Continuity & Uniform Continuity

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Continuity

A function is said to be continuous if it is 'smooth'. For example, a function for the trajectory of a ball would be continuous, however a function of someone's bank balance would be discontinuous, since large sums are taken/added suddenly, making the function not 'smooth'. It can also be said, a function is continuous if small changes in the input variable, results in small changes in the output. The formal $\epsilon - \delta$ definition:

A function f is said to be continuous at c if given any $\epsilon > 0$ there exists a $\delta > 0$ such that $\forall x \in \mathbb{R}$, with $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$. If f is not continuous at c, it is said to be discontinuous.

This statement says that if the x and c values are close (δ close to be precise), then the value of the function ($f(x)$ and $f(c)$), must also be sufficiently close. This ensures that the value around $f(c)$ stay in a small neighborhood. See the below image for an example:

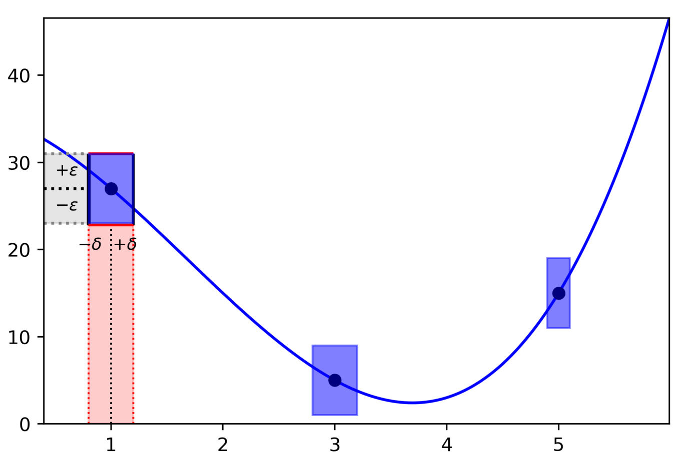

Figure 1: $f(x) = (x - 2)(x + 2)(x - 5) + 15$. ϵ is chosen as 4, δ for the first 2 points is 0.2, and 0.1 for the last.

In figure 1, the $\epsilon = 4$. If the function is continuous, there must be a δ that works for any point c. It can be seen that the $\delta = 0.2$ works for the first two examples, however, for the last, δ was reduced to ensure $|f(x) - f(c)| < \epsilon$, this is because the function starting growing faster. Below are some examples of how to prove continuity.

Example x^2

We can use this definition to prove that a function is continuous. Take x^2, which we know to be a smooth function, so intuitively it would be continuous. To prove it, take any $c \in \mathbb{R}$, and given any $\epsilon > 0$, let $0 < |x - c| < \delta$ then we have,

$$\begin{aligned}
|f(x) - f(c)| &= \left|x^2 - c^2\right| \\
&= |(x - c)(x + c)| \\
&< \delta\, |x - c + 2c| \\
&\leq \delta(|x - c| + 2\,|c|) \\
\text{Choosing } \delta = \min\{1, \frac{\epsilon}{1 + 2\,|c|}\}, \\
&< \delta(1 + 2\,|c|) \\
&= \frac{\epsilon}{1 + 2\,|c|}(1 + 2\,|c|) \\
&= \epsilon
\end{aligned}$$

Note that δ may depend on both ϵ (how close $f(x)$ and $f(c)$ values may be) AND c i.e. the point of interest. This is not the case for uniform continuity.

Example $\sin(x)$

The method is similar to above. Given any $\epsilon > 0$, let $0 < |x - c| < \delta$, then choosing $\delta = \epsilon$, will give us:

$$\begin{aligned}
|\sin(x) - \sin(c)| &= \left|2\cos(\frac{x + c}{2})\sin(\frac{x - c}{2})\right| \\
&\leq 2\left|\sin(\frac{x - c}{2})\right| \quad (|\cos(\alpha)| \leq 1) \\
&\leq 2\left|\frac{x - c}{2}\right| \quad (|\sin(x)| \leq |x|\,,\ \forall x) \\
&< \delta \\
&= \epsilon
\end{aligned}$$

$\therefore$ $|\sin(x) - \sin(c)| < \epsilon$, and $\sin(x)$ is continuous everywhere on the $\mathbb{R}$.
Note here that δ does not depend on c.

Example $\frac{1}{x}(x > 0)$

Here $\frac{1}{x}$ will be continuous for all $x \neq 0$, since $f(x)$ is not defined for $x = 0$. Given any $\epsilon > 0$, let $0 < |x - c| < \delta$, and choosing $\delta = \frac{\epsilon c^2}{1+\epsilon c}$ gives,

$$\begin{aligned}
|f(x) - f(c)| &= \left|\frac{1}{x} - \frac{1}{c}\right| \\
&= \frac{|c - x|}{|xc|} \\
&< \frac{\delta}{c(c - \delta)} \quad \text{Noting that equivalently, } |x - c| = |c - x| < \delta \\
&= \epsilon
\end{aligned}$$

Hence, $\frac{1}{x}$ is continuous $\forall x > 0$. In fact, $\frac{1}{x}$ is continuous everywhere, except $x = 0$, where it isn't even defined.

Uniform Continuity

Uniform continuity is a more restrictive form of continuity in that it not only means the fucntion is 'smooth', but doesn't grow extremely fast. This requires δ to be independent of our point of interest 'c'. Hence, restricting our δ to be constant given a choice of ϵ. Formally:

$$\text{For any } \epsilon > 0, \exists \delta > 0, \text{ s.t. } \forall x, y \in \mathbb{R} \text{ with } |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

The only difference in definition is that y, which is now a variable, is included instead of the constant c.

Example x^2

Before we saw x^2 was continuous, however δ depended on c, so the correct guess would be that x^2 isn't uniformly continuous. If we can show that given a $\epsilon > 0$ and $|x - y| < \delta$ (for any choice of δ), we get $|f(x) - f(y)| > \epsilon$, then f will not be uniformly continuous. The proof is similar to the previous examples, however we will choose x and y to be sequences, which can be thought of as just a subset of all the points x and y.

$$\text{Take } \epsilon = 2, x_n = n, \text{ and } y_n = n + \frac{1}{n}.$$

$$\text{Then for any choice of } \delta \text{ we can choose an n big enough, so that } |x_n - y_n| = \left|\frac{1}{n}\right| < \delta.$$

$$\text{However, } |f(x_n) - f(y_n)| = \left|n^2 - n^2 - 2 - \frac{1}{n^2}\right| = \left|2 + \frac{1}{n^2}\right| > 2 = \epsilon$$

Contradiction. We chose an ϵ, where there did not exist a δ that could squeeze x and y close enough that $|f(x) - f(y)| < \epsilon$. note that x^2 is an example of a function that is continuous but not uniformly continuous (on $\mathbb{R}$), however any bounded set will make x^2 uniformly continuous. More generally, by the Heine-Borel theorem: any continuous function is uniformly continuous on a closed and bounded set.

Example $\sin(x)$

Since we found that δ didn't depend on c, $\sin(x)$ must be uniformly continuous. Which of course also makes sense, since 'sin' is cyclical, and bounded from above and below. The proof is therefore exactly the same as with continuity, just replace c with y.

Example $\frac{1}{x}$

Before, we found that $\frac{1}{x}$ was continuous (cts.) on $\mathbb{R}\backslash\{0\}$, but δ depended on c so we would correctly say that $\frac{1}{x}$ is not uniformly cts. Lets look at the interval $(0,1)$ where $\frac{1}{x}$ is known to be cts.:

$$\begin{aligned}
&\text{Take } \epsilon = \frac{1}{2}, x_n = \frac{1}{n}, y_n = \frac{1}{n+1}, \\
&\text{Then, for large enough } n, |x_n - y_n| = \left|\frac{1}{n(n+1)}\right| < \delta, \text{ for any } \delta \\
&\text{But, } |f(x_n) - f(y_n)| = 1 > \frac{1}{2} = \epsilon \quad \text{Contradiction.}
\end{aligned}$$

Hence, $\frac{1}{x}$ is not uniformly cts. on the interval $(0,1)$.

Note:
There is an important distinction between a closed and open interval. Heine-Borel theorem states that the interval which f is defined on must be **closed and bounded**. On the interval $(0,1)$, $\frac{1}{x}$ is cts. and the interval is bounded. So we have a cts. fucntion on a bounded interval, it must be uniformly cts. right? No, since the function grows infinitely when approaching 0, uniform continuity doesn't hold true. We can easily see, that on the interval $[0,1]$, $\frac{1}{x}$ is not cts., hence using the Heine-Borel theorem, we make the correct conclusion that $\frac{1}{x}$ is not uniformly cts. This is why we must have a closed interval, as well as the other conditions.

Code Used to Produce Plot

```
#graph (x-2)(x+2)(x-5)+15
#add in the boxes for epsilon-delta, allowing delta to change
#Add 3 of these boxes
#Box at 1, 3, 5
#window [0,6]

import matplotlib.pyplot as plt
import matplotlib.patches as patches
import numpy as np
#Function
x = np.arange(0.4, 6, 0.01)
y = (x-2)*(x+2)*(x-5)+15


fig = plt.figure()
axis = plt.gca()
plt.plot(x, y, color='blue')
plt.plot([1,3,5], [27,5,15], 'o', color='black')

#The first point and the lines surrounding the coloured rectangle
plt.vlines(x=0.8, ymin= 22.8, ymax= 31, color = 'black', linewidth= 2)
plt.vlines(x=1.2, ymin= 22.8, ymax= 31, color = 'black')
plt.vlines(x=1, ymin= 0, ymax= 22.8, color = 'black', linewidth= 1, linestyle= ':')
plt.vlines(x=0.8, ymin= 0, ymax= 22.8, color = 'red', linewidth= 1, linestyle= ':')
plt.vlines(x=1.2, ymin= 0, ymax= 22.8, color = 'red', linewidth= 1, linestyle= ':')
axis.fill_between(x, 0, 22.8, where = (x<1.21) & (x>0.8), alpha=0.2, color='red')


plt.hlines(y=22.8, xmin = 0.8, xmax = 1.2, color='red')
plt.hlines(y=31, xmin = 0.8, xmax = 1.2, color='red')
plt.hlines(y=27, xmin = 0.4, xmax = 0.8, color='black', linestyle=':')
plt.hlines(y=31, xmin = 0.4, xmax = 0.8, color='grey', linestyle=':')
plt.hlines(y=23, xmin = 0.4, xmax = 0.8, color='grey', linestyle=':')
axis.fill_between(x, 23, 31, where = (x<0.8) & (x>0.4), alpha=0.2, color='grey')

#The annotations
axis.annotate('$-\delta$  +$\delta$', xy = (1,27), xytext = (0.70, 20), size=9)
axis.annotate('$-\epsilon$', xy = (0.5,27), xytext = (0.50, 24.5), size=9)
axis.annotate('$+\epsilon$', xy = (0.5,27), xytext = (0.50, 28.5), size=9)

#Rectangles added around the 3 points
plt.gca().add_patch(patches.Rectangle((1-0.2,27-4), 0.4, 8, fill='true',
                                      color='blue', alpha = 0.5,
                                      zorder = 100, figure=fig))
plt.gca().add_patch(patches.Rectangle((3-0.2,5-4), 0.4, 8, fill='true',
                                      color='blue', alpha = 0.5,
                                      zorder = 100, figure=fig))
plt.gca().add_patch(patches.Rectangle((5-0.1,15-4), 0.2, 8, fill='true',
                                      color='blue', alpha = 0.5,
                                      zorder = 100, figure=fig))
#margins
plt.margins(y=0, x=0)
plt.savefig('epsilon-delta-visual.png', dpi=400)
plt.show()
```