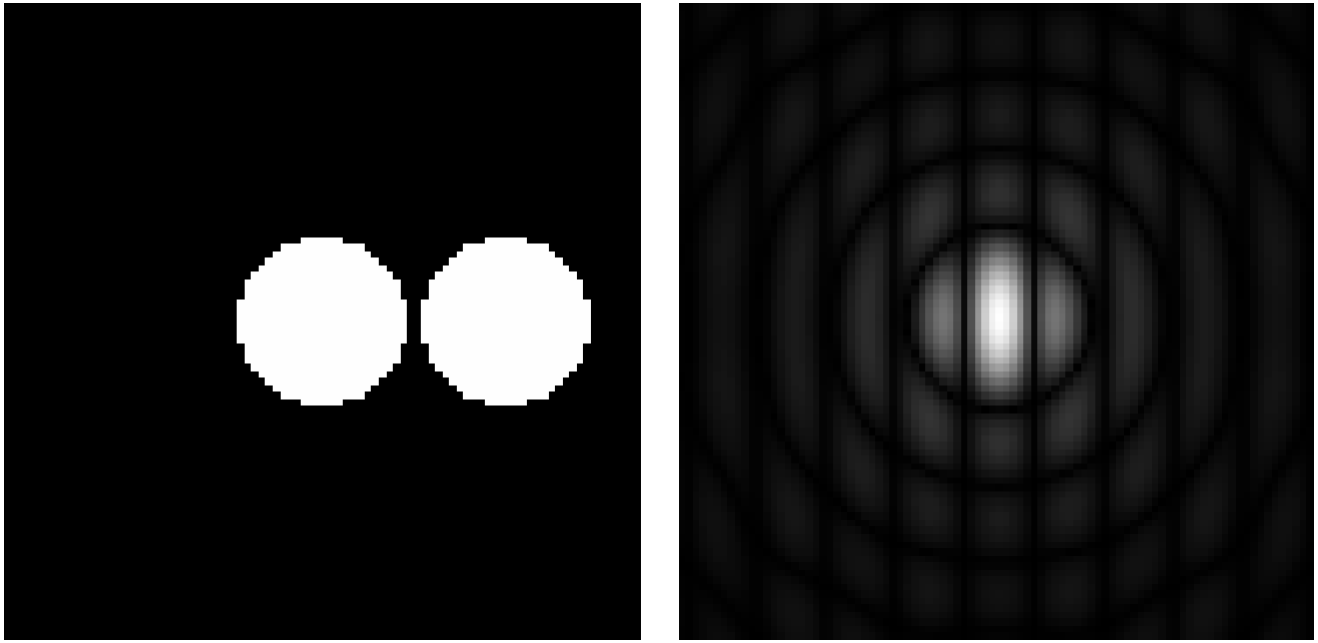
Far field
= Focal plane Image

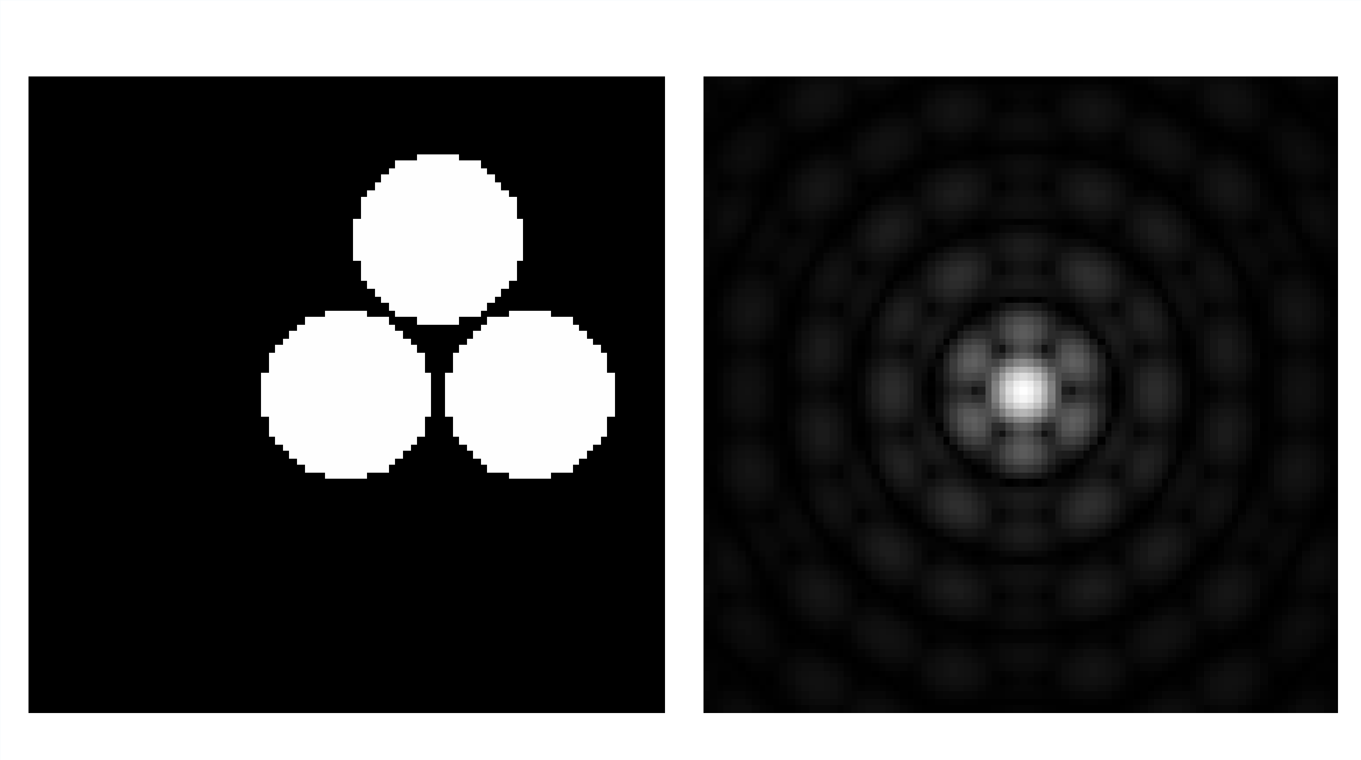
Image cross section

… AND IMAGING TO INTERFEROMETRY

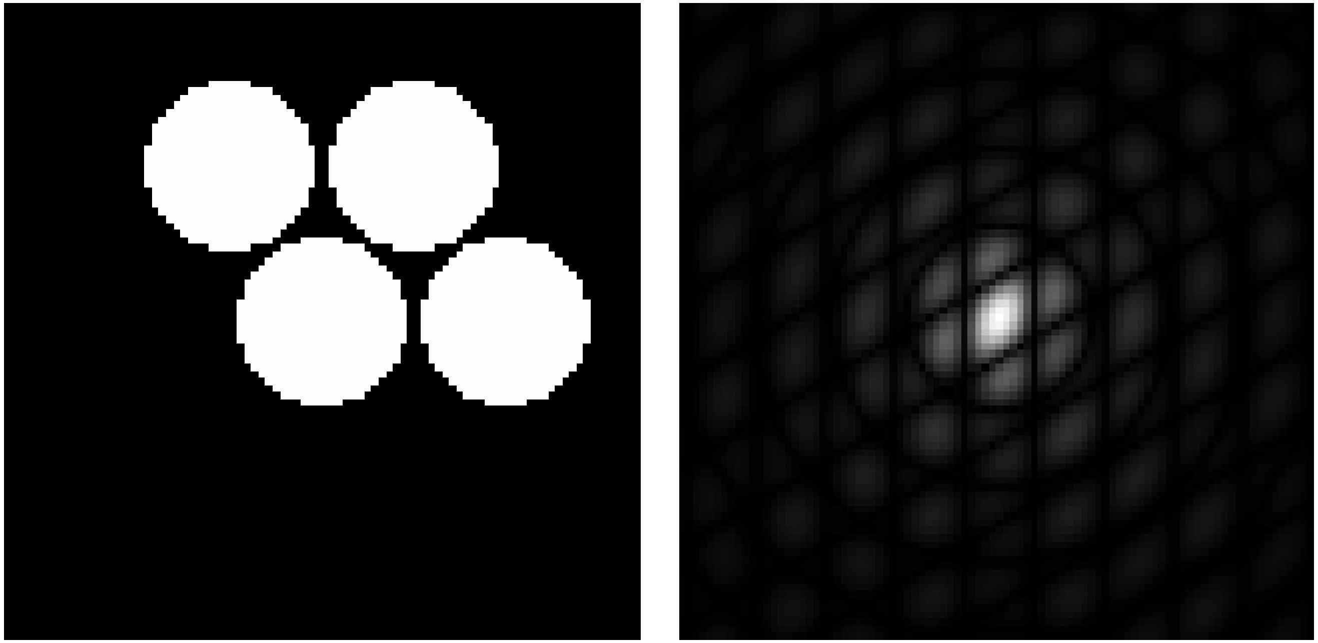


... AND IMAGING TO INTERFEROMETRY

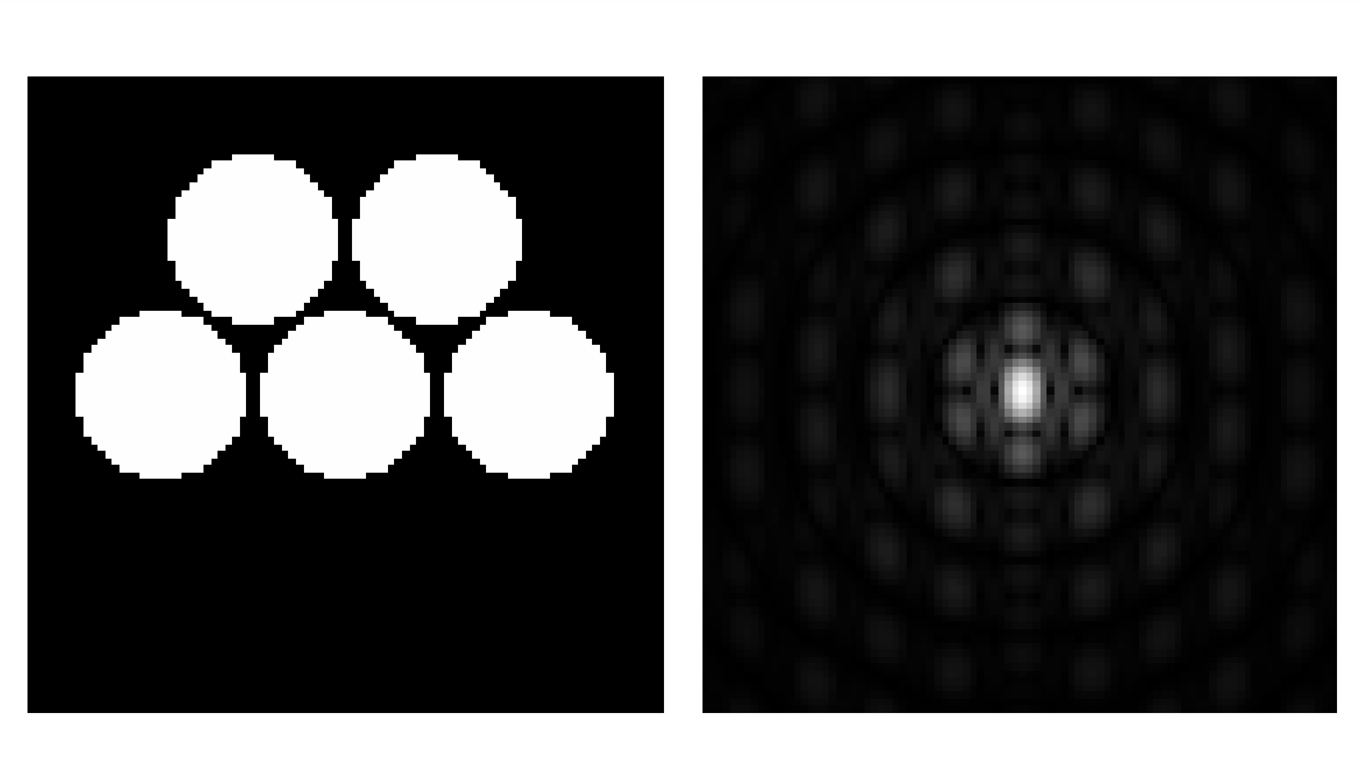




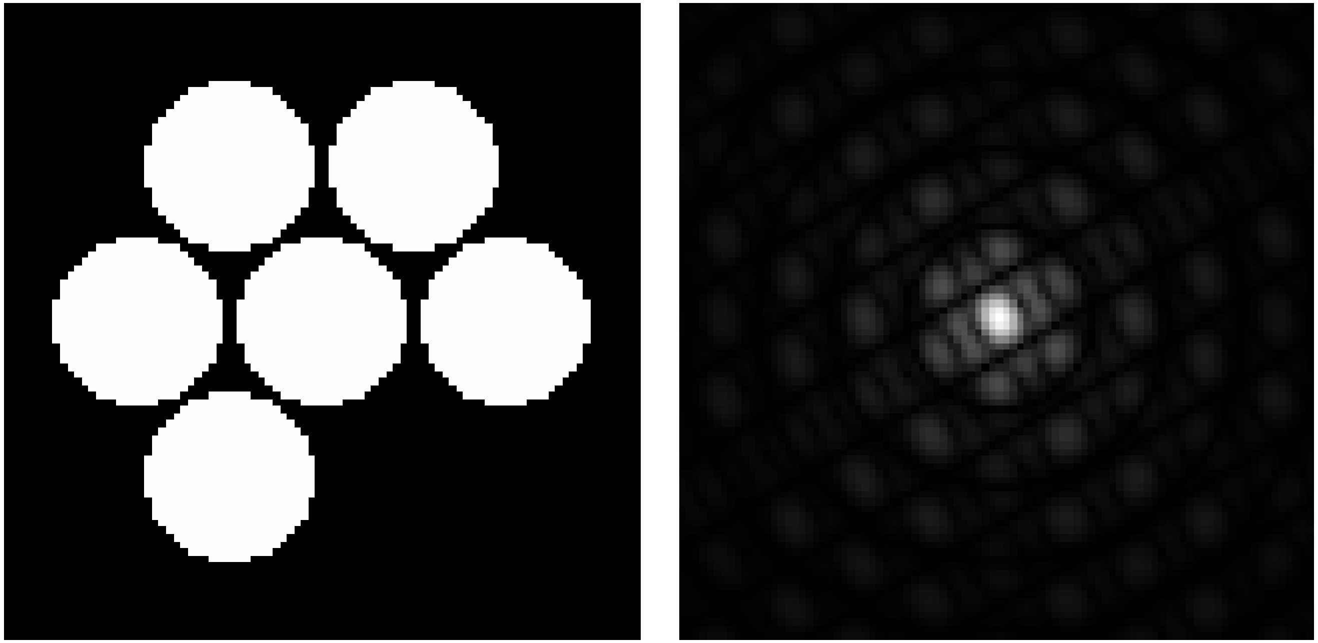
… AND IMAGING TO INTERFEROMETRY



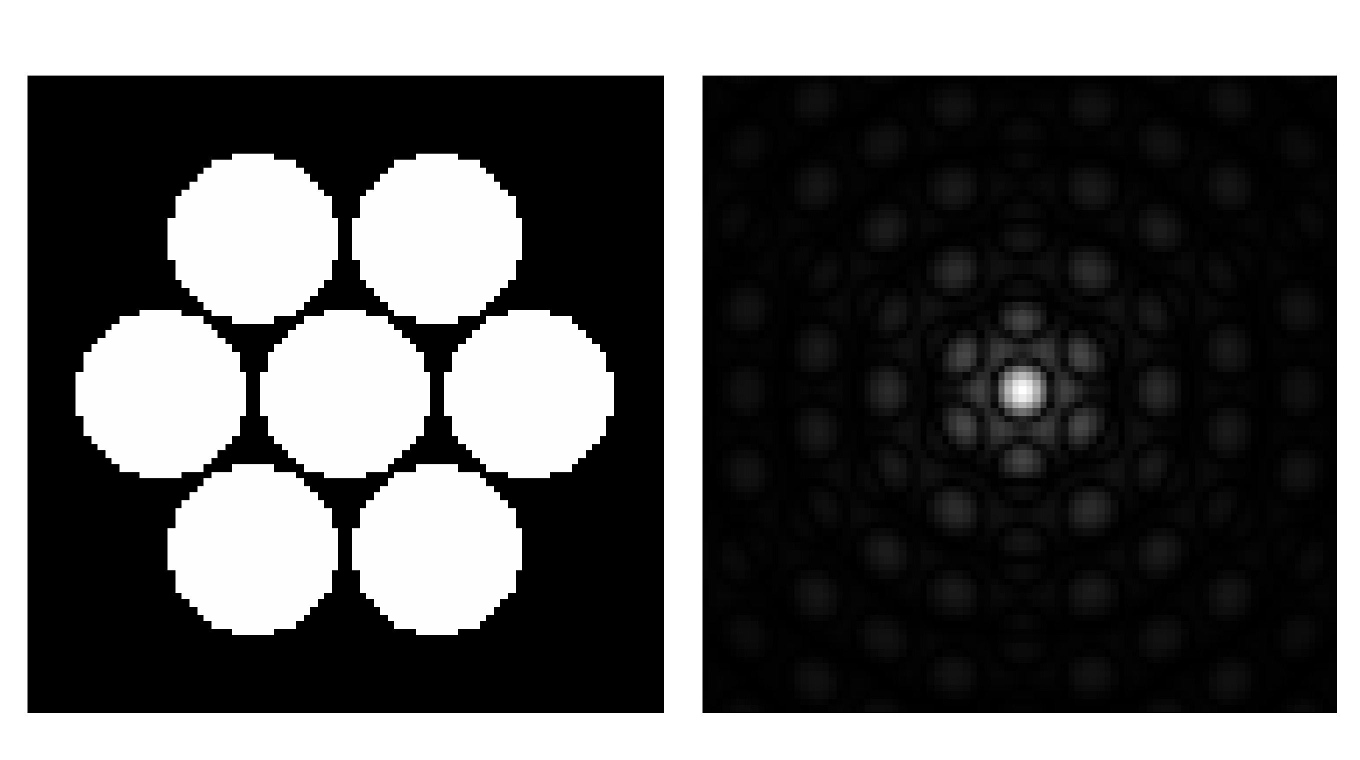
… AND IMAGING TO INTERFEROMETRY



… AND IMAGING TO INTERFEROMETRY



… AND IMAGING TO INTERFEROMETRY



ELEMENTS OF SAMPLING THEORY

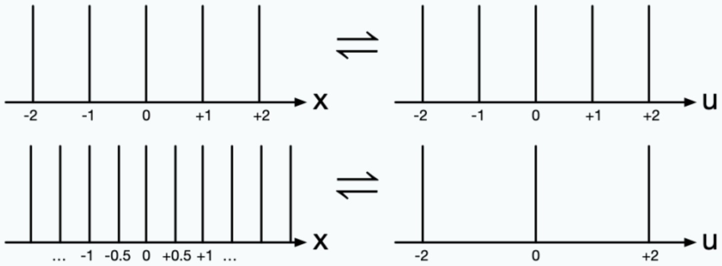
SAMPLING & ALIASING (SHANNON/NYQUIST)

> *If a continuous, band-limited function $f(x)$ contains no frequency component higher than f_c, then it can be fully specified by a set of samples at frequency of $2 \times f_c$ or larger.*

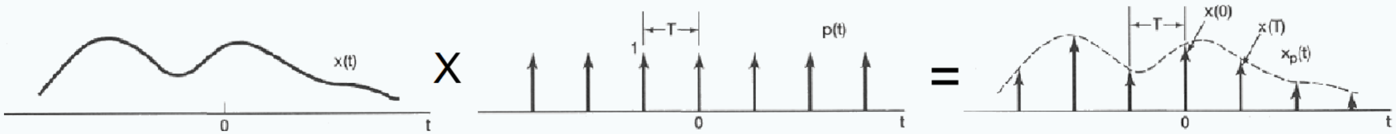
THE SHAH FUNCTION $\text{III}(X)$

1. $\text{III}(x)$ is its own Fourier transform

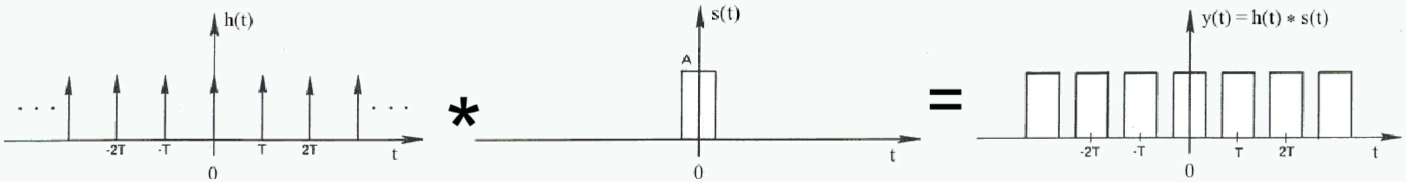
Similarity theorem $\mathcal{F}\left[f(ax)\right] = \frac{1}{|a|}\hat{f}\left(\frac{u}{a}\right) \Rightarrow$



2. The act of sampling is taking value at discrete points $\equiv$ Multiplication by III

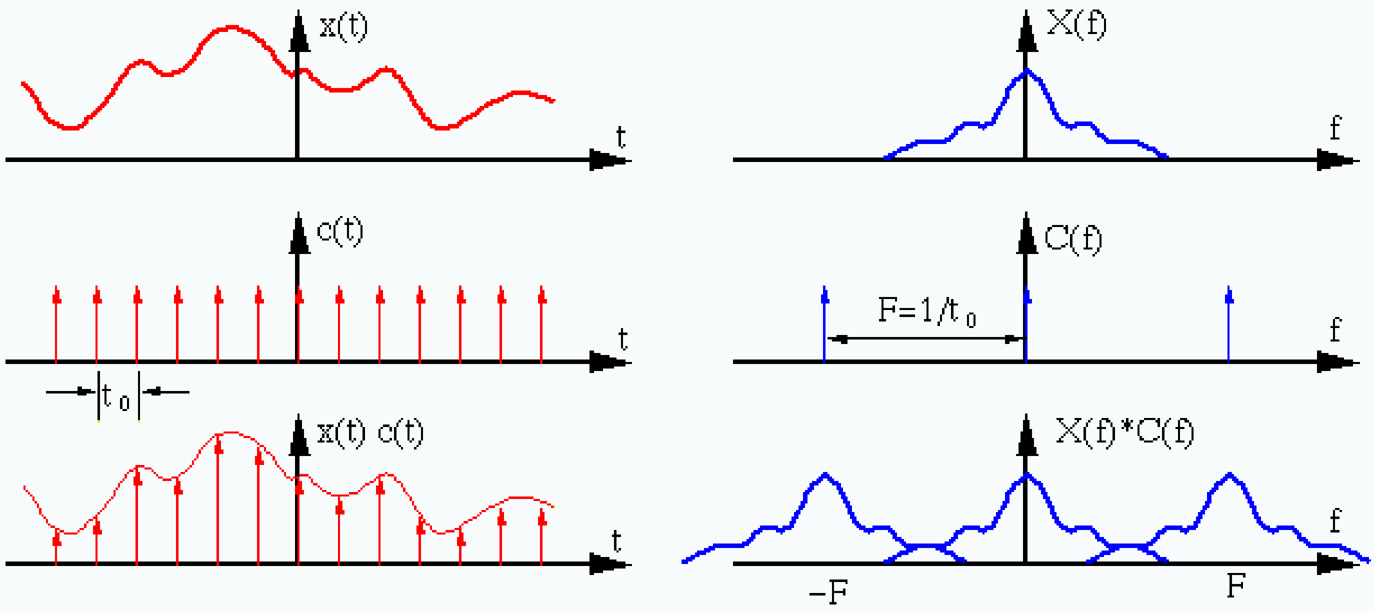


3. Convolution by III is equivalent to creating an infinite number of shifted replicas of the original functions



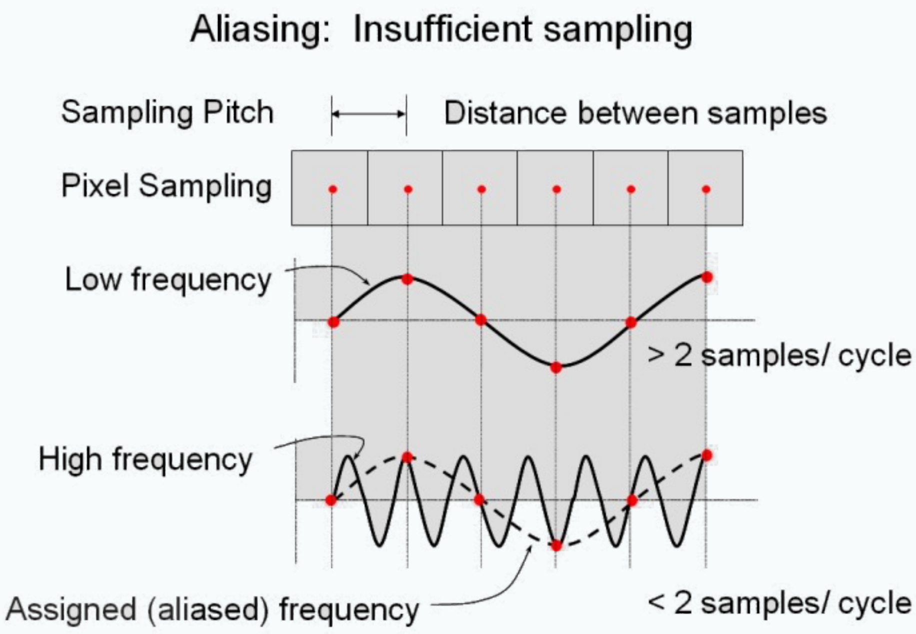
ALIASING

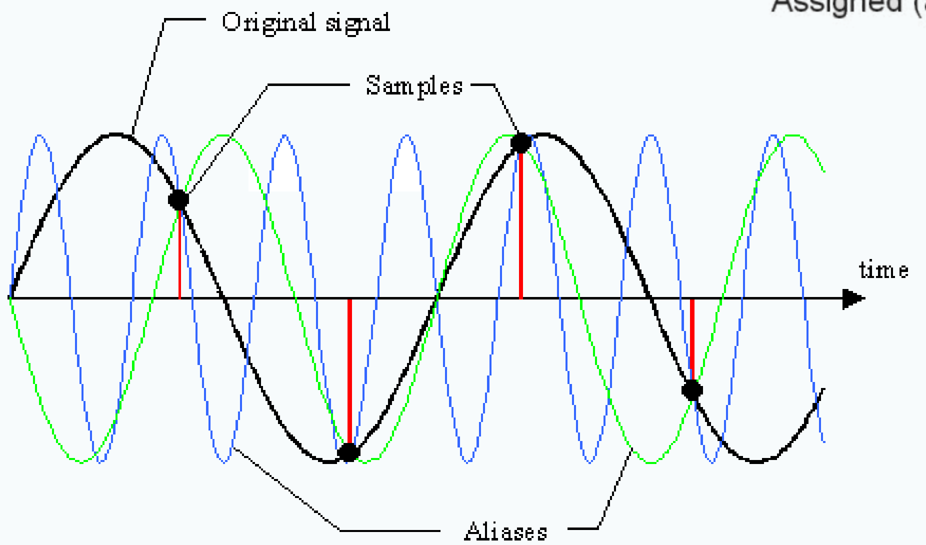
Fourier view: In the DFT/FFT space, the function spectrum is replicated at intervals $2 \times f_c$. If the spectrum spills over $\pm f_c$, then the spectrum replicas will overlap, resulting in a mixed signal (original lost).

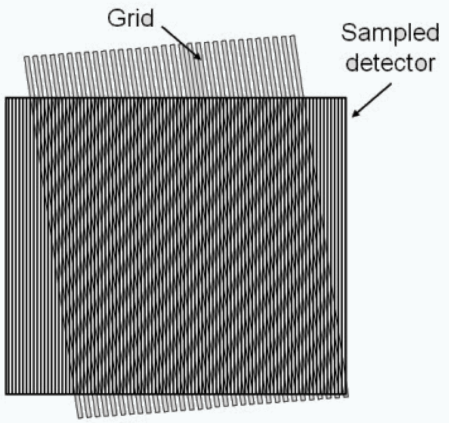


ALIASING

- Can be spatial, temporal, angular, etc
- Can be solved/mitigated by pre-filtering the signal before sampling







PROOF OF THE SAMPLING THEOREM

From $f(x)$ we obtain the sampled $f_s(x)$ with sampling interval τ by:

$$f_s(x) = \text{III}\left(\frac{x}{\tau}\right).f(x)$$

In the Fourier domain:

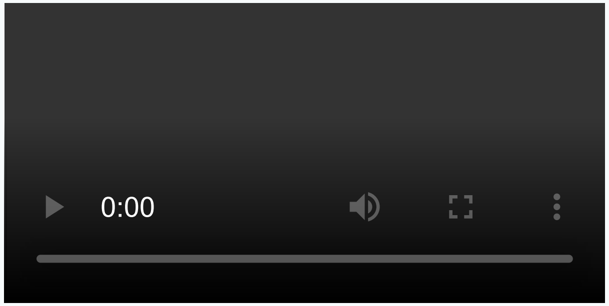
$$\hat{f}_s(u) = \tau\, \text{III}(\tau u) * \hat{f}(u) = \tau \sum_{-\infty}^{+\infty} \hat{f}\left(u - \frac{n}{\tau}\right)$$

The spectrum of the sampled function consists of an infinite sum of replicas of $\hat{f}(u)$. If $\tau^{-1} < 2f_c$, the replicas are separated by distances larger than their width and do not overlap (if not, they do and it creates in aliasing). Hence the information on $\hat{f}(u)$ and thus on $f(x)$ is preserved if the sampling condition $\tau \leq 1/(2f_c)$ is met. We can retrieve the original spectrum by multiplying $\hat{f}(u)$ by a rectangle function (gate) $\Pi(\tau u)$ in order to eliminate all replicas but one:

$$\left[\tau\, \text{III}(\tau u) * \hat{f}(u)\right] \times \Pi(\tau u) = \hat{f}(u)$$

which yields by inverse Fourier transform

$$[\text{III}(x/\tau).f(x)] * \tau^{-1}\text{sinc}(x/\tau) = f_s(x) * \tau^{-1}\text{III}(x/\tau) = f(x)$$

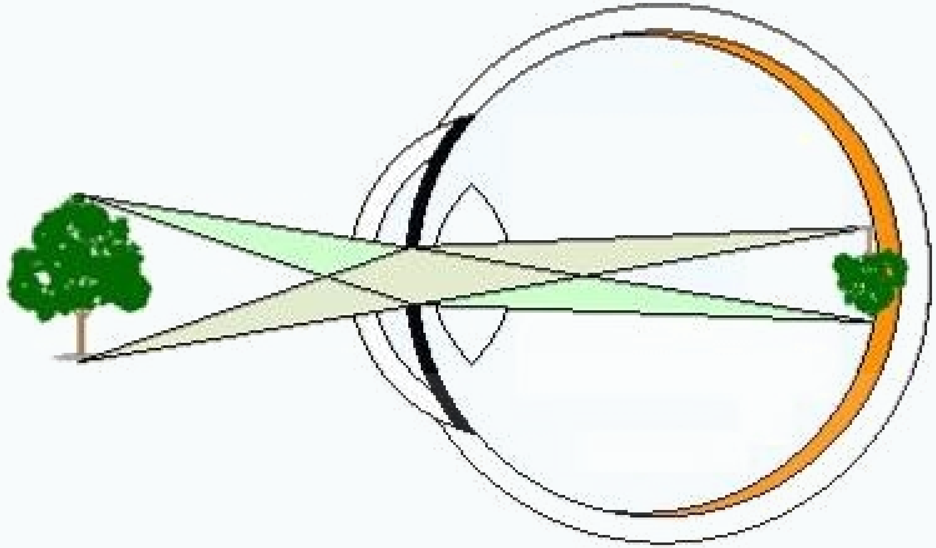


MAIN POINTS OF PAST TWO LECTURES

- Basic understanding of the **Fourier transform** and its properties, **sampling and aliasing** issues
- In Fourier Optics, light is described by a scalar field $\Psi = A \exp^{i\varphi}$
- In **Fraunhofer diffraction**, the **far and near field** complex amplitudes are linked by a Fourier transform $\Psi(P) = \mathcal{F}(\Psi(M))$
- An optical system can be characterised by its **impulse function** H. The impulse function is $H = |\mathcal{F}\left(\Psi(x)\right)|^2$
- Object O and image $\mathcal{I}$ are linked by the relation $\mathcal{I} = O * H$
- The Optical Transfer Function of a system characterises its spatial frequencies filtering properties $\mathrm{OTF} = \mathcal{F}(H) = \Psi * \Psi^\star$

IMAGE METRICS, ABERATIONS

PRACTICAL OPTICAL SYSTEMS



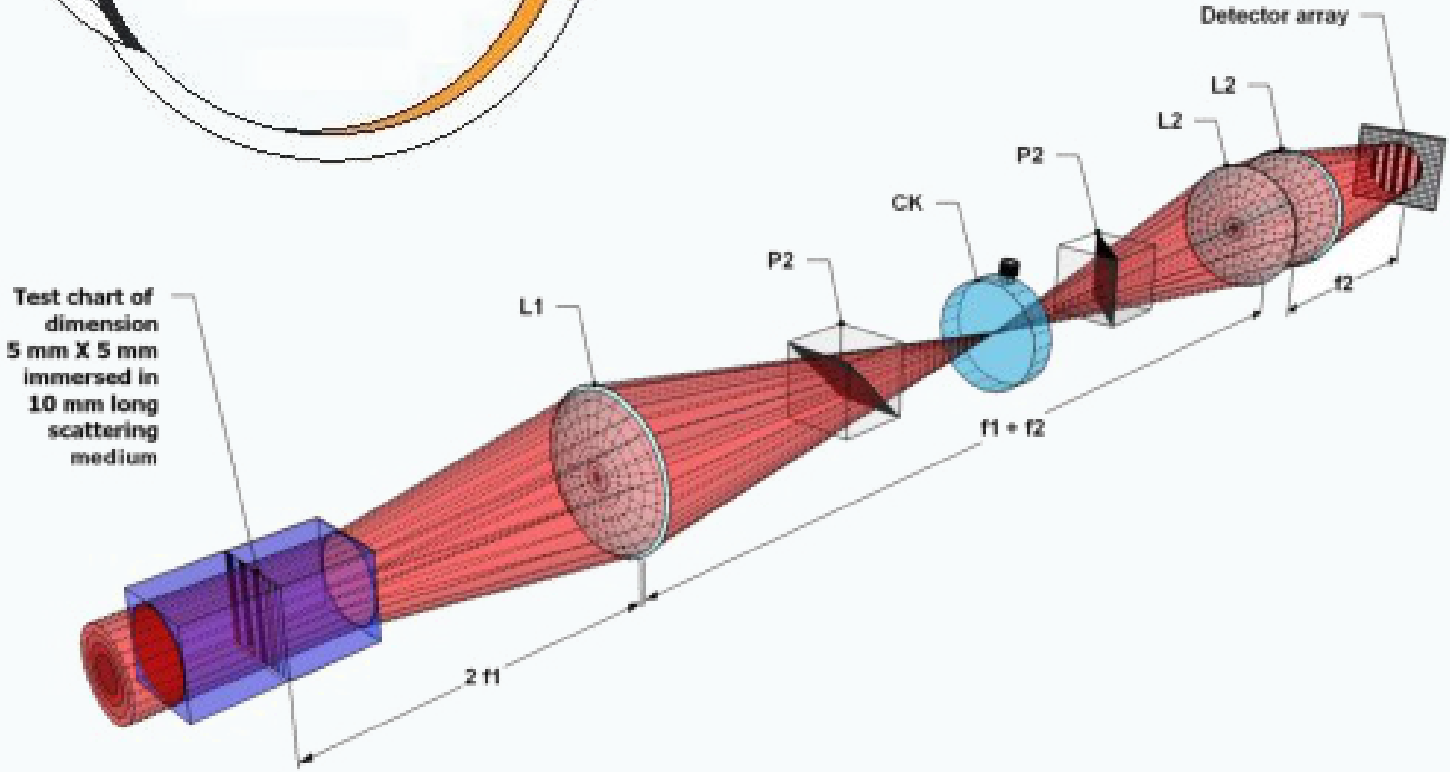
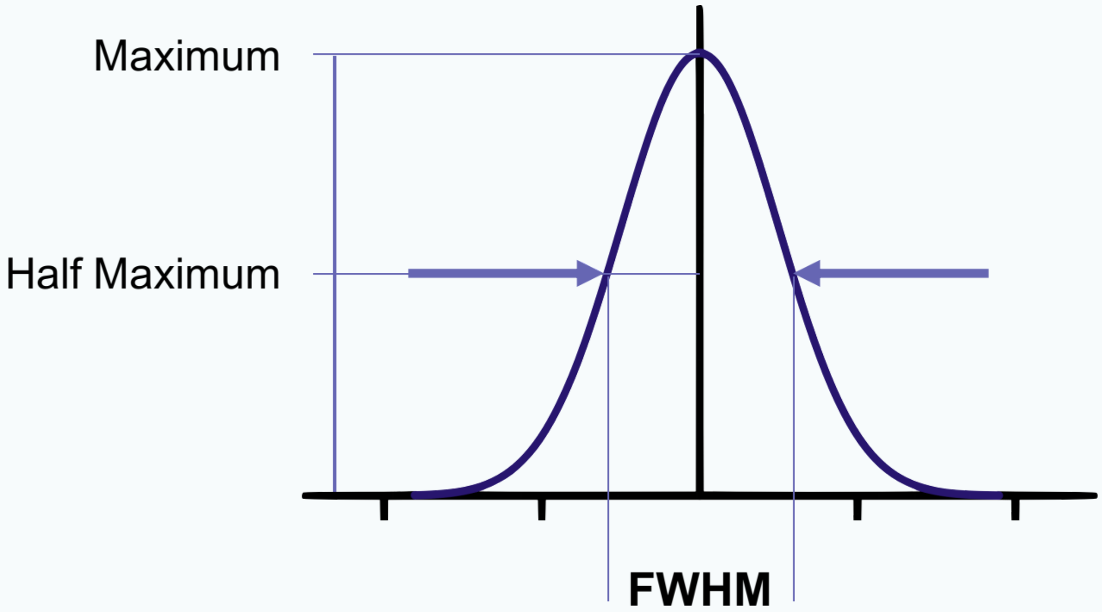
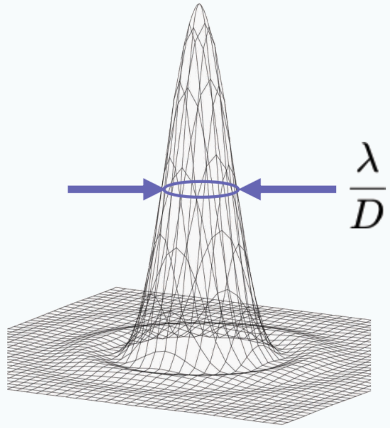


IMAGE METRIC: FULL-WIDTH AT HALF-MAXIMUM

- The width of the image at half its maximum. Often written FWHM
- For instance, a cross section of a gaussian image

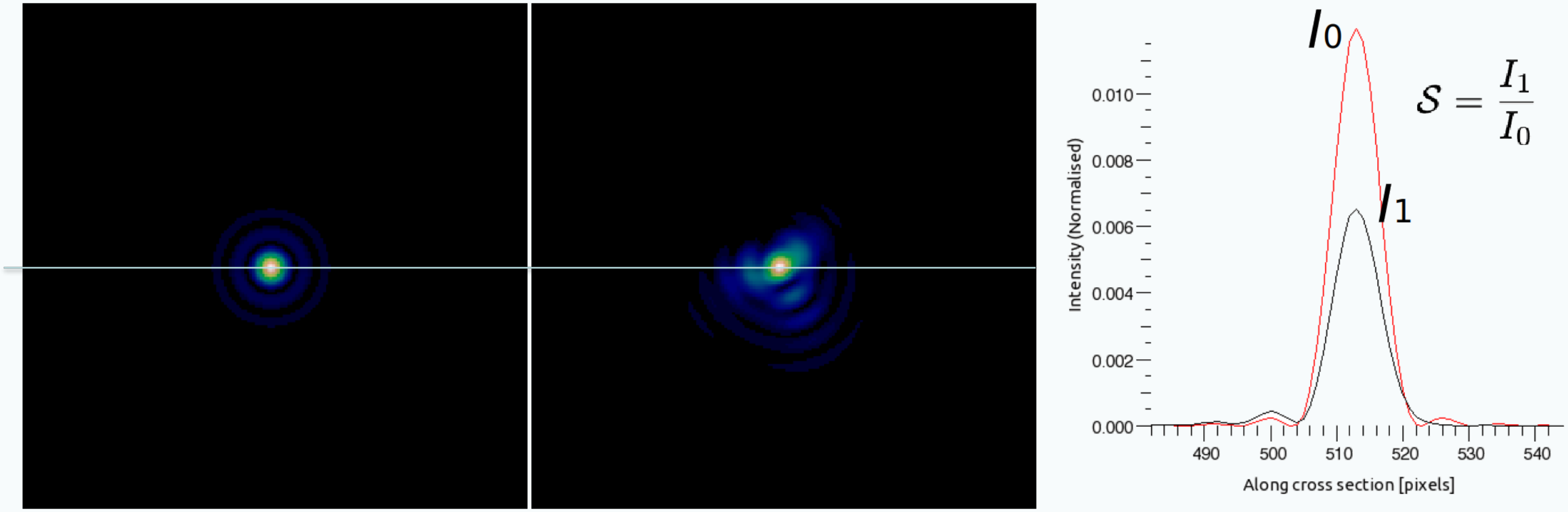




- The FWHM is often naturally expressed as an angle (e.g. arcsec) or a distance (e.g. mm), as it often characterise a resolution

IMAGE METRIC: STREHL RATIO $\mathcal{S}$

- The ratio between the maximum intensity in the actual image to the maximum in a diffraction limited image.



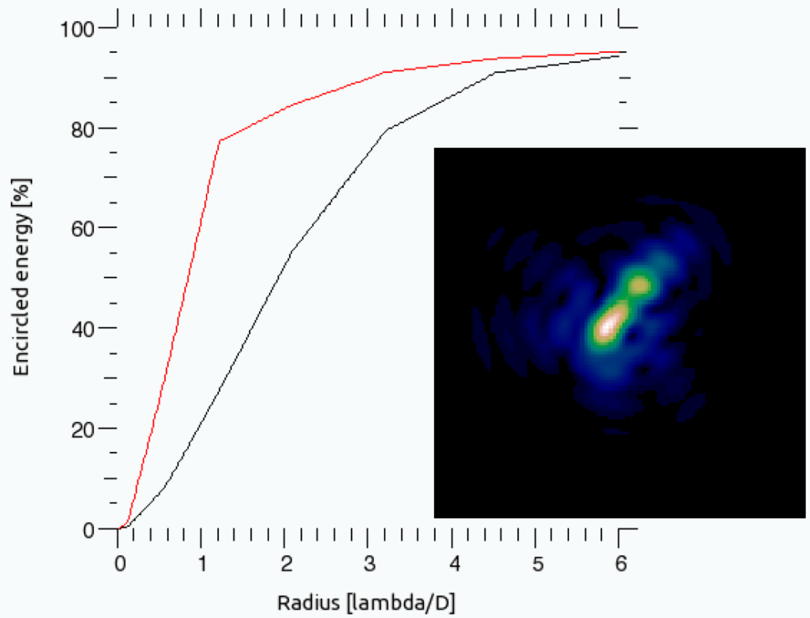
- A measure of how much energy is in the diffraction limited core
- For $\mathcal{S} \geq 0.2$, $\mathcal{S} = \exp(-\sigma_\varphi^2)$
- $0 \leq \mathcal{S} \leq 1$. The Strehl ratio is often expressed in % ($0 < \mathcal{S} < 100\%$)

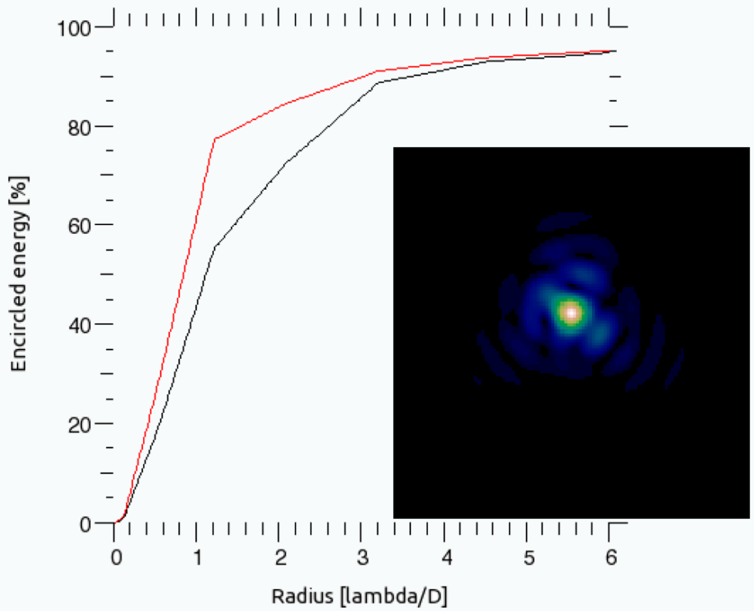
$$\sigma_\varphi^2 = \frac{1}{S} \iint_S (\varphi(x,y) - \bar{\varphi})^2 \, ds \text{ is the phase variance}$$

IMAGE METRIC: ENCIRCLED ENERGY

- Intensity within a certain radius normalised by total intensity of the image

$$\varepsilon(r) = \frac{\int_{\theta=0}^{2\pi} \int_{\rho=0}^{r} \mathcal{I}(\rho,\theta)\rho \, d\rho \, d\theta}{\int_{\theta=0}^{2\pi} \int_{\rho=0}^{\infty} \mathcal{I}(\rho,\theta)\rho \, d\rho \, d\theta}$$

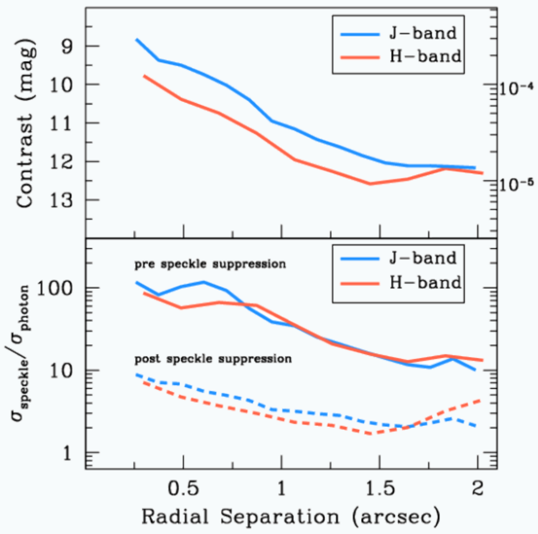




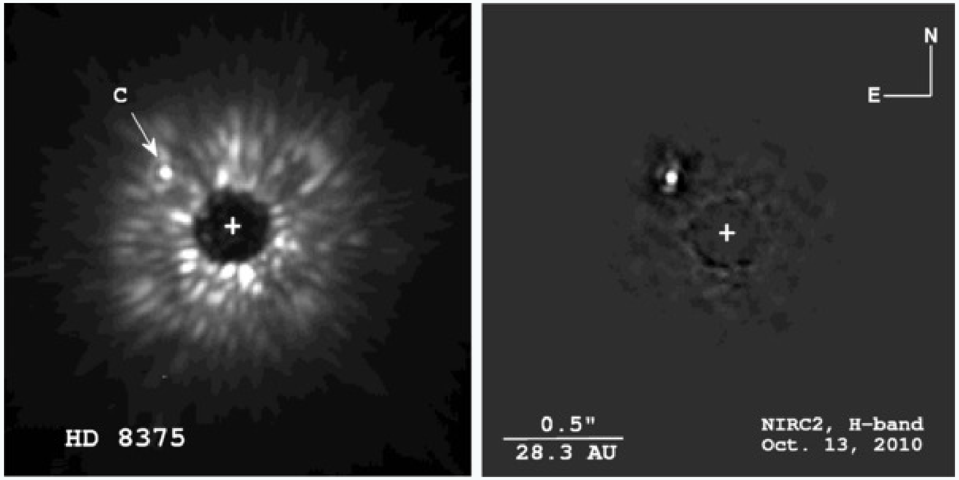
BEYOND "SIMPLE" METRICS

- High contrast imaging required the development of new metrics and new techniques to improve contrast performance
- Speckle control (next page is one of them)

Contrast plot



HD8375. J.Crepp et al 2013



HR8799. T.Currie et al 2012